Graph theory

Class Test 2

24 January 2024

The class test consists of two parts: "Exercises" and "Questions".
*Please attempt **BOTH** Exercises and **TWO** Questions.*

Refer to any results you are using by name
(or state them if you don't remember their name).

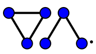
Duration of the class test: 90 minutes.

Questions

Please attempt ***TWO*** *Questions (out of 3).*

Each Question is worth 7 points.

Question 1

Show that $R(H, K_3) = 8$, where H is a disjoint union of K_3 and P_2, i.e. the graph .

[*Recall*: the Ramsey number $R(G, H)$ is the smallest integer $n \geq 1$ such that any blue/orange edge colouring of K_n contains a blue G or an orange H.]

Question 2

Given two graphs G and H with non-empty vertex sets, their *product* is the graph $G \cdot H$ with vertex set $V(G \cdot H) = V(G) \times V(H)$, such that two vertices $(v, v'), (w, w') \in G \cdot H$ are adjacent if and only if either $v \sim_G w$ and $v' \sim_H w'$, or $v = w$ and $v' \sim_H w'$, or $v \sim_G w$ and $v' = w'$. Show that for any graphs G and H we have $\alpha(G \cdot H) \leq R(\alpha(G) + 1, \alpha(H) + 1) - 1$. By considering $C_5 \cdot C_5$, show also that $\leq$ cannot be replaced with $<$.

[*Recall*: the independence number $\alpha(G)$ of G is the largest cardinality of an independent subset $A \subseteq V(G)$, i.e. a subset A such that no two vertices of A are adjacent.]

Question 3

Let $p \in (0, 1)$ be a constant, and let $k \colon \mathbb{N} \to \mathbb{N}$ be such that $k(n) = o(n/\ln(n))$. By considering common neighbours of pairs of vertices in G, show that almost every $G \in \mathcal{G}(n, p)$ is $k(n)$-connected.

Your name: ....

Exercises

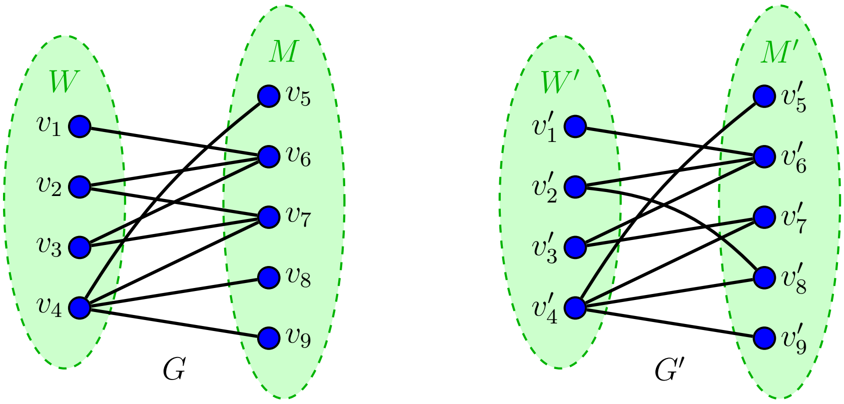
*Please attempt **BOTH** Exercises (A and B) displayed below.*

Wherever explanations are needed, please give precise *reasons*
(by explicitly writing down specific subgraphs, collections of vertices/edges, etc).

The Exercises are worth 6 points in total.

Exercise A

Consider the following bipartite graphs G and G':



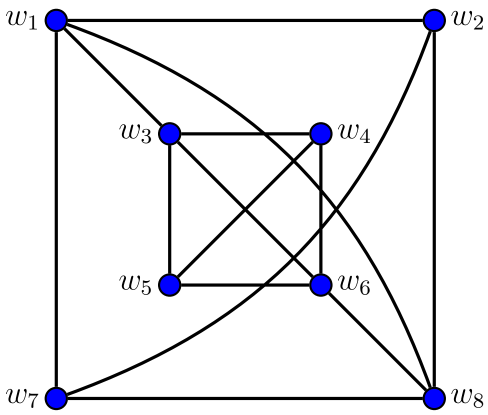
Decide whether or not each of these graphs has a matching between the given vertex classes. Explain your answers.

Solution for G (from W to M):	Solution for G' (from W' to M'):

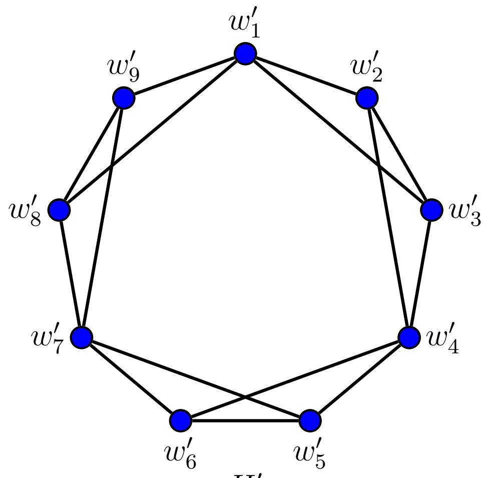
[Please turn over]

Exercise B

Consider the following graphs H and H':



H



H'

For each of these graphs, write down the value of the largest $k \geq 0$ such that the graph is k-edge-connected, and give a reason why it is not $(k+1)$-edge-connected.

Solution for H:	Solution for H':