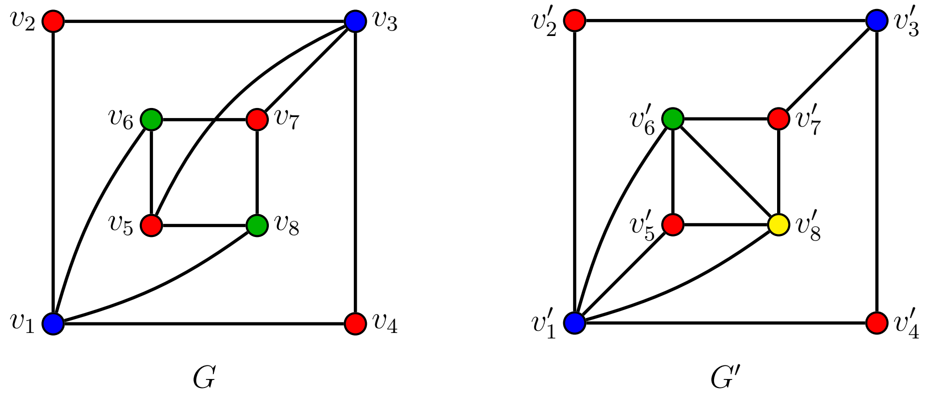
Graph theory

Class Test 1

Solutions

Exercise A

Consider the following graphs G and G':



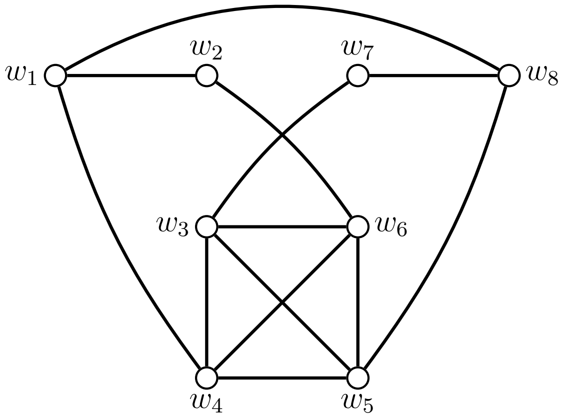
Find the chromatic numbers of both of these graphs. Explain your answers.

Solution for G:
$\boldsymbol{\chi(G) = 3}$**:** the colouring above shows that $\chi(G) \leq 3$.
On the other hand, G contains a cycle $v_1v_2v_3v_7v_6v_1$ of odd length, and so G is not bipartite, i.e. $\chi(G) \geq 3$.

Solution for G':
$\boldsymbol{\chi(G') = 4}$**:** the colouring above shows that $\chi(G') \leq 4$.
On the other hand, vertices v'_1, v'_5, v'_6 and v'_8 span a complete subgraph $H \cong K_4$ of G', and so $\chi(G') \geq \chi(H) = 4$.

Exercise B

Consider the following graph H:



H

Determine whether or not H is Hamiltonian and/or Eulerian. Explain your answers.

[*Recall that a graph G is said to be* Eulerian *if there exists a closed walk in G passing through each edge exactly once.*]

Is H Hamiltonian?

Yes: the cycle $w_1w_2w_6w_3w_7w_8w_5w_4w_1$ is a Hamilton cycle in H.

Is H Eulerian?

No: H has a vertex of odd degree, e.g. $d(w_1) = 3$, so H cannot be Eulerian by the characterisation of connected Eulerian graphs.

Question 1

Let G be a bipartite graph with vertex classes W and M, such that $|W| = |M|$. Show that the following statements are equivalent:

(i) for all $A \subseteq V(G)$, the graph $G - A$ has at most $|A|$ isolated vertices (i.e. vertices of degree 0);

(ii) for all $A \subseteq V(G)$, the graph $G - A$ has at most $|A|$ connected components of odd order;

(iii) G has a matching from W to M.

Solution: We will show (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii): Let $B \subseteq W$, and let $A = N_G(B)$. Then every vertex of B is an isolated vertex of the graph $G - A$, implying by our assumption that $|B| \leq |A|$. Therefore, we have $|B| \leq |N_G(B)|$ for all $B \subseteq W$, and so (G, W) satisfies the Hall's Condition. By Hall's Marriage Theorem, the graph G then has a matching from W to M, as required.

(iii) $\Rightarrow$ (ii): Let $P = \{wv_w \mid w \in W\} \subseteq E(G)$ be a matching from W to M in G. Since $|W| = |M|$ and since $v_w \neq v_{w'}$ whenever $w, w' \in W$ and $w \neq w'$, it follows by the pidgeonhole principle that every vertex of M is "covered" by P, i.e. for all $u \in M$ there exists a (unique) $w \in W$ such that $u = v_w$. In particular, every vertex of G is an endpoint of a unique edge in P.

Now let $A \subseteq V(G)$, and consider the set

$$U = \{u \in G - A \mid ut \in P \text{ for some } t \in A\}.$$

Since P consists of independent edges, it is clear that $|U| \leq |A|$. Moreover, if we have $ut \in P$ for some $u, t \in (G - A) - U$, then $u \sim_{G-A} t$ and so u and t belong to the same connected component of $G - A$. It follows that if a connected component C of $G - A$ has no vertices in U, then $V(C)$ is a disjoint union of sets of the form $\{u, t\}$ for edges $ut \in P$ (since every vertex of G is incident to a unique edge of P), implying that C has even order. In particular, if $\mathcal{C}$ is the set of connected components of $G - A$ and the map $f: U \to \mathcal{C}$ sends $u \in U$ to the component containing u, then $f(U)$ contains every component of $G - A$ of odd order. This shows that $G - A$ has at most $|f(U)| \leq |U| \leq |A|$ connected components of odd order, as required.

(ii) $\Rightarrow$ (i): Since any isolated vertex of a graph forms a connected component of order 1 (so, in particular, a connected component of odd order), the number of isolated vertices of $G - A$ is bounded above by the number of connected components of $G - A$ of odd order. This implies the result.

Question 2

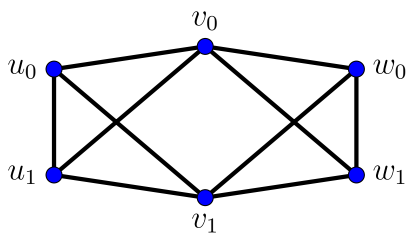
Let $k \geq 0$, and let G be an incomplete graph. Show that if $\Delta(G) \leq 3$, then G is k-connected if and only if G is k-edge-connected. Give an example (with justification) of a graph G with $\Delta(G) = 4$ such that G is 3-edge-connected but not 3-connected.

Solution: Suppose first that $\Delta(G) \leq 3$. Since G is incomplete, by Menger's Theorem, G is k-connected if and only if for all $v, w \in G$ with $v \neq w$ there exists a collection of k independent (v, w)-paths in G. Furthermore, by the edge version of Menger's Theorem, G is k-edge-connected if and only if for all $v, w \in G$ with $v \neq w$ there exists a collection of k edge-disjoint (v, w)-paths in G. It is thus enough to show that, given $v, w \in G$ with $v \neq w$ and a collection of edge-disjoint (v, w)-paths in G, this collection of paths is independent if and only if it is edge-disjoint.

Thus, let $v, w \in G$ with $v \neq w$ and let $\mathcal{Q}$ be a collection of edge-disjoint (v, w)-paths in G. If $\mathcal{Q}$ is a collection of independent paths, then it is clearly also a collection of edge-disjoint paths.

Conversely, suppose the paths in $\mathcal{Q}$ are not independent, and so there exist two paths $P, P' \in \mathcal{Q}$ and a vertex $u \notin \{v, w\}$ such that $u \in V(P) \cap V(P')$. Then we can write $P = v \cdots u_1 u u_2 \cdots w$ and $P' = v \cdots u_3 u u_4 \cdots w$ for some vertices $u_1, u_2, u_3, u_4 \in N_G(u)$. Since $|N_G(u)| = d_G(u) \leq \Delta(G) \leq 3$, it follows that $u_i = u_j$ for some $i \neq j$. However, since P is a path we have $u_1 \neq u_2$, and since P' is a path we have $u_3 \neq u_4$. It thus follows that $u_i = u_j$ for some $i \in \{1, 2\}$ and $j \in \{3, 4\}$, and therefore $\{u_1 u, u_2 u\} \cap \{u_3 u, u_4 u\} \neq \varnothing$. This implies that $E(P) \cap E(P') \neq \varnothing$ and therefore the paths in $\mathcal{Q}$ are not edge-disjoint, as required.

Now consider the following graph G with $\Delta(G) = 4$:



It is clear that $G - \{v_0, v_1\}$ is a disjoint union of the edges $u_0 u_1$ and $w_0 w_1$, and is therefore not connected; thus G is not 3-connected. However, we claim that if $E' \subseteq E(G)$ with $|E'| \leq 2$, then $G - E'$ is still connected, and therefore G is 3-edge-connected. Indeed, for any $i, j \in \{0, 1\}$, the three (u_i, v_j)-paths $u_i v_j$, $u_i u_{1-i} v_j$ and $u_i v_{1-j} w_0 v_j$ are edge-disjoint, meaning that at least one of these three paths survives in $G - E'$; therefore u_i and v_j are in the same connected component of $G - E'$. Similarly, w_i and v_j are in the same connected component of $G - E'$. In particular, v_0 is in the same connected component of $G - E'$ as u_0, u_1, w_0 and w_1, which are in the same connected component of $G - E'$ as v_1; thus $G - E'$ is connected, as required.

Question 3

Show that $t_r(n) \le \frac{n^2}{2}(1-\frac{1}{r})$ for any $n \ge r \ge 1$. Deduce that any graph G of order n has a complete subgraph of order $\ge \frac{n}{n-d(G)}$. Use this to show that any graph H of order n has an independent subset $A \subseteq V(H)$ such that $|A| \ge \frac{n}{d(H)+1}$, where we define a subset $A \subseteq V(H)$ to be *independent* if $e(H[A]) = 0$.

Solution: We can write $n = qr + s$, where $q = \lfloor n/r \rfloor$, so that $0 \le s < r$. Then the Turán graph $T_r(n)$ has s large vertex classes containing $q+1$ vertices of degree $n-q-1$ each, and $r-s$ small vertex classes containing q vertices of degree $n-q$ each (if $s=0$ then we consider all vertex classes to be small). Therefore, using the Handshaking Lemma we get

$$\begin{aligned} 2t_r(n) &= \sum_{v \in T_r(n)} d(v) = s(q+1)(n-q-1) + (r-s)q(n-q) \\ &= s(q+1)(n-q) - s(q+1) + (r-s)q(n-q) \\ &= (sq + s + rq - sq)(n-q) - s(q+1) = (qr+s)(n-q) - s(q+1) \\ &= n(n-q) - s(q+1) = n^2 - \big[(n+s)q + s\big]. \end{aligned}$$

Furthermore, since $q = \frac{n-s}{r}$ and $0 \le s < r$, we obtain

$$(n+s)q + s = \frac{(n+s)(n-s)}{r} + \frac{sr}{r} = \frac{n^2 - s^2 + sr}{r} = \frac{n^2 + s(r-s)}{r} \ge \frac{n^2}{r}$$

and therefore $2t_r(n) \le n^2 - \frac{n^2}{r}$. This implies that $t_r(n) \le \frac{n^2}{2}(1-\frac{1}{r})$, as required.

Now let G be a graph of order n, and let $r = \lceil \frac{n}{n-d(G)} \rceil - 1$. We then have $r < \frac{n}{n-d(G)}$, implying (using the Handshaking Lemma) that

$$2e(G) = n \cdot d(G) = n^2 - (n - d(G))n > n^2 - \frac{n}{r}n = n^2\left(1 - \frac{1}{r}\right),$$

and therefore $e(G) > \frac{n^2}{2}(1-\frac{1}{r}) \ge t_r(n)$ by the argument above. Hence, by Turán's Theorem, G contains a complete subgraph of order $r+1 = \lceil \frac{n}{n-d(G)} \rceil \ge \frac{n}{n-d(G)}$, as required.

Finally, let H be a graph of order n and let $G = \overline{H}$, the complement of H. We then have $e(H) = \binom{n}{2} - e(G) = \frac{n}{2}(n-1) - e(G)$ and therefore $d(H) = \frac{2e(H)}{n} = (n-1) - \frac{2e(G)}{n} = n - 1 - d(G)$. By the argument above, there exists a subset $A \subseteq V(G) = V(H)$ such that $G[A]$ is complete and $|A| \ge \frac{n}{n-d(G)} = \frac{n}{d(H)+1}$. Since $G[A]$ is complete, it follows by the definition of $G = \overline{H}$ that $H[A]$ has no edges and so A is independent in H, as required.