Graph theory

Class Test 1

13 December 2023

The class test consists of two parts: "Exercises" and "Questions".
Please attempt ***BOTH*** *Exercises and* ***TWO*** *Questions.*

Refer to any results you are using by name
(or state them if you don't remember their name).

Duration of the class test: 90 minutes.

QUESTIONS

*Please attempt **TWO** Questions (out of 3).*

Each Question is worth 7 points.

Question 1

Let G be a bipartite graph with vertex classes W and M, such that $|W| = |M|$. Show that the following statements are equivalent:

(i) for all $A \subseteq V(G)$, the graph $G - A$ has at most $|A|$ isolated vertices (i.e. vertices of degree 0);

(ii) for all $A \subseteq V(G)$, the graph $G - A$ has at most $|A|$ connected components of odd order;

(iii) G has a matching from W to M.

Question 2

Let $k \geq 1$, and let G be an incomplete graph. Show that if $\Delta(G) \leq 3$, then G is k-connected if and only if G is k-edge-connected. Give an example (with justification) of a graph G with $\Delta(G) = 4$ such that G is 3-edge-connected but not 3-connected.

Question 3

Show that $t_r(n) \leq \frac{n^2}{2}(1 - \frac{1}{r})$ for any $n \geq r \geq 1$. Deduce that any graph G of order n has a complete subgraph of order $\geq \frac{n}{n-d(G)}$. Use this to show that any graph H of order n has an independent subset $A \subseteq V(H)$ such that $|A| \geq \frac{n}{d(H)+1}$, where we define a subset $A \subseteq V(H)$ to be *independent* if $e(H[A]) = 0$.

Your name: ....

EXERCISES

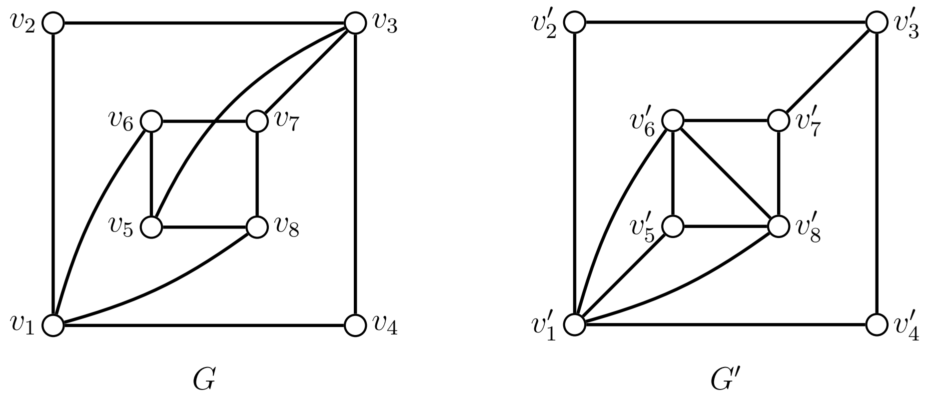
*Please attempt **BOTH** Exercises (A and B) displayed below.*

Wherever explanations are needed, please give precise *reasons*
(by explicitly writing down specific subgraphs, collections of vertices/edges, etc).

The Exercises are worth 6 points in total.

Exercise A

Consider the following graphs G and G':



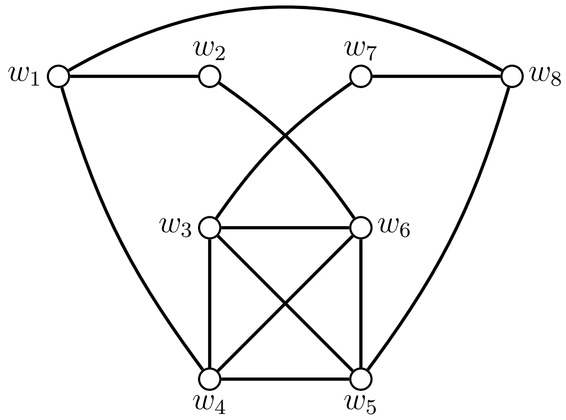
Find the chromatic numbers of both of these graphs. Explain your answers.

Solution for G:	Solution for G':

[Please turn over]

Exercise B

Consider the following graph H:



H

Determine whether or not H is Hamiltonian and/or Eulerian. Explain your answers.

[*Recall that a graph G is said to be* Eulerian *if there exists a closed walk in G passing through each edge exactly once.*]

Is H Hamiltonian?	Is H Eulerian?