SIAM J. MATH. ANAL.
Vol. 23, No. 4, pp. 965-969, July 1992

011

ORTHOGONAL POLYNOMIALS AND A DISCRETE BOUNDARY VALUE PROBLEM II*

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Abstract. Let $\{P_n\}_{n=0}^{\infty}$ be a system of polynomials orthogonal with respect to a measure μ on the real line. Then P_n satisfy the three-term recurrence formula $xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}$. Conditions are given on the sequence α_n, β_n, and γ_n under which any product $P_n P_m$ is a linear combination of P_k with positive coefficients. The result is applied to the measures $d\mu(x) = (1-x^2)^{\alpha}|x|^{2\beta+1}\,dx$ and $d\mu(x) = |x|^{2\alpha+1}e^{-x^2}\,dx$, α, $\beta > -1$. As a corollary, a Gasper result is derived on the Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $\alpha \geqq \beta$ and $\alpha + \beta + 1 \geqq 0$.

Key words. orthogonal polynomials, recurrence formula

AMS(MOS) subject classifications. primary 33A65, 39A70

The present paper is a continuation of our earlier work [9]. We were concerned in part I with the following question. Given a probability measure μ on the real line $\mathbb{R}$ such that all its moments are finite, let $\{P_n\}_{n=0}^{\infty}$ be a system of orthogonal polynomials obtained from the sequence of consecutive monomials $1, x, x^2, \cdots$ by the Gram-Schmidt procedure. We do not impose any special normalization upon P_n except that its leading coefficient be positive. The product $P_n P_m$ is a polynomial of degree $n+m$ and it can be expressed as

$$P_n P_m = \sum_{k=|n-m|}^{n+m} c(n, m, k) P_k \tag{1}$$

with some real coefficients $c(n, m, k)$. We are asking when $c(n, m, k)$ are nonnegative for any $n, m, k \in \mathbb{N}$. The coefficients $c(n, m, k)$ from (1) are called the *linearization coefficients* of $\{P_n\}$ and if they are nonnegative we simply say that the linearization coefficients are nonnegative.

It is well known that P_n that P_n obey a three-term recurrence formula of the form

$$xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}, \tag{2}$$

where α_n, γ_n are positive, except $\alpha_0 = 0$, and β_n are real. In [9, Thm. 1], we proved that if $\{\alpha_n\}$, $\{\beta_n\}$, $\{\alpha_n + \gamma_n\}$ are increasing sequences and $\gamma_n \geqq \alpha_n$, for $n = 0, 1, 2, \cdots$, then the linearization coefficients of $\{P_n\}$ are nonnegative.

Our aim now is to get rid in some way of the condition of the monotonicity of the sequence $\{\beta_n\}$. Roughly the idea consists in reducing the problem to the case $\beta_n = 0$. This can be done in the following way. Consider first polynomials P_n satisfying

$$xP_n = \gamma_n P_{n+1} + \alpha_n P_{n-1}, \qquad P_0 = 1. \tag{3}$$

Then, of course, P_{2n} are even functions while P_{2n+1} are odd ones. Equivalently, this means that the corresponding measure, which orthogonalizes $\{P_n\}$ (and which exists by the Favard theorem [5]) is symmetric with respect to zero. An easy calculation gives the following:

$$\begin{aligned} x^2 P_{2n}(x) = {} & \gamma_{2n+1}\gamma_{2n} P_{2n+2}(x) + (\alpha_{2n+1}\gamma_{2n} + \alpha_{2n}\gamma_{2n-1}) P_{2n}(x) \\ & + \alpha_{2n}\alpha_{2n-1} P_{2n-2}(x). \end{aligned} \tag{4}$$

* Received by the editors May 1, 1990; accepted for publication August 16, 1991.

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Let us define the polynomials Q_n by

$$Q_n(y) = P_{2n}(\sqrt{y}). \tag{5}$$

Then by (4) the polynomials Q_n satisfy

$$yQ_n(y) = \gamma_{2n+1}\gamma_{2n}Q_{n+1}(x) + (\alpha_{2n+1}\gamma_{2n} + \alpha_{2n}\gamma_{2n-1})Q_n(x) + \alpha_{2n}\alpha_{2n-1}Q_{n-1}(x). \tag{6}$$

Observe that (6) is again a three-term recurrence formula. Moreover, if the polynomials P_n have nonnegative linearization coefficients, then by (5) the polynomials Q_n do as well.

We can go the other way around. Assume we are given a sequence of polynomials Q_n orthogonal with respect to a measure ν supported on $[0, +\infty)$. Instead of studying the Q_n we can examine the polynomials P_n satisfying (3) and (5) with regard to the question of nonnegative linearization coefficients. Those are easier to handle, because in (3) the coefficients β_n are missing, unlike in the recurrence formula for Q_n.

First we will sharpen Theorem 1 from [9] in case of symmetric measures.

THEOREM 1. *Let orthogonal polynomials P_n satisfy*

$$xP_n = \gamma_n P_{n+1} + \alpha_n P_{n-1}, \qquad n = 0, 1, 2, \cdots, \tag{7}$$

where $\alpha_0 = 0$, $\alpha_n, \gamma_n \geqq 0$. Assume that the sequences $\{\alpha_{2n}\}$, $\{\alpha_{2n+1}\}$, $\{\alpha_{2n} + \gamma_{2n}\}$, $\{\alpha_{2n+1} + \gamma_{2n+1}\}$ are increasing and $\alpha_n \leqq \gamma_n$ for $n = 0, 1, 2, \cdots$. Then the linearization coefficients of P_n are nonnegative.

Proof. As in [9], Remark 1, we can renormalize P_n (i.e., multiply each P_n by a positive number σ_n) so as to satisfy

$$xP_n = \alpha_{n+1}P_{n+1} + \gamma_{n-1}P_{n-1}. \tag{8}$$

Of course, it does not affect the conclusion of the theorem, so we introduce no new symbols for the renormalized polynomials. Let μ be a symmetric probability measure that orthogonalizes the polynomials P_n. Then by (1)

$$c(n, m, k) \int P_k^2 \, d\mu = \int P_n P_m P_k \, d\mu. \tag{9}$$

Hence the quantity $c(n, m, k) \int P_k^2 \, d\mu$ is invariant under permutations of n, m, k. Since μ is symmetric, then $c(n, m, k) = 0$ if n, m, k are all odd numbers. Thus if $c(n, m, k) \neq 0$ then one of n, m, k is an even number. By invariance, we can always assume that k is such. Collecting all of the above it suffices to show that in the formulas

$$\begin{aligned} P_{2n}P_{2m} &= \sum c(2n, 2m, 2k)P_{2k}, \\ P_{2n+1}P_{2m+1} &= \sum c(2n+1, 2m+1, 2k)P_{2k} \end{aligned} \tag{10}$$

the coefficients $c(2n, 2m, 2k)$ and $c(2n+1, 2m+1, 2k)$ are nonnegative. It automatically implies that they are also nonnegative in the formula

$$P_{2n+1}P_{2m} = \sum c(2n+1, 2m, 2k+1)P_{2k+1}. \tag{11}$$

Let L be the linear operator acting on the sequences $\{a_n\}_{n=0}^{\infty}$ by

$$La_n = \alpha_{n+1}a_{n+1} + \gamma_{n-1}a_{n-1}. \tag{12}$$

Let L_n and L_m denote the linear operators acting on the matrices $\{u(n, m)\}_{n,m=0}^{\infty}$ as the operator L does but according to the n or m variable (cf. [9]). Fix $k \in \mathbb{N}$ and consider the matrix $u(n, m) = c(n, m, k)$. By (8) and (9) (cf. [9]) we have $(L_n - L_m)u = 0$. Moreover, $u(n, 0) = 1$ for $n = 2k$ and $u(n, 0) = 0$ otherwise. Hence the following maximum principle would complete the proof.

LEMMA 1. *Let the matrix* $u(n, m)$, $n, m = 0, 1, 2, \cdots$ *satisfy*

$$(13)\qquad \begin{gathered}(L_n - L_m)u = 0\\ u(2n, 0) \geqq 0, \quad u(2n+1, 0) = 0, \quad n = 0, 1, 2, \cdots.\end{gathered}$$

Then (*under the assumptions of Theorem* 1) $u(n, m) \geqq 0$ *for* $n \geqq m$.

For the proof of Lemma 1 we refer the reader to [9] (the proof of Theorem 3). It suffices to observe that (10) and (11) imply $u(n, m) = 0$ whenever $n + m$ is an odd number. Hence, scanning the proof of Theorem 3 from [9], we can observe that the coefficients $c_{s,t}$, which are computed there, have the property that $s + r$ is an even number.

Combining Theorem 1, (4), (5), and (6) immediately gives the following corollary.

COROLLARY 1. *Let the orthogonal polynomials* $Q_n(y)$ *satisfy the recurrence formula*

$$yQ_n = \tilde{\gamma}_n Q_{n+1} + \tilde{\beta}_n Q_n + \tilde{\alpha}_n Q_{n-1}.$$

Assume that there exist sequences α_n, γ_n *of nonnegative numbers* ($\alpha_0 = 0$) *and a real constant* β *such that*

$$(14)\qquad \tilde{\gamma}_n = \gamma_{2n+1}\gamma_{2n}, \quad \tilde{\alpha}_n = \alpha_{2n}\alpha_{2n-1}, \quad \tilde{\beta}_n = \alpha_{2n+1}\gamma_{2n} + \alpha_{2n}\gamma_{2n-1} + \beta,$$

and α_n, γ_n *satisfy the assumptions of Theorem* 1. *Then the linearization coefficients of* Q_n *are nonnegative.*

Before giving applications of Corollary 1 let us study the relation between orthogonal polynomials P_n and Q_n connected by (3) and (5). Let μ be a measure that orthogonalizes the polynomials P_n. Then

$$0 = \int_{-\infty}^{+\infty} P_{2n}(x)P_{2m}(x)\,d\mu(x) = 2\int_0^{+\infty} P_{2n}(x)P_{2m}(x)\,d\mu(x)$$
$$= 2\int_0^{+\infty} Q_n(y)Q_m(y)\,d\mu(\sqrt{y}).$$

Hence Q_n are orthogonal with respect to the measure $d\nu(y) = 2d\mu(\sqrt{y})$, $y \geqq 0$. Note that the measure μ can be recovered back from ν by $d\mu(x) = \frac{1}{2}d\nu(x^2)$, $x \geqq 0$, and $d\mu(-x) = d\mu(x)$.

It is worthwhile to look at the polynomials R_n defined by

$$S_n(y) = \frac{1}{\sqrt{y}} P_{2n+1}(\sqrt{y}).$$

Then

$$0 = \int_{-\infty}^{+\infty} P_{2n+1}(x)P_{2m+1}(x)\,d\mu(x)$$
$$= 2\int_0^{+\infty} x^2 \frac{P_{2n+1}(x)}{x}\frac{P_{2m+1}(x)}{x}\,d\mu(x)$$
$$= 2\int_0^{+\infty} S_n(y)S_m(y)y\,d\mu(\sqrt{y}).$$

Hence the measure that orthogonalizes the S_n is $2y\,d\mu(\sqrt{y})$ or simply $y\,d\nu(y)$.

THEOREM 2. *Let* $\{P_n\}_{n=0}^{\infty}$ *be the system of polynomials orthogonal with respect to the measure* $d\mu(x) = (1-x^2)^{\alpha}|x|^{2\beta+1}\,dx$, $x \in (-1, 1)$, $\alpha, \beta > -1$. *If* $\alpha \geqq \beta$ *and* $\alpha + \beta + 1 \geqq 0$, *then the coefficients* $c(n, m, k)$ *in* $P_nP_m = \sum c(n, m, k)P_k$ *are nonnegative.*

Proof. It suffices to find a three-term recurrence formula for P_n so as to fulfill the assumptions of Theorem 1.

LEMMA 2. *The polynomials* $\{P_n\}_{n=0}^{\infty}$ *satisfying*

$$xP_{2n} = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1} P_{2n+1} + \frac{n}{2n+\alpha+\beta+1} P_{2n-1}, \tag{15}$$

$$xP_{2n-1} = \frac{n+\alpha}{2n+\alpha+\beta} P_{2n} + \frac{n+\beta}{2n+\alpha+\beta} P_{2n-2} \tag{16}$$

for $n=0,1,2,\cdots$, $(P_0=1)$ *are orthogonal with respect to the measure* $d\mu(x) = (1-x^2)^{\alpha}|x|^{2\beta+1}\,dx$.

Proof of Lemma 2. Let $R_n^{(\alpha,\beta)}(y)$ denote the Jacobi polynomials normalized by $R_n^{(\alpha,\beta)}(1)=1$. Let

$$\tilde{Q}_n(y) = R_n^{(\alpha,\beta)}(2y-1). \tag{17}$$

Then $\tilde{Q}_n$ are orthogonal with respect to the measure $d\nu(y) = (1-y)^{\alpha}y^{\beta}\,dy$. By the recurrence formula for $R_n^{(\alpha,\beta)}$ (see [6, (4) p. 172] or [4, (3) and (11), p. 169]), $\tilde{Q}_n$ satisfy

$$\begin{aligned} y\tilde{Q}_n &= \frac{(n+\alpha+\beta+1)(n+\alpha+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} \tilde{Q}_{n+1} \\ &\quad + \frac{1}{2}\left(1 + \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right)\tilde{Q}_n \\ &\quad + \frac{n(n+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)} \tilde{Q}_{n-1}. \end{aligned}$$

Let P_n be the polynomials satisfying (13). Then by (4) and (6) the polynomials $Q_n(y) = P_{2n}(\sqrt{y})$ satisfy the same recurrence formula as $\tilde{Q}_n$ do. Indeed, in both recurrence formulas the coefficients of Q_{n+1}, Q_{n-1} *and* $\tilde{Q}_{n+1}$, $\tilde{Q}_{n-1}$ coincide. Then the coefficients of Q_n, $\tilde{Q}_n$ must also coincide because in both formulas the sum of coefficients is equal to 1 (for $\tilde{Q}_n = R_n^{(\alpha,\beta)}(1) = 1$ and $Q_n(1) = P_{2n}(1) = 1$ by (14)). Hence we have just proved that $Q_n = \tilde{Q}_n$. This means Q_n are orthogonal with respect to the measure $d\nu(y) = (1-y)^{\alpha}y^{\beta}\,dy$. Thus by the reasoning of Corollary 1 the polynomials P_n are orthogonal with respect to the measure $d\mu(x) = \frac{1}{2}d\nu(x^2) = (1-x^2)^{\alpha}|x|^{2\beta+1}\,dx$, as was required.

Let us return to the proof of Theorem 2. From Lemma 1 we can easily see that if $\alpha \geqq \beta$ *and* $\alpha+\beta+1 \geqq 0$ then the assumptions of Theorem 1 are satisfied. This completes the proof.

COROLLARY 2 (Gasper [6]). *Let* $R_n^{(\alpha,\beta)}$ *be the Jacobi polynomials normalized so that* $R_n^{(\alpha,\beta)}(1)=1$. *If* $\alpha \geqq \beta$ *and* $\alpha+\beta+1 \geqq 0$ *then*

$$R_n^{(\alpha,\beta)} R_m^{(\alpha,\beta)} = \sum_{k=|n-m|}^{n+m} c(n,m,k) R_k^{(\alpha,\beta)}$$

with nonnegative coefficients $c(n,m,k)$.

Proof. Let P_n be the polynomials orthogonal with respect to the measure $d\mu(x) = (1-x^2)^{\alpha}|x|^{2\beta+1}\,dx$ and satisfying (15) and (16). Then by Theorem 2 we have $P_nP_m = \sum d(n,m,k)P_k$, where $d(n,m,k) \geqq 0$. From the proof of Lemma 2 we know that $P_{2n}(\sqrt{y}) = R_n^{(\alpha,\beta)}(2y-1)$. Hence we get $R_n^{(\alpha,\beta)}R_m^{(\alpha,\beta)} = \sum d(2n,2m,2k)R_k^{(\alpha,\beta)}$, where $d(2n,2m,2k) \geqq 0$.

COROLLARY 3. *Let* $\alpha \geqq \beta$ *and* $\alpha+\beta+1 \geqq 0$. *Then*

$$(y+1)R_n^{(\alpha,\beta+1)}R_m^{(\alpha,\beta+1)} = \sum_{k=|n-m|}^{n+m} c(n,m,k)R_k^{(\alpha,\beta)},$$

$$R_n^{(\alpha,\beta)}R_m^{(\alpha,\beta+1)} = \sum_{k=|n-m|}^{n+m} d(n,m,k)R_k^{(\alpha,\beta+1)},$$

where $c(n,m,k)$ *and* $d(n,m,k)$ *are nonnegative coefficients.*

Proof. Let P_n be the orthogonal polynomials corresponding to the measure $d\mu(x) = (1-x^2)^\alpha |x|^{2\beta+1}\,dx$. Then, as we have seen in the proof of Lemma 2, $P_{2n}(\sqrt{y}) = R_n^{(\alpha,\beta)}(2y-1)$. Let the polynomials $S_n(y)$ be defined as $S_n(y) = (1/\sqrt{y})\,P_{2n+1}(\sqrt{y})$. By the considerations following Corollary 1 we know that $S_n(y)$ are orthogonal with respect to the measure $2y\,d\mu(\sqrt{y}) = (1-y)^\alpha y^{\beta+1}\,dy$ and $S_n(1)=1$. This yields $S_n(y) = R_n^{(\alpha,\beta)}(2y-1)$. Now both required formulas coincide with (10) and (11). The latter have nonnegative coefficients if $\alpha \geqq \beta$ and $\alpha+\beta+1 \geqq 0$.

Now we turn to the so called generalized Hermite polynomials.

THEOREM 3. *Let* P_n *be the polynomials orthogonal with respect to the measure* $d\mu(x) = |x|^{2\alpha+1} e^{-x^2}\,dx$, $\alpha > -1$. *Then the* P_n *have nonnegative linearization coefficients.*

Proof. First we show that P_n satisfy the following recurrence formulas.

$$xP_{2n} = (n+\alpha+1)P_{2n+1} + nP_{2n-1}, \tag{18}$$

$$xP_{2n-1} = P_{2n} + P_{2n-2}. \tag{19}$$

Indeed, let P_n satisfy (18) and (19). Then

$$x^2 P_{2n} = (n+\alpha+1)P_{2n+2} + (2n+\alpha+1)P_{2n} + nP_{2n-2}.$$

Hence, putting $Q_n(y) = P_{2y}(\sqrt{y})$ gives

$$yQ_n = (n+\alpha+1)Q_{n+1} + (2n+\alpha+1)Q_n + nQ_{n-1}.$$

Therefore, the polynomials Q_n coincide with the Laguerre polynomials $(-1)^n L_n^{(\alpha)}$, so they are orthogonal with respect to the measure $d\nu(y) = y^\alpha e^{-y}\,dy$. This implies that P_n are orthogonal with respect to the measure $d\mu(x) = \frac{1}{2}d\nu(x^2) = |x|^{2\alpha+1} e^{-x^2}\,dx$. Combining (18), (19) and Theorem 2 yields the conclusion.

REFERENCES

[1] R. ASKEY, *Linearization of the product of orthogonal polynomials*, in Problems in Analysis, R. Gunning, ed., Princeton University Press, Princeton, NJ, 1970, pp. 223–228.

[2] ———, *Orthogonal polynomials and special functions*, Regional Conference Series in Applied Mathematics 21, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1975.

[3] R. ASKEY AND G. GASPER, *Linearization of the product of Jacobi polynomials*, III, Canad. J. Math., 23 (1971), pp. 119–122.

[4] A. ERDÉLVI, *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York, 1953.

[5] J. FAVARD, *Sur les polynômes de Tchebycheff*, C.R. Acad. Sci. Paris, 200 (1935), pp. 2052–2055.

[6] G. GASPER, *Linearization of the product of Jacobi polynomials*, I, Canad. J. Math., 22 (1970), pp. 171–175.

[7] ———, *Linearization of the product of Jacobi polynomials*, II, Canad. J. Math., 22 (1970), pp. 582–593.

[8] G. SZEGÖ, *Orthogonal Polynomials*, Fourth ed., Amer. Math. Soc. Colloq. Publ. 23, American Mathematical Society, Providence, RI, 1975.

[9] R. SZWARC, *Orthogonal polynomials and a discrete boundary value problem*, I, SIAM J. Math. Anal., this issue (1992), pp. 959–964.