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CONSTRUCTIVE APPROXIMATION

A Counterexample to Subexponential Growth of Orthogonal Polynomials

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Abstract. We construct an example of polynomials p_n orthonormal with respect to a measure μ such that the sequence $p_n(x)$ has an exponential lower bound for a point x in the support of μ.

1. Introduction

Let μ be a probability measure on $\mathbf{R}$ with all moments finite. If $\{p_n\}_{n=0}^{\infty}$ is a system of orthonormal polynomials obtained by the Gram–Schmidt procedure from $1, x, x^2, \ldots,$ then

$$xp_n = \lambda_{n+1}p_{n+1} + \beta_n p_n + \lambda_n p_{n-1},$$

where λ_n are positive coefficients and β_n are real ones.

In [Sz1] we showed that if a point z does not belong to the support of μ, then

$$\liminf_{n\to\infty} \left(\left|p_n^2(z)\right| + \left|p_{n+1}^2(z)\right|\right)^{1/n} > 1,$$

provided that the sequence λ_n is bounded.

There are several results suggesting that the converse should also hold. Namely, if the sequences λ_n and β_n are asymptotically periodic and λ_n is bounded away from 0, then

$$\limsup_{n\to\infty} |p_n(x)|^{1/n} \le 1$$

uniformly for $x \in \operatorname{supp}\mu$. This result is due to Zhang [Z1] (see also [Sz2] for a simple alternate proof). Zhang's proof is a refinement of [NTZ] where the case of convergent coefficients was considered (see also [LN] and [Nev]).

In this paper we show that if the coefficients are not asymptotically periodic, but still bounded and bounded away from 0, then it is possible to have a point x in $\operatorname{supp}\mu$ for which

$$\liminf_{n\to\infty} |p_n(x)|^{1/n} > 1.$$

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Our results are the following.

Theorem 1. *Let p_n be orthonormal polynomials satisfying the recurrence formula*

$$xp_n = \lambda_{n+1} p_{n+1} + \lambda_n p_{n-1}, \qquad n = 0, 1, \ldots, \tag{1.1}$$

where $p_{-1} = 0$, $p_0 = 1$, and

$$\lambda_n = \begin{cases} \dfrac{1}{2} & \text{for} \quad n \neq 2^k, \\ \dfrac{a}{2} & \text{for} \quad n = 2^{2k}, \\ \dfrac{1}{2a} & \text{for} \quad n = 2^{2k+1}, \end{cases}$$

where $a > 0$ is fixed. Let μ be the corresponding spectral measure. Then

(i) $\operatorname{supp} \mu \subset [-\frac{1}{2}(a + a^{-1}), \frac{1}{2}(a + a^{-1})]$,
(ii) $\pm\frac{1}{2}(a + a^{-1}) \in \operatorname{supp} \mu$,
(iii) $\liminf_{n\to\infty} p_n^{1/n}(\frac{1}{2}(a + a^{-1})) \geq a^{\pm 1/12} > 1$, *and*
(iv) $[-1, 1] \subset \operatorname{supp} \mu$.

Theorem 2. *Let r_n be orthonormal polynomials satisfying the recurrence formula*

$$xr_n = \frac{1}{2} r_{n+1} + \beta_n r_n + \frac{1}{2} r_{n-1}, \qquad n = 0, 1, \ldots, \tag{1.2}$$

where $r_{-1} = 0$, $r_0 = 1$, $\beta_0 = -\frac{1}{2} b(1 + \sqrt{1 + b^{-2}})$, and

$$\beta_n = \begin{cases} 0 & \text{for} \quad n \neq 2^k, \\ b & \text{for} \quad n = 2^{2k}, \\ -b & \text{for} \quad n = 2^{2k+1}, \end{cases}$$

where b is a fixed nonzero real number. Let ϱ be the corresponding spectral measure. Then

(i) $\operatorname{supp} \varrho \subset [-\sqrt{b^2 + 1}, \sqrt{b^2 + 1}]$,
(ii) $\pm\sqrt{b^2 + 1} \in \operatorname{supp} \varrho$,
(iii) $\liminf_{n\to\infty} r_n^{1/n}(\sqrt{b^2 + 1}) \geq (|b| + \sqrt{b^2 + 1})^{1/12} > 1$,
(iv) $\liminf_{n\to\infty} r_n^{1/n}(-\sqrt{b^2 + 1}) \leq (|b| + \sqrt{b^2 + 1})^{-1/3}$,
(v) $[-1, 1] \subset \operatorname{supp} \varrho$, *and*
(vi) $\operatorname{supp} \varrho$ *is symmetric about* 0.

The proofs of Theorems 1 and 2 are contained in Section 2. We point out that in both cases the polynomials p_n and r_n are very close to the case of so-called asymptotically periodic coefficients. Indeed, in Theorem 1 although the sequence λ_n is not asymptotically periodic itself, the limit

$$\lim_{n\to\infty} (\lambda_1 \lambda_2 \cdots \lambda_n)^{1/n} = \frac{1}{2}$$

exists. In Theorem 2 the coefficients β_n are not convergent but their average tends to 0. In both theorems the coefficients λ_n are bounded away from 0. Without the latter it is not hard to obtain large growth at a point in the support of μ (see Example 1).

2. Proofs of the Theorems

We begin with polynomials q_n satisfying the recurrence relation

$$\begin{aligned} xq_{2n} &= \gamma_n q_{2n+1} + \alpha_n q_{2n-1}, \\ xq_{2n+1} &= \alpha_{n+1} q_{2n+2} + \gamma_n q_{2n}, \end{aligned} \tag{2.1}$$

where γ_n, α_n are positive numbers for $n \geq 1$, $\alpha_0 = 0$, $\gamma_0 = 1$, $q_{-1} = 0$, and $q_0 = 1$. Since the corresponding Jacobi matrix is symmetric and its diagonal entries are equal to 0, the polynomials q_n are orthonormal with respect to a measure θ, which is symmetric about the origin. Notice that since $\gamma_0 = 1$, we have $\int x^2\, d\theta(x) = 1$. By (2.1) the polynomials $q_{2n}(x)$ and $q_{2n+1}(x)$ involve only even or odd powers of x, respectively. Hence the following are polynomials of exact degree n:

$$\begin{aligned} W_n(y) &= (-1)^n q_{2n}(y^{1/2}), \\ V_n(y) &= (-1)^n y^{-1/2} q_{2n+1}(y^{1/2}). \end{aligned} \tag{2.2}$$

The next lemma is an immediate consequence of (2.2).

Lemma 1. (i) *The polynomials $W_n(y)$ are orthonormal with respect to the measure $d\omega(y) = 2\, d\theta(y^{1/2})$, $y \geq 0$.*
(ii) *The polynomials $V_n(y)$ are orthonormal with respect to the measure $d\nu(y) = 2y\, d\theta(y^{1/2})$, $y \geq 0$.*

Substituting $x = y^{1/2}$ into (2.1) and using (2.2) give

$$\begin{aligned} W_n &= \gamma_n V_n - \alpha_n V_{n-1}, \\ yV_n &= \gamma_n W_n - \alpha_{n+1} W_{n+1}. \end{aligned} \tag{2.3}$$

Let $w_n = W_n(0)$ and $v_n = V_n(0)$. Then (2.3) gives

$$\begin{aligned} w_n &= \gamma_n v_n - \alpha_n v_{n-1}, \\ \alpha_{n+1} w_{n+1} &= \gamma_n w_n. \end{aligned} \tag{2.4}$$

By the induction argument the following can be deduced:

Lemma 2.

$$w_n = \frac{\gamma_{n-1}\gamma_{n-2}\cdots\gamma_0}{\alpha_n\alpha_{n-1}\cdots\alpha_1},$$

$$v_n = \frac{1}{\gamma_0\gamma_n w_n}\sum_{m=0}^{n} w_m^2.$$

Our plan is now the following. We want to find bounded sequences of positive coefficients $\{\alpha_n\}$ and $\{\gamma_n\}$ such that:

(i) $\liminf_{n\to\infty}(w_n^2+w_{n+1}^2)^{1/n} \le 1$. By Theorem 1 of [Sz1] this implies that $0 \in \operatorname{supp}\omega$.
(ii) $\sum_{n=0}^{\infty} w_n^2 = +\infty$. This implies $\omega(\{0\}) = 0$, hence 0 is an accumulation point of $\operatorname{supp}\omega$. Therefore $0 \in \operatorname{supp}\nu$, where $d\nu(y) = y\,d\omega(y)$.
(iii) $\liminf_{n\to\infty} v_n{}^{1/n} > 1$.

After this plan is carried out the polynomials p_n satisfying Theorem 1 will be obtained by affine transformation of the polynomials V_n, so the point 0 will be mapped onto $-\frac{1}{2}(a+a^{-1})$.

Fix $a > 1$. We skip the case $0 < a < 1$ which can be dealt with similarly. Let $n_m = 2^m$, for $m \ge 2$, and $n_1 = 0$. Let

$$\alpha_n = \begin{cases} a & \text{for} \quad n_{2m-1} < n \le n_{2m}, \\ 1 & \text{for} \quad n_{2m} < n \le n_{2m+1}, \end{cases} \tag{2.5}$$
$$\gamma_{n-1} = \frac{a}{\alpha_n}.$$

Thus the ratio $\frac{\gamma_{n-1}}{\alpha_n}$ takes values a^{-1} or a according to whether n falls into the interval $(n_{2m-1}, n_{2m}]$ or $(n_{2m}, n_{2m+1}]$. Using Lemma 2 we then get

$$\begin{aligned} w_{n_{2m}} &= \prod_{j=1}^{n_{2m}} \left(\frac{\gamma_{j-1}}{\alpha_j}\right) \\ &= a^{-(n_{2m}-n_{2m-1})+(n_{2m-1}-n_{2m-3})-\cdots-(n_2-n_1)} = a^{-n_{2m}-2\sum_{j=1}^{2m-1}(-1)^j n_j}. \end{aligned}$$

Similarly we obtain

$$w_{n_{2m+1}} = a^{n_{2m+1}-2\sum_{j=1}^{2m}(-1)^j n_j}.$$

Substituting $n_m = 2^m$ for $m \ge 2$ and $n_1 = 0$ gives

$$w_{2^{2m}} = a^{-(2^{2m}+8)/3}, \tag{2.6}$$

$$w_{2^{2m+1}} = a^{(2^{2m+1}-8)/3}. \tag{2.7}$$

Since the ratio $\frac{w_{n+1}}{w_n}$ is bounded (2.6) yields

$$\begin{aligned} \liminf_{n\to\infty}(w_n^2+w_{n+1}^2)^{1/n} &= \liminf_{n\to\infty}\left\{1+\left(\frac{w_{n+1}^2}{w_n^2}\right)\right\}^{1/n}(w_n^2)^{1/n} \\ &= \liminf_{n\to\infty}(w_n)^{2/n} \le a^{-2/3} < 1. \end{aligned}$$

Moreover, (2.7) implies

$$\sum_{n=0}^{\infty} w_n^2 = +\infty.$$

Now we check that property (iii) of the plan is also satisfied. To this end observe that (2.4), (2.5), and the fact that $a > 1$ imply

$$\begin{aligned} w_{n-1} &> w_n \quad \text{if} \quad n_{2m-1} < n \le n_{2m}, \\ w_{n-1} &< w_n \quad \text{if} \quad n_{2m} < n \le n_{2m+1}. \end{aligned}$$

Now by Lemma 2 we have ($\gamma_0 = 1$)

$$\gamma_n v_n \ge w_n^{-1} \max\{w_0^2, \ \ldots, \ w_n^2\}.$$

Fix n. There is an m such that $n_{2m-1} < n \le n_{2m+1}$. We consider two cases:

(a) $w_n \le w_{n_{2m-1}}$. Then

$$\gamma_n v_n \ge w_n^{-1} w_{n_{2m-1}}^2 \ge w_{n_{2m-1}}.$$

(b) $w_n > w_{n_{2m-1}}$. Then

$$\gamma_n v_n \ge w_n^{-1} w_n^2 = w_n > w_{n_{2m-1}}.$$

Hence

$$(\gamma_n v_n)^{1/n} \ge (w_{n_{2m-1}})^{1/n} \ge (w_{n_{2m-1}})^{1/n_{2m+1}}.$$

Now combining (2.7), the fact that $n_m = 2^m$ and $\gamma_n^{1/n} \to 1$ gives

$$\liminf_{n\to\infty} v_n^{1/n} \ge a^{1/12}.$$

Summarizing what we have done so far: V_n are polynomials orthonormal with respect to the measure ν such that

(i) $\operatorname{supp} \nu \subset [0, +\infty)$,
(ii) $0 \in \operatorname{supp} \nu$, and
(iii) $\liminf_{n\to\infty} V_n^{1/n}(0) \ge a^{1/12} > 1$.

Using (2.3) we can derive the recurrence formula for V_n:

$$yV_n = -\alpha_{n+1}\gamma_{n+1}V_{n+1} + (\alpha_{n+1}^2 + \gamma_n^2)V_n - \alpha_n\gamma_n V_{n-1}.$$

In view of (2.5) we get that the coefficient of V_n is constant. Namely,

$$\alpha_{n+1}^2 + \gamma_n^2 = 1 + a^2.$$

Also the sequence $\alpha_n\gamma_n$ can be easily determined:

$$\alpha_n\gamma_n = \begin{cases} a & \text{if} \quad n \ne n_m, \\ a^2 & \text{if} \quad n = n_{2m}, \\ 1 & \text{if} \quad n = n_{2m-1}. \end{cases}$$

Let p_n be defined by

(2.8) $$p_n(x) = (-1)^n V_n(2ax + (a^2 + 1)).$$

If μ denotes the corresponding orthogonalizing measure, then

(2.9) $$d\mu(x) = d\nu(2ax + (a^2 + 1)).$$

Using the recurrence relation for V_n it is easy to verify that p_n satisfies (1.1). Hence the corresponding measure μ is symmetric about the origin. Also by properties (i)–(iii) of the polynomials V_n we obtain

(i) $\operatorname{supp}\mu \subset [-\frac{1}{2}(a+a^{-1}), +\infty)$,
(ii) $-\frac{1}{2}(a+a^{-1}) \in \operatorname{supp}\mu$, and
(iii) $\liminf_{n\to\infty} \{p_n(-\frac{1}{2}(a+a^{-1}))\}^{1/n} \ge a^{1/12} > 1$.

Now Theorem 1(i)–(iii) follows from the fact that μ is symmetric about 0.

It remains to show Theorem 1(iv). The proof is based entirely on the recurrence formula (1.1). It is well known that the support of μ can be identified with the spectrum of the difference operator

$$(La)_n = \lambda_{n+1}a_{n+1} + \lambda_n a_{n-1}$$

on the Hilbert space $\ell^2(\mathbf{N})$ of square summable sequences. Fix a real number x. We will show that $\cos x$ is in the spectrum of L. To this end it suffices to find a sequence of vectors v_m in $\ell^2(\mathbf{N})$ such that $\|v_m\| = 1$ and $\|(\cos x - L)v_m\| \to 0$, when $m \to \infty$. The sequence v_m is called an approximate eigenvector. Let

$$u_m(n) = \begin{cases} e^{inx} & \text{if} \quad 2^{2m}+1 \le n < 2^{2m+1}-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is not hard to compute (see the proof of Theorem 1 of [Sz1]) that

$$\|(\cos x - L)u_m\| = 1 \quad \text{and} \quad \|u_m\| = (2^{2m}-2)^{1/2}.$$

Hence $v_m = \frac{u_m}{\|u_m\|}$ is an approximate eigenvector corresponding to the eigenvalue $\cos x$.

Remark. We were unable to determine if the entire interval $[-\frac{1}{2}(a+a^{-1}), \frac{1}{2}(a+a^{-1})]$ is contained in $\operatorname{supp}\mu$. Theorem 1 implies only that the endpoints are accumulation points of $\operatorname{supp}\mu$. Perhaps methods of [GHV] might be used to determine the spectrum of μ.

Remark. The sequence $n_m = 2^m$ can be replaced by $n_m = [\beta^m]$, where $\beta > 1$ and $[\cdot]$ denotes the greatest integer value of a number. Then the estimate in Theorem 1(iii) becomes

$$\liminf_{n\to\infty} p_n^{1/n}(\tfrac{1}{2}(a+a^{-1})) \ge a^{\pm(\beta-1)/(\beta^2(\beta+1))} > 1.$$

If we do not require that λ_n is bounded away from 0, then it is much easier to get an exponential lower bound on the support of μ as the following example shows.

Example 1. Let p_n satisfy

$$\begin{aligned} xp_n &= 2^{-4n-1}p_{n+1} + (2^{-4n} + 2^{2-4n})p_n + 2^{3-4n}p_{n-1}, \qquad n \ge 1, \\ xp_0 &= 2^{-1}p_1 + 2^{-2}p_0. \end{aligned}$$

Since the coefficients in the recurrence formula converge to 0, the support of μ is a countable set with 0 as the only accumulation point. Hence $0 \in \operatorname{supp}\mu$. By induction it can be shown that $p_n(0) = (-2)^n$. Thus

$$\lim_{n\to\infty} |p_n(0)|^{1/n} = 2 > 1.$$

Now we turn to the proof of Theorem 2. We consider only the case $b > 0$. Then there is a unique number $a > 1$, satisfying $b = \frac{1}{2}(a - a^{-1})$. Moreover, we have $\sqrt{b^2+1} = \frac{1}{2}(a + a^{-1})$ and $a = b + \sqrt{b^2+1}$.

Let W_n be the polynomials defined by (2.1), (2.2), and (2.5) for this value of a. By (2.3) we get a recurrence relation for the polynomials W_n:

$$yW_n = -\alpha_{n+1}\gamma_n W_{n+1} + (\alpha_n^2 + \gamma_n^2)W_n - \alpha_n\gamma_{n-1}W_{n-1}.$$

By (2.5) we have $\alpha_n\gamma_{n-1} = a$. Also by (2.5) we can deduce that

$$\alpha_n^2 + \gamma_n^2 = \begin{cases} 1 & \text{if } n = 0, \\ a^2 + 1 & \text{if } n \neq n_m, \\ 2a^2 & \text{if } n = n_{2m}, \\ 2 & \text{if } n = n_{2m+1}. \end{cases}$$

Define the polynomials $r_n(x)$ by

$$r_n(x) = (-1)^n W_n(2ax + (a^2+1)). \tag{2.10}$$

Then it can be readily checked that r_n satisfy (1.2). If ϱ denotes the corresponding orthogonalizing measure, then

$$d\varrho(x) = d\omega(2ax + (a^2+1)) \tag{2.11}$$

(see Lemma 1). Also, since the supports of ω and ν are equal (see the proof of Theorem 1), so are the supports of ϱ and μ (see (2.9), (2.11)). This gives Theorem 2(i), (ii), (v), and (vi). Now by (2.3), (2.8), and (2.10) we get

$$r_n = \gamma_n p_n + \alpha_n p_{n-1}.$$

Hence

$$\liminf_{n\to\infty} r_n^{1/n}\left(\frac{1}{2}\left(a + a^{-1}\right)\right) \geq \liminf_{n\to\infty} p_n^{1/n}\left(\frac{1}{2}\left(a + a^{-1}\right)\right) \geq a^{1/12} > 1.$$

We used the fact that the polynomials p_n take positive values at the point $\frac{1}{2}(a + a^{-1})$ as the support of the corresponding measure lies to the left of this point. By (2.6) and (2.10) we obtain

$$\liminf_{n\to\infty} r_n^{1/n}\left(-\frac{1}{2}\left(a + a^{-1}\right)\right) \leq a^{-1/3}.$$

This completes the proof of Theorem 2. ■

Condition (iii) of Theorem 1 implies that the integrals

$$\int_{-\gamma(a)}^{\gamma(a)} \frac{d\mu(x)}{x \pm \gamma(a)}$$

are finite, where $\gamma(a) = \frac{1}{2}(a + a^{-1})$. Indeed, it follows from the next proposition.

Proposition 1. *Let μ be a probability measure supported in $[0,+\infty)$. Let $\{p_n\}_{n=0}^{\infty}$ denote the system of corresponding orthonormal polynomials. If the integral $\int_0^{+\infty} x^{-1} d\mu(x)$ is divergent, then*

$$\liminf_{n\to\infty} (\beta_n p_n^2(0))^{1/n} \le 1,$$

where $\beta_n = \int_0^{+\infty} x p_n^2(x)\, d\mu(x)$.

Proof. We can assume that the support of μ is infinite because otherwise there are only finitely many nonzero p_n. Being orthonormal polynomials, p_n satisfy a recurrence formula

$$x p_n = \lambda_n p_{n+1} + \beta_n p_n + \lambda_{n-1} p_{n-1}. \tag{2.12}$$

Let

$$g_n = -\frac{\lambda_{n-1} p_{n-1}(0)}{\beta_n p_n(0)}. \tag{2.13}$$

The numbers g_n are well defined because

$$\beta_n = \int_0^{+\infty} x p_n^2(x)\, d\mu(x) > 0.$$

Since $\operatorname{supp}\mu \subset [0,+\infty)$, the numbers $p_n(0)$ have alternating signs. Hence $g_n > 0$ and by (2.12) we have

$$g_{n+1}(1-g_n) = \frac{\lambda_n^2}{\beta_n \beta_{n+1}}. \tag{2.14}$$

This means that the right-hand side of (2.14) is a chain sequence with g_n as its parameter sequence (see Chapter III.5 of [Ch]). Since the integral $\int_0^{\infty} x^{-1}\, d\mu(x)$ is infinite then by Theorem 1 of [Sz3] g_n is determined uniquely by (2.14). Thus g_n is also a maximal parameter sequence (see Theorem III.5.3 of [Ch]). By Theorem III.6.2 of [Ch] we get

$$\limsup_{n\to\infty} \left(\prod_{i=1}^{n} \frac{g_i}{1-g_i} \right)^{1/n} \ge 1.$$

By (2.13) and (2.14) we get that

$$\prod_{i=1}^{n} \frac{g_i}{1-g_i} = -\frac{\beta_0}{\lambda_n p_n(0) p_{n+1}(0)}.$$

Observing that

$$-\lambda_n p_n(0) p_{n+1}(0) \le \beta_n p_n^2(0)$$

yields the conclusion. ■

In order to apply Proposition 1 to our example we can shift polynomials by $\frac{1}{2}(a+a^{-1})$ to the right and observe that $\beta_n = \frac{1}{2}(a+a^{-1}) > 0$.

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