Kaczmarz algorithm in Hilbert space

by

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Abstract. The aim of the Kaczmarz algorithm is to reconstruct an element in a Hilbert space from data given by inner products of this element with a given sequence of vectors. The main result characterizes sequences of vectors leading to reconstruction of any element in the space. This generalizes some results of Kwapień and Mycielski.

1. Introduction. Let $\{e_n\}_{n=0}^{\infty}$ be a sequence of unit vectors in a Hilbert space $\mathcal{H}$. For a given $x \in \mathcal{H}$ we have the numbers $\{\langle x, e_n \rangle\}_{n=0}^{\infty}$. We want to reconstruct x from these numbers. The sequence $\{e_n\}_{n=0}^{\infty}$ should be linearly dense. Define

$$x_0 = \langle x, e_0 \rangle e_0,$$
$$x_n = x_{n-1} + \langle x - x_{n-1}, e_n \rangle e_n.$$

We are interested in when $x_n \to x$ for any $x \in \mathcal{H}$. The sequences $\{e_n\}_{n=0}^{\infty}$ for which this holds will be called *effective*.

The formula is called the *Kaczmarz algorithm*. In 1937 Kaczmarz (see [1]) considered this problem in the finite-dimensional case. He proved that if $\dim \mathcal{H} < \infty$ and the sequence $\{e_n\}_{n=0}^{\infty}$ is linearly dense and periodic then it is effective.

Let P_n be the orthogonal projection onto $e_n^{\perp}$. Then we have

$$x_n = x_{n-1} + (I - P_n)(x - x_{n-1}),$$
(1)
$$x - x_n = P_n(x - x_{n-1}),$$
$$x - x_n = P_n P_{n-1} \cdots P_1 P_0 x.$$

Therefore the sequence $\{e_n\}_{n=0}^{\infty}$ is effective if and only if the operators $P_n P_{n-1} \cdots P_1 P_0$ tend to zero strongly. Since the norms of these operators are bounded it suffices to get pointwise convergence on a linearly dense subset of vectors, e.g. on members of the sequence $\{e_n\}_{n=0}^{\infty}$.

The Kaczmarz theorem can now be proved as follows. Let $\dim \mathcal{H} < \infty$ and $\{e_n\}_{n=0}^{\infty}$ be N-periodic. For $A = P_{N-1} \cdots P_1 P_0$ it suffices to show that A^n tends to zero. We claim that $\|A\| < 1$. If not, there is a vector x such that $\|Ax\| = \|x\| = 1$. Then $\|P_0 x\| \geq \|Ax\| = \|x\|$, hence $P_0 x = x$. Similarly $P_1 x = x, \ldots, P_{N-1} x = x$, which implies that $x \perp e_0, e_1, \ldots, e_{N-1}$. Since the vectors $\{e_n\}_{n=0}^{N-1}$ are linearly dense we get $x = 0$.

We now turn to the infinite-dimensional case. We recall some basic properties of the algorithm which can be found in [2]. By construction the vector x_n is a linear combination of $e_0, e_1, \ldots, e_n$. It can be shown that

$$(2) \qquad x_n = \sum_{i=0}^{n} \langle x, g_i \rangle e_i,$$

where the vectors g_n are given by the recurrence relation

$$(3) \qquad g_0 = e_0, \qquad g_n = e_n - \sum_{i=0}^{n-1} \langle e_n, e_i \rangle g_i$$

or

$$(4) \qquad \sum_{i=0}^{n} m_{ni} g_i = e_n, \qquad m_{ni} = \langle e_n, e_i \rangle.$$

By (2) we have

$$x - x_{n-1} = x - x_n + \langle x, g_n \rangle e_n.$$

Since by (1) the vectors $x - x_n$ and e_n are orthogonal we get

$$\|x\|^2 = \|x - x_0\|^2 + |\langle x, g_0 \rangle|^2,$$
$$\|x - x_{n-1}\|^2 = \|x - x_n\|^2 + |\langle x, g_n \rangle|^2, \qquad n \geq 1.$$

Summing up these equalities gives

$$\|x\|^2 - \lim_{n \to \infty} \|x - x_n\|^2 = \sum_{n=0}^{\infty} |\langle x, g_n \rangle|^2.$$

Therefore the sequence $\{e_n\}_{n=0}^{\infty}$ is effective if and only if

$$(5) \qquad \|x\|^2 = \sum_{n=0}^{\infty} |\langle x, g_n \rangle|^2 \qquad \text{for any } x \in \mathcal{H}.$$

This equation means that $\{g_n\}_{n=0}^{\infty}$ is a tight frame with constant 1 as was already mentioned in [2]. We have noticed before that it suffices to check formula (5) on vectors $\{e_n\}_{n=0}^{\infty}$, provided they form a linearly dense subset in $\mathcal{H}$.

2. Characterization of effective sequences. The formula (3) implies that g_n is a linear combination of the vectors $e_0, e_1, \ldots, e_n$, i.e.

$$(6) \qquad \sum_{i=0}^{n-1} c_{ni} e_i + e_n = g_n$$

for some coefficients c_{nj}. By (4) these coefficients can be obtained by taking the algebraic inverse matrix to the lower triangular matrix $I + M$ where

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ m_{10} & 0 & 0 & 0 & 0 & \cdots \\ m_{20} & m_{21} & 0 & 0 & 0 & \cdots \\ m_{30} & m_{31} & m_{32} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \qquad m_{ij} = \langle e_i, e_j \rangle.$$

Namely $(I + M)^{-1} = I + U$, where

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ c_{10} & 0 & 0 & 0 & 0 & \cdots \\ c_{20} & c_{21} & 0 & 0 & 0 & \cdots \\ c_{30} & c_{31} & c_{32} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

The matrix U and the coefficients c_{ni} will play a crucial role in all what follows. Since the matrices $I + M$ and $I + U$ are inverse to each other we get

$$(7) \qquad\qquad UM = MU = -M - U,$$
$$(8) \qquad\qquad U^* M^* = M^* U^* = -M^* - U^*.$$

Our first simple result shows that the matrix U is a contraction in the Hilbert space $\ell^2(\mathbb{N}_0)$.

PROPOSITION 1. *Let U and M be strictly lower triangular matrices such that $MU = UM = -U - M$. Then U is a contraction if and only if the matrix $M + M^* + I$ is positive definite. In that case there is a Hilbert space $\mathcal{H}$ and vectors $\{e_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ such that $M + M^* + I$ is the Gram matrix of these vectors.*

Proof. Let M_n and U_n denote the truncated matrices given by

$$M_n = \begin{pmatrix} 0 & & & & \\ m_{10} & 0 & & & \\ \vdots & \ddots & 0 & & \\ m_{n0} & \cdots & m_{n,n-1} & 0 & \\ 0 & \cdots & 0 & & \ddots \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}, \qquad U_n = \begin{pmatrix} 0 & & & & \\ c_{10} & 0 & & & \\ \vdots & \ddots & 0 & & \\ c_{n0} & \cdots & c_{n,n-1} & 0 & \\ 0 & \cdots & 0 & & \ddots \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}.$$

Then M_n and U_n are bounded on $\ell^2(\mathbb{N}_0)$ and by assumption $M_n U_n = U_n M_n = -U_n - M_n$. Assume the matrix $M + M^* + I$ is positive definite. Then the matrix $M_n + M_n^* + I$ corresponds to a positive bounded operator on $\ell^2(\mathbb{N}_0)$. Thus

$$0 \le (U_n^* + I)(M_n + M_n^* + I)(U_n + I) = I - U_n^* U_n.$$

Hence $\|U_n\| \le 1$, where $\|\cdot\|$ denotes the operator norm. Consequently, we obtain $\|U\| \le 1$. The converse implication follows from

$$(M_n^* + I)(I - U_n^* U_n)(M_n + I) = M_n + M_n^* + I.$$

Indeed, if $\|U\| \le 1$ then $\|U_n\| \le 1$. Therefore the matrix $M_n + M_n^* + I$ is positive definite, which implies that so also is $M + M^* + I$. It is then well known that there exist a Hilbert space $\mathcal{H}$ and vectors $\{e_n\}_{n=0}^{\infty}$ such that

$$m_{ij} = \langle e_i, e_j \rangle, \quad i > j. \ \blacksquare$$

Now we can state the main result of our paper.

THEOREM 1. *The sequence $\{e_n\}_{n=0}^{\infty}$ is effective if and only if it is linearly dense and U is a partial isometry, i.e. U^*U is an orthogonal projection.*

Proof. Assume $\{e_n\}_{n=0}^{\infty}$ is effective. By (5) and by the polar identity we get

$$(9) \qquad\qquad \langle x, y \rangle = \sum_{n=0}^{\infty} \langle x, g_n \rangle \langle g_n, y \rangle$$

for any $x, y \in \mathcal{H}$. In particular

$$(10) \qquad\qquad m_{ij} = \langle e_i, e_j \rangle = \sum_{n=0}^{\infty} \langle e_i, g_n \rangle \langle g_n, e_j \rangle.$$

We want to state the formula (10) in terms of matrices on $\ell^2(\mathbb{N}_0)$. Let δ_i denote the sequence in $\ell^2(\mathbb{N}_0)$ whose ith entry is 1 and all other entries are 0. We have the following.

LEMMA 1.

$$\langle g_n, e_j \rangle = \langle (UM^* + M^* + I)\delta_j, \delta_n \rangle_{\ell^2(\mathbb{N}_0)}$$

Proof of Lemma 1. Set $c_{nn} = 1$. Then by (6) we have

$$\langle g_n, e_j \rangle = \begin{cases} \displaystyle\sum_{i=0}^{n} c_{ni} m_{ij} & \text{for } j > n, \\ \displaystyle\sum_{i=0}^{j-1} c_{ni} m_{ij} + \sum_{i=j}^{n} c_{ni} m_{ij} & \text{for } j \le n. \end{cases}$$

Since $(I + U)(I + M) = I$ we get

$$\sum_{i=j}^{n} c_{ni} m_{ij} = \delta_{jn}, \quad \text{for } j \leq n.$$

Therefore

$$\langle g_n, e_j \rangle = \Big\langle \sum_{i=0}^{j-1} \overline{m}_{ji} \delta_i, \sum_{i=0}^{n} \overline{c}_{ni} \delta_i \Big\rangle_{\ell^2} + \langle \delta_j, \delta_n \rangle_{\ell^2}$$
$$= \langle M^* \delta_j, (U^* + I) \delta_n \rangle_{\ell^2} + \langle \delta_j, \delta_n \rangle_{\ell^2} = \langle (UM^* + M^* + I) \delta_j, \delta_n \rangle_{\ell^2}. \quad \blacksquare$$

Let $A = UM^* + M^* + I$. Applying Lemma 1 to (10) and using the Parseval identity gives

$$(11) \qquad m_{ij} = \sum_{n=0}^{\infty} \langle A\delta_j, \delta_n \rangle \langle \delta_n, A\delta_i \rangle_{\ell^2} = \langle A\delta_j, A\delta_i \rangle_{\ell^2}.$$

Let $A_n = U_n M_n^* + M_n^* + I$. Unlike A, the matrices A_n correspond to bounded operators on $\ell^2(\mathbb{N}_0)$. Since $M_n^* \delta_k = M^* \delta_k$ for $n \geq k$ and $U_n \xrightarrow{n} U$ strongly we have

$$(12) \qquad \lim_{n \to \infty} \langle A_n^* A_n \delta_j, \delta_i \rangle_{\ell^2} = \lim_{n \to \infty} \langle A_n \delta_j, A_n \delta_i \rangle_{\ell^2} = \langle A\delta_j, A\delta_i \rangle_{\ell^2}.$$

On the other hand, the relation $M_n U_n = -U_n - M_n$ implies

$$A_n^* A_n = M_n U_n^* U_n M_n^* - M_n(U_n^* + M_n^*) + M_n U_n^*$$
$$- (U_n + M_n)M_n^* + M_n M_n^* + M_n + U_n M_n^* + M_n^* + I$$
$$= M_n + M_n^* + I + M_n U_n^* U_n M_n^* - M_n M_n^*.$$

Hence

$$(13) \;\; \lim_{n \to \infty} \langle A_n^* A_n \delta_j, \delta_i \rangle_{\ell^2} = m_{ij} + (UM^* \delta_j, UM^* \delta_i)_{\ell^2} - (M^* \delta_j, M^* \delta_i)_{\ell^2}.$$

Combining (11)–(13) yields

$$(14) \qquad (UM^* \delta_j, UM^* \delta_i)_{\ell^2} = (M^* \delta_j, M^* \delta_i)_{\ell^2}.$$

Let $\mathcal{F}(\mathbb{N}_0) = \operatorname{span} \{\delta_0, \delta_1, \dots\}$. Formula (14) states that the operator U is isometric on

$$\mathcal{H}_0 = \overline{M^*(\mathcal{F}(\mathbb{N}_0))}.$$

It suffices to show that U vanishes on $\mathcal{H}_0^\perp$. To this end observe that the matrices U^* and M^* leave the subspace $\mathcal{F}(\mathbb{N}_0)$ invariant. The formula $M^*(U^* + I) = -U^*$ implies that

$$U^*(\mathcal{F}(\mathbb{N}_0)) \subset \mathcal{H}_0.$$

Taking orthogonal complements of both sides results in

$$\mathcal{H}_0^\perp \subset \ker U,$$

which completes the proof that U is a partial isometry.

Conversely, let U be a partial isometry. Hence U is isometric on $\mathcal{H}_0 = \overline{U^*(\mathcal{F}(\mathbb{N}_0))}$. The formula $U^*(M^* + I) = -M^*$ implies that U is isometric on $M^*(\mathcal{F}(\mathbb{N}_0))$, which is equivalent to (14). Now tracking backwards the proof of the first part implies the formula (10). In particular for $i = j$ we obtain

$$\|e_i\|^2 = \sum_{n=0}^{\infty} |\langle e_i, g_n \rangle|^2$$

for any $i \geq 0$. This implies effectivity (see comments at the end of the Introduction). ∎

REMARK 1. Theorem 1 can be used to construct examples of effective sequences of vectors. It suffices to come up with a strictly lower triangular partial isometry U. Then one has to compute the algebraic inverse $I + M$ of $I + U$. As a result a positive definite matrix $M + M^* + I$ is constructed. The corresponding vectors form a sequence effective in the closed linear span of these vectors.

REMARK 2. Proposition 1 and Theorem 1 can be interpreted as follows. There are as many effective sequences among sequences of unit vectors as there are partial isometries among strictly lower triangular contractions on $\ell^2(\mathbb{N}_0)$. This can be compared with a result of Kwapień and Mycielski who showed that if we choose the sequence of unit vectors at random then almost surely we end up with an effective sequence. More precisely, fix a probability Borel measure μ on the unit sphere of $\mathcal{H}$, such that the support of μ is linearly dense. Then drawing consecutive vectors independently with respect to that measure almost surely yields an effective sequence of vectors.

3. Strongly effective sequences. For a partial isometry U the *carrier space*, i.e. $\operatorname{Im} U^*$, may vary from the trivial space to the whole space $\mathcal{H}$. For instance when the system $\{e_n\}_{n=0}^{\infty}$ is orthonormal we get $M = U = 0$. The other extreme case is when $\operatorname{Im} U^* = \mathcal{H}$, which is equivalent to $U^*U = I$. While proving Theorem 1 we showed that for an effective sequence $\{e_n\}_{n=0}^{\infty}$ the carrier space for U coincides with $\overline{M^*(\mathcal{F}(\mathbb{N}_0))}$. Hence $U^*U = I$ if and only if $\overline{M^*(\mathcal{F}(\mathbb{N}_0))} = \ell^2(\mathbb{N}_0)$. This occurs exactly when the rows of the matrix M form a linearly dense subset of $\ell^2(\mathbb{N}_0)$. For example this is the case when $m_{n+1,n} \neq 0$ in the matrix

$$\begin{pmatrix} 0 & & & & \\ m_{10} & 0 & & & \\ * & m_{21} & 0 & & \\ * & * & m_{32} & 0 & \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}.$$

It turns out that sequences $\{e_n\}_{n=0}^{\infty}$ with $U^*U = I$ have many interesting properties. Let

$$(15) \qquad v_{in} = \sum_{j=i}^{n} \bar{c}_{nj} e_j, \quad n > i.$$

The next lemma is related to a formula obtained in the proof of Theorem 2 of [2].

LEMMA 2. *For $0 \le i, j < n$ the following equalities hold:*

$$\langle v_{in}, v_{jn} \rangle = \langle (I - U_n^*U_n)\delta_j, \delta_i \rangle_{\ell^2}.$$

Proof. Without loss of generality we may assume that $j \le i$. Then using the fact that $I + M$ and $I + U$ are inverse to each other we get

$$\langle v_{in}, v_{jn} \rangle = \Big\langle \sum_{k=i}^{n} \bar{c}_{ki} e_k, \sum_{l=j}^{n} \bar{c}_{lj} e_l \Big\rangle = \sum_{k=i}^{n} \bar{c}_{ki} \sum_{l=j}^{n} m_{kl} c_{lj} = \sum_{k=i}^{n-1} \bar{c}_{ki} \sum_{l=j}^{n} m_{kl} c_{lj}$$

$$= \sum_{k=i}^{n-1} \bar{c}_{ki} \sum_{l=j}^{k} m_{kl} c_{lj} + \sum_{k=i}^{n-1} \bar{c}_{ki} \sum_{l=k+1}^{n} m_{kl} c_{lj}$$

$$= \sum_{k=i}^{n-1} \bar{c}_{ki} \delta_{jk} + \sum_{l=i+1}^{n} c_{lj} \sum_{k=i}^{l-1} \bar{m}_{lk} \bar{c}_{ki} = \delta_{ij} - \sum_{l=i+1}^{n} c_{lj} \bar{c}_{li}$$

$$= \langle (I - U_n^*U_n)\delta_j, \delta_i \rangle_{\ell^2}. \quad \blacksquare$$

The next corollary should be compared with Remark 2 of [2].

COROLLARY 1. *Assume $U^*U = I$. Then for any $j \ge 0$,*

$$(16) \qquad e_j = - \sum_{i=j+1}^{\infty} \bar{c}_{ij} e_i.$$

Proof. By Lemma 2 we get $v_{jn} \to 0$ as $n \to \infty$. Hence

$$\sum_{i=j}^{\infty} \bar{c}_{ij} e_i = 0.$$

Since $c_{jj} = 1$ we get the conclusion. $\blacksquare$

DEFINITION 1. The sequence $\{e_n\}_{n=0}^{\infty}$ will be called *strongly effective* if $\{e_n\}_{n=k}^{\infty}$ is effective for each $k \ge 0$.

In particular dropping finitely many vectors from $\{e_n\}_{n=0}^{\infty}$ does not spoil linear density. Hence this is a highly nonorthogonal case.

THEOREM 2. *Assume the sequence $\{e_n\}_{n=0}^{\infty}$ is linearly dense in a Hilbert space $\mathcal{H}$. Then $\{e_n\}_{n=0}^{\infty}$ is strongly effective if and only if $U^*U = I$.*

Proof. By Corollary 1 for any k the sequence $\{e_n\}_{n=k}^{\infty}$ is linearly dense. Let $M^{(k)}$ and $U^{(k)}$ denote truncated matrices obtained by removing the first k rows and the first k columns from M and U, respectively. These matrices correspond to the sequence $\{e_n\}_{n=k}^{\infty}$. Also $U^*U = I$ implies $(U^{(k)})^*U^{(k)} = I$ for any k. Hence $U^{(k)}$ is a partial isometry. Now we can use Theorem 1 to get the conclusion.

Conversely, suppose that $\{e_n\}_{n=0}^{\infty}$ is strongly effective. Let Q_k denote the orthogonal projection from $\ell^2(\mathbb{N})$ onto the orthogonal complement of $\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$. Let $U_{(k)} = UQ_k$. Then $U_{(k)} = 0_k \oplus U^{(k)}$, where 0_k denotes the $k \times k$ zero matrix. Hence $U_{(k)}$ are partial isometries just as U is. But this is possible only if U^*U and Q_k commute. On the other hand, if U^*U commutes with Q_k for any k then U^*U must be diagonal. Assume that $U^*U \neq I$. Then $U\delta_j = 0$ for some j. This implies that $M\delta_j = 0$ and consequently e_j is orthogonal to all the vectors e_i, $i > j$. Hence $\{e_n\}_{n=0}^{\infty}$ cannot be strongly effective. ∎

LEMMA 3. *Assume* $U^*U = I$. *Then*

$$U^*M = -M - I,$$
(17) $$U^*(M + M^* + I) = -(M + M^* + I).$$

Proof. By taking the inner product with e_k in (16) we obtain

$$m_{jk} = -\sum_{i=j+1}^{\infty} \overline{c}_{ij} m_{ik}$$

for any $k \geq 0$. Let $k \leq j$. Then

(18) $$m_{jk} = -\sum_{i=j+1}^{\infty} \overline{c}_{ij} m_{ik} = (U^*M)_{jk}.$$

For $k > j$ we get

(19) $$0 = \sum_{i=j}^{\infty} \overline{c}_{ij} m_{ik} = \sum_{i=j}^{k} \overline{c}_{ij}\, \overline{m}_{ki} + \sum_{i=k+1}^{\infty} \overline{c}_{ij} m_{ik}$$
$$= \overline{((M + I)U)_{jk}} + (U^*M)_{jk} = (U^*M)_{jk}.$$

Combining (18) and (19) gives the first equality. The second equality now follows by applying $U^*M^* = -U^* - M^*$. ∎

THEOREM 3. *Assume the matrix* U *associated with the sequence* $\{e_n\}_{n=0}^{\infty}$ *satisfies* $U^*U = I$. *Then*

$$\sum_{i=0}^{\infty} |\langle e_i, e_j \rangle|^2 = \infty$$

for any j. In particular the conclusion holds if the sequence $\{e_n\}_{n=0}^{\infty}$ is effective and the rows of M are linearly dense in $\ell^2(\mathbb{N}_0)$.

Proof. Let $G = M + M^* + I$. Observe that G is nothing other than the Gram matrix of the vectors $\{e_n\}_{n=0}^{\infty}$. Assume that

$$\sum_{i=0}^{\infty} |\langle e_i, e_j \rangle|^2 < \infty$$

for some j. Then $v = G\delta_j \in \ell^2(\mathbb{N}_0)$. By (17) we get $U^* v = -v$ and

$$(U^*)^n v = (-1)^n v.$$

But $\|U^*\| \leq 1$ and U^* is strictly upper triangular. Hence $(U^*)^n$ tends to zero strongly, which implies $v = 0$. This gives a contradiction because $v(j) = 1$. ∎

4. Stationary case. Assume

$$\langle e_{i+1}, e_{j+1} \rangle = \langle e_i, e_j \rangle.$$

Then the matrix M is constant on diagonals:

$$M = \begin{pmatrix} 0 & & & & \\ a_1 & 0 & & & \\ a_2 & a_1 & 0 & & \\ a_3 & a_2 & a_1 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By the Herglotz theorem there is a measure μ on the unit circle such that

$$\langle e_{i+n}, e_i \rangle = a_n = \int_{\mathbb{T}} z^n \, d\mu(z).$$

Kwapień and Mycielski showed that the sequence $\{e_n\}_{n=0}^{\infty}$ is effective if and only if either μ is the Lebesgue measure (orthogonal case) or it is singular with respect to the Lebesgue measure. We now reprove this result by applying our Theorem 1.

Also U is constant on diagonals, i.e. it is a Toeplitz operator,

$$U = \begin{pmatrix} 0 & & & & \\ u_1 & 0 & & & \\ u_2 & u_1 & 0 & & \\ u_3 & u_2 & u_1 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is then unitarily equivalent to the multiplication operator on $H^2(\mathbb{T})$ with the function

$$u(z) = \sum_{n=1}^{\infty} u_n z^n.$$

Moreover

$$\|U\| = \|u(z)\|_{H^\infty(\mathbb{T})} \le 1.$$

Now multiplication with $u(z)$ is a partial isometry if and only if the boundary values of $|u(z)|$ are 0 or 1. By the F. Riesz and M. Riesz theorem, $u(e^{i\theta}) \equiv 0$ or $|u(e^{i\theta})| \equiv 1$. The first case corresponds to the orthogonal case, because $M = 0$. The second case is equivalent to the singularity of the spectral measure μ. Indeed, for $a(z) = \sum_{n=1}^\infty a_n z^n$ we have $(a(z)+1)(u(z)+1) = 1$. Therefore

$$1 + a(z) + \overline{a(z)} = \frac{1 - |u(z)|^2}{|u(z)+1|^2}.$$

By the Fatou theorem

$$\lim_{r \to 1^-} [1 + a(re^{i\theta}) + \overline{a(re^{i\theta})}] = \frac{d\mu}{d\theta}.$$

Therefore μ is singular if and only if $|u(e^{i\theta})| \to 1$ almost everywhere.

References

[1]   S. Kaczmarz, *Approximate solution of systems of linear equations*, Bull. Acad. Polon. Sci. Lett. A 35 (1937), 355–357 (in German); English transl.: Internat. J. Control 57 (1993), 1269–1271.

[2]   S. Kwapień and J. Mycielski, *On the Kaczmarz algorithm of approximation in infinite-dimensional spaces*, Studia Math. 148 (2001), 75–86.

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