ASYMPTOTIC INVARIANTS OF INFINITE GROUPS

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§0. Introduction

A group Γ with a given system of generators $\{\gamma_i\}_{,i\in I,}$ carries a unique *maximal* left invariant distance function for which

$$\mathrm{dist}(\gamma_i, id) = \mathrm{dist}(\gamma_i^{-1}, id) = 1,\ i \in I\ .$$

This distance function, called the *word metric* associated to the generating set $\{\gamma_i\} \subset \Gamma$, makes Γ a subject to a geometric scrutinity as any other metric space.

This space may appear boring and uneventful to a geometer's eye since it is discrete and the traditional local (e.g. topological and infinitesimal) machinery does not run in Γ. To regain the geometric perspective one has to change his/her position and move the observation point far away from Γ. Then the metric in Γ seen from the distance d becomes the original distance divided by d and for $d \to \infty$ the points in Γ coalesce into a connected continuous solid unity which occupies the visual horizon without any gaps or holes and fills our geometer's heart with joy. For example, an Abelian group Γ with a finite generating set $\{\gamma_i\}$ and the corresponding family of metric, $\mathrm{dist}_{\{\gamma_i\}}/d,\ d > 0$, turns in the limit for $d \to \infty$ into a real linear space L of dimension $n = \mathrm{rank}\ \Gamma$ with a *Minkowski metric* (also called a Banach norm) whose unit ball around the origin is a convex centrally symmetric polyhedron in L.

Instead of passing to the limit of metric spaces,

$$\lim_{d\to\infty} (\Gamma, \mathrm{dist}/d)\ ,$$

(technically speaking, one appeals here to the topology in the set of "all" metric spaces coming along with the *Hausdorff metric*; if the ordinary limit does not exist, one resorts to ***ultralimits***, see 2.A), one may remain in the original metric space $(\Gamma,\ \mathrm{dist}_{\{\gamma_i\}})$ and concentrate on the ***asymptotic*** properties of Γ which are expressed in terms of distances between variable points in Γ as these distances $\to \infty$.

0.1. Example: the growth function. Let Γ be a discrete metric space and consider the concentric balls of radii d around a chosen point $\gamma_0 \in \Gamma$,

$$B(d) = \{\gamma \in \Gamma \mid \mathrm{dist}(\gamma, \gamma_0) \le d\}\ .$$

To make the discussion meaningful, we assume that the balls $B(d)$ are finite (subsets) for all d (which is obviously the case for the word metrics of finitely generated groups) and then we have the growth function of Γ that is

$$N(d) = \mathrm{card}\, B(d)\ .$$

For small values of d the function $N(d)$ strongly depends on γ_0 and it is oversensitive to perturbations of the metric in Γ. On the other hand, the behaviour of $N(d)$ for large $d \to \infty$ is essentially independent of γ_0 (under mild assumptions on Γ which are satisfied in all examples we are concerned with in this article) and this behaviour is also rather stable under reasonable changes of the metric.

0.1.A. Subexample: growth of an Abelian group. Let Γ be an Abelian group with the word metric corresponding to a finite generating set. Then (this is almost obvious) $N(d)$ has polynomial growth of degree $n = \mathrm{rank}\,\Gamma$, i.e.

$$A_1 d^n \le N(d) \le A_2 d^n + 1\ , \qquad (*)$$

where A_1 and A_2 are some positive constants depending on the chosen system of generators. It is also not hard to show that there exists a limit

$$A = \lim_{d \to \infty} d^{-n} N(d) , \qquad (**)$$

which is an improvement over the above inequality $(*)$ for large d. (In fact, the convergence in $(**)$ is quite fast, $A - d^{-n} N(d) = O(d^{n-1})$, and it is known to some people in certain quarters when $N(d)$ is actually an honest polynomial in d, compare [Ehr], [Bens], [McM], [Ka-Kho].

0.2. Large-scale equivalence relations between metric spaces. Our "asymptotic" attitude obliges every such equivalence relation to be strong enough to make every *bounded* space X equivalent to a single point (or, at least to an arbitrarily small space). Recall that a metric space X is called *bounded* if

$$\operatorname{Diam} X \underset{\text{def}}{=} \sup_{x_1, x_2} \operatorname{dist}(x_1, x_2) < \infty .$$

Here is the weakest relation of this sort used in geometry

0.2.A. Hausdorff equivalence between metric spaces. Write

$$X \underset{Hau}{\sim} Y ,$$

where X and Y are metric spaces, if there exists a metric on the disjoint union Z of X and Y, such that dist_Z on X equals the original metric dist_X on X and similarly $\operatorname{dist}_Z \mid Y = \operatorname{dist}_Y$, such that the distance functions

$$\delta(x) = \operatorname{dist}_Z(x, Y) \underset{\text{def}}{=} \inf_{y \in Y} \operatorname{dist}_Z(x, y)$$

and

$$\delta(y) = \operatorname{dist}_Z(y, X)$$

are *bounded*, i.e.

$$\sup_{x \in X} \delta(x) < \infty \qquad \text{and} \qquad \sup_{y \in Y} \delta(y) < \infty .$$

Recall that the maximum of the above two suprema is called the *Hausdorff distance* (between subsets X and Y in Z) and the *infimum* of these distances over all metrics on Z which restrict to dist_X on $X \subset Z$ and dist_Y on $Y \subset Z$ is called the (abstract) *Hausdorff distance between metric spaces* X and Y. Thus the relation $X \underset{\text{Hau}}{\sim} Y$ expresses the finiteness of $\operatorname{dist}_{\text{Hau}}(X, Y)$. (Our discussion on the limit of spaces at the beginning of this introduction refers to the convergence of unbounded spaces, $X_i \to X$, $i = 1, 2, \dots$, with respect to the Hausdorff distance between appropriately chosen *bounded* subsets $B_i \subset X_i$ and $B_i' \subset X$. Then the Hausdorff convergence $X_i \to X$, does not preclude the infinite Hausdorff distance between X and every X_i, $i = 1, 2, \dots,$. This is similar to the uniform convergence of functions on bounded or compact subsets of a fixed infinite space, such as $\mathbb{R}^n$, for instance.)

0.2.A$_1$. *Example.* Let Γ be a free Abelian group of rank n and $\{\gamma_1, \dots, \gamma_n\}$ be a (free) system of generators. Then Γ with the corresponding word metric is $\underset{\text{Hau}}{\sim}$ to the n-dimensional Euclidean space $\mathbb{R}^n$ with the so-called ℓ_1-metric

$$\operatorname{dist}(x, y) = \sum_{i=1}^{n} |x_i - y_i| .$$

In fact, the homomorphism $\Gamma \to \mathbb{R}^n$ extending

$$\gamma_1 \mapsto (1, 0, \dots, 0), \ \gamma_2 \mapsto (0, 1, 0, \dots, 0), \dots ,$$

is an isometry and every point of $\mathbb{R}^n$ lies at most distance one from the image of Γ.

0.2.A$_2$. Long-range connectedness. Here is the simplest instance of redefining a standard topological notion in the large-scale terms. A metric space X is called *long-range* (or large-scale)*connected* if there exists a constant $d > 0$ such that every two points x and y in X can be joined by a finite chain of points

$$x_0 = x,\ x_1,\ x_2, \ldots, x_n = y\ ,$$

such that

$$\mathrm{dist}(x_i, x_{i-1}) \leq d,\ i = 1, \ldots, n\ .$$

It is clear, that the long range connectedness in invariant under $\underset{\mathrm{Hau}}{\sim}$. In fact, X is l.r. connected if and only if it is $\underset{\mathrm{Hau}}{\sim}$ to a path connected space. (*Idea of the proof*: add to X the edges between all pairs of points with mutual distances $\leq d$ and extend the metric from X to the resulting space $X_d \supset X$ of paths.

Example. If $X = (\Gamma,$ word metric) then X_1 equals the Cayley graph of Γ which, as we know, is always connected.)

0.2.A$'_2$. L.r. connectedness at ∞. The idea of l.r. connectedness becomes interesting in the group theoretic context when it applies not to a group Γ directly, but to some auxiliary space or a sequence of spaces. An instance of that is l.r. *connectedness at infinity* defined as follows.

A metric space X is called l.r. *disconnected* at infinity if for every $d > 0$ there exist two subsets X_1 and X_2 in X such that

(i) $\mathrm{dist}(X_1, X_2) \geq d$ which means by the definition of this dist between subsets that

$$\mathrm{dist}(x_1, x_2) \geq d \quad \text{for all} \quad x_1 \in X_1, \quad \text{and} \quad x_2 \in X_2\ .$$

(ii) X_1 and X_2 cover almost all X, i.e. the complement $X - (X_1 \cup X_2)$ is bounded.

Then X is called *l.r.connected at* ∞ if for some d the above X_1, X_2 do not exist.

Similarly, using k different X_i instead of two, one defines the *number* of l.r. connected components at ∞ which agrees with the usual notion of the *ends* of groups.

A remark relevant to our discussion is the invariance of the number of ends (i.e. l.r. components at ∞) under the Hausdorff equivalence.

0.2.B. Terminology: "asymptotic", "long-range", "large-scale". These expressions are used interchangeably and the choice of a particular one depends on what kind of associations we want to carry along with a formal argument. Thus "asymptotic" awakens an analyst in our minds, "large scale" shifts the discussion into a more geometric vein and "long range" appeals to whatever is left in us of a physicist.

0.2.C. Lipschitz equivalence and quasi-isometry. Two metrics on the same space, say dist_1 and dist_2, are called (Lipschitz) *equivalent* if the ratios $\mathrm{dist}_1 / \mathrm{dist}_2$ and $\mathrm{dist}_2 / \mathrm{dist}_1$ are ***bounded*** when they are considered as functions on the Cartesian square of the space minus the diagonal. Then two different metric spaces X_1 and X_2 are called *(bi-)Lipschitz equivalent* if there exists a bijection $X_1 \to X_2$ which brings the metric from X_1 to a metric on X_2 which is equivalent to the original metric on X_2.

Example. If dist_1 and dist_2 are word metrics on Γ corresponding to two finite generating sets then they are (obviously) equivalent. Consequently, isomorphic finitely generated groups are $\underset{\mathrm{Lip}}{\sim}$ (this is an abbreviation of "Lipschitz equivalent") for their respective word metrics.

Remark. One can alternatively define the Lipschitz equivalence as an isomorphism in the *category* of *metric spaces* and *Lipschitz map* where a map $f : X_1 \to X_2$ is called Lipschitz if there exists a (Lipschitz) constant $\lambda \geq 0$, such that

$$\mathrm{dist}(f(x),\ f(y)) \leq \lambda\ \mathrm{dist}\,(x, y) \quad \text{for all} \quad x, y \in X_1\ .$$

Notice, that every homomorphism between finitely generated groups is Lipschitz.

Now we use both relation $\underset{\text{Hau}}{\sim}$ and $\underset{\text{Lip}}{\sim}$ and generate with them what is called the *quasi-isometry* equivalence between metric spaces X and Y. In fact, X and Y are quasi-isometric if and only if there exist X' and Y', such that

$$X \underset{\text{Hau}}{\sim} X' \underset{\text{Lip}}{\sim} Y' \underset{\text{Hau}}{\sim} Y .$$

0.2.C₁. *Basic example.* Let X be a Riemannian manifold and let Γ be a finitely generated group *properly* and *isometrically* acting on X. (An action of a discrete group is *proper* if for every compact subset $B \subset X$ the intersection $B \cap \gamma(B)$ is empty for almost all, i.e. for all but finitely many $\gamma \in \Gamma$.) Next, a proper action is called *cocompact* if the quotient space X/Γ is compact. This is equivalent (for the proper actions) to the existence of a compact subset $B \subset X$ whose Γ-translates cover all of X, i.e. $\Gamma B = X$.

The following obvious proposition-example constitutes the major link between the asymptotic group theory and the large-scale Riemannian geometry.

If the action of Γ on X is proper and cocompact then Γ is quasi-isometric to X.

(Here and in future, Γ is given the word metric associated to some *finite* generating set.)

Corollary. *There exist quasi-isometric groups Γ_1 and Γ_2 which are not commensurable.* (Recall that Γ_1 and Γ_2 are *commensurable* if there exist subgroups of finite index, $\Gamma_1' \subset \Gamma_1$ and $\Gamma_2' \subset \Gamma_2$ such that Γ_1' is isomorphic to Γ_2'.)

For example, the product of two hyperbolic planes, $X = H^2 \times H^2$, admits an irreducible cocompact proper action of a discrete group Γ, where "irreducible" means that the induced action of Γ (or rather of the subgroup $\Gamma' \subset \Gamma$ of index ≤ 2 which does not interchange the Cartesian components of $H^2 \times H^2$) on each H^2 is non-proper. Such a Γ is quasi-isometric to the product $\Gamma_1 \times \Gamma_2$ of two surface groups (as $\Gamma_1 \times \Gamma_2$ obviously act on $H^2 \times H^2$) but one can easily show that Γ is not commensurable to $\Gamma_1 \times \Gamma_2$. (The only truly non-trivial point in the above discussion is the existence of an irreducible Γ. This is constructed by arithmetic means, see [Gr-Pa] for an elementary discussion on the matter.)

0.2.C₂. Let us indicate a non-Riemannian version of the above example. Take an arbitrary locally compact group G and consider two discrete subgroups Γ_1 and Γ_2 in G. *Then, if Γ_1 and Γ_2 are finitely generated and cocompact in G then they are quasi-isometric.* Instead of giving a proof (which is trivial anyway) we indicate a further generalization which is motivated by the following features of our picture

(i) The left action of Γ_1 on G commutes with the right action of Γ_2;

(ii) both actions are cocompact on G.

Now we state the following

0.2.C′₂. Topological criterion for quasi-isometry. *Two finitely generated groups Γ_1 and Γ_2 are quasi-isometric if and only if there exist proper actions of Γ_1 and Γ_2 on some locally compact topological space X such that*

(i) *the actions commute;*

(ii) *both actions are cocompact.*

Idea of the proof. We only indicate here how to produce an X starting from a quasi-isometry between Γ_1 and Γ_2. To simplify the matter we assume a Lipschitz equivalence rather than a quasi-isometry which is given by a bi-Lipschitz bijection $f : \Gamma_1 \to \Gamma_2$. Then we consider the space F of *all* maps $\Gamma_1 \to \Gamma_2$ with the pointwise convergence (topologically, this is a countable union of Cantor sets) and observe that the natural actions of Γ_1 and Γ_2 on F are proper and they commute. Then we take the closure X of the $(\Gamma_1 \times \Gamma_2)$-orbit of our $f \in F$ and leave to the reader checking that the actions of Γ_1 and Γ_2 on X are co-compact.

0.2.D. Why Lipschitz? Let us try to relax further our equivalences. Say that two metrics dist_1 and dist_2 on X are *uniformly equivalent on the large-scale* (or l.s.u. *equivalent*) if there exists a real function $\lambda(d)$, $d > 0$, such that

$$\mathrm{dist}_1(x,y) \le \lambda\big(\mathrm{dist}_2(x,y)\big) \quad \text{for all } x \text{ and } y \text{ in } X$$

and conversely,

$$\mathrm{dist}_2 \le \lambda(\mathrm{dist}_1)\ .$$

Then one defines the *l.s.u. equivalence* between metric spaces X and Y by mixing the above with the Hausdorff equivalence. This may appear significantly more general than quasi-isometry but it is not quite so because of the following trivial

Lemma. *If the spaces X and Y are quasi-geodesic (see the definition below) then l.s.u. equivalence between X and Y is the same thing as quasi-isometry.*

Definition. A metric space X is called *quasi-geodesic* if there exist positive constants d and λ, such that for every two points x and y in X there exists a finite chain of points in X,

$$x_0 = x,\ x_2, \dots, x_n = y\ ,$$

such that

$$\mathrm{dist}\,(x_i, x_{i-1}) \le d, \quad i = 1, \dots, n\ , \tag{$*$}$$

and

$$\sum_{i=1}^{n} \mathrm{dist}\,(x_i, x_{i-1}) \le \lambda\ \mathrm{dist}\,(x,y)\ . \tag{$\substack{*\\*}$}$$

Examples. (a) Every group Γ with a word metric is (obviously) quasi-geodesic. In fact it is almost geodesic as one can satisfy $(*)$ and $(\substack{*\\*})$ with $d = 1$ and $\lambda = 1$. (For truly geodesic one asks for an arbitrarily small $d > 0$ in $(*)$.)

(b) Let X be a connected Riemannian manifold. Then it is quasi-geodesic almost by definition as $\mathrm{dist}(x,y)$ appears as the infimum of the lengths of paths in X between x and y. If X is complete as a metric space, then X is truly geodesic as the above infimum is actually achieved by some curve between x and y. (Notice, that this does not exclude manifolds with boundaries which are metrically complete but are not complete in a certain more technical sense.)

(c) Let $\Gamma_1 \subset \Gamma_2$ be a finitely generated subgroup in a finitely generated group Γ. Then the word metric dist_2 restricted to Γ_1 is not, in general, quasi-geodesic in Γ_1. The simplest instance of that is seen in the nilpotent group $\Gamma_2 = \{a, b, c \,|\, [a,b] = c,\ [a,c] = [b,c] = 1\}$ for $\Gamma_1 = \mathbb{Z}$ generated by the (central) element c. Here one immediately sees that the commutator $[a^n, b^n]$ lies in Γ_1 and is equal to c^{n^2}. Thus $\mathrm{dist}_2 \mid \Gamma_1 \underset{\mathrm{Lip}}{\sim} (\mathrm{dist}_1)^{\frac{1}{2}}$, and so dist_1 and dist_2 are uniformly equivalent on Γ_1 but by no means are Lipschitz equivalent.

0.3. From groups to spaces. Take a finitely generated group Γ and let dist be a word metric. Now we try to forget the structure of the group in Γ and look on (Γ, dist) as on a metric space. (Forgetting the structure is not quite complete at this stage as Γ appears as a subgroup in the full isometry group $\mathrm{Iso}\,(\Gamma, \mathrm{dist})$; moreover, $\Gamma = \mathrm{Iso}(\Gamma, \mathrm{dist})$ in most cases.) Furthermore, as we are interested in the large-scale geometry of (Γ, dist) we want our analysis of Γ to be stable under quasi-isometries. In other words our (geo)metric invariants should remain unchanged if we pass to a metric space $(\Gamma', \mathrm{dist}')$ quasi-isometric to (Γ, dist). Now it is not at all easy to recognize Γ by looking at Γ', yet a variety of characteristics of Γ can be reconstructed in terms of Γ' ! These are precisely the asymptotic (or large scale) invariants we are after. In fact, there are certain cases (e.g. $\Gamma = \mathbb{Z}^n$) where one can recapture the group Γ itself up to commensurability.

Given a discrete metric space Γ, one can make it more palatable by adding some meat to Γ in the form of edges and higher dimensional simplices with vertices in Γ, without changing the quasi-isometry type. For example, if Γ is finitely presented, then there is a finite 2-dimensional polyhedron P with $\pi_1(P) = \Gamma$ and the universal covering $\widetilde{P}$ gives us a nice tasty thickening of Γ as $\widetilde{P}$ is connected and simply connected. There is not, in general, any distinguished metric on $\widetilde{P}$ quasi-isometric to Γ, but there is a reasonable class of such metrics which are invariant under the deck transformation group Γ. A geometrically oriented reader may prefer another version of this construction where instead of P one takes a compact Riemannian manifold V (possibly with a boundary) having the fundamental group $\pi_1(V) = \Gamma$ and then passes to the universal cover $\widetilde{V}$ with the induced Riemannian metric. This is a geodesic metric space which is connected and simply connected and where Γ acts properly and cocompactly. So again $\widetilde{V}$ is quasi-isometric to Γ. Thus the large-scale (or asymptotic) geometry of finitely presented group embedds into a more general theory, that is the quasi-isometric geometry of non-compact Riemannian manifolds with no group acting anywhere.

Here one may start feel rather uncomfortable by realizing how much structure has been lost as one passed from Γ to the quasi-isometry class of (Γ, word metric). Indeed one barters here a rigid crystalline beauty of a group for a soft and flabby chunk of geometry where all measurements have built in errors. But something amazing and unexpected happens here as was discovered by Mostow in 1968: the quasi-isometric (or large-scale) geometry turns out by far more rich and powerful than appears at first sight. In fact, one believes nowadays that most essential invariants of an infinite group Γ are quasi-isometry invariant. Well, even so, why should we go through all the pains of reconstructing the group structure from geometry if nobody forces us to leave the pure group theoretic world in the first place? Here are several reasons to do so.

I. The group theoretic structure appears too rigid and limits one to formal combinatorial and algebraic manipulations with no room for transcendental (i.e. the analysis of infinity). This is similar to the elementary theory of metric spaces where the only admissible maps are isometries. It is fruitful to include into the category more morphisms, such as Lipschitz maps, continuous maps, measurable maps etc, thus bringing analysis into play.

II. Even in purely group theoretic questions the geometric *language* may tremendously clarify the picture. For example, from a geometer's (even a topologist's) viewpoint the *free subgroup theorem* ("a subgroup of a free group is free") appears as a painful way of expressing (in a special case) the obvious feature of covering maps $\widetilde{Y} \to Y$,

$$\dim Y = 1 \Rightarrow \dim \widetilde{Y} = 1 \ .$$

(If you have ever tried and failed to drag yourself through the notational rigours of an algebraic proof you must share my relief at the realization that the difficulty there stemmed not from mathematics but from a non-adequate language. I still feel thankful to Dima Kazhdan who explained the matter to me many years ago.) Similar linguistic aberrations can be observed (at least by a geometer) in all corners of the traditional geometric group theory, such as the theory of free products (with and without amalgamations), small cancellation theory etc. (The adherence to the combinatorial language comes from an instinctive mistrust most algebraists feel toward geometry which they regard "non-rigorous".)

II$'$. ***Example: Hyper-Euclidean groups.*** Here is an instance of a useful notion which naturally pops up in the geometric setting and which would become a major nuisance once one committed oneself to a purely algebraic language.

Definition. A group Γ is called *hyper-Euclidean in dimension n* if it admits a proper isometric action on a connected oriented n-dimensional Riemannian manifold X without boundary which admits a *proper Lipschitz* map $f : X \to \mathbb{R}^n$ of *degree one*. It is sometimes desirable to vary this definition (a) by requesting the action to be cocompact, (b) by allowing the action to be quasi-isometric, (c) by admitting maps f of degree ≥ 1, (d) by insisting that X should be contractible. (The hyper-Euclidean conception appears in geometry and topology in the study of positive scalar curvature, see [Gr-Law] and the Novikov higher signature conjecture, see [Fa-Hs] and [C-G-M].)

III. The geometric language brings along a variety of concepts, constructions and ideas unimaginable in the world of pure algebra (such as the above "hyper-Euclidean"). Thus, geometry suggests an impressive number of potentially useful asymptotic invariants of groups about which one may ask the following standard questions,

(A) When and how can one compute such an invariant for a given group? (E.g. how to decide if a given group is hyper-Euclidean.)

(B) What are relations between different invariants?

(C) Which values of an invariant can be realized by some group Γ ? (E.g. when does a given function $f(d)$ appear as the growth function of some finitely generated group Γ ? Compare 0.1.)

(D) How large is the class of groups with a given value of an invariant? (E.g. is every group (of finite cohomological dimension) hyper-Euclidean in dimension n for a given n ?)

IV. When we go from groups to spaces we mentally change the class of essential examples. The most important manifolds studied by geometers are symmetric and locally symmetric spaces (of finite and infinite dimension) and other homogeneous spaces. Besides being remarkably attractive objects in their own right these spaces may serve as measuring rods for the study of more general spaces and groups. A typical instance of that is the above definition of "hyper-Euclidean" where a general manifold is compared in a certain way with $\mathbb{R}^n$.

V. The last but not the least argument in favour of geometry is applicability of geometric ideas (and very rarely of techniques) to the solution of some group theoretic problems. Unfortunately, this is an exception rather than the rule but the situation will probably change with the development of the field.

(I do not know how convincing the above evidence truly is. After all, the actual reason why one approaches a problem from a geometric angle is because one's mind is bent this way. No amount of rationalization can conceal the truth.)

0.4. About this paper. Our purpose here is to demonstrate the efficiency of the geometric language for defining invariants and isolating interesting properties of groups. In many cases we just specialize the standard notions of the asymptotic geometry to groups in order to make them known to the group theorists. We do not attempt a serious study of our invariants and leave the standard questions wide open. At some occasions we treat simple examples lying immediately on the surface. Often we speculate on the possible outcome of the game only not to lose reader's attention, even when we have no inkling of a viable approach to the solution. Thus the reader should not expect new theorems (even not half proven) in this article but he/she may come across an amusing problem.

Remarks on the language. We develop many of our notions in the geometrically friendly surroundings of Riemannian manifolds and similar spaces. This immediately applies to groups in-so-far as the quasi-isometry invariance of the concepts in question is insured. Namely, in order to attribute some geometric property Pr to a group Γ, we just require Pr for some (and thus every) manifold X quasi-isometric to Γ, where, in addition, we may impose some specific condition on X (e.g. being simply connected, contractible etc) if this is needed for the introduction of Pr. On the other hand if we do not want to bother with the quasi-isometry invariance we have to make our choice: either we insist Pr is satisfied for all X (with some specified conditions) quasi-isometric to Γ (sometimes we must insist on a proper isometric action of Γ on X) or we only require the existence of some X quasi-isometric to Γ which has Pr. Of course, when a quasi-isometry invariance of some property is unknown it adds a problem to our list.

0.5. Random historical remarks. The first distinctively asymptotic ideas in the geometric group theory appeared in the mid-fifties in the papers by Efremovic [Ef], Folner [Fo] and Ŝvarc [Ŝv]. Folner gave a geometric criterion for *amenability* of a finitely generated group Γ. The notion of amenability comes from the ergodic theory where a group Γ (which may be infinitely generated) is called *amenable* if every continuous action of Γ on a compact space has an invariant measure.

0.5.A. Folner Criterion. Γ is amenable if and only if there exists an exhaustion of Γ by finite (Folner) subsets $F_1 \subset F_2 \subset \ldots \subset F_i \subset \ldots \Gamma$, such that for every $d > 0$ the *d-boundary* $\partial_d F_i$ (defined below) of F_i has asymptotically a smaller number of elements than F_i,

$$\limsup_{i \to \infty} \operatorname{card}(\partial_d F_i)/\operatorname{card} F_i = 0$$

0.5.A$_1$. *Definition.* The d-boundary of a subset F in a metric space Γ consists of the points $x \in F$ whose distance to the complement $\Gamma - F$ does not exceed d. (An alternative definition which is as good for the present purpose is where $\partial_d\Gamma$ consists of the points in $\Gamma - F$ within distance $\leq d$ from F.)

0.5.A$'_1$. *Example.* Folner's criterion immediately shows that every finitely generated Abelian group is amenable. On the other hand the standard example of a non-amenable group is the free group $\mathbb{F}_2$ on two generators. Some people naively believed for some time that every finitely generated non-amenable group should contain a copy of $\mathbb{F}_2$ but to day there are counterexamples which are infinitely presented (see [Ols]). One still has no construction of a finitely presented non-amenable group containing no $\mathbb{F}_2$.

It is useful to reformulate the Folner criterion with the emphasis on non-amenability.

0.5.A$_2$. Isoperimetric form of Folner criterion. A group Γ is non-amenable if and only if there exist positive constants d and C, such that every finite subset $F \subset \Gamma$ satisfies

$$\operatorname{card} F \leq C \operatorname{card} \partial_d F \tag{$*$}$$

This inequality immediately brings to one's mind the classical linear isoperimetric inequality for bounded domains Ω in the hyperbolic space H^n,

$$\operatorname{Vol}_n \Omega \leq \operatorname{const} \operatorname{Vol}_{n-1} \partial\Omega \,. \tag{${}^*_*$}$$

In fact, the similarity between $(*)$ and $({}^*_*)$ can be made precise as these inequalities are equivalent for quasi-isometric spaces satisfying the following *bounded geometry* conditions.

0.5.A$_3$. *Definitions.*

(a) A discrete metric space Γ is said to be *uniformly quasi-locally bounded* (u.q.-l.b.) if there exists a function $N(d)$, $d \geq 0$, such that every ball $B \subset \Gamma$ of radius d has

$$\operatorname{card} B \leq N(d) \,.$$

(b) A Riemannian manifold X has *locally bounded geometry* (l.b.g.) if there exist positive constants ε and λ such that every ε-ball in X is λ-bi-Lipschitz equivalent to the ε-ball $B_0 \subset \mathbb{R}^n$. (This means the existence of a bi-Lipschitz map $B \to B_0$ with the implied constant λ, compare 0.2.C.)

0.5.A$_4$. *Example.* Every finitely generated group Γ is u.q.-l.b. Every Riemannian manifold X without boundary whose full isometry group is cocompact on X has l.b.g. (If X has a boundary the definition needs a minor adjustment.)

0.5.A$_5$. *Proposition.* *Let Γ be a discrete* u.q.-l.b. *space and X a Riemannian manifold having* l.b.g. *If X is quasi-isometric to Γ then the inequality $(*)$ for Γ* (i.e. *for all finite subsets $F \subset \Gamma$) implies $({}^*_*)$ for X* (i.e. for all bounded domains $\Omega \subset X$) *where the constant in $({}^*_*)$ depends on C in $(*)$ as well as on the implied quasi-isometry. Conversely, $({}^*_*)$ for X implies $(*)$ for Γ.*

The proof appears obvious to a geometrically oriented mind and nowadays even the hard core group theorists are coming to the agreement with this view.

0.5.A$_6$. *Corollary.* *Let a discrete group Γ admit a proper cocompact action on X. Then Γ is non-amenable if and only if X satisfies the (linear isoperimetric) inequality $({}^*_*)$.*

This applies, in particular, to the universal covering X of a compact manifold V with $\pi_1(V) = \Gamma$.

0.5.B. Also in the fifties Efremovich [Ef] observed that the growth-rate of the volume of the balls in the universal covering X of V, i.e.

$$\operatorname{Vol} B(d) \quad \text{for} \quad d \to \infty ,$$

depends only on the fundamental group Γ of V but not on the particular choice of V. In fact he pointed out (now it looks totally obvious) that $\operatorname{Vol} B(d)$ for $d \to \infty$ grows essentially with the same rate as the corresponding function $N_\Gamma(d)$ defined in 0.1,

$$N_\Gamma(d) = \operatorname{card} B_\Gamma(d)$$

for the balls $B_\Gamma(d) \subset \Gamma$.

The ideas of the growth of balls, Folner sets and sets of conjugacy classes in groups (especially in fundamental groups of manifolds of negative curvature, see $[\text{Mar}]_1$ $[\text{Mar}]_2$) were quite popular in the sixties among ergodic theorists in Moskow and Leningrad. (Much of these ideas I learned at the time from A. Vershik, D. Kazhdan and G. Margulis.) Then the geometers took a part in the story and related the growth to curvature. The first results here for non-negative curvature are due to A. Ŝvarc [Ŝv]. Similar results were obtained independently by J. Milnor (see [Mil]) who stated the following

0.5.B$_1$. *Conjecture.* The growth function $N_\Gamma(d)$ of a finitely generated group Γ is either *polynomial* (i.e. $N(d) \leq 1 + Cd^n$ for some positive C and n) or *exponential*, which means

$$N(d) \geq A^d \quad \text{for some } A > 1 .$$

This conjecture is known to be true for linear groups (i.e. subgroups in GL_N) by the work of Tits who proved the following

0.5.B$_2$. Freedom theorem (see $[\text{Tit}]_1$). *Every finitely generated linear group Γ is either virtually solvable* (i.e. contains a solvable subgroup of finite index) *or contains a copy of $\mathbb{F}_2$, the free group on two generators.* This implies the conjecture, for the groups $\Gamma \supset \mathbb{F}_2$ obviously have exponential growth; furthermore, the residually solvable groups Γ have $N_\Gamma(d)$ exponential unless they are virtually nilpotent. The latter are known to have polynomial growth and are, in fact, characterized by this property, see $[\text{Tit}]_2$ and references therein.

0.5.B$_3$. Milnor's conjecture is still open for *finitely presented* groups but recently Grigorchuk found a remarkable class of finitely generated infinitely presented groups of *intermediate* growth where $N(d)$ behaves as A^{d^α}, $0 < \alpha < 1$. (Grigorchuk's groups Γ act on an infinite regular tree fixing a vertex and therefore are residually finite without being linear. The essential feature of Γ responsible for the intermediate growth is the existence of mutually isomorphic subgroups $H \subset \Gamma$ and H' is the Cartesian product $\Gamma\times\Gamma\times\Gamma\times\Gamma\times\Gamma\times\Gamma\times\Gamma\times\Gamma$. See [Gri] for a comprehensive survey of the growth theory.) The current version of the growth conjecture due to Grigorchuk reads

There exists $\alpha > 0$ (possibly $\alpha = \frac{1}{2} - \varepsilon$) such that either $N_\Gamma(d)$ grows faster than A^{d^α} or Γ has polynominal growth (and, hence, is virtually nilpotent).

0.5.B$_4$. There is a simple link between growth and amenability.

If Γ is non-amenable then it has exponential growth.

This immediately follows by applying the (linear isoperimetric) inequality (∗) to the concentric balls $B(d) \subset \Gamma$.

Thus Grigorchuk's examples provide a new class of amenable groups. Prior to his work all known amenable groups were obtained from finite and Abelian groups (which are easily seen to be amenable) by the following three operations.

1. *Extensions*: Here one uses the fact that if in the exact sequence $1 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$ the groups Γ_1 and Γ_3 are amenable, then so is Γ_2.

2. *Infinite unions*: If Γ is a union of an increasing family of amenable subgroups then Γ is amenable.

3. *Taking subgroups and factorgroups*: Every subgroup of an amenable group is amenable and so is every factorgroup.

Notice that in the course of such a construction one may have intermediate groups infinitely generated even if the final result is f.g., as was pointed out, I believe, by H. Bass. Also recall that Grigorchuk's groups are *not* finitely presented and one has still no ways to produce finitely presented amenable groups apart from 1, 2 and 3.

0.5.C. The main source of infinite groups in the differential geometry is provided by manifolds of non-positive sectional curvature, $K \leq 0$. One of the first asymptotic results here is the following result by A. Avez (see [Av] and §6).

0.5.C$_1$. Non-amenability theorem. *Let V be a compact manifold without boundary and $K(V) \leq 0$. Then the fundamental group of V is non-amenable unless V is flat (and then $\pi_1(V)$ is virtually Abelian).*

The proof suggested by Avez is based on the following

0.5.C$'_1$. Non-amenability criterion. *Let X be an n-dimensional Riemannian manifold which admits a vector field Z with the following two properties*

(i) *the length of Z is uniformly bounded*

$$\sup_{x \in X} \|Z(x)\| < \infty ,$$

(ii) *the divergence of Z is strictly positive,*

$$\inf_{x \in X} \operatorname{div} Z(x) > 0 .$$

Then every bounded domain Ω in X with a smooth boundary satisfies

$$\operatorname{Vol}_n \Omega \leq \operatorname{const} \operatorname{Vol}_{n-1} \partial\Omega . \qquad (+)$$

Furthermore, the conclusion remains valid if we replace (ii) *by the following weaker condition* (ii)$_0$ *and additionally assume that X has locally bounded geometry* (see 0.5.A$_3$).

(ii)$_0$ div $Z(x) \geq 0$ for all $x \in X$ *and there exist positive numbers d and ε such that for every ball $B \subset X$ of radius d the integrated divergence of Z over B is at least ε,*

$$\int_B \operatorname{div} Z(x) dx \geq \varepsilon .$$

Idea of the proof. Integrate div Z over Ω and apply Stokes' theorem.

Avez applies this criterion to the gradients Z of *horofunctions* in the universal covering X of V. Recall that a horofunction $h : X \to \mathbb{R}$ is a limit of a sequence of additively normalized distance functions $h_i(x) = \operatorname{dist}(x, x_i) - c_i$, where $x_i \in X$ is a sequence of points going to infinity and c_i is a sequence of constants. If $K \leq 0$ then horofunctions h (as well as distance functions) are (known to be) convex and so div grad $h \geq 0$. In general, the strict inequality

$$\operatorname{div} \operatorname{grad} h \geq \varepsilon > 0$$

needs strictly negative curvature,

$$K(X) \leq \kappa < 0 ,$$

but in the case where X covers a compact non-flat manifold V Avez produces a horofunction h whose gradient satisfies (ii)$_0$.

0.5.C$_1''$. *Remark.* There is a version of the non-amenability criterion for discrete spaces X (e.g. for groups Γ) where "field" signifies a map $Z : X \to X$ and the conditions (i) and (ii) are replaced by

(i)' $\mathrm{dist}(Z, \mathrm{id}) < \infty$, i.e.

$$\sup_{x \in X} \mathrm{dist}\,(x, Z(x)) < \infty \ ,$$

and

(ii)' Z is strictly compressing, in the following sense

$$\mathrm{card}\left(Z^{-1}(x)\right) \geq 2 \quad \text{for all } x \in X \ .$$

(There is a more general version of this criterion where Z is a multi-valued map which assigns to each $x \in X$ a probability measure on Z.)

0.5.C$_2$. *Problem.* Manifolds with sectional curvature $K \leq 0$ and their fundamental groups have been extensively studied for the last 20 years. Yet the following fundamental question remains unresolved.

Let Γ be a finitely generated group of finite (cohomological) dimension which means for us the existence of a finite dimensional aspherical polyhedron with the fundamental group isomorphic to Γ. (Recall, that "aspherical" here amounts to the universal covering being contractible.) *Does there exist a complete Riemannian manifold V without boundary with $K(V) \leq 0$ whose fundamental group is isomorphic to* Γ ?

0.5.D. The fundamental group influences the geometry of the underlying manifold V most profoundly if V has strictly negative curvature, $K(V) \leq \kappa < 0$. In fact, even if we change the metric of V the basic asymptotic geometry of the universal covering $\widetilde{V}$ retains the distinctively hyperbolic behaviour. This was discovered 70 years ago by M. Morse for compact surfaces V of genus ≥ 2 (see [Mor]) and then extended to the higher dimensional closed manifolds V with $K(V) < 0$ by Klingenberg (see [Kli]). (*Warning*: the main result claimed in [Kli] is incorrect but the background discussion is valid and interesting.)

The hyperbolic geometry took a new turn in 1968 when G. Mostow (see [Most]$_1$) discovered his amazing asymptotic proof of the rigidity of lattices in $O(n,1)$. (The quasi-isometric structure of Mostow's arguments was crystallized two years later by Margulis [Mar]$_3$ and then extended by Mostow to other rank one groups in his book [Most]$_2$ which contains a wealth of hyperbolic and semi-hyperbolic ideas.)

0.5.E. The quasi-isometric approach to infinite groups has a measure theoretic (or probabilistic) counterpart, where groups are studied modulo the *measure equivalence* defined as follows. First we introduce the standard (proper cofinite) action of Γ on a measure space X: the space X is isomorphic to the union of the copies of the unit interval $I = [0,1]$ indexed by $\gamma \in \Gamma$, i.e. $X = \Gamma \times I$ and Γ acts by

$$\gamma(\gamma', t) = (\gamma\gamma', t) \ .$$

Now comes our

0.5.E$_1$. ***Definition.*** Γ_1 and Γ_2 are called measure equivalent if there exists a measure theoretic isomorphism $\Gamma_1 \times I \to \Gamma_2 \times I$ which brings the action of Γ_1 from $\Gamma_1 \times I$ to an action on $\Gamma_2 \times I$ which *commutes* with the action of Γ_2. (Compare 0.2.C$_2'$.)

0.5.E$_2$. ***Example.*** Let Γ_1 and Γ_2 be (cocompact or not) lattices in a locally compact separable group G (Γ is a *lattice* if $mes(G/\Gamma) < \infty$). Then Γ_1 and Γ_2 are measure equivalent. In fact the left Γ_1-translations on G commute with the right Γ_2-translations while for every lattice $\Gamma \subset G$ the measure space G acted by Γ is Γ-equivalently isomorphic to $\Gamma \times I$.

The measure theoretic study of groups is often conducted in the language of random walks on groups where the first results relating amenability to the *heat decay* and the *spectrum of the (combinatorial) Laplace operator* are due to H. Kesten (see [Kes]). Then H. Furstenberg has introduced the key notion of the (Furstenberg) boundary of Γ (see [Fur]) and the development of these ideas has culminated in Margulis' super-rigidity and arithmeticity theorems in the mid-seventies (see [Mar]$_{4,5}$). Notice that a parallel study of the measure equivalence for groups has been carried over by ergodic theorists under the banner of orbit equivalence starting from a seminal paper by Dye [Dye]. One remarkable conclusion of this study is the measure equivalence between every two amenable groups, see [C-F-W].

0.5.F. Finally we indicate a topological source of inspiration in the geometric group theory. From our asymptotic viewpoint the most important starting events were J. Stallings' theorem on the *ends of groups* (see [Sta]$_1$) and Mishchenko's approach (see [Mis]) to the *Novikov higher signature conjecture* which followed the earlier remarkable work by G. Lusztig. (A special case of the Novikov conjecture states that every homotopy equivalence between two closed *aspherical* manifolds carries the rational Pontryagin classes of the first manifolds to the Pontryagin classes of the second one. Mishchenko's theorem implies that this is the case if one of the manifolds admits a Riemannian metric with non-positive sectional curvature. Despite the recent progress one still cannot prove or disprove the full Novikov conjecture even in the above mentioned case of aspherical manifolds.) Mishchenko's method is based on a construction of certain *Fredholm representations* of the fundamental groups Γ of manifolds of non-positive curvature. (A Fredholm representation of Γ is given by a pair of actual unitary representations of Γ, say on Hilbert spaces H_1 and H_2, which are related by a *Fredholm* operator $f : H_1 \to H_2$ commuting with the Γ actions on H_1 and H_2 *modulo compact operators.*

Example: Take $\Gamma = \mathbb{Z}$, $H_1 = H_2 = \ell_2(\mathbb{Z})$ and f the multiplication by the function $\varphi : \mathbb{Z} \to \mathbb{R}$ which equals -1 for $z < 0$ and $+1$ for $z \geq 0$). One may think that the infinite dimensional representation theory provides the third asymptotic way in group theory parallel to the quasi-isometric and measure theoretic ones. In fact there are some remarkable connections between representations and the geometry of groups stemming from *Kazhdan's T-property* (discovered by Kazhdan during the Moskow Congress of Mathematicians in 1966. Instead of giving the definition we recall that by Kazhdan's theorem every lattice in $SL_n\mathbb{R}$ (e.g. $SL_n\mathbb{Z}$) for $n \geq 3$ has property T). For example, (this was pointed out to me by Kazhdan) *if Γ is T and $\Gamma' \subset \Gamma$ is a subgroup for which the quotient space Γ/Γ' has subexponential growth (in an obvious sense) then, in fact, Γ/Γ' is a finite set.* (This must be clear for those who know the definition of T. Just look on the obvious representation of Γ in $\ell_2(\Gamma/\Gamma')$ and observe that the subexponential growth delivers the trivial representation weakly into $\ell_2(\Gamma/\Gamma')$ in the same way as such growth of the group itself makes it amenable.)

0.5.F$_1$. The "zero in the spectrum" conjecture. Let us indicate an honestly asymptotic (i.e. quasi-isometry invariant) characteristic of infinite groups Γ belonging to the circle of ideas surrounding the Novikov conjecture. We discuss here only the simplest case where Γ admits a proper cocompact action on a *contractible* manifold X without boundary, for example, $\Gamma = \pi_1(V)$ for a closed aspherical manifold V (where X equals the universal covering of V). Fix an equivariant triangulation of X and let $\ell_2^i = \ell_2^i(X)$ be the space of square summable i-dimensional cochains (these are functions on the set Σ_i of i-simplices and we require them to be in $\ell_2(\Sigma_i)$). Both boundary and coboundary operators ∂ and δ are ℓ_2-bounded and they are mutually adjoint. One defines the combinatorial Laplace operator $\Delta_* = \Delta_0 \oplus \Delta_1 \oplus \cdots \oplus \Delta_n$, $n = \dim X$ on $\ell_2^* = \ell_2^0 \oplus \cdots \oplus \ell_2^n$ by $\Delta_* = \partial\delta + \delta\partial$ (compare §8 where we use slightly different notations). This is a bounded self-adjoint operator on ℓ_2^* which is non-strictly positive. The non-strictness may be due to the *ℓ_2-cohomology* in some dimension i, that is the kernel of $\Delta_i : \ell_2^i \to \ell_2^i$, denoted $\overline{\ell_2 H}^i(X)$ (which is independent on the choice of a Γ-invariant triangulation). But it may happen that $\overline{\ell_2 H}^*$ is zero but yet Δ_* is *not* strictly positive, i.e. spec $\Delta_* \ni 0$. This happens when there exists an "almost kernel" of Δ_i for some i. More precisely, there exists a sequence $\varphi_j \in \ell_2^i$, $j = 1, 2, \ldots$, with $||\varphi_j||_{\ell_2} = 1$, for all j, such that

$$||\Delta_i \varphi_j||_{\ell_2} \to 0 \quad \text{for } j \to \infty .$$

Examples. (a) If we look at the Δ_0-component of Δ_* which corresponds to the ordinary Laplace operator on functions then (obviously) $\overline{\ell_2 H}^0 = 0$ (provided Γ is infinite and X is connected) and that spec $\Delta_0 \ni 0$ if and only if Γ is amenable (compare [Kes]).

(b) Let X be a symmetric space of non-compact type. Then one knows (this is non-trivial) that $\overline{\ell_2 H}^i = 0$ unless $i = \frac{1}{2}\dim X$. Yet spec Δ_i always contains zero for some i (see §8). For instance, if $X = H^{2k+1}$ the odd dimensional hyperbolic space (of constant negative curvature), then $\overline{\ell_2 H}^*(X) = 0$ but spec $\Delta_i \ni 0$ for $i = k,\ k+1$. (If $X = H^{2k}$, then $\overline{\ell_2 H}^k(X) \neq 0$).

It is not hard to show that $\overline{\ell_2 H}^*$ depends only on Γ and that the vanishing or non-vanishing of $\overline{\ell_2 H}^i$ for a given i is a quasi-isometry invariant of Γ. Similarly, the strict positivity of Δ_i (i.e. non-inclusion $0 \notin \operatorname{spec}\Delta_i$) is a q.-i. invariant of Γ.

Notice that the strict positivity of Δ_i is equivalent to the bound

$$\langle \Delta_i \varphi, \varphi \rangle \geq \lambda \|\varphi\| \qquad (+)$$

for all $\varphi \in \ell_2^i$ and a fixed $\lambda > 0$. The upper bound of the possible λ's marks the lower bound of $\operatorname{spec}\Delta_i$ and so (+) is equivalent to $\operatorname{spec}\Delta_i \geq \lambda$. On the other hand, the inclusion $0 \in \operatorname{spec}\Delta_i$ expresses the *non-strictness* of positivity of Δ_i, which means that for every $\lambda > 0$ there is a cochain $\varphi \in \ell_2^i(X)$ violating (+). (These φ for $\lambda \to 0$ constitute the "almost kernel" of Δ_i mentioned earlier.) Then the strict positivity of Δ_* (or $0 \notin \operatorname{spec}\Delta_*$) signifies $\Delta_i > 0$ for all $i = 0, 1, \ldots, n = \dim X$. This can be equivalently expressed in terms of δ rather than Δ_*. Namely one looks at the ℓ_2-cochain complex (ℓ_2^*, δ),

$$0 \to \ell_2^0 \xrightarrow{\delta} \ell_2^1 \xrightarrow{\delta} \ldots \xrightarrow{\delta} \ell_2^n \to 0$$

and observes that (ℓ_2^*, δ) is *chain contractible* in the category of *topological* vector spaces and *bounded* operators if and only if $\Delta_* > 0$. (This suggests a continuation of the present discussion to the ℓ_p-category for $p \neq 2$, see §8.)

To clarify further the idea of the ℓ_2-cohomology and of $\operatorname{spec}\Delta_*$ near zero we turn again to the case $\Gamma = \pi_1(V)$ for a compact V and let $\widetilde{V}_k \to V$ $k = 1, 2, \ldots$, be a sequence of finite coverings which converges to the universal covering $\widetilde{V}$ of V in the following sense. For every *compact* subset $B \subset \widetilde{V}$ there exists k such that the (natural covering) map $\widetilde{V} \to \widetilde{V}_k$ is injective on B. (Algebraically, one could say that the intersection of the covering subgroup $\pi_1\left(\widetilde{V}_k\right) \subset \pi_1(V)$ consists of {id}.) Denote by $\widetilde{b}_i\left(\widetilde{V}_k\right)$ the *normalized* Betti numbers

$$\widetilde{b}_i\left(\widetilde{V}_k\right) = \operatorname{rank} H^i\left(\widetilde{V}_k, \mathbb{R}\right) / \operatorname{ind}_k ,$$

where $\operatorname{ind}_k = \operatorname{card}\left(\pi_1(V)/\pi_1(\widetilde{V}_k)\right)$, that is the number of the sheets of the covering $\widetilde{V}_k \to \widetilde{V}$. These numbers are related to the ℓ_2-cohomology of $X = \widetilde{V}$ by the following

0.5.F$'_1$. Kazhdan's criterion (see [Kaz]$_2$, [Ch-Gr]$_2$). *If*

$$\limsup_{k \to \infty} \widetilde{b}_i\left(\widetilde{V}_k\right) > 0 , \qquad (*)$$

(i.e. *if* rank $H^i\left(\widetilde{V}_k, \mathbb{R}\right)$ *grows proportionally to the number of sheets), then the ℓ_2-cohomology of $\widetilde{V}$ does not vanish,*

$$\overline{\ell_2 H}^i(\widetilde{V}) \neq 0 .$$

Proof (compare 8.A$_3$). Using (*) one can (easily) produce *harmonic* i-cochains $\widetilde{c}_k$ on $\widetilde{V}_k$ (harmonic means $\Delta_i \widetilde{c}_k = 0$) such that $\|\widetilde{c}_k\|_{\ell_2} = 1$ and such that the value of $\widetilde{c}_k$ on some simplex $\widetilde{\sigma}_i \subset \widetilde{V}_k$ satisfies

$$|\widetilde{c}_k(\widetilde{\sigma}_i)| \geq \varepsilon > 0$$

for some ε *independent* of k. Then these cochains (obviously) converge (or rather subconverge) to a *non-zero* harmonic ℓ_2-cochain on $\widetilde{V}$. Q.E.D.

Question. Can one reverse Kazhdan's criterion? Namely, suppose $\ell_2 H^i(\widetilde{V}) \neq 0$. Does it follow that $H^i\left(\widetilde{V}_k, \mathbb{R}\right)$ grows proportionally to ind_k (or grows at all for $k \to \infty$) ?

Next, we observe that Kazhdan's criterion generalizes to the spectrum of Δ_* as follows. Denote by $N_\lambda\left(\widetilde{V}_k, \widetilde{\Delta}_i\right)$ the number of the eigenvalues of the Laplace operator $\widetilde{\Delta}_i$ on $\widetilde{V}_k$ in the interval $[0, \lambda]$.

Extended Kazhdan's criterion. *If*

$$\limsup_{k\to\infty} N_\lambda\left(\widetilde{V}_k, \widetilde{\Delta}_i\right) / \operatorname{ind}_k > 0 ,$$

for every $\lambda > 0$ then the Laplace operator Δ_i on the universal covering $\widetilde{V}$ contains zero in the spectrum,

$$\operatorname{spec} \Delta_i \ni 0 .$$

Notice that this criterion is (easily) reversible. Namely,

$$\textit{If } 0 \in \operatorname{spec} \Delta_i \textit{ on } \widetilde{V}, \text{ then } \liminf_{k\to\infty} N_\lambda\left(\widetilde{V}_k, \widetilde{\Delta}_i\right) / \operatorname{ind}_k > 0 \quad \text{for every } \lambda > 0 .$$

Finally, we observe that everything we have said has a de Rham counterpart for the Hodge-Laplace operator on the differential forms. In fact, the de Rham L_2-cohomology of X (e.g. of $\widetilde{V}$) are canonically isomorphic to the combinatorial ones and the strict positivity of Δ_i on i-forms is equivalent to that for the combinatorial Δ_i.

Now we are ready to formulate the *zero in the spectrum problem* for Γ acting on a contractible manifold X without boundary with X/Γ compact.

0.5.F_1''. *Problem*: *Does the spectrum of Δ_* on $\ell_2^*(X)$ always contain zero?*

The positive answer is known for a variety of groups, first of all for those, where X is hyper-Euclidean, i.e. admits a proper Lipschitz map $X \to \mathbb{R}^n$, $n = \dim X$, of non-zero degree (see $[\text{Gro}]_{13}$ and §8).

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Contents.

§**7.** Hyperbolic groups and $K < 0$; hierarchy in the hyperbolic ranks; pinching and conformal geometry at infinity; round trees and strongly branched polyhedra; Kähler and anti-Kähler groups; uniform embeddings and a-T-menability.

§**8.** L_p-cohomology and conformal dimensions at infinity; L_2-Betti numbers and spectrum near zero; vanishing and non-vanishing of $L_p H^*(X)$.

Apppendix to §**8**: Conformally hyperbolic groups and spaces.

§**9.** Density of random groups and other speculations.

§1. Nets and thickenings; asymptotic connectedness and contractibility; large-scale dimension and (co)homology; width and filling invariants.

Let us indicate some simple operations which allow one to go back and forth from discrete spaces to "continuous" ones in the same Hausdorff equivalence class (see 0.2.A).

1.A. Nets. A subset Y in a metric space X is called a *net* if

$$\operatorname{dist}(x,Y) \underset{\text{def}}{=} \inf_{y\in Y} \operatorname{dist}(x,y) \leq c \tag{$*$}$$

for a fixed c independent of X. If we want to keep track of the constants this is called a *c-net* (and if for some reason the constant is perceived as a small one, we speak of ε-nets).

Y is called *δ-separated* if

$$\operatorname{dist}(y_1,y_2) \geq \delta \quad \text{for all } y_1 \neq y_2 \ .$$

Obviously, for every $c > 0$ there exists a $2c$-separated c-net in X. (For example, every *maximal* $2c$-separated subset $Y \subset X$ is a c-net.)

1.A′. Lipschitz on the large-scale. Using nets one can give another version of the definition of quasi-isometry (which is obviously equivalent to the one in 0.2.C): X_1 is quasi-isometric to X_2 if and only if there exist nets $Y_1 \subset X_1$ and $Y_2 \subset X_2$, which are related by a bi-Lipschitz homeomorphism $Y_1 \leftrightarrow Y_2$. More generally, call $f : X_1 \to X_2$ a *λ-Lipschitz map* on *d-scale* if

$$\operatorname{dist}_{X_2}\left(f(x)\, f(x')\right) \leq \lambda \operatorname{dist}_{X_1}(x,x') + d \ ,$$

for all $x_1,\ x_1' \in X$. If the constants λ and d are suppressed we say that f is a *large-scale Lipschitz* map. Clearly, every such map is honestly Lipschitz on every separated net $Y_1 \subset X_1$.

One can think of quasi-isometries as of isomorphisms in the category of metric spaces and large-scale Lipschitz maps. To make it precise one should identify certain maps. Namely, call $f,\ g : X_1 \to X_2$ *parallel* if

$$\operatorname{dist}_{X_2}(f,g) \underset{\text{def}}{=} \sup_{x\in X_1} \operatorname{dist}_{X_2}\left(f(x),g(x)\right) < \infty \ .$$

This (obviously) is an equivalence relation and we take the equivalence classes of large-scale Lipschitz maps for the morphisms in our category. Then isomorphisms in this category are quasi-isometries.

Questions. One may think that for separated (i.e. δ-separated for $\delta > 0$) spaces quasi-isometry is the same as bi-Lipschitz equivalence but, in general, this is not so. Yet, the examples I have in mind are rather artificial and I do not know of a practical criterion on a metric space X that would insure a bi-Lipschitz equivalence between every two separated nets in X. This property of nets seems to be unknown even for $X = \mathbb{R}^n$, $n \geq 2$ (if $X = \mathbb{R}$, then every two separated nets are, obviously, bi-Lipschitz) as well as for the hyperbolic spaces H^n, $n \geq 2$, and infinite regular trees. Similarly, one does not know when two subgroups of finite index in a finitely generated groups Γ are mutually bi-Lipschitz equivalent. For example, are the free groups $\mathbb{F}_2$ and $\mathbb{F}_3$ bi-Lipschitz? When is Γ bi-Lipschitz to $\Gamma \times \Gamma_0$ for a finite group Γ_0 ?

1.B. Thickenings. This is a passage from a net to an ambient space. Namely a *c-thickening* of a metric space X is a larger metric space $Z \supset X$ whose distance function extends that of X and such that X is a c-net in Z.

With this notion we can give a different version of the Hausdorff equivalence (denoted $\underset{\text{Hau}}{\sim}$). $X_1 \underset{\text{Hau}}{\sim} X_2$ *if and only if* X_1 *and* X_2 *possess isometric thickenings.* This is obvious.

Examples of thickenings. (a) Join every two points x_1 and x_2 in X with $\operatorname{dist}(x_1,x_2) = d < 2c$ by a copy of the segment $[0,d]$ and equip the resulting space $Z = \{[0,d]_{x_1,x_2}\}, x_1, x_2 \in X$ with the *obvious* metric. We mean the *maximal* metric whose restriction to X is $\leq \operatorname{dist}_X$ and whose restriction to each copy of [0,d] is $\leq$ the standard metric on $[0,d]$. (From now on we call metrics of that kind "obvious" or "maximal" and do not say more, compare $[\text{Gro}]_{14}$.) Clearly this Z is a c-thickening of X. For example, if X is a group Γ with a word metric then this Z for $d=1$ is the Cayley graph of the group Γ.

(b) The above construction can be generalized by replacing $[0,d]$ by a k-dimensional Euclidean ball B or radius $c=d/2$ and by attaching to X the copies of this ball using all distance *decreasing maps* $\varphi:\partial B\to X$ (where ∂B denotes the boundary sphere). This is again a c-thickening of X. For example, if X is the Cayley graph of a finitely presented group Γ where all words in the presentation have length $\leq \pi c$, then this thickening Z for $k=2$ contains the universal covering of the standard cell 2-complex associated to the presentation of Γ. Notice that this Z is simply connected.

(c) *The Rips complex.* Assume for simplicity's sake that X is a discrete space and let $P_d(X)$ be the simplicial complex whose vertex set equals X and where a subset S is spanned by a simplex if and only if the mutual distances between the points in S are all $\leq d$. Here again there is a natural metric in $P_d(X)$ which makes it a thickening of X. This is named after Ilia Rips since he made the following simple and beautiful observation: *If Γ is a word hyperbolic group with the word metric then $P_d(\Gamma)$ is contractible for all sufficiently large d.* (See $[\text{Gro}]_{14}$. This is the only case I know of where there is *canonical contractible* thickening of a metric space.)

(d) *L_∞-thickenings.* Embed X into the space $L_\infty(X)$ of bounded functions φ on X, with the sup-metric: $\text{dist}(\varphi,\psi)=\sup\limits_X|\varphi-\psi|$, by the embedding $x\mapsto\varphi(x')=\text{dist}_X(x,x')-\text{dist}_X(x_0,x')$ for a fixed point $x_0\in X$. This embedding is (obviously) isometric and the c-neighbourhoods of $X\hookrightarrow L_\infty(X)$ are c-thickenings of X which are universal in a certain sense.

1.C. Large-scale k-connectedness. We have already discussed 0-connectedness in $0.2.\text{A}_2$. Now we generalize and say that X is *large-scale k-connected* if for every thickening $Y\supset X$ there exists yet a bigger thickening $Z\supset Y$ which is k-connected in the usual sense, i.e. Z is path connected and

$$\pi_i(Z)=0,\ i=1,...,k\ .$$

Warning. A path connected space is automatically large-scale connected but this is not so with the k-connectedness for $k\geq 1$ (to see this look at Fig. 3 in $1.\text{D}_1$).

$1.\text{C}_1$. ***Example.*** *A finitely generated group Γ is large-scale 1-connected* (i.e. *simply connected) if and only if it is finitely presentable.* This is obvious.

$1.\text{C}_1'$. One may try to modify the (word) metric of an infinitely presented Γ in order to recapture the (asymptotic) 1-connectedness. For example, when attaching (to the Cayley graph) the 2-disks corresponding to the relations one may take the (maximal) metric for which every disk has metric $\leq$ the *unit* cone over the boundary which is a circle of the length $=$ number of the letters in the word w in the relation $w=1$ associated to the disk in question. (An essentially equivalent way to organize the new metric is by insisting that every disk has diameter ≤ 2.) Every piece of the above word w in this new metric has diameter ≤ 2 which makes Γ locally infinite. This can be corrected by somehow ordering the relations (e.g. according to their lengths) $r_1, r_2\dots r_i\dots$, and making a new distance for which diam D_i grows as a given sequence φ_i satisfying $\varphi_i\to\infty$ for $i\to\infty$. The geometry of such metrics may be interesting.

$1.\text{C}_2$. The above example $(1.\text{C}_1)$ obviously extends to all dimensions.

A finitely generated group Γ is k-connected on the large-scale if and only if there exists a proper simplicial action of Γ on a $(k+1)$-dimensional k-connected simplicial complex X with compact quotient X/Γ.

The proof is (nearly) obvious in view of (b) in 1.A′. In fact, every thickening of a $(k-1)$-connected space to a k-connected one can be achieved by attaching $(k+1)$-balls and this can be done Γ-equivariantly if required. What is more interesting is the following

$1.\text{C}_2'$. ***Corollary.*** *The existence of the above X acted upon by Γ is a* q.-i. *(quasi-isometry) invariant* of Γ.

Example. Let Γ have $H^3(\Gamma, \mathbb{R})$ infinite dimensional (see [Sta]$_1$). Then it is not 2-connected on the large-scale and is not quasi-isometric to any group $\Gamma' = \pi_1(P)$ where P is a finite polyhedron having $\pi_2(P) = 0$.

1.C$_3$. Combing. A family of maps $f_t : X \to Y$, $t \in T$, is called *δ-continuous (on t)* if the jumps at the discontinuity points t are uniformly bounded by δ. More precisely, for every bounded subset $B \subset X$, every $t \in T$ and every $\delta' > \delta$ there exists a neighbourhood $S \subset T$ of t, such that

$$\operatorname{dist}_Y \left(f_t(x) - f_s(x) \right) \le \delta'$$

for all $x \in B$ and $s \in S$.

Example. Let $f_t : X \to Y$, $t \in \mathbb{R}_+$ be a continuous family which is, moreover, uniformly continuous on bounded subsets in X, i.e. f_t is 0-continuous according to the above definition. Take a net $Y' \subset Y$ and let $p' : Y \to Y' \subset Y$ be some map parallel to the identity map $id : Y \to Y$, (e.g. p' move each $y \in Y$ to a nearest point in Y). Then the composed family of map $f'_t = p' \circ f_t$ clearly is δ-continuous for some δ depending on $\varepsilon = \operatorname{dist}(id, p') < \infty$.

A family $f_t : X \to X$, $t \in \mathbb{R}_+$, is called a *combing* (see [E-C-H-P-T], [Ger]$_6$) or a *(uniformly Lipschitz) contraction on the large-scale* if

(a) there exist λ, $d \ge 0$, such that f_t is λ-Lipschitz on the scale d for every $t \in \mathbb{R}_+$,

(b) f_t is δ-continuous in t, for some $\delta \ge 0$,

(c) $f_0 = \mathrm{id} : X \to X$,

(d) f_t converges to a constant map for $t \to \infty$. More precisely,

$$\operatorname{dist}(f_t(x), x_0) \to 0, \ t \to \infty ,$$

where $x_0 \in X$ is a fixed point and the convergence is uniform on the bounded subsets in X.

Basic example. Let $f_t : X \to X$, $t \in \mathbb{R}_+$, be an actual (continuous) contraction of X uniform on bounded subsets, such that every f_t, $t \in \mathbb{R}_+$ is a contracting, i.e. 1-Lipschitz map. (Take, for instance, the maps $f_t : \mathbb{R}^n \to \mathbb{R}^n$ for $f_t(x) = e^{-t}x$.) Now, if we replace X by a quasi-isometric space X', then what becomes of f_t is a *combing* of X'. (In the above example $f_t(x) = e^{-t}x$ the corresponding combing of X' is Lipschitz on both variables t and x', see §9 in [E-C-H-P-T].) Conversely, suppose we are given an X with a combing and want to produce another space X' quasi-isometric to X, such that X admits an actual (Lipschitz) contraction. Probably, such X' does not, in general, exist but one can easily see that there always exists a thickening $X' \supset X$ such that the inclusion of X to X' is a contractible map. In fact one could *define* the combing as a Lipschitz contraction of X in some thickening $X' \supset X$. (This is how combing comes up in most cases of the real life.) Finally, we observe that the above contraction of X in X' trivially implies that a combable space is large-scale k-contractible for all $k \ge 0$. In the case of a combable group Γ this leads (see 1.C$_2$) to the existence of an aspherical polyhedron P with $\pi_1(P) = \Gamma$, such that for all k the k-skeleton of P is a finite polyhedron. This statement appears as Theorem 9.5.6 in [E-C-H-P-T] where the reader may find a *detailed* discussion on combing and related matters.

1.D. Uniform connectedness. Consider a k-connected (in the ordinary topological sense) metric space X and denote by $R_x(r)$, for $x \in X$ and $r \in \mathbb{R}_+$, the minimal (better to say "infimal" if you want to be pedantic) number $R \ge r$, such that the inclusion between the concentric balls in X around x

$$B_x(r) \subset B_x(R)$$

is a *k-connected map*, i.e. every continuous map of k-dimensional polyhedron into $B_x(r)$ is contractible in $B_x(R)$. We say that X is *uniformly k-connected* if

$$\overline{R}_k(r) \underset{\text{def}}{=} \sup_{x \in X} R_x(r) < \infty \quad \text{for all } r \in \mathbb{R}_+ .$$

1.D₁. *Basic example.* Let X be a k-connected locally finite polyhedron which admits a proper *cocompact* action of a group Γ. Then (obviously) X is uniformly k-connected.

1.D′₁. *Counterexamples.* (a) In general, a k-connected (even contractible) space does not have to be uniformly k-connected. Take for example the Euclidean space $\mathbb{R}^n$ with a metric dist which makes $\mathbb{R}^n$ isometric to the cylinder $S^{n-1} \times \mathbb{R}_+$ away from some ball, see Fig. 1 below.

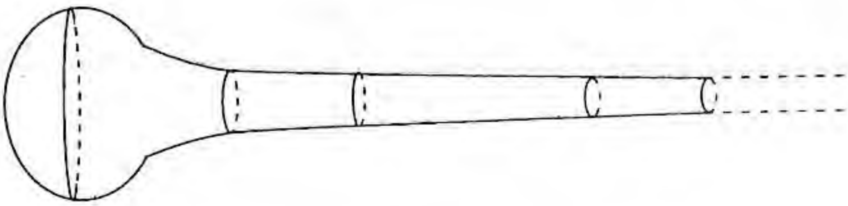

Figure 1

Clearly this $(\mathbb{R}^n, \text{dist})$ is *not* uniformly $(k-1)$-connected (though it is uniformly $(k-2)$-connected).

(b) In the above example the space $(\mathbb{R}^n, \text{dist})$, not being uniformly $(n-1)$-connected, still admits a contractible uniformly n-connected thickening. Here is how to make the non-uniformness of connectedness stable under thickening. Let S_{ij} be the n-dimensional Euclidean sphere of radius i from which we remove an n-ball of radius j as in the Fig. 2 below.

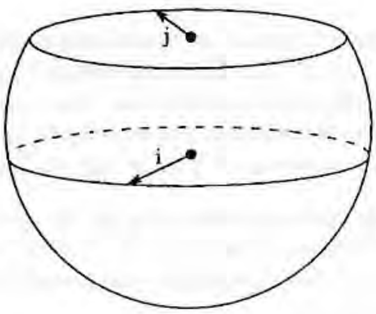

Figure 2

Topologically, every S_{ij} is homeomorphic to the n-ball, but the boundary spheres are not uniformly contractible when $i \to \infty$ for j being kept bounded. Now we order in some way all S_{ij} for all $i \geq j = 1, 2, \ldots$, and put them on the line as in Fig. 3 below.

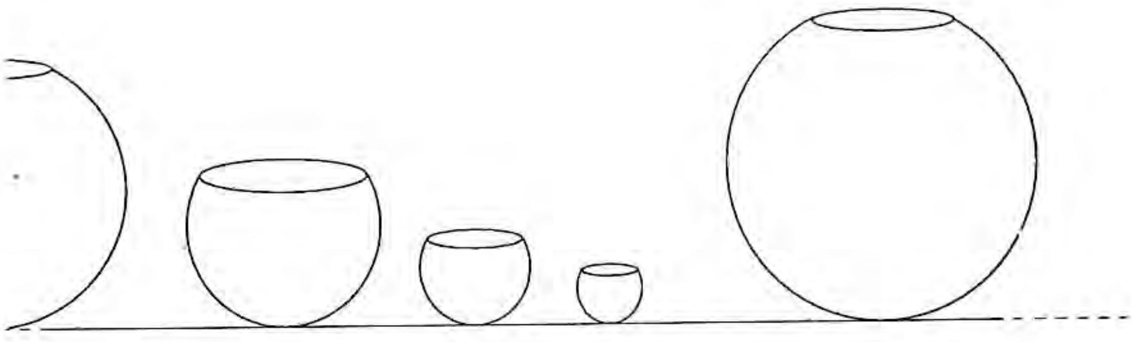

Figure 3

The resulting space is contractible but no thickening of it is uniformly $n-1$-connected.

1.D$_2$. *Exercise.* Let X be a uniformly k-connected and (non-uniformly) $(k+1)$-connected metric space. Show X is $(k+1)$-connected on the large-scale.

1.D$_3$. *Definition.* X is called *uniformly k-connected on the large-scale* if it admits a uniformly k-connected thickening.

In view of the above exercise the *uniform* k-connectedness on the large- scale implies the earlier defined (non-uniform) one. On the other hand, if $X = (\Gamma$, word metric) then, conversely, the (non-uniform) large-scale k-connectedness (obviously) implies the uniform one. For example, every finitely presented group Γ is uniformly 1-connected on the large-scale and this connectedness is reflected in the implied function $\overline{R}_1(r)$. This function provides an interesting invariant of Γ on which we look more closely in §4.

Remark. The notion of uniform connectedness aims at the geometric characterization of the universal coverings of compact aspherical manifolds. Namely if $\widetilde{V} \to V$ is such a covering, then $\widetilde{V}$ is uniformly k-connected for all k and this property (called *geometric contractibility* in [Gro]$_{10}$) seems to provide a strong low bound on the "size" of $\widetilde{V}$ (see [Gro]$_{13}$).

1.E. Large-scale dimension. We start with the most natural but not very flexible definition. First we say that a metric space Y is *d-disconnected* (or has *dimension zero on d-scale*) if it decomposes into the union $Y = \bigcup_{i \in I} B_i$, where

(a) $\sup_{i \in I} \operatorname{Diam} B_i \leq D < \infty$,

(b) $\operatorname{dist}(B_i, B_j) \geq d$, for all $i \neq j$, where

$$\operatorname{dist}(A, B) \underset{\text{def}}{=} \inf_{\substack{a \in A \\ b \in B}} \operatorname{dist}(a, b) .$$

Then we define the large-scale (or asymptotic) dimension of a metric space X as the minimal number n, such that for every $d > 0$ the space X can be decomposed into a union of $n+1$ subsets, $X = \bigcup_{k=0}^{n} X_k$, where each X_k is d-disconnected. We denote this dimension by $\operatorname{as\,dim}_+ X$ (where "+" serves to distinguish this from another as dim defined later on). One can also define $\operatorname{as\,dim}_+$ as the minimal number n, such that for every d there is a covering of X by bounded subsets, $X = \bigcup_{i \in I} B_i$, such that the *d-multiplicity* (see below) of this covering is bounded by $n+1$. Here the inequality

$$d\text{-multiplicity} \leq n+1$$

signifies that no ball in X of radius d meets more than $n+1$ subsets among B_i. (It is obvious that the new dimension is $\leq$ than the one we started with; then an obvious large-scale rendition of the standard dimension theoretic play with coverings yields the equality of the two dimensions.)

Remark. Besides the integer $n = \operatorname{as\,dim}_+ X$ the above discussion provides an extra quasi-isometry invariant of X. Namely, for each $p \geq n$ we define $D_p(d)$ as the minimal D in the above (a) for which X can be covered by $p+1$ d-disconnected subsets X_k satisfying (a) and (b) with these D and d. The new invariant is the asymptotics of $D_p(d)$ for $d \to \infty$.

Examples. (a) If $X = \mathbb{R}^n$, then it is easy to show (with the classical dimension theory) that $\operatorname{as\,dim}_+ = n$ and $D_p(d) = \text{const}_{n,p}\, d$.

(b) If X is an infinite tree, then, obviously, $\operatorname{as\,dim}_+ = 1$ and $D_p(d) = p^{-1}d$.

(c) Let X be the hyperbolic plane H^2. Then $\operatorname{as\,dim}_+ = 2$ and $D_2(d) \leq \text{const}\,d$. This is seen with concentric horospheres and coverings by bounded subsets indicated in Fig. 4 below (compare 1.E).

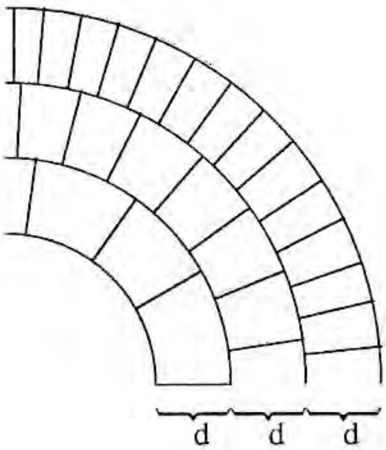

Figure 4

1.E₁. In order to get a better grip on $\operatorname{as\,dim}_+$ we invoke the usual maps of X to the (extended) *nerves* of our coverings of X by B_i, $i \in I$. This is done with the functions $\varphi_i : X \to \mathbb{R}_+$, such that

$$\varphi_i(x) = 1 \quad \text{for} \quad x \in B_i \ ,$$
$$\varphi_i(x) = 0 \quad \text{for} \quad \operatorname{dist}(x, B_i) \geq d \ ,$$
$$\varphi_i(x) = 1 - d^{-1} \operatorname{dist}(x, B_i) \quad \text{for} \quad \operatorname{dist}(x, B_i) \leq d \ .$$

These functions map X to the infinite dimension Euclidean space $\mathbb{R}^I$ and we compose this map with the radial projection to the unit simplex $\Delta^I \subset \mathbb{R}^I_+$ defined by $\sum_{i\in I} x_i = 1, x_i \geq 0$. If the d-multiplicity of our covering is $\leq n+1$ then this map lands in an n-dimensional subcomplex of Δ^I and the resulting map, call it $\Phi : X \to N$, is λ-Lipschitz for $\lambda \approx d^{-1}$, where N is given the Euclidean metric induced from $\mathbb{R}^I \supset N$. Now, the essential property of our Φ is the *uniform coboundness* (defined below). To have this property we assume that N is the *minimal* subcomplex of Δ^I containing the image $\Phi(X)$, i.e. the interior of each simplex σ in N meets $\Phi(X)$. Then, clearly

$$\operatorname{Diam} \Phi^{-1}(\sigma) \leq 2(D+d) < \infty$$

for all simplices σ in N, where D and d are the constants characterizing our covering. This implies the uniform coboundness of Φ which we express here by

$$\operatorname{Diam} \Phi^{-1}(M) \leq Cm$$

where M is an arbitrary connected finite subcomplex in N consisting of m simplices and $C = C_\Phi$ is a positive constant.

Thus we arrive at the third definition of $\operatorname{as\,dim}_+$, i.e. the minimal number n, such that for every $\lambda > 0$ there exists an n-dimensional simplicial polyhedron N and a λ-Lipschitz uniformly cobounded map $\Phi : X \to N$, where "Lipschitz" refers to the metric in N whose restriction to every finite subcomplex K is induced from some $\mathbb{R}^m$ by the standard simplicial embedding $K \to \mathbb{R}^m$.

Remark. Every simplex of N is isometric to the unit Euclidean simplex and there is a unique maximal metric on N with such simplices. The map Φ is not in general Lipschitz for this metric. Yet it is such if X is a quasi-geodesic space. In fact, the map Φ is a quasi-isometry in this case.

We leave to the reader the (straightforward) check-up of the equivalence of the third definition to the two earlier ones and pass to an example where the third definition makes the computation of $\operatorname{as\,dim}_+$ easy.

1.E$'_1$. *Example.* Let X be a complete simply connected manifold without boundary with bounded strictly negative sectional curvatures,

$$-\infty < -c \leq K(X) \leq -c' < 0 .$$

Then we claim that

$$\operatorname{as\,dim}_+ X = \dim X .$$

The proof of the inequality $\operatorname{as\,dim}_+ \geq \dim$ follows from the standard dimension theory (compare Example (a) in 1.E) and we concentrate here on showing that $\operatorname{as\,dim}_+ \leq \dim$. We take a family of concentric horospheres $H_t \subset X$, $t \in (-\infty, +\infty)$, (compare Fig. 4 above) and first observe that

$$\operatorname{as\,dim}_+ H_t = \dim H_t .$$

In fact, every H_t admits a cover by uniformly bounded subsets with d-multiplicity $n = \dim H_t$ for some (small) $\delta > 0$. Then we use the normal projection $H_{t+d} \to H_t$, and observe that the cover of H_t pulls back to a cover of H_{t+d} of d'-multiplicity n where $d' \to \infty$ for $d \to \infty$. (Our convention about the direction of t is such that the projections $H_{t'} \to H_t$ are contracting for $t' > t$.) Thus, every H_t maps to some $(n-1)$-dimensional polyhedron N_t according to the third definition. On the other hand, since $K(X) \leq 0$, there are natural maps $N_t \to X$ once we send each vertex of N_t to a point in the subset of H_t corresponding to this vertex. Namely, each simplex from N_t goes to the *geodesic span* of its vertices lifted to X. We compose every such map $N_t \to X$ with the projection to H_{t-d} for some $d \geq 0$ followed by the map $H_{t-d} \to N_{t-d}$. Thus we get maps, say $p_{t,d} : N_t \to N_{t-d}$ which are clearly λ-Lipschitz with $\lambda \to 0$ for $d \to \infty$. Therefore, for large d one can approximate $p_{t,d}$ by simplicial maps and construct the simplicial complex N by joining the cylinders of the maps $p_{i,d}$, for $i = 0, \pm d, \pm 2d, \ldots, \ldots \to N_{2d} \to N_d \to N_0 \to N_{-d} \to N_{-2d} \to \ldots$. Now it is easy to map X to N by a λ-Lipschitz uniformly cobounded map, where λ can be made as small as we wish. We leave doing this in detail to the reader.

Remarks. With the above example we see that the fundamental group Γ of a closed manifold V of negative curvature has

$$\text{as}\dim_+ \Gamma = \dim V .$$

If Γ is a general word hyperbolic group our argument shows that $\text{as}\dim_+ \Gamma < \infty$ and by looking closely one can probably see that

$$\text{as}\dim_+ \Gamma \leq \dim \partial_\infty \Gamma + 1 ,$$

where ∂_∞ is the ideal boundary. (The natural conjecture reads $\text{as}\dim_+\Gamma = \dim \partial_\infty \Gamma + 1$). Also, our argument seems to yield the equality

$$\text{as}\dim_+ X = \dim X$$

for the symmetric spaces with $K(X) \leq 0$ via the usual $X = A\,N$ representation (corresponding to $K\,A\,N$ of the underlying Lie group). But the evaluation of $\text{as}\dim_+$ looks harder in the general (non-symmetric) case of $K(X) \leq 0$ as well as for the solvable Lie groups (generalizing $A \cdot N$ associated to the symmetric spaces). We shall see in the next section how to bypass these problems with another notion of the asymptotic dimension (compare [Swi]).

1.F. Dimension and width. The *k-width* of a metric space defined by Uryson measures a certain distance from X to some k-dimensional polyhedron N where the distance is understood as the infimal δ for which there exists a proper continuous map $f : X \to N$ satisfying

$$\text{Diam}\, f^{-1}(\nu) \leq \delta, \text{ for all } \nu \in N .$$

(If X is not locally compact one should slightly refine this definition, compare $[\text{Gro}]_{10,15}$.) Then the large-scale (or asymptotic) dimension $\text{as}\dim X$ is defined as the minimal n, such that every ***admissible*** thickening X' of X has

$$\text{width}_n X' < \infty,$$

where "admissible" means that X' is, in fact, a finite dimensional locally finite polyhedron. (This somewhat restricts the class of spaces X to which this definition effectively applies but it is good enough for finitely generated groups Γ which are the major characters of our narrative.)

Remark. As in the case of $\text{as}\dim_+$ there is a finer invariant behind the above definition, namely the asymptotics of $\text{width}_k B(R)$, $k < \text{as}\dim X$, for the R-balls in X', as $R \to \infty$.

1.F$_1$. ***Basic example.*** *Let X be a uniformly contractible* (i.e. ***uniformly k-connected for all $k = 0, 1, \ldots$***) ***manifold without boundary.*** Then

$$\text{as}\dim X = \dim X .$$

Proof. Let $X' \supset X$ be a polyhedral thickening of dimension k. Then there is a retraction $r : X' \to X$, ***parallel*** to the identity map $X' \to X'$ (i.e. $\text{dist}(x', r(x')) \leq C \leq \infty$) as easily follows from the uniform k-connectedness of X. Thus

$$\text{as}\dim X \leq \dim X .$$

Now, let $f : X \to N$ be a proper map with $\text{Diam}\, f^{-1}(\nu) \leq C < \infty$. We take the cylinder $C_f \supset X$ of this map and construct as above a retraction $C_f - X$ which moves each point by a uniformly bounded amount. Then by composing this retraction with the obvious map $C_f \to N$ we obtain a map, call it $h : X \to X$, which is parallel to the identity and which factors via our $f : X \to N$. Using again the uniform contractibility of X we observe that *parallel to the identity $\Rightarrow$ properly homotopic to the identity* and so h cannot factor through N if $\dim N < \dim X$. Thus

$$\text{as}\dim X \geq \dim X .$$

Q.E.D. (See $[\text{Gro}]_{10,16}$ for a more detailed discussion of this kind.)

Corollary. *Let Γ be the fundamental group of a closed aspherical manifold V. Then*

$$\text{as}\dim \Gamma = \dim V .$$

In particular, dim V *is a quasi-isometry invariant of* Γ.

Remarks. (a) It is clear that $\operatorname{as dim}_+ \geq \operatorname{as dim}$ but the equality seems unlikely even in the above example.

(b) In the obvious cases the k-width of the R-balls in Γ for $k < \operatorname{as dim} \Gamma$ grows linearly as $R \to \infty$ but I do not know what happens in general.

1.6. Asymptotic homology. It has been known for a long time (compare [Tih]) that a discrete space (e.g. a group) X may harbour a non-trivial topology at infinity, such as H_0 for example, i.e. the set of *the ends of* X, as well as the fundamental group at ∞ and the higher dimensional (co)-homology and homotopy. The simplest definition of (co)homology at ∞ is as follows. First, we consider the complements of concentric R-Balls in X, denoted $X^R \subset X$, which form a decreasing family and for every topological invariant, i.e. a functor H, define $H(X^\infty)$ as the limit (direct if H contravariant and inverse if H is covariant) of $H(X^R)$. (In the case of $H = \pi_1$ one should take an extra care of the base points.) Then we consider the (category of the) thickenings of X and go to the limit again. Here we may take a representative sequence of thickenings

$$X \subset X_1 \subset X_2 \subset \dots$$

and take $\lim H(X_i^\infty)$ for $i \to \infty$, denoted $L.S.H(X^\infty)$. More recently, J. Roe (see $[\mathrm{Roe}]_{1-4}$) focused attention on the relative invariants of this kind, such as the cohomology of X with bounded supports (i.e. $\lim\limits_{R\to\infty} H^*(X, X^R)$), denoted $H^*(X,\infty)$ and the corresponding large-scale cohomology $L.S.H^*(X,\infty)$ (which J. Roe christened "exotic cohomology").

1.G₁. *Examples.* (a) Let X be a uniformly contractible n-dimensional manifold without boundary. Then the fundamental class of X defines a (non-zero) element in $L.S.H_n(X,\infty)$ whose boundary is non-zero in $L.S.H_{n-1}(X^\infty)$. Cohomologically, the fundamental class of X is also non-zero in $L.S.H^n(X,\infty)$ and it appears as the coboundary of somebody from $L.S.H^{n-1}(X^\infty)$. This follows by the argument indicated in $1.\mathrm{F}_1$. In fact, non-vanishing of $L.S.H^k$ (or $L.S.H_k$) insures the inequality $\operatorname{as dim} \geq k$.

(b) Let X be an Euclidean (Bruhat-Tits) building of dimension n. Then every n-flat gives a non trivial element (infinite cycle) in $L.S.H_n(X,\infty)$ and thus $\operatorname{as dim} X = n$.

1.G₂. Filling invariants. The large-scale cohomology is accompanied by a variety of finer numerical invariants measuring the "size" of a chain needed to *fill in* a given cycle. (See §5 and also $[\mathrm{Gro}]_{10}$.)

1.H. *Summary.* We have shown in this § how the standard *topological* invariants can be adjusted to measure discrete spaces on the large-scale. Basically, we have two means, thickenings (e.g. Rips complexes) and nerves of coverings by large but uniformly bounded subsets. The first objective of the study is proving (whenever possible) that the ordinary cohomological invariants of groups Γ are, in fact, quasi-isometry invariant. (We have shown q.i. invariance of homological dimension of Γ in some special cases, but the question is still open in full generality.) Next, we want to have a more systematic and coherent theory and to evaluate our invariants for as many groups as possible. Another direction is an analysis of the behaviour of our invariants under non-quasi-isometric maps, e.g. *uniform embeddings* (see §7 and $[\mathrm{Gro}]_{15}$).

§2. Ultrafilters and the asymptotic cones Con_∞ X and Con_ω X; connectivity of Con_∞ for Lie groups and lattices; geometry of Con_∞ and the space of the word metrics.

The invariants of X in the previous §1 were produced by performing standard topological constructions on the large scale. Here we want to proceed differently by defining a canonical *asymptotic cone of X at ∞* whose ordinary topological invariants will serve as large scale invariants of X. This cone $Con_\infty X$ captures the geometry rather than the topology of X on the large scale and it can be described informally as follows. Let us imagine an observer who moves away from a metric space X and from time to time makes an observation consisting in measuring finitely many distances between certain points in X. When the observer is located distance d away from X he/she concentrates on the distances which appear to him/her of bounded magnitude (as he/she has a limited field of vision) which correspond to distances proportional to d on the real scale. As the observer goes further away at some distance d' which is much greater than d, the points seen earlier from the distance d become indistinguishable; but now he/she is concerned with points within distance about d' in X. It may happen by pure accident that the result of the d'-observation is similar to the d-observation despite the fact that in reality (i.e. in X) the observed objects have nothing in common. (Namely, X may contain two subsets of points $\{x_1, \dots, x_k\}$ and $\{x'_1, \dots, x'_k\}$ such that $\mathrm{dist}_X(x_i, x_j)$ are of order d, the distances $\mathrm{dist}_X(x'_1, x'_k)$ are about d' and the *normalized* distance $d^{-1}\ \mathrm{dist}_X(x_i, x_j)$ are very close to $(d')^{-1}\ \mathrm{dist}_X\ (x'_i, x'_j)$ for all i and j.) Moreover, as the observer goes further and further away from X he/she increases the number of the observed points, say he/she measures the distances between k-points when he/she positioned d_k-away from X. If all these visual distances converge when k and d_k go to infinity (i.e. there are points x_i^k in X, $i = 1, \dots, k$, $k = 1, 2, \dots$, such that there exist limits

$$\lim_{k\to\infty}\ d_k^{-1}\ \mathrm{dist}(x_i^k, x_j^k)$$

for all (i, j)). Then the observer will see in the limit certain (infinite) space Y. (If the observer is unaware that his/her position changes and if all of the field of vision was eventually scrutinized and recorded, then he/she may be convinced that Y is the real thing out there.)

If the R-balls $B_0(R) \subset X$ around fixed points with the metrics $R^{-1}\ \mathrm{dist}_X$ are *uniformly precompact* for $R \to \infty$ then the above measurement process does converge (or subconverge) in the Hausdorff topology on bounded subsets (see [G-L-P], $[\mathrm{Gro}]_6$). But in the general case one has to resort to *ultralimits*.

2.A. Definition of Con_∞ X via ultralimits (compare [Dr-Wi]). First we recall that a *non-principal ultrafilter* is a finitely additive measure ω (or probability) defined on *all* subsets $A \subset \mathbb{N}$, such that

(a) $\omega(A)$ equals 0 or 1 for all $A \subset \mathbb{N}$;

(b) $\omega(A) = 0$ for all finite subsets $A \subset \mathbb{N}$.

Given a bounded function $\varphi : \mathbb{N} \to \mathbb{R}$ one defines with ω the *(ultra)limit of φ* with respect to ω denoted $\lim\limits_{\omega} \varphi(i)$ or $\varphi(\omega)$ or $\varphi(\infty)$ (as ω is thought of as an ideal infinite point attached to $\mathbb{Z}$). This limit is uniquely characterized by the following condition: for every $\varepsilon > 0$ the subset $I \subset \mathbb{N}$ where φ is ε-close to $\varphi(\omega)$, i.e.

$$A_I = \{i \in \mathbb{N} |\ |\varphi(i) - \varphi(\omega)| \le \varepsilon\}$$

has $\omega(A_I) = 1$. (It is easy to see that the correspondence $\varphi \mapsto \varphi(\omega)$ is a (order preserving) homomorphism of the ring of bounded real functions on $\mathbb{N}$ into $\mathbb{R}$ and so ω can be thought of as a point in the Cech-Stone compactification of $\mathbb{N}$.)

Now we take an arbitrary metric space X, fix a point $x_0 \in X$ and consider the set of maps $f : \mathbb{N} \to X$ such that $\mathrm{dist}(x_0, f(i)) \le \mathrm{const}_f\ i$. For every two such functions we define $\mathrm{dist}(f_1, f_2)$ as $\varphi(\omega)$ for $\varphi(i) = i^{-1}\ \mathrm{dist}(f_1(i), f_2(i))$, where ω refers to the chosen ultrafilter. This dist_ω is not truly a distance as it may vanish on some distinct points, but we just factorize the space of maps $\mathbb{N} \to X$ by the relation $\mathrm{dist}_\omega = 0$ and call the resulting metric space the asymptotic *ω-cone* $\mathrm{Con}_\omega X$. If we do not care to specify ω, we write $\mathrm{Con}_\infty X$ and call it an *asymptotic cone of X* (at infinity). Intuitively, it is the maximal space Y an observer may see from infinitely far away where the ultrafilter sifts and selects an infinitely rare subsequence from d_i such that all measurements made by our friend observer converge.

A remarkable feature of Con_∞ is that for many concrete spaces (e.g. groups) X the cone $\mathrm{Con}_\omega X$ is not as disgustingly large as one could a priori expect. For example, this Con_ω is *finite dimensional* for many X (though it is usually not locally compact). Furthermore, the (essential) geometry and topology of $\mathrm{Con}_\omega X$ seem independent of ω for many X.

2.B. ***Examples.*** (a) *Abelian groups.* Let X be a finitely generated Abelian group of rank r. Then the tangent cone is unique and isometric to $\mathbb{R}^r$ with some Minkowski (Banach) metric. This is nearly obvious and one can use here the Hausdorff limit without ultrafication (compare $2.C_1$ and [G-L-P]).

(a′) *Nilpotent groups.* Let X be a finitely generated nilpotent group. Then again the ultralimit reduces to an ordinary Hausdorff limit which turns X into a nilpotent Lie group G with a dilation. This is similar to the Abelian case and quite easy (compare $[\mathrm{Gro}]_6$). What is more difficult to show is the ***uniqueness*** of the limit, i.e. an actual convergence (rather than subconvergence) of $(X,\ d^{-1}\,\mathrm{dist})$. This is due to P. Pansu (see $[\mathrm{Pan}]_2$).

(b) *Hyperbolic spaces.* If X is a hyperbolic space then every tangent cone is a tree (i.e. an $\mathbb{R}$-tree). Notice that the ordinary Hausdorff limit does not exist here (apart from a few trivial examples) and the "ultradefinition" of Con_∞ is a sheer necessity. Also observe that the limit formula

$$\mathrm{Con}_\infty X = \mathrm{tree}$$

can be used as a definition of hyperbolicity (see $[\mathrm{Gro}]_{14}$, [Pou]).

(c) *Log-metrics.* Start with an arbitrary metric dist and let $\mathrm{dist}' = \log(1+\mathrm{dist})$. Then $\mathrm{Con}_\infty(X,\mathrm{dist}')$ is a totally disconnected space. Moreover it is an *ultrametric* space,

$$\mathrm{dist}(a,c) \le \max\big((\mathrm{dist}(a,b),\ \mathrm{dist}(b,c)\big)\ ,$$

and so all triangles in this Con_∞ are *isosceles*. This is true in general for every metric dist (rather than log dist) where the large triangles are nearly isosceles, i.e. if for every three point x, y and z in X

$$\mathrm{dist}(x,z) \le (1+\varepsilon)\max\big(\mathrm{dist}(x,y), \mathrm{dist}(y',z)\big)\ ,$$

where ε is some function in $d = \mathrm{dist}(x,z)$ which goes to 0 for $d \to \infty$. For example, the metric on a horosphere of a hyperbolic space has this property.

(c′) *Doubled horoballs.* Take two horoballs in a non-elementary hyperbolic space (e.g. in H^n for $n \ge 2$) and glue them across the boundary (horosphere). Then Con_∞ of the resulting space X is obtained by gluing two $\mathbb{R}$-trees across subsets of extremal points. Thus $\mathrm{Con}_\infty X$ is a 1-dimensional space with a huge (uncountably generated) π_1.

(d) *Baumslag-Solitar group.* Consider the group Γ presented by $\{a, b \mid a^b = a^2\}$, where a^b is an abbreviation for $b\,a\,b^{-1}$. There is a natural metric of piecewise constant negative curvature on the corresponding 2-polyhedron P, which is the mapping cylinder for the map $S^1 \to S^1$ by $z \to z^2$. Namely, we give the cylinder $S^1 \times [0,1]$ the constant curvature metric which is S^1-invariant and where the S^1-orbits lift to horocircles in the universal covering. Moreover, we make our choice, such that

$$\mathrm{length}(S^1 \times 0) = 2\ \mathrm{length}(S^1 \times 1)$$

and then we glue the two ends together by the obvious (locally isometric) map $S^1 \times 0 \to S^1 \times 1$. The universal covering $\widetilde{P}$ of the resulting polyhedron P is obtained by gluing together countably many horodisks along horospheres in a treelike fashion. In fact $\widetilde{P}$ is naturally fibered over the infinite triadic tree sketched in Fig. 5 below

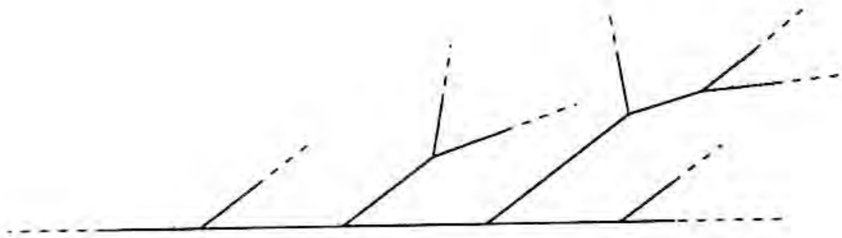

Figure 5

with the fibers isometric to the horospheres in H^2 with induced metrics, and these fibers are "metrically parallel" in $\widetilde{P}$ (as a family of concentric horospheres in H^2). It follows that $\mathrm{Con}_\infty \widetilde{P}$ is naturally continuously mapped onto an $\mathbb{R}$-tree (which is Con_∞ of the above triadic tree) with all fibers totally discontinuous. Hence, $\dim \mathrm{Con}_\infty \widetilde{P} = 1$. Furthermore, $\widetilde{P}$ contains infinitely many pairs of horodisks glued across their boundaries and so $\pi_1(\mathrm{Con}_\infty \widetilde{P})$ is uncountably generated. (Our picture of $\widetilde{P}$ gives a clear geometric view on the isoperimetric and the isodiametric properties of $\widetilde{P}$ established by Gersten: the isoperimetric function is exponential and the isodiametric one is linear. The extremal disks displaying such features are obtained by doubling truncated horodisks across horocircles, compare the above (c′) and §5.)

(e) *Solvable Lie groups.* Consider the semi-direct product $S = \mathrm{Sol}_{n+1}$ of $\mathbb{R}^n$ and $\mathbb{R}$ for the action of $\mathbb{R}$ on $\mathbb{R}^n$ given by the diagonal matrix with the diagonal entries $e^{\gamma_1 t}, \ldots, e^{\gamma_n t}$. There is a natural (left invariant for the group structure in S) Riemannian metric on $S = \mathbb{R}^n \rtimes \mathbb{R}$ whose restriction to $H_t = \mathbb{R}^n \times t$ equals the standard Riemannian (i.e. Euclidean) metric on $\mathbb{R}^n \times 0 = \mathbb{R}^n$ transported by the above diagonal matrix. For example, if $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda \neq 0$, then S is (isometric to) the usual hyperbolic space of constant curvature with the family H_t of concentric horospheres in S. As we know (compare (c)) the distance function (metric) of S restricted to $H_0 \subset S$ (or any other H_t for this matter) looks like log(Euclidean distance in H_0).

(e′) **Lemma.** *If $\lambda_i \neq 0$, $i = 1, \ldots, n$, then (H_0, dist_S) is quasi-isometric* to the *Cartesian product of n horocycles (i.e. horospheres in the hyperbolic plane).*

Proof. Denote by $h_0^i \subset H_0 = \mathbb{R}^n$ the i-th 1-dimensional coordinate subspace in $\mathbb{R}^n = H_0$ (i.e. the eigenspace corresponding to $e^{\lambda_i t}$) and let $S_i \subset S$ be the 2-dimensional subgroup $h_0^i \times \mathbb{R} \subset S = H_0 \times \mathbb{R}$. This S_i, as we know, is isometric to the hyperbolic plane where h_0^i serves as a horocycle. Then we look at the map $S \to S_i$ induced by the orthogonal projection of $H_0 = \mathbb{R}^n$ to $h_0^i \subset H_0$. This projection commutes with our action of $\mathbb{R}$ on $\mathbb{R}^n$ and then it is clear that the map $S \to S_i$ decreases the Riemannian metric. Hence, it is distance decreasing; consequently, the projection $H_0 \to h_0^i$ is distance decreasing for the metric $\mathrm{dist}_S |H_0$. Now it follows that $(H_0, \mathrm{dist}_S |H_0)$ is (quasi)-isometric to the Cartesian product of $\left(h_0^i, \mathrm{dist}_S h_0^i\right)$, $i = 1, \ldots, n$, Q.E.D.

Using this lemma we conclude as earlier that $\mathrm{Con}_\infty(H_0 | \mathrm{dist}_S H_0)$ is totally disconnected and that

$$\dim \mathrm{Con}_\infty S = 1 .$$

If all $\lambda_i > 0$ (or all < 0) then S is hyperbolic and Con_∞ is a tree. Otherwise it has a huge (uncountably generated) fundamental group. To see this it suffices to consider the case $n = 2$. Let us exhibit curves in $S = \mathbb{R}^2 \rtimes \mathbb{R}$ which nontrivially contribute to $\pi_1(\mathrm{Con}_\infty S)$. Take the square $\square_d$ with vertices $(0,0)$, $(0,d)$, $(d,0)$ and (d,d) in $\mathbb{R}^2$ (this square has size about $\log d$ in S) and join the pairs of the adjacent vertices by minimal segments in S. One should think of each edge of $\square_d \subset \mathbb{R}^2$ as an arc of a horocircle while the minimal segments in S are given by geodesic segments in the corresponding hyperbolic planes. Here is an attempt to make a picture.

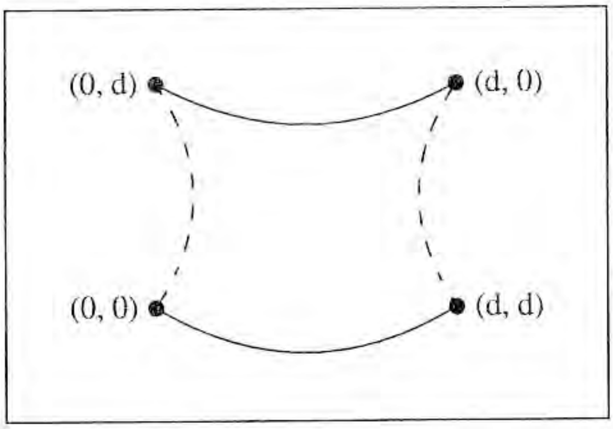

Figure 6

Thus we obtain a closed curved $\square'_d$ in S of length about $\log d$ which meets $\mathbb{R}^2 \times 0 \subset S$ at the four vertex points. It is not hard to see that the opposite edges of this square of geodesics in S never come close together, i.e. they are separated by the distance of order $\log d$, because the ambient (parallel) hyperbolic planes are that far apart. It follows, that the sequence $(\square'_d, (\log d)^{-1} \operatorname{dist}_S)$ gives us in the ultralimit a curve $\square_\infty$ in $\operatorname{Con}_\infty S$ which is Lipschitz equivalent to the unit circle. Clearly, this $\square_\infty$ is noncontractible (since $\dim \operatorname{Con}_\infty S = 1$) and by moving d and/or $\square_d$ we obtain a continium of such curves $\square_\infty$ in $\operatorname{Con}_\infty S$.

(f) *Solvable groups S of $\mathbb{R}$-rank $k \geq 2$.* Consider a diagonal action of $\mathbb{R}^k = (\mathbb{R}_+^\times)^k$ on $\mathbb{R}^n$ and let $S = \mathbb{R}^n \rtimes \mathbb{R}^k$ be the corresponding semi-direct product. For example, if $k = n$ and $\mathbb{R}^k = (\mathbb{R}_+^\times)^k$ consists of the diagonal $(k \times k)$-matrices with positive entries then the resulting S is the Cartesian product of $k = n$ copies of the two-dimensional solvable group S_2 which is isometric to the hyperbolic plane H^2 (see (e)). In this case $\operatorname{Con}_\infty S$ is the Cartesian product of k $\mathbb{R}$-trees.

Our next example is where $k = n - 1$ and $\mathbb{R}^k = (\mathbb{R}_+^\times)^k$ consists of the diagonal $(n \times n)$-matrices with determinant one acting on $\mathbb{R}^n$.

Claim. *The asymptotic cone of the solvable group $S = \mathbb{R}^n \rtimes \mathbb{R}^{n-1}$ is simply connected for $k \geq 2$, while $\pi_{n-1}(\operatorname{Con}_\infty S)$ is an infinite (uncountably generated) group.*

Idea of the proof. Let $\mathbb{R}_i^{n-1} \subset \mathbb{R}^n$ denote the i-th coordinate hyperplane in $\mathbb{R}^n$ and $S_i = \mathbb{R}_i^{n-1} \rtimes \mathbb{R}^{n-1} \subset S$. Every S_i, as we already know, is isometric to $\underbrace{H^2 \times H^2 \times \ldots \times H^2}_{n-1}$ and the orthogonal projections $\mathbb{R}^n \to \mathbb{R}_i^n$, $i = 1, \ldots, n$, induce *Lipschitz* projections $S \to S_i$. If we translate $\mathbb{R}_i^n \subset \mathbb{R}^n$ in the normal direction by t we accordingly move S_i and thus fiber S into the translates $S_i(t)$, $t \in \mathbb{R}$, where each $S_i(t)$ is isometric to $H_2 \times \ldots \times H_2$ and receives a Lipschitz retract of S. We have altogether n such fibrations for $i = 1, \ldots, n$, we call their fibres *semi-hyperbolic slices* in S, and geodesic segments in these slices are called s.h. *segments* in S.

Lemma. *Every two points x and y in S can be joint by a curve (broken geodesic line) a consisting of two s.h. segments joint at a single point such that*

$$\text{length } a \leq \text{const dist}(x, y) \qquad (*)$$

for some universal "const" *(independent of x and y).*

Proof. There obviously exist two semi-hyperbolic slices $S_{i_1}(t_1)$ and $S_{i_2}(t_2)$ whose union U contains both points x and y. This U is a Lipschitz retract in S, since the union of two perpendicular hyperplanes in $\mathbb{R}^n$ is such a retract. It follows that the shortest curve a in U between x and y satisfies $(*)$.

Corollary. *For every closed curve C in S there exists another closed curve C' built of $2N$ s.h. segments such that*

(a) $N \leq$ const,

(b) length $C' \leq$ const length C ,

(c) C' meets C at N-points,

and such that the resulting N closed curves (built by the pairs of segments from C and C') have

$$\text{length } \leq \frac{1}{2} \text{ length } C .$$

Proof. Subdivide C into N segments of equal lengths and join their ends by broken s.h. lines.

Now, to prove that $\mathrm{Con}_\infty S$ is simply connected it is enough to show that (sequences of) broken curves C' consisting of a fixed number of s.h. segments give rise to *contractible* curves in $\mathrm{Con}_\infty S$ (compare 5.A_3). To see this we observe that every our C' is contained in some union U' of $N' \leq$ const s.h. slices. If $n-1 \geq 3$, we can add extra slices, so that the total number N'' will be still bounded by some universal constant and such that the new union $U'' \supset U'$ is simply connected. For this we must just make simply connected the corresponding union of hyperplanes in $\mathbb{R}^{n-1}$, that is $V'' = U'' \cap (0 \times \mathbb{R}^{n-1}) = 0 \times \mathbb{R}^{n-1} = \mathbb{R}^{n-1}$, since $U'' = \mathbb{R}^n \times V''$. The rest of the proof is taken care of by the following

Final Lemma. *The curve C'' bounds a disk in U'' whose size is controlled by the length of C'' in the following sense: there is a λ-Lipschitz map of the unit disk D, such that ∂D parametrizes C'' by this map and such that $\lambda \leq$ const length D. (In the terminology* of §5

$$\mathrm{Fill\,Span}\, C'' \lesssim \text{ length } C'') .$$

The proof of the lemma is straightforward as we are in a complete command over the geometry of the s.h. slices constituting U''.

With this lemma we see (compare 5.A_3) that the curves C'' give trivial contribution to $\pi_1(\mathrm{Con}_\infty S)$ which is exactly what was needed.

It still remains to show that $\pi_{n-2}(\mathrm{Con}_\infty S) \neq 0$. To see that we take the boundary $\square_d^{n-2} \subset \mathbb{R}^{n-1}$ $(= (\mathbb{R}_+^\times)^{n-1})$ of the standard cube of size d and let $\square_d^+ = \mathbb{R}^n \times \square_d^{n-2} \subset S$.

Then we homotop $\square_d^{n-2}$ inside $\square_d^+$ to another spherical surface, say $\square' \subset \square_d^+ \subset S$, such that the implied map $\square_d^{n-2} \to \square'$ is λ-Lipschitz with $\lambda \lesssim \log d$. To do this we observe that every $(n-2)$-face of $\square_d^{n-2}$ is contained in a unique semi-hyperbolic slice while every lower dimensional face lies in an intersection of such slices. In particular, every edge of $\square_d^{n-2}$ lies in a hyperbolic plane. The first stage of our homotopy consists in replacing the edges by s.h. segments with the same ends. Then we fill in the boundaries of the 2-faces by minimal surfaces in the corresponding slices etc. Thus we obtain $\square' \subset S$ similar to that considered earlier in (e) and the non-contractibility of the corresponding sphere(s) in $\mathrm{Con}_\infty C$ appearing in the limit for $d \to \infty$ is proven along the same lines as in (e).

Remarks (a). The above consideration is inspired by the discussion in §12.4 of [E-C-H-P-T], where the reader may find another construction of $\square'$.

(b) It seems that our argument leads to the vanishing of $\pi_i(\mathrm{Con}_\infty S)$ for $i = 1, \ldots, n-3$, but I did not check the details 100%.

2.B_1. Open questions. Our study of the asymptotic cones probably extends with little effort to all connected (algebraic ?) Lie groups as well as p-adic groups. It seems that these groups G have pretty looking finite dimensional cones $\mathrm{Con}_\omega G$ which are (essentially) independent of the choice of the ultrafilter

ω. (Besides Riemannian left invariant metrics on C one may use those induced from some ambient groups $G' \supset G$ and look at $\mathrm{Con}_\omega (G|\,\mathrm{dist}_{G'} G)$, compare §3.)

A next manageable class of groups is provided by non-cocompact lattices in (real and p-adic) Lie groups. For example for $\Gamma = SL_n\mathbf{Z}$, $n \geq 3$, one expects $\pi_i\,(\mathrm{Con}_\infty \Gamma) = 0$ if $i = 1, 2, \ldots, n-3$, and one knows that $\pi_{n-2}(\mathrm{Con}_\infty \Gamma) \neq 0$, as this follows from the analysis in chapter 12 of [E-C-H-P-T]. (The vanishing of π_1 for $n \geq 4$ seems to follow from our discussion on $\mathbf{R}^n \ltimes \mathbf{R}^{n-1}$ and the considerations in chapter 12 of [E-C-H-P-T]. In fact, we expect the filling span function (defined in §5) for these Γ to be linear, $FS(\ell) \sim \ell$ and this is, I suspect, is known to the authors of [E-C-H-P-T].) Now we suggest some questions concerning $\mathrm{Con}_\infty \Gamma$ for (more) general groups.

(c) Find an example of a finitely generated (better, finitely presented) Γ where $\mathrm{Con}_{\omega_1} \Gamma$ is not homeomorphic (or, at least not bi-Lipschitz) to $\mathrm{Con}_{\omega_2} \Gamma$ for two ultrafilters ω_1 and ω_2. *

(d) Does the non-vanishing of $\pi_1(\mathrm{Con}_\infty \Gamma)$ automatically imply that this π_1 is infinitely generated? (Observe that the group Iso of isometries of $\mathrm{Con}_\infty \Gamma$ is *transitive* on $\mathrm{Con}_\infty \Gamma$ and for every non-trivial $\alpha \in \pi_1(\mathrm{Con}_\infty \Gamma)$ the elements $i_*(\alpha)$, $i \in$ Iso may span in π_1 an infinitely generated subgroup.)

(e) How non-free can $\pi_1(\mathrm{Con}_\infty \Gamma)$ be? For example, when can a given (finitely generated) group be embedded into $\pi_1(\mathrm{Con}_\infty \Gamma)$ for some Γ ?

One could continue indefinitely with such questions but this appears premature at this stage as the major task lies in an actual determination of $\mathrm{Con}_\infty \Gamma$ for various groups Γ, for example, for amalgamated products and HNN-extensions. (This is easy for the free products; for the amalgamated products $G_1 *_H G_2$ the results depend on how much H is distorted in G_1 and in G_2, compare §3.)

An asymptotically attractive class of spaces more general than infinite groups is provided by leaves of foliations of compact spaces. For example let X be a manifold of negative curvature and $S \subset X$ be a horosphere with the induced Riemannian metric. One asks what is the structure of $\mathrm{Con}_\infty S$, for example when this Con_∞ has finite (Hausdorff) dimension. (This would follow from the boundness of the ratios of the volumes of R-balls in S for $R \to \infty$, that is $\mathrm{Vol}\, B(R, s_1)/\mathrm{Vol}\, B(R, s_2)$.) Of course, one does not expect much unless X has some degree of homogeneity, e.g. $V = X/\mathrm{Iso}\, X$ is compact and S appears as leaf of the horospherical foliations of V. In this (compact) case one also wants to understand how Con_∞ depends on a chosen leaf in this foliation and what happens when one deforms the metric in X. Similar questions arise in the case of discrete (i.e. time $= \mathbf{Z}$) Anosov systems where the cones Con_∞ (stable leaves) are essentially determined by the dynamics.

2.C. Fine structures on Con_∞. The space $\mathrm{Con}_\infty X$, as we defined it, encodes the geometry of the configurations of points $x_1, \ldots, x_k$ with mutual distances $\mathrm{dist}(x_i, x_j)$ having same order of magnitude as these distances all go to infinity. But now let us try to include into the picture of Con_∞ more general configurations of points, such, for example, as x_i^μ, $i = 1, \ldots, k$, $\mu = 1, \ldots \rho$, where the distances $\mathrm{dist}(x_i^\mu, x_j^\nu)$ have order d, for $i \neq j$ whilst the distances $\mathrm{dist}(x_i^\mu, x_i^\nu)$ are much smaller, say about $\sqrt{d}$ for all i, μ and ν. As we pass to the limit for $d \to \infty$ and normalize the distances by dividing by d, we wish to keep track of $\mathrm{dist}(x_i^\mu, x_i^\nu)$ or at least of their mutual ratios. This can be achieved by using the full power of the ultralimits where $\mathrm{Con}_\omega X$ comes along with the structure of a *non-standard* metric space whose distances take values in the field of non-standard real numbers. To continue the discussion in this direction we need some evidence that the non-standard $\mathrm{Con}_\omega X$ may carry an additional useful information about X but at the present moment we have no convincing example.

2.C$_1$. Geometry of Con_∞ and the space of the word metrics. Look again at the most elementary example of $\Gamma = \mathbf{Z}^n$ with a word metric or with a metric coming from a Riemannian manifold X where Γ acts discretely and cocompactly. (In the latter case the orbit map $\gamma \mapsto \gamma x_0$ *embeds* $\Gamma \hookrightarrow X$ for a *generic* x_0

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* M. Sageev indicated to me that this question should be treated in the general framework of the model theory.

and we speak of the distance function induced on Γ from X.) The asymptotic cone for Γ with such metric is unique and it can be described elementarily as follows. Rescale the original distance function on Γ by

$$\operatorname{dist}_i(\gamma_i, \gamma_2) = i^{-i} \operatorname{dist}(\gamma_1^i, \gamma_2^i)$$

and observe for $\Gamma = \mathbb{Z}^n$ that the function $d_i(\gamma) = \operatorname{dist}_i(\mathrm{id}, \gamma)$ is (obviously) asymptotically subadditive, i.e.

$$d_{i+j} \leq d_i + d_j \ + \text{const} \ ,$$

and therefore, there exists a limit $\operatorname{dist}_\infty = \lim_{i\to\infty} \operatorname{dist}_i$. This limit metric is (obviously) scale invariant and hence extends to a Minkowski metric (associated to a Banach norm) on $\mathbb{R}\Gamma = \Gamma \otimes_{\mathbb{Z}} \mathbb{R} (\approx \mathbb{R}^n)$. It is now clear that $\mathbb{R}\Gamma$ with this extended metric $\operatorname{dist}_\infty$ is canonically isometric to $\operatorname{Con}_\infty \Gamma$.

Let us indicate a simple corollary of the above discussion which indicates possible generalizations to non-Abelian groups Γ.

Asymptotic Lipschitz distance between metric spaces. Given spaces X and Y we define AL $\operatorname{Dist}(X, Y)$ as the infimum of logarithms of those numbers $\lambda \geq 1$ for which there exists a λ-Lipschitz on the large-scale quasi-isometry $X \to Y$ whose inverse $Y \to X$ is also λ-Lipschitz on the large-scale. (To be precise one should make our map a bijection between some nets in X and Y.)

This AL Dist is not quite a metric since $AL \operatorname{Dist}(X, Y) = 0$ does not imply that X is isometric to Y. But we agree to identify the spaces with AL Dist $= 0$ and thus every set of metric spaces becomes a metric space.

Example. Consider the word metrics in a group Γ corresponding to all finite generating subsets $S \subset \Gamma$. The resulting metric space $\{(\Gamma, \operatorname{dist}_S)\}$ is denoted by $WM\Gamma$.

Remark. One could modify the definition of $WM\Gamma$ in several ways. For example, one could enlarge the class of the metrics by allowing those induced by Riemannian metrics in suitable X acted upon by Γ. Another (more serious) modification consists in changing the definition of AL Dist by restricting to those quasi-isometries $(\Gamma, \operatorname{dist}_S) \leftrightarrow (\Gamma, \operatorname{dist}_{S'})$ which are isomorphisms of Γ or, at least, isomorphisms between subgroups of finite index in Γ.

Now we return to $\Gamma = \mathbb{Z}^n$ and state the promised

Corollary. *The metric completion of the space $WM\mathbb{Z}^n$ is isometric to the Banach-Mazur space BM_n of n-dimensional Banach spaces. In particular, the completed $WM\mathbb{Z}^n$ is a compact infinite dimensional connected space.*

Proof. The convergence $\operatorname{dist}_i \to \operatorname{dist}_\infty$ shows that

$$AL \operatorname{Dist}\big((\mathbb{Z}^n, \operatorname{dist}),\ (\mathbb{Z}^n, \operatorname{dist}_\infty)\big) = 0$$

which easily yields the Corollary.

Remark. If we modify the definition of $WM\mathbb{Z}^n$ according to the above remark we shall still arrive at the same *completed* space of metrics, isometric to BM_n. Notice that BM_n appears as the quotient of the space of norms on $\mathbb{R}^n$, say $BM_n = N_n/GL_n\mathbb{R}$, (where N_n is quasi-isometric to $GL_n\mathbb{R}$) and $WM\mathbb{Z}^n$ can be described in a similar way. (Here one should take into account not only the action $SL_n\mathbb{Z}$ on the space of metrics on $\mathbb{Z}^n$ but also isomorphisms between subgroups of finite index whose major representative is $\gamma \mapsto \gamma^i$.)

Questions. Is it possible (and worthwhile) to determine $WM\Gamma$ for (more) general Γ ? For example, when is $WM\Gamma$ compact, bounded, has finite asymptotic dimension, connected, etc. ?

Examples. (a) *Nilpotent groups* (compare $[\mathrm{Pan}]_2$, $[\mathrm{Bab}]_{1,2}$). If Γ is a finitely generated nilpotent group, then by Pansu's theorem there exists a unique asymptotic cone $\operatorname{Con}_\infty \Gamma$ which is a nilpotent Lie group G with a (Finsler) Carnot-Caratheodory left invariant metric. Moreover, $(G, \operatorname{dist}_\infty)$ admits a selfsimilarity by some automorphism (dilation) of G. The space of such $(G, \operatorname{dist}_\infty)$ is quite similar to the Banach-Mazur space and one may expect that $WM\Gamma$ is rather simple (e.g. bounded) in this case. Here, one does not know, in general, if Γ is quasi-isometric to $(G, \text{dist}_\infty) = \text{Con}_\infty \Gamma$ which causes certain problems. Yet, such a quasi-isometry does exist if Γ itself admits a dilation, i.e. a distance strictly increasing endomorphism $e : \Gamma \to \Gamma$. Such an endomorphism replaces $\gamma \mapsto \gamma^i$ for $\Gamma = \mathbb{Z}^n$ and makes Γ in all respects very similar to $\mathbb{Z}^n$.

(b) *Hyperbolic groups.* If Γ is a non-elementary hyperbolic group, e.g. $\Gamma = \mathbf{F}_2$, then the space $WM\Gamma$ looks very large. For example, if we fix some generators $\gamma_1 \dots \gamma_k$ in Γ and take the word metrics for $\{\gamma_1, \dots, \gamma_k, \gamma\}$ with variable γ, then the resulting image of Γ in $WM\Gamma$ (for $\gamma \mapsto$ metric) seems unbounded. (This is especially easy to prove for groups Γ having few quasi-isometries, e.g. for cocompact lattices in $Sp(n,1)$.) On the other hand, the (ergodic) probabilistic considerations suggest that different word metrics on Γ are "very similar" on "very large" subsets in Γ but one does not know how to make this precise *without* fixing in advance some measure on a pertinent probability space (of finite or infinite words in Γ). Yet one still hopes, inspired by the Mostow rigidity theorem (in the shape of [Mar]$_3$), that there is some natural equivalence relation on $WM\Gamma$ (for most hyperbolic Γ) whose quotient is a pretty little space.

(c) *Groups of higher rank.* Let Γ be a cocompact lattice in a non-compact simple Lie group G of $\mathbf{R}$-rank $n \geq 2$. Then Γ is full of free Abelian subgroups of rank n. If we take one such subgroup A and restrict a word metric from Γ to (a non-word metric on) A we obtain a map $WM\Gamma \to BM_n (= WMA)$. It is easy to see that this map does not depend on A, as any two flats in the symmetric space $G/(\text{Max.Comp})$ admit a third flat asymptotic to both of them. This can also be seen in $\text{Con}_\infty \Gamma$ which is built of uncountably many copies of $\text{Con}_\infty A$. Thus, although $WM\Gamma$ may be uncomfortably large, what remains in $\text{Con}_\infty \Gamma$, say $WM\,\text{Con}_\infty \Gamma$, is not bigger than BM_n.

(d) In order to make a large space $WM\Gamma$ more manageable one may use some exhaustion by smaller (e.g. compact or bounded) subsets corresponding to metrics on Γ restricted in a certain way. For example, if Γ is a hyperbolic group, a good class of metrics consists of those which are δ-hyperbolic for a given δ and such that the number of elements in the R-balls satisfy $Nmb_R \leq \exp CR$ for a given C. Now we may exhaust $WM\Gamma$ by the subsets $WM_\rho\Gamma$ built out of metrics on Γ with $C + \delta^{-1} \leq \rho$. For most hyperbolic groups the subsets $WM_\rho\Gamma$ are precompact in $WM\Gamma$ for every $\rho > 0$. (See [Gro]$_{14}$ on some remarks about the proof.)

Now we return to $\Gamma = \mathbb{Z}^n$ with a metric dist as in the beginning of this section and state the following recent result by D. Yu. Burago (see [Bur]).

2.C_1'. Burago's Theorem. *The asymptotic cone* $\text{Con}_\infty(\Gamma) = (\mathbf{R}\Gamma, \text{dist}_\infty)$ *has finite Hausdorff distance from* (Γ, dist), i.e.

$$\Gamma \underset{\text{Hau}}{\sim} \text{Con}_\infty \Gamma \,.$$

Ingredients of the proof. The basic technical observation by Burago is the following property of our metric dist on Γ.

2.C_1''. Burago's inequality.

$$\text{dist}(\text{id}, \gamma) \leq \frac{1}{2} \text{dist}(\text{id}, \gamma^2) + \text{const} \,. \qquad (*)$$

Burago proves this inequality by constructing a path from id to γ out of n-pieces of the shortest path from id to γ^2. (Burago states and proves all his results for Γ freely acting on a Riemannian manifold X homeomorphic to $\mathbf{R}^n$, but this is just because he does not use the invariant language.) This is done with another pretty geometric lemma attributed by Burago to F.L. Nazarov,

2.C_1'''. *Let* $p : [0,1] \to \mathbf{R}^n$ *be a* C^1*-path. Then there exists an open subset* $A \subset [0,1]$, *such that*

$$(1) \qquad \int_A p'(t)dt = \frac{1}{2} \int_0^1 p'(t)dt \left(= \frac{1}{2} p(1) \right),$$

where p' *denotes the derivative of* p.

(2) *The number of connected components (intervals) of A is at most n.*

(3) *The length (i.e. the measure) of A is at most $\frac{1}{2}$.*

Proof (due to Burago and Perelman). Every point $s = (t_1, \dots t_n) \in \mathbb{R}^n$ lying on the (topological) sphere $\sum_{i=1}^{n} |t_i| = 1$ defines a partition of $[0,1]$ into two subsets, $[0,1] = A_+^s \cup A_-^s$, as $[0,1]$ is covered by the n intervals

and, by definition, $t \in A_+^s$ if it is contained in such interval, say

$$\left[|t_1| + \dots + |t_i|,\ |t_1| + \dots + t_{i+1}|\right] ,$$

where $t_{i+1} \geq 0$ and $t \in A_-^s$ if t_{i+1} corresponding to t is negative. Then we have a (continuous!) map of our sphere to $\mathbb{R}^n$ given by

$$s \mapsto \int_{A_+^s} p'(t)dt - \int_{A_-^s} p'(t)dt ,$$

which by the Borsuk-Ulam theorem vanishes at some s in the sphere. Then we take the smallest of the two sets $A_+^{s_0}$ and $A_-^{s_0}$ for A.

2.C₂. *Remarks.* (a) The above discussion suggests a variety of generalizations to non-Abelian groups Γ. For example, if Γ is nilpotent we also have a unique $\mathrm{Con}_\infty \Gamma$ and we may ask when this Con_∞ is $\underset{\text{Hau}}{\sim} \Gamma$. This is quite unlikely to be true in the general case, but not impossible if Γ admits a dilation (e.g. Γ is the Heisenberg group, $\{a, b, c \mid [ac] = [bc] = 1, [ab] = c\}$). Even if the Hausdorff distance from Γ to $\mathrm{Con}_\infty \Gamma$ is infinite one may look at the distances from R-balls $B(R, X)$ in Γ to R-balls in $\mathrm{Con}_\infty \Gamma$. Any bound on these distances better than

$$\mathrm{dist}_{\mathrm{Hau}}(B(R;\Gamma), B(R, \mathrm{Con}_\infty \Gamma)) \leq \mathrm{const}\, R, \quad \text{for } R \to \infty ,$$

would be a pleasure to have in our possession, even in the special case of word metrics on Γ.

(b) Burago's inequality $(*)$ makes sense for an arbitrary (non-Abelian) group Γ and is well known (and obvious) for non-torsion γ in *hyperbolic* groups Γ, and one may conjecture this for more general semi-hyperbolic groups of some kind. Furthermore, it is not hard to prove $(*)$ for non-torsion elements γ in the cocompact lattices in semi-simple (real and p-adic) Lie groups. In fact the minimal path (in X) from id (i.e. $x_0 \in X$) to γ (i.e. $\gamma(x_0) \in X$) lies near the centralizer $C_\gamma \subset \Gamma$ and $(*)$ for $\gamma \in \Gamma$ follows from $(*)$ for $\gamma \in C_\gamma$. It seems this argument works for the fundamental groups of closed manifolds (and singular spaces) V with $K(V) \leq 0$, but for the small cancellation group the situation is less clear.

(b′) One may look further for inequalities more general than $(*)$ of the form $\sum_i \mathrm{dist}(\mathrm{id}, w_i) \leq \mathrm{const}$ for some particular configuration of words $\{w_i\}$ in Γ. Here, besides $(*)$ (and similar relations concerning $w_i = \gamma^i$) one knows a variety of such inequalities for *hyperbolic groups*, starting with the inequality *defining* δ-hyperbolicity in $[\mathrm{Gro}]_{14}$.

(b″) In the case of nilpotent groups Γ with a dilation, i.e. an expanding endomorphism $e : \Gamma \to \Gamma$, one asks for inequalities generalizing $(*)$ where $\gamma \mapsto \gamma^2$ is replaced by $\gamma \mapsto e(\gamma)$.

(c) If Γ is non-nilpotent the space $\mathrm{Con}_\infty \Gamma$ lies very far from Γ in the Hausdorff sense. Yet one may expect a version of Burago's theorem in the following setting. Suppose we have two manifolds X_1 and X_2

acted upon by Γ discretely isometrically and cocompactly, such that $AL\,\mathrm{Dist}(X_1, X_2) = 0$. The question is whether $X_1 \underset{\mathrm{Hau}}{\sim} X_2$ and if not what is the behaviour of the Hausdorff distance between the balls $B(R)$ in X_1 and X_2 as $R \to \infty$.

(d) One wonders if Lemma 2.C_1''' extends to some non-Abelian groups Γ. For example one may take a (long) word w in Γ, and ask what are possible values of the product $w_1 w_2 \dots w_n \in \Gamma$ for the n-tuples of disjoint subwords $w_1, \dots, w_n$ in w.

§3. Extrinsic geometry of subgroups and submanifolds; distortion and retra-curvature of subgroups in Lie groups; polynomial, exponential and non-recursive distortion.

Consider an embedding between finitely generated groups, $\Gamma_0 \subset \Gamma$, such that the generating set of Γ_0 includes into that of Γ. Then, clearly

$$\mathrm{dist}_{\Gamma_0} \geq \mathrm{dist}_\Gamma |\Gamma_0$$

and we want to understand by how much dist_{Γ_0} is greater than $\mathrm{dist}_\Gamma |\Gamma_0$. The relation between the two distances can be expressed numerically if we take the concentric balls $B(R) \subset \Gamma$ around the identity and look at the *distortion* function

$$\mathrm{disto}\,(R) = R^{-1}\ \ \mathrm{Diam}_{\Gamma_0}\,(\Gamma_0 \cap B(R))\ .$$

If the metric dist_{Γ_0} is Lipschitz equivalent to $\mathrm{dist}_\Gamma |\Gamma_0$ then the distortion is bounded; otherwise it $\to \infty$ for $R \to \infty$. For example, if Γ is a *free group* or a free Abelian group, then the distortion is (obviously) bounded for every $\Gamma_0 \subset \Gamma$. But we shall see in 3.J and 3.K that the distortion sometimes grows faster than any recursive function.

The group theoretic discussion has a Riemannian counterpart. For every submanifold $X_0 \subset X$ one compares the distance in Y induced from X and the distance corresponding to the induced Riemannian metric. Obviously again $\mathrm{dist}_{X_0} \geq \mathrm{dist}_X |X_0$ and the distortion is measured as earlier by the growth of $\mathrm{Diam}_{X_0}(X_0 \cap B(R, X))$. (Here, this diameter may depend on the center of the ball B in X and, if one wishes to come up with a number, one should take sup of the distortion function on X.)

The typical example of the bounded distortion is $X_0 = \mathbb{R}^k \subset \mathbb{R}^n = X$. More generally one takes a totally geodesic subspace Y in a complete simply connected space X of non-positive curvature. In this case the bounded distortion (obviously) comes from the following stronger property: *X_0 is a Lipschitz retract in X.* In fact, the normal projection $X \to Y$ decreases the Riemannian metric and, hence, it decreases the distances as well.

We have already met an instance of *exponential distortion*, namely that of a horosphere X_0 in the hyperbolic space $X = H^n$ where the *extrinsic* distance $\mathrm{dist}_X |X_0$ is about $\log \mathrm{dist}_{X_0}$.

On the other hand, if a subset in a (general) hyperbolic space has sub-exponential distortion, then, in fact, the distortion is bounded (see §7 in $[\mathrm{Gro}]_{14}$).

As on the previous occasions the group theoretic situation reduces to the Riemannian one whenever Γ acts discretely, isometrically and cocompactly on X, such that $X_0 \subset X$ is invariant under Γ_0 whose action on X_0 is cocompact. A particular instance of that is a cocompact lattice Γ in a Lie group G such that the intersection of Γ with a connected subgroup $G_0 \subset G$ is cocompact in G_0.

What follows is a series of examples where we evaluate the distortion.

3.A. Subgroups in GL_n. The simplest instance here is that of a cyclic subgroup generated by a single $n \times n$ matrix $g \in G = GL_n\mathbb{R}$. If some eigenvalue, say λ of g has $|\lambda| \neq 1$, then the subgroup $\{g^i\}$ has bounded distortion in G. In other words

$$\mathrm{dist}_G\,(id, g^i) \geq \mathrm{const}\, i\ ,$$

as everybody knows. (Here and below G comes along with a left invariant Riemannian metric.) On the other hand if all eigenvalues of g have absolute value 1, then the distortion is unbounded. Moreover, if such a g is *semi-simple*, it has

$$\mathrm{dist}_G\,(\mathrm{id}, g^i) \leq \mathrm{const} < \infty\ ,$$

which means *infinite* distortion in our terminology. If g is not semi-simple, then

$$\mathrm{dist}_G\,(\mathrm{id}, g^i) \approx \log i\ \text{ for }\ i \to \infty$$

and so the distortion is exponential. For example, this is the case for every *unipotent* $g \neq \mathrm{id}$ in G, such as the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in GL_2.

3.A₁. *Remark.* The natural embeddings $GL_n\mathbf{R} \subset GL_n\mathbf{C}$ and $GL_n\mathbf{C} \subset GL_{2n}\mathbf{R}$ have bounded distortion as well as the corresponding embeddings of the SL_n-groups. This is seen in the SL_n-case by looking at the corresponding symmetric spaces where the embeddings are geodesic. (In fact, the embeddings between semi-simple Lie groups always have bounded distortion.) Then the GL-case easily reduces to SL. Thus we do not have to care much about the ground field being $\mathbf{R}$ or $\mathbf{C}$. (In fact, the above discussion extends to p-adic fields as well.)

3.A₂. Algebraic subgroups. Let $G_0 \subset G = GL_n\mathbf{R}$ be a topological connected component of a real algebraic subgroup. We claim that the distortion is *at most exponential* in this case. To see that we first observe that the left invariant dist_G is related to the (Euclidean) matrix norm $\|g\| = \sum_{i,j=1}^{n} |g_{ij}|^2$ by

$$\mathrm{dist}_G(g, \mathrm{id}) \leq \mathrm{const}\, |\log \|g\|| \ .$$

(In fact, $\mathrm{dist}_G(g, \mathrm{id}) \approx \log(1 + \|g - \mathrm{Id}\|)$.) Secondly, we notice that a connected (component of a) real algebraic subvariety $X_0 \subset \mathbf{R}^N$, always has *bounded* distortion measured with balls $B(R) \subset \mathbf{R}^N$ around a *fixed* point in $\mathbf{R}^N$. This follows by an elementary real algebraic argument (compare p. 124 in [Gro]$_{12}$). In particular, our subgroup $G_0 \subset GL_n\mathbf{R} \subset \mathbf{R}^{n^2}$ has bounded Euclidean distortion and, therefore, the distortion with respect to the left invariant metric dist_G is at most exponential.

3.A′₂. Non-algebraic connected subgroups $\mathbf{G_0}$ in $\mathbf{G = GL_n}$. All these have at most exponential distortion but the obvious proof is non-conceptual. One uses here the Levi decomposition and takes a cocompact solvable subgroup $S_0 \subset G_0$ (whose distortion in G is essentially the same as that of G_0). Then, working over $\mathbf{C}$, one has S_0 inside of the group G' of triangular matrices whose geometry is seen with the decomposition $G' = \Delta N$, where Δ is the group of the diagonal matrices and N is the nilpotent (normal in G') subgroup of the unipotent triangular matrices (compare 3.A₃). We leave an actual proof to the reader (compare 3.F).

3.A₃. Unipotent subgroups. Let G_0 be a connected unipotent subgroup in GL_n. Then, over $\mathbf{C}$, it can be realized by triangular matrices with the units on the diagonal. It easily follows that G_0 has exponential distortion in the following sharp sense: *The induced metric* $\mathrm{dist}_G |G_0$ *is quasi-isometric to* $\log(1 + \mathrm{dist}_{G_0})$ (compare (c) in 2.B). In fact this is already true for the distortion of G_0 inside G', the subgroup of triangular matrices in G (see 3.D).

3.B. Subgroups in nilpotent groups. Let G_0 be a connected subgroup in a simply connected nilpotent Lie group G. Then, as is well known, the distortion of G_0 in G is (at most) *polynomial*, i.e.

$$\mathrm{dist}_G |G_0 \geq C\, (\mathrm{dist}_{G_0})^{\alpha}$$

for some constants $C,\ \alpha > 0$.

3.B₁. *Example: Heisenberg group.* This is the only non-Abelian three-dimensional nilpotent group G. It contains, as a cocompact lattice, the discrete group $\{a, b, c \mid [a, b] = c, [a, c] = [b, c] = 1\}$. The center $C \subset G$ is one-dimensional, $C = \mathbf{R}$, and

$$\mathrm{dist}_G(\mathrm{id}, t) \sim \sqrt{|t|},\ t \in \mathbf{R} = C\ ,$$

since $[a^n, b^n] = c^{n^2}$. On the other hand, all one parameter subgroups in this G apart from C have bounded distortion.

3.B₂. One-parameter subgroups G_0. Every such $G_0 \subset G$ has

$$\mathrm{dist}_G \mid G_0 \sim (\mathrm{dist}_{G_0})^{\frac{1}{k}}\ ,$$

where k is the maximal integer, such that

$$G_0 \subset \underbrace{[[G,G],\ldots,G]}_{k} .$$

3.B$_3$. Polynomial similarity between nilpotent groups and $\mathbb{R}^n$. Every simply connected nilpotent group G is algebraic and biregular equivalent to $\mathbb{R}^n$, for $n = \dim G$. In fact the exponential map of the Lie algebra Lie $G = \mathbb{R}^n$ to G provides such an equivalence. Furthermore, the multiplication map

$$G \times G \to G, \text{ for } (g,h) \to gh^{-1} ,$$

is polynomial (for G identified with $\mathbb{R}^n$ via the exponential map). On the geometric side, the exponential map $e : \mathbb{R}^n \to G$ has the following "polynomially Lipschitz" property:

The Lipschitz constant of e on the balls $B(R) \subset \mathbb{R}^n$ around zero grows (at most) as R^α, for some $\alpha > 0$ (in truth, $\alpha \leq n$).

In fact, our exponential e is an equivalence in the category of such maps: it is invertible and e^{-1} is also "polynomially Lipschitz".

This property allows a "polynomial transplantation" of many features of $\mathbb{R}^n$ to nilpotent groups G. For example, every connected algebraic subvariety $X_0 \subset G$ (not necessarily a subgroup) has at most polynomial distortion (where the implied balls $B(R) \subset G$ are concentric around a *fixed* center). Next, G satisfies the polynomial isoperimetric inequality in dimension $k = 2$ (which was proven earlier by Gersten by more complicated means). In fact, G satisfies the k-dimensional inequality for all $k \leq \dim G$, but here the argument requires a little bit more besides the "polynomial Lipschitz" equivalence between G and $\mathbb{R}^n$ (namely, some general properties of filling from $[\mathrm{Gro}]_{10}$).

3.C. Strict exponential distortion of G_0 in G. This signifies the asymptotic relation for the distances of far away points

$$\mathrm{dist}_G \mid G_0 \sim \log \mathrm{dist}_{G_0} \qquad (*)$$

for dist $\to \infty$. The starting example is that of the 2-dimensional solvable group G which splits into semi-direct product $G = A \cdot N$, such that $a_0\, t\, a_0^{-1} = 2t$ for all $t \in \mathbb{R} = N$ and a fixed $a_0 \in A (= \mathbb{R})$. Then, obviously $G_0 = N$ satisfies the above $(*)$. In fact, our G is isometric to the hyperbolic plane and N represents a horocycle.

A similar picture is seen in the discrete version of the above group,break i.e. $\{a, b \mid a\,ba^{-1} = b^2\}$ (which *does not* sit *discretely* in the above G), where the cyclic subgroup $\{b^i\}$ has strict exponential distortion.

3.C$_1$. Let, more generally, N be a connected subgroup in a Lie group G, such that there exists some $a_1, \ldots, a_k \in G$ which *expand* N in the following sense: for every sufficiently large R the subsets

$$B^i(R) = \left(a_i B(R,N) a_i^{-1}\right) \cap N \subset N$$

contain in their group-theoretic product, denoted

$$B' = \left\{b_1 b_2, \ldots, b_k \mid b_i \in B^i(R)\right\} ,$$

the ball $B(\lambda R, N)$ for a fixed $\lambda > k$, i.e.

$$B' \supset B(\lambda R, N) ,$$

where $B(R,N)$ denotes the R-ball in N around id for the metric dist_N. Then, straightforwardly, N is strictly exponentially distorted in G.

3.C$'_1$. *Example.* Let G be a semi-simple group with the usual decomposition $G = KAN$. The N is strictly exponentially distorted in G and in AN. Geometrically speaking, one has the symmetric space

$X = G/R = AN$ and N appears as a normal orbit to a family of mutually asymptotic flats in X corresponding to A. For example, if $\operatorname{rank} X = 1$, N is as just a horosphere in X.

3.C$''_1$. Remarks on the asymptotic cone $\mathrm{Con}_\infty \mathbf{X}$. Let X be a symmetric space of non-compact type which we view as the above solvable group, $X = AN$.

Claim.

$$\dim \mathrm{Con}_\infty X = \operatorname{rank} X \underset{\text{def}}{=} \dim A\ .$$

Proof. Project $X \to A$ by $a\nu \mapsto \nu$ and observe that this map, say $p : X \to A$ is distance decreasing. Since the metric dist_X is logarithmic on the orbits $p^{-1}(a)$, $a \in A$, it becomes totally disconnected when we pass to Con_∞ (compare 2.B). Thus $\mathrm{Con}_\infty X$ fibers over $\mathrm{Con}_\infty A = \mathbb{R}^k$, $k = \operatorname{rank} X$, with totally disconnected fibers. Q.E.D.

Remark. The above argument shows that *maximal flats* in $\mathrm{Con}_\infty X$ coming from those in distance decreasing X are 1-Lipschitz retracts (of quite special kind) in $\mathrm{Con}_\infty X$ which suggests a picture of an *Euclidean* $\mathbb{R}$*-building* for $\mathrm{Con}_\infty X$.

Questions. Let X be a complete simply connected manifold without boundary of non-positive sectional curvature. Is $\mathrm{Con}_\infty X$ finite dimensional? Does the dimension of $\mathrm{Con}_\infty X$ equal that of the maximal flat in $\mathrm{Con}_\infty X$ (or in X) if the isometry group $\mathrm{Iso}\, X$ is cocompact on X ? What is the relation between $\mathrm{Con}_\infty X$ and the *Tits boundary* of X defined in [B-G-S]? One asks similar questions for singular spaces X with $K \le 0$, where one looks first at the 2-polyhedra with $K \le 0$. Then one asks these questions for "semi-hyperbolic" groups and especially for the small cancellation groups. (See §6 for a more extensive discussion on these problems.)

3.D. Non-distortion of certain solvable subgroups. The geometry of horospheres dramatically changes when we go from spaces X with rank $= 1$ (i.e. $K(X) \le -c < 0$) to those of rank ≥ 2 (where $K \le 0$ and somewhere $= 0$). Namely, we shall see presently that a *generic* horosphere in a symmetric space X of rank ≥ 2 has (amazingly!) *bounded* distortion.

3.D$_1$. *Example.* Let X be the product of two hyperbolic planes H^2. Here "generic" geodesics are the *non-singular* ones, i.e. those which are neither vertical nor horizontal. (In other words the *singular* geodesics are those lying in the fibers $h \times H^2 \subset H^2 \times H^2$ and in $H^2 \times h$, for $h \in H^2$.) A horosphere normal to such a geodesic is intrinsically isometric to a 3-dimensional solvable Lie group. In fact, if we think of H^2 as a 2-dimensional solvable group AN (see 3.C) our product $H^2 \times H^2$ is a 4-dimensional group where our horosphere lies as a subgroup. Despite a strong convexity, this horosphere is undistorted in the ambient space.

3.D$_2$. *Explanation.* The following simple proposition strips off the mystery and amazement from the above discussion.

Let G be a simply connected solvable group, $h : G \to A = \mathbb{R}^k$ a split homomorphism into an Abelian group, let $A_0 \subset A$ be a connected subgroup (i.e. a linear subspace in $\mathbb{R}^k$) and $G_0 \subset G$ the pull-back of A_0, i.e.

$$G_0 = h^{-1}(A_0)\ .$$

If the kernel $N = h^{-1}(0) \subset G_0$ has strictly exponential distortion in G_0, then G_0 has bounded distortion in G.

Proof. The exponential distortion condition implies that $\mathrm{dist}_{G_0} N$ is quasi-isometric to $\log(1 + \mathrm{dist}_N)$. By the same token the (even smaller) metric $\mathrm{dist}_G N$ is also quasi-isometric to $\log(1 + \mathrm{dist}_N)$. Now the metric in G is determined in a simple way by distance functions $\mathrm{dist}_G N$ and dist_A. To see that, fix a lift $A \hookrightarrow G$ and write $G = AN$. Then $\mathrm{dist}_G(\mathrm{id}, a\nu)$ is (bi-Lipschitz) equivalent to

$$\mathrm{dist}_G(\mathrm{id}, \nu) + \mathrm{dist}_A(\mathrm{id}, a)\ .$$

This equally applies to $G_0 = A_0N$ and shows the equivalence of $\mathrm{dist}_G \mid G_0$ to dist_{G_0}.

3.D$'_2$. *Remark.* Let $G_0 = A_0N$ be a solvable group as above and let us additionally assume that the adjoint action of every element $a_0 \in A_0$ on the Lie algebra Lie N has at least one eigenvalue λ satisfying $|\lambda| \neq 1$. Then it is not hard to see that for any Lie group extension $G \supset G_0$ the distortion of G_0 in G is bounded.

Final remarks on the distortion of Lie groups. It is probably possible (and not difficult) to give a complete description of the distortion of a connected Lie group G_0 inside another one. Apparently, there are only three mutually exclusive cases: bounded distortion, polynomial distortion and exponential distortion and our examples suggest algebraic criteria to distinguish these cases. In fact one expects here finer results. For example, given a left invariant metric on G_0 one should tell if it comes, up to bi-Lipschitz equivalence, from some $G \supset G_0$.

3.E. Lipschitz non-retractability of G to G_0. Although the above G_0 appears undistorted in G as far as the distance function is concerned the embedding $G_0 \subset G$ may greatly (in fact, exponentially) distort areas and volumes (see §5). This may lead, in particular, to non-existence of a Lipschitz (and even large-scale Lipschitz) retraction $G \to G_0$ unlike the situation one has for the inclusion between semi-simple Lie groups. There is yet another way to prove this non-retraction property using the asymptotic cones Con_∞. For example, look at an embedding of the $(n+1)$-dimensional solvable group $G_0 = S = AN$ for $A = \mathbb{R}$ and $N = \mathbb{R}^n$ into the Cartesian product of n copies of S_2 (= Sol_2 in the notations of (e) of 2.B), where

$$G = S_2 \times S_2 \times \ldots \times S_2 = A' \cdot N, \text{ for } A' = \mathbb{R}^n .$$

Recall (see (e) in 2.B) that the group $S = \mathrm{Sol}_{n+1}$ is determined by the action of $A = \mathbb{R}$ on $N = \mathbb{R}^n$ which we assume diagonal (in the Euclidean basis of $\mathbb{R}^n$) with the diagonal entries $e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}$, $t \in A = \mathbb{R}$ (compare (e) in 2.B). In particular S_2 is determined by a single $\lambda \in \mathbb{R}$ which we assume non-zero. Thus S_2 is isometric to the hyperbolic plane H^2 of constant negative curvature. Now, every linear embedding of $A = \mathbb{R}$ to $A' = \mathbb{R}^n$, given by $t \mapsto \underline{t}' = \{c_i t\}$, $i = 1, \ldots, n$, defines a map of $S = AN$ to $G = A'N$ as $N \overset{\mathrm{id}}{\mapsto} N$. Furthermore, the induced group structure equals that of S if $\lambda_i = a_i\lambda$ for all $i = 1, \ldots, n$. Thus by choosing $c_i = \lambda_i\lambda^{-1}$ we can always realize S by a subgroup in G.

If we assume as earlier that all $\lambda_i \neq 0$, then the above embedding has bounded distortion and, hence, induces a bi-Lipschitz embedding of the asymptotic cones

$$\mathrm{Con}_\infty S \to \mathrm{Con}_\infty G = \mathrm{Con}_\infty \underbrace{H^2 \times \ldots \times H^2}_{n} .$$

The latter cone, as we know, is the Cartesian product of n trees while $\mathrm{Con}_\infty S$ may be non-contractible. In fact, it has infinitely generated π_1 if $\lambda_1 > 0$ and $\lambda_2 < 0$. In this case there is no Lipschitz (not even topological) retraction $\mathrm{Con}_\infty G \to \mathrm{Con}_\infty S$ and hence $S = \mathrm{Sol}_{n+1}$ is not a Lipschitz (not even large-scale Lipschitz) retract in $G = \mathrm{Sol}_2 \times \mathrm{Sol}_2 \times \ldots \mathrm{Sol}_2$.

Question. Suppose all λ_i are positive. Does then $H^2 \times \ldots \times H^2$ admit a Lipschitz retraction on the corresponding S ?

Remark. The embedded cone $\mathrm{Con}_\infty(S) \subset \mathrm{Con}_\infty(H^2 \times \ldots \times H^2)$ does admit a Lipschitz retraction in this case, because $\mathrm{Con}_\infty S$ is a tree. In fact, every tree (or Cartesian product of finitely many trees, $\mathbb{R}^k$ for instance) is a Lipschitz retract in an arbitrary ambient metric space where it sits with bounded distortion (everybody knows that). Yet one does not (seem to) know if the hyperbolic space H^{n+1} for $n \geq 1$ has this property. (Notice that S for $\lambda_1 = \lambda_2 \ldots = \lambda_n$ is isometric to H^{n+1}.)

3.F. Retra-Curvature and distortion. One can measure the distortion (or the asymptotic extrinsic curvature) of $X_0 \subset X$ by the expansion rate of suitable retractions $X \to X_0$.

3.F$_1$. Nilpotent Example. Let $G_0 \subset G$ be an inclusion between simply connected nilpotent groups. Then there exists a retraction $\rho : G \to G_0$ with the following two properties

(1) *Polynomially Lipschitz.* The Lipschitz constant $\lambda = \lambda(R)$ of ρ restricted to every *R-neighbourhood* of G_0, i.e.

$$U_R(G) \underset{\text{def}}{=} \{u \in G \mid \text{dist}(u, G_0) \leq R\} ,$$

is bounded by

$$\lambda(R) \leq C = R^\alpha$$

for some $C, \alpha > 0$.

(2) *Equivariance.* The map ρ commutes with the (left) action of G_0 on G.

Proof. If the required retractions ρ exist for $G_0 \subset G_1$ and for $G_1 \subset G_2$ then ρ also exists for $G_0 \subset G_2$. Thus we can use induction on codim$(G_0 \subset G)$ and assume that G_0 is a *maximal* subgroup in G. If this is the case, then necessarily codim$\, G_0 = 1$ and moreover, G_0 is a *normal* subgroup. (This easily follows from the nilpotency of G.) Now every one-parameter subgroup $A_0 \subset G$ transversal to G_0 defines a unique G_0-equivariant retraction $\rho : G \to G_0$ satisfying $\rho^{-1}(\text{id}) = A_0$. Then the polynomial nature of the (nilpotent!) group G (see 3.B.3) takes over and provides a polynomial bound on the Lipschitz characteristic $\lambda(R)$ of this ρ.

Remark. The inverse of the exponential map $e : \mathbb{R}^n = \text{Lie}(G) \to G$, which establishes a polynomially Lipschitz equivalence between G and $\mathbb{R}^n$, sends $G_0 \subset G$ onto a linear subspace $L_0 \subset \mathbb{R}^n$. Thus, linear projections $\mathbb{R}^n \to L_0$ are transported by e^{-1} to polynomially Lipschitz retractions. But these retractions are *not*, in general, G_0-equivariant.

3.F$_2$. Split extensions. Let $G_0 \subset G$ be a normal subgroup, such that the quotient homomorphism splits

$$q : G \to G/G_0 = H \overset{s}{\rightleftarrows} H_0 \subset G .$$

Then $G = H_0 G_0$ and we have two retractions $G \to H_0$ and $G \to G_0$ for $h_0 g_0 \mapsto h_0$ and $h_0 g_0 \to g_0$ respectively.

Claims. The first retraction $G \to H_0$ is Lipschitz; the second retraction, $G \to G_0$, is *exponentially Lipschitz*, i.e. the function $\lambda(R)$ measuring the Lipschitz constant of our retraction on $U_R(G_0) \subset G$ satisfies

$$\lambda(R) \leq C^R \text{ for some } C > 1 .$$

Proof. The first claim is obvious, as q and s are Lipschitz. (Recall, that any homomorphism between Lie groups or finitely generated discrete groups is Lipschitz for our metrics.)

Now, we want to show that the second map, call it $\rho : G \to G_0$ is (at most) exponentially Lipschitz for left invariant Riemannian metrics in our Lie groups (or for the word metrics in the case of finitely generated groups). For this we need the following obvious

Lemma. *Let a group H act by automorphisms on G. Then*

$$|h(g)|_G \leq |g|_G \, \exp(C|h|_G) ,$$

where $|\cdot|$ denotes the distance from id *(or the minimal word length) in the group in question.*

Back to our $\rho : G \to G_0$, we recall that $\rho(h g_0) = g_0$ and so $\rho(g_0 h) = \rho(hh^{-1}gh) = h^{-1}g_0 h$. Thus $|\rho(hg_0)|_{G_0} \leq |g_0|_{G_0} \exp(C|h|_H)$, and our claim follows as $\text{dist}_G(\text{id}, h) = \text{dist}_H(\text{id}, h)$.

3.F$'_2$. ***Corollary.*** *The distortion of G_0 in G is (at most) exponential.*

3.F$''_2$. *Remark.* If $G \supset G_0$ is a *non-split* normal extension of finitely generated groups, then the distortion may be by far greater than exponential (see 3.K$''_3$). Also notice, that an extension always splits if the quotient group G/G_0 is free and then our exponential bounds remains valid.

3.F$_3$. Non-homogeneous case. An important example where the above applies is that of a symmetric space X of non-compact type which can be thought of as a (split) solvable group, $X = AN$. In particular, if

$$\operatorname{rank} X \underset{\text{def}}{=} \dim A = 1$$

and thus $K(X) < 0$, the (nilpotent) group N appears as a horosphere in X and our claim amounts to the well-known (and obvious) exponential bound on the Jacobi fields along geodesics normal to a horosphere. Such a bound remains valid for an arbitrary complete simply connected Riemannian manifold X with

$$-\infty < -c_1 \leq K(X) \leq 0 \ ,$$

and therefore, the normal retraction of X on every horosphere $S \subset X$ is exponentially Lipschitz. As a corollary we obtain the exponential bound on the 2-dimensional isoperimetric (filling, compare §5) function in S,

Every closed curve of length ℓ in S bounds a disk of area $\leq \exp C\ell$.

Proof. Our curve bounds a disk D of area $\leq \ell^2$ in $X \supset S$. Moreover, there is such a (minimal) disk within distance ℓ from our curve and, hence, from S. Then the retraction of D to S has the required exponential bound.

Remark. The above argument applies to the higher dimensional subvarieties and yields the exponential bound on minimal fillings of cycles in S of all dimensions.

3.F$_4$. Quasi-nilpotent extension. Let G be a connected Lie group and $N \subset G$ a simply connected nilpotent normal subgroup. Moreover, we assume that the adjoint action of N on Lie(G) is unipotent. The relevant consequence of that is the following bound on the norm of $Ad_\nu \ell$,

$$||Ad_\nu \ell|| \leq (|\nu|^\alpha + C)\,||\ell|| \ , \tag{$*$}$$

where $|\nu| = \operatorname{dist}_N(\mathrm{id}, \nu)$. The basic example one should keep in mind is where G is the group of upper triangular matrices and N consists of the matrices with unit diagonal entries.

Next, we take a connected subgroup G_0 in G, such that the quotient homomorphism $G \to H = G/N$ sends G_0 *onto* H and such that the group $N_0 = G_0 \cap N$ is *connected*. (Notice that N_0 is connected provided H is simply connected.)

Claim. *There exists a G_0-equivariant retraction $\rho : G \to G_0$ such that*

(a) *ρ is exponentially Lipschitz,* (i.e. *$\rho|U_R(G_0)$ is $\lambda(R)$-Lipschitz with $\lambda(R) \leq \exp CR$ for $R \to \infty$*).

(b) *The restriction* of *ρ to N is polynomially Lipschitz with respect to a left invariant Riemannian metric in N. Moreover, the norm of the differential* of *ρ on $T(G)$ is bounded on N* by *$||D_\rho(\nu)|| \leq |\nu|^\beta + B$ for all $\nu \in N$.*

Proof. According to 3.F$_1$, there exists a polynomially Lipschitz N_0-equivariant retraction $N \to N_0$ which then uniquely (and obviously) extends to a G_0-equivariant retraction $G \to G_0$. Using ($*$) one easily obtains the polynomial bound on $D_\rho(\nu)$ in (b). Finally, one obtains (a) by noticing that N has in G at most exponential distortion.

3.F$_5$. Solvable extensions. Let G and $N \subset G$ be as in 3.F$_4$ and let us additionally assume that

(i) the factorgroup $H = G/N$ is Abelian,

(ii) the extension $G \supset N$ splits, i.e. the group H embedds into G transversally to N.

Proposition. *Let $G_0 \subset G$ be a connected subgroup for which the fibration of G over the quotient space,* i.e. *$G \to G/G_0$, admits a continuous section (this is equivalent to the existence of a continuous G_0-equivariant retraction $G \to G_0$). Then there exists an exponentially Lipschitz G_0-equivariant retraction $G \to G_0$.*

Proof. Denote by $H_0 \subset H$ the image of G_0 in H and let $N_0 = G_0 \cap N$. Then we consider two intermediate subgroups,

(1) $\overline{G}_0 \subset G$, which is the minimal connected subgroup whose intersection with N is connected. This $\overline{G}_0$ is generated by G_0 and the connected hull of N_0 in N.

(2) G_1, which is just the pull-back of $H_0 \subset H$ under the quotient map $G \to H = G/N$.

Clearly,

$$G \supset G_1 \supset \overline{G}_0 \supset G_0 .$$

Now we retract $G \to G_1$ according to 3.F_2 and $G_1 \to \overline{G}_0$ according to 3.F_4. The composition of the two retractions is a $\overline{G}_0$-equivariant retraction $G \to \overline{G}_0$ which is exponentially Lipschitz as follows from 3.F_2 and (b) (not (a) !) in 3.F_4.

Since G_0 is a net in $\overline{G}_0$ this automatically gives us a "large-scale exponentially Lipschitz" retraction $G \to G_0$ without any extra assumptions on G_0, but if we want to drop "large-scale" we use our assumption, i.e. the existence of some continuous G_0-equivariant retraction $G \to G_0$ and hence, of $\overline{G}_0 \to G_0$. One can easily go from "continuous" to "smooth" which implies Lipschitz for the latter retraction as $\overline{G}_0$ lies within bounded distance from G_0. We conclude the proof by composing $G \to \overline{G}_0$ with $\overline{G}_0 \to G_0$.

3.F_5'. *Corollary.* *Let $G = GL_n$ and $G_0 \subset G$ a simply connected solvable group. Then G_0 is an exponentially Lipschitz G_0-equivariant retract in G.*

Proof. Working over $\mathbf{C}$ we have G_0 in the group $G_1 \subset GL_n\mathbf{C}$ of upper triangular matrices to which the proposition applies and immediately yields our corollary.

3.F_6. Remarks and open questions. (a) Let $G \supset G_0$ be an arbitrary extension of connected Lie groups. We can always assume that the maximal compact subgroup $K_0 \subset G_0$ is contained in $K \subset G$ and then we have a G_0-equivariant inclusion between the Riemannian homogeneous spaces

$$G_0/K_0 = X_0 \subset X = G/K .$$

It seems very likely that there always exists a G_0-equivariant exponentially Lipschitz retraction $X \to X_0$. Notice, that our proposition applies to a cocompact solvable subgroup $G_0' \subset G_0$ and thus yields an exponentially Lipschitz G_0'-equivariant retraction $X \to X_0$, provided G is a subgroup in some GL_n. (In the general case one should be aware of groups like the universal covering of $SL_2\mathbf{R}$ but they do not seem to be troublesome.)

(b) Our corollary 3.F_5' becomes especially useful if one recalls that $SL_n\mathbf{R}$ is quasi-isometric to the symmetric space $X = SL_n/SO_n$ which has a rather straightforward geometry because $K(X) \leq 0$. In particular we see that the usual isoperimetric inequalities in X imply exponential inequalities in G_0 (compare 3.F_3 and $[\text{Ger}]_{1,3,4,6}$).

3.G. Non-uniform lattices of rank 1. Let Γ be a *non*-cocompact lattice in a simple Lie group of rank 1. More generally, we may include in our discussion discrete groups Γ of isometries of complete simply connected manifolds X with negative curvature $K(X)$, such that

(i) $-\infty < -c_1 \leq K(X) \leq -c_2 < 0$,

(ii) $\text{Vol}\, X/\Gamma < \infty$ but the quotient space is non-compact. It is well known that such a Γ is finitely generated (see $[\text{Mar}]_4$, $[\text{Gro}]_2$) and we want to indicate the following properties of the corresponding word metric.

(1) *Every orbit map $\Gamma \to X$ for $\gamma \mapsto \gamma x_0$ has unbounded distortion. In fact, the distortion is strictly exponential on certain cyclic subgroups $C \subset \Gamma$ which themselves are undistorted in Γ. (In the Lie theoretic case this applies to the distortion of $\Gamma \subset G$.)*

(2) *The distortion of every nilpotent subgroup in Γ is at most polynomial.*

(3) *The asymptotic cone of* Γ *has* $\dim \mathrm{Con}_\infty \Gamma = \dim X - 1$.

Proof. One knows, that there exists a "core" $X_0 \subset X$ that is a Γ-invariant submanifold with boundary obtained by removing from X a disjoint union of horoballs, such that X_0/Γ is compact. Thus the boundary of X_0 consists of a union of disjoint horospheres and the orbit maps $\Gamma \to X_0$ are quasi-isometric.

Take one horosphere in ∂X_0, say $H \subset \partial X_0 \subset X_0 \subset X$ and recall (this is a well known result by Margulis and Heintze and Eberlein, see [B-G-S]) that there is a nilpotent subgroup $N \subset \Gamma$ mapping H into itself, such that H/N is compact. Since H is strictly exponentially distorted in X so is N. On the other hand N is undistorted in Γ because H is undistorted in X_0. In fact there is an obvious (normal) retraction $X_0 \to H$ which is 1-Lipschitz.

Now let N_0 be a torsionless nilpotent subgroup in Γ. Then one knows that either Γ preserves some horosphere, say $H \subset \partial X_0$, or Γ is infinite cyclic preserving some geodesic in X. In the latter case N_0 is a Lipschitz retract in X and hence in Γ, which makes it undistorted. Next, if Γ preserves H (which, as we know, is a Lipschitz retract in X_0) then the distortion of N_0 in Γ is the same as that of N_0 in the maximal N preserving H, which is at most polynomial.

Finally, to understand $\mathrm{Con}_\infty \Gamma$ (which is bi-Lipschitz to $\mathrm{Con}_\infty X_0$) one should invoke the notion of the relative hyperbolicity from $[\mathrm{Gro}]_{14}$. Then one can see that $\mathrm{Con}_\infty \Gamma$ is contractible; moreover it is a kind of a $(n-1)$-dimensional "tree", $n = \dim X$, whose "branches" are copies of $\mathrm{Con}_\infty N$ (or $\mathrm{Con}_\infty H$ which is bi-Lipschitz to N). In other words $\mathrm{Con}_\infty \Gamma$ looks like (probably homeomorphic to) the asymptotic cone of a free product of several nilpotent groups. (The above is by no means a proof which needs a more careful analysis of relatively hyperbolic groups.)

3.H. Lattices of Q-rank one. Here X is a symmetric space of rank ≥ 2 and Γ is a lattice with $\mathrm{rank}_{\mathbf{Q}} \Gamma = 1$. The basic feature of such a Γ (which we can take as a definition of $\mathrm{rank}_{\mathbf{Q}} = 1$) is the existence of a Γ-invariant core $X_0 \subset X$ where Γ acts cocompactly and whose boundary is a union of disjoint horospheres in X. Now, unlike the previous case, the subgroups in Γ preserving the boundary horospheres are not nilpotent (or virtually nilpotent) anymore but they remain (virtually) solvable in some cases.

3.H₁. *Example.* Let X be a Cartesian product of hyperbolic planes $X = H^2 \times \ldots \times H^2$. Then there are the famous *Hilbert modular groups* Γ of $\mathrm{rank}_{\mathbf{Q}} \Gamma = 1$ acting on this X (which are by no means products of any groups discretely acting on the hyperbolic plane). In the simplest case of $X = H^2 \times H^2$ the boundary horospheres of X_0 are identical to the solvable group $S = \mathrm{Sol}_3$ which is the semi-direct product for the diagonal action of $\mathbf{R}$ on $\mathbf{R}^2$ with the diagonal entries e^t and e^{-t} (compare (e) in 2.B). This group as we know (see 3.D) is undistorted in $X = H^2 \times H^2$. It follows that the inclusion $X_0 \subset X$ also has bounded distortion and so the distortion of Γ in the ambient Lie group $G = \mathrm{Iso}\, X$ (which is a finite extension of $PSL_2\mathbf{R} \times PSL_2\mathbf{R}$) is bounded as well. (This drastically contrasts with the case $\mathrm{rank}\, X = 1$.)

What we have just said about Γ acting on $H^2 \times H^2$, probably, applies to the general $\mathbf{Q}$-rank $= 1$ case (remember, we assume $\mathrm{rank}\, X \geq 2$) and shows that Γ is undistorted in $G = \mathrm{Iso}\, X$. Yet, there is no large-scale Lipschitz retraction $G \to \Gamma$ (or, equivalently $X \to X_0$) because the solvable subgroups in G corresponding to the horospheres on the boundary of X_0 are not Lipschitz retracts in G. On the other hand each "boundary subgroup" S_Γ is a Lipschitz retract in Γ (but not in G) since each horosphere $H \subset \partial X_0$ is such a retract in X_0.

3.H₂. Corollary. *If* S_Γ *is solvable then* Γ *contains an infinite cyclic subgroup which is strictly exponentially distorted in* Γ.

In fact, each S_Γ contains such subgroups as easily follows from the geometry of lattices in solvable groups, while S_Γ itself is undistorted in Γ.

Remark. (Pointed out to me by Enrico Leuzinger.) There are many groups Γ with $\mathbf{Q}$-rank one where the corresponding horosphere contains an essential semisimple factor and so S_Γ is far from solvable. Yet the non-distortion property seems to remain valid in these examples as well.

3.H$_3$. Application to $\mathrm{Con}_\infty \Gamma$. The above discussion shows that $\mathrm{Con}_\infty \Gamma$ embedds into $\mathrm{Con}_\infty X$ and so

$$\dim \mathrm{Con}_\infty \Gamma \leq \dim \mathrm{Con}_\infty X = \operatorname{rank} X .$$

Then there are infinitely many maximal flats in X_0 (or in some R-neighbourhood $U_R(X_0) \subset X$) which eventually embed into $\mathrm{Con}_\infty \Gamma$ and so

$$\dim \mathrm{Con}_\infty \Gamma = \operatorname{rank} X .$$

Furthermore, since $\mathrm{Con}_\infty \Gamma$ continuously (even Lipschitz) retracts to $\mathrm{Con}_\infty S_\Gamma \subset \mathrm{Con}_\infty \Gamma$ the fundamental group of $\mathrm{Con}_\infty S_\Gamma$, for each "boundary subgroup" $S_\Gamma \subset \Gamma$, injects into $\pi_1(\mathrm{Con}_\infty \Gamma)$. Moreover, one can show (using the concavity of ∂X_0) that the images (of $\pi_1(\mathrm{Con}_\infty S_\Gamma)$ for the "boundary subgroups" S_Γ and their conjugates in Γ) *generate* $\pi_1(\mathrm{Con}_\infty \Gamma)$ (and in fact, generate it freely). In particular, if $X = H^2 \times H^2$, then $\pi_1(\mathrm{Con}_\infty \Gamma)$ is an uncountably generated (free) group and this remains true whenever $\operatorname{rank} X = 2$. On the other hand, if $X = \underbrace{H^2 \times \ldots \times H^2}_{n}$ for $n \geq 3$, then $\pi_1(\mathrm{Con}_\infty \Gamma) = 0$ as follows from (e) and (f) in 2.B, but $\pi_{n-1}(\Gamma) \neq 0$ (in fact, it is uncountably generated). It seems, in general, that if $\operatorname{rank} X = n$ and $\operatorname{rank}_{\mathbb{Q}} \Gamma = 1$ then $\pi_1(\mathrm{Con}_\infty X) = 0$ for $i = 0, 1, \ldots, n-2$ and $\pi_{n-1}(\mathrm{Con}_\infty X)$ is uncountable.

3.H$_4$. Exponentially Lipschitz retraction $\mathbf{X} \to \mathbf{X_0}$. There is an obvious such retraction, coming from the normal projection of X to the (horospherical) boundary $\partial X_0 \subset X_0$. It follows that Γ is an exponential retract in X and thus satisfies the exponential isoperimetric inequalities. (In fact, Γ has the same isoperimetric profile as the boundary horospheres. For example, for $X = H^2 \times H^2$ the horosphere is S_3 and has exponential isoperimetric (Dehn)function according to $[\mathrm{Ger}]_{1,3,4,6}$; hence, the isoperimetric (Dehn)function of Γ is also exponential in this case.)

3.I. Arithmetic groups of $\mathbb{Q}$-rank ≥ 2. Let Γ be an irreducible lattice in a semi-simple Lie group G without compact factors and with $\operatorname{rank}_{\mathbb{R}} G \geq 2$. Then Γ is *arithmetic* by the Margulis theorem, the $\mathbb{Q}$-rank of Γ makes sense and we are now interested in the case where it is ≥ 2 (as the case $\operatorname{rank}_{\mathbb{Q}} \Gamma = 1$ has already been studied). First, we look at the quotient space $W = G/\Gamma$ and we express in geometric terms the basic facts of the standard *reduction* theory.

3.I$_1$. *There exists a unique asymptotic cone $C = \mathrm{Con}_\infty W$ which, in fact, is an actual cone over a finite simplicial polyhedron, and*

$$\dim C = \operatorname{rank}_{\mathbb{Q}} \Gamma .$$

Furthermore, $C \underset{\mathrm{Hau}}{\sim} W$, i.e. W lies within finite Hausdorff distance from C. (Actually, C is built of Weyl chambers associated to the maximal $\mathbb{Q}$-split torus in the $\mathbb{Q}$-group corresponding to Γ.)

Examples. (a) If $\operatorname{rank}_{\mathbb{Q}} \Gamma = 1$, then C consists of finitely many rays $\mathbb{R}_+$ joint at zero. (These rays correspond to different cusps of W.)

(b) Let $\Gamma = SL_n\mathbb{Z} \subset SL_n\mathbb{R}$. Then W is the space of unimodular lattices in $\mathbb{R}^n$. Denote by $\Delta \subset W$ the (sub)set consisting of the lattices spanned by certain multiples of the basis in $\mathbb{R}^n$, namely by $(\lambda_1, 0, 0, \ldots, 0)$, $(0, \lambda_2, 0, \ldots, 0) \ldots$, $(0, 0, \ldots, \lambda_n)$, such that

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$$

and $\lambda_1 \lambda_2 \ldots \lambda_n = 1$. This Δ is a *net* in W and the geometry of Δ in the induced metric is (essentially) conical. To see this geometry we use the logarithmic coordinates in Δ, that are $\ell_i = \log \lambda_i$ which embed Δ onto a cone (Weyl chamber) in the hyperplane $\sum_{i=1}^{n} \ell_i = 0$ in $\mathbb{R}^n$.

(b′) Let $\Gamma' \subset \Gamma = SL_n\mathbb{Z}$ be a subgroup of finite index. Then $W' = SL_n\mathbb{R}/\Gamma$ appears as a finite covering of W and $\Delta \subset W$ lifts to a union $\widetilde{\Delta}$ of finitely many copies of Δ in W'. Then $\mathrm{Con}_\infty W' = \mathrm{Con}_\infty \widetilde{\Delta}$ which is a simplicial cone built of several copies of Δ (see [Hat] for a better description).

3.I$_2$. *There exists a smooth function $f : W \to \mathbb{R}_+$ such that*

(a) *$f(w)$ is equivalent to the distance function* $\operatorname{dist}(w_0, w)$ *for $w \to \infty$ and a fixed point $w_0 \in W$ (recall, that "equivalent" means a uniform bound on f^{-1}* dist *and $f(\text{dist})^{-1}$ as $w \to \infty$).*

(b) *The gradient of f is bounded from below outside a compact subset $W_0 \subset W$,* i.e. $\|\operatorname{grad} f(w)\| \geq \varepsilon > 0$ for $w \in W - W_0$.

(c) *The Hessian of f is bounded, which means a uniform bound on first and second (covariant) derivatives of f.*

Idea of the proof. We shall work in the symmetric space $X = G/$maximal compact subgroup and in the corresponding space. $V = X/\Gamma$ (we always write the quotient on the right). Notice that V is a *manifold* if Γ has no torsion and an orbifold otherwise. We construct our function on V rather than on W (which makes no essential difference) by patching together several *horofunctions*. Recall that a horofunction (or Busemann function) h associated to a *geodesic ray* $r = r(t)$ in a Riemannian manifold Y, i.e. an isometric embedding $r : \mathbb{R}_+ \to Y$, is defined by

$$h(y) = \lim_{t \to \infty} \operatorname{dist}(y, r(t)) - t \; .$$

(The existence of the limit is obvious. Also notice that $h(y) = \operatorname{dist}(r(0), y)$ for $y = r(t)$.)

Now we take the rays in the cone $\operatorname{Con}_\infty V = \operatorname{Con}_\infty W$ issuing from the origin and meeting the base B of the cone in the vertices of some simplicial subdivision of B. (Recall that B is a simplicial complex; the subdivision we need must be sufficiently fine but, in fact, the barycentric subdivision will do.) Then we take geodesic rays, say $r_i, i = 1, \ldots, k$ (where k is the number of the vertices in our subdivision of B) in V lying within finite (Hausdorff) distance from the rays in $\operatorname{Con}_\infty V$. (To avoid undue complications one may assume V is a manifold.) We denote by h_i the horofunctions attached to r_i and set

$$f_0(v) = \min_{i=1,\ldots,k} \left(-h_i(v)\right) .$$

The function f_0 is not smooth but it has nice corners. To smooth the corners, we lift f_0 to X and smooth this lift using a smoothing operator with a smooth kernel of the form $S\Big(\big(\operatorname{dist}(x_1, x_2)\big)^2\Big)$ where $S(d)$ is a smooth positive function in $d \geq 0$ supported in a small interval $[0, \varepsilon] \subset [0, \infty]$. The smoothed function, say $\tilde{f}$ on X, descends to the required function on V. (In fact, instead of working on V we could use Γ-invariant functions on X.)

Corollary. *There exists a compact submanifold (core) $W_0 \subset W$ (or suborbifold $V_0 \subset V$, if you wish), which is homotopy retract in W. Moreover, W_0 is an exponentially Lipschitz retract in W.*

Proof. Take $W_0 = f^{-1}[0, t_0]$ for large enough $t_0 \in \mathbb{R}_+$. As f has no critical point at infinity, W_0 contracts to W_0. In fact, (a), (b) and (c) show that the gradient flow delivers the exponentially Lipschitz retraction we want. Moreover we get a Γ-equivariant retraction of X to $X_0 \subset X$ covering V_0.

3.I$'_2$. *Corollary.* *An irreducible lattice Γ in a semi-simple Lie group G satisfies exponential isoperimetric inequalities in all dimensions.*

Remark. This answers a question pointed out to me by Steve Gersten who proved the above for $\Gamma = SL_n\mathbb{Z}$ (see $[\text{Ger}]_{1,3,4,6}$).

Question. Let Γ be the fundamental group of a complete manifold V without boundary, such that $-\infty \leq -c \leq K(V) \leq 0$ and $\operatorname{Vol} V < \infty$. One knows that Γ is finitely generated in many cases, e.g. if $K < 0$ (see $[\text{Gro}]_2$ and [B-G-S]) but an exponentially Lipschitz retraction (on the large scale) of $V \to \Gamma$ seems hard to get. Yet one may expect such a retraction with a suitable perturbation of the original metric in V and also the exponential isoperimetric inequalities for Γ appear plausible.

3.I$_3$. Kazhdan's question. For lattices $\Gamma \subset G$ as above D. Kazhdan conjectured several years ago that they have bounded distortion in G. (We confirmed this conjecture for $1 = \operatorname{rank}_{\mathbb{Q}} \Gamma < \operatorname{rank}_{\mathbb{R}} G$ in 3.H.)

The conjecture implies that every parabolic element $\gamma \in \Gamma$ is *algebraically parabolic* in Γ which means, by definition, strict exponential distortion of the cyclic groups $\{\gamma^i\}$ in Γ, i.e. $|\gamma^i| \lesssim \log i$. In particular, Γ should always contain algebraic parabolics. The latter conclusion seems easy to prove independently. For example, $SL_n\mathbf{Z}$, $n \geq 3$, contains the (obvious) semi-direct product Γ' of $\mathbf{Z}^{n-1}$ by $SL_{n-1}\mathbf{Z}$ where all γ coming from $\mathbf{Z}^{n-1}$ are algebraic parabolic. (Similar subgroups Γ' seem to sit inside all our groups Γ.)

A natural approach to Kazhdan's problem is as follows. First show that the relevant horospheres in X are undistorted (this part is easy) and then prove the same for the region $X_0 \subset X$ bounded by horospheres, where X_0/Γ is compact. (The difficulty resides in the intersections between the horospheres in ∂X_0 which do not appear for $\operatorname{rank}_{\mathbf{Q}} \Gamma = 1$.)

3.I_3'. If Kazhdan's conjecture is true, then $\operatorname{Con}_\infty \Gamma$ embeds into a k-dimensional space $\operatorname{Con}_\infty G$ for $k = \operatorname{rank}_{\mathbf{R}} G$ and then one can easily show that

$$\dim \operatorname{Con}_\infty \Gamma = k .$$

Probably, $\operatorname{Con}_\infty \Gamma$ is non-contractible.

Finally notice that the first open case for Kazhdan's conjecture is $\Gamma = SL_3\mathbf{Z}$. (The conjecture was recently settled by Raghunathan for $SL_n\mathbf{Z}, n \geq 6$.)

3.I_4.P-adic remark. All we have said applies to lattices in p-adic algebraic groups where the (expected) situation is similar to the real case and the proofs should be even easier.

3.J. On distortion of discrete linear groups Γ which are not lattices. Discrete subgroups $\Gamma \subset GL_n$ of geometric origin (seem to) have at most exponential distortion. The simplest example of actual exponential distortion is $\Gamma = \{\gamma^i\}$ where λ is parabolic. A more interesting picture emerges for totally *degenerate Kleinian groups* but it seems unknown if they all have at most exponential distortion. (This is clear for the Jorgensen example of an exponentially distorted surface group Γ inside a cocompact lattice in $SL_2\mathbf{C}$.) In general, the distortion tends to increase when we go to the boundary of the space of discrete faithful representations $\Gamma \to GL_n$, but nothing specific is known in this regard. Here one may try first to understand the distortion of *free* subgroups Γ. (The most sophisticated geometric example up to date is due to Margulis, see [Mar]$_6$; this is a free group $\mathbf{F}_2$ discretely acting on $\mathbf{R}^3$ by affine transformations, but it does not show much distortion in $G = \operatorname{Aff} \mathbf{R}^3$, as was explained to me by Bill Goldman.)

Michailova's groups. Let some group Γ' be generated by k-elements, $p : \mathbf{F}_k \to \Gamma'$ be the corresponding surjective homomorphism and $\Gamma = \Delta_{\Gamma'} \subset \mathbf{F}_k \times \mathbf{F}_k$ be the pull-back of the diagonal subgroup $\{\gamma', \gamma'\} \subset \Gamma' \times \Gamma'$ under the homomorphism $p \times p : \mathbf{F}_k \times \mathbf{F}_k \to \Gamma' \times \Gamma'$. If Γ' admits a finite presentation, say $\{x_1, \dots, x_k \mid w_1, \dots, w_\ell\}$, then Γ is finitely generated. In fact, it is generated by (x_i, x_i), $i = 1, \dots, k$, and (id, w_j), $j = 1, \dots, \ell$, (see §4 of chapter IV in [Ly-Sch]).

Now it is clear that the distortion of Γ in $\mathbf{F}_k \times \mathbf{F}_k$ is of the order of the complexity of the word problem in Γ', since $u = v$ in Γ' if and only if (u, v) is contained in Γ (compare [Pit]). In particular, if the word problem in Γ' is unsolvable, the group Γ is unrecursively badly distorted in $\mathbf{F}_k \times \mathbf{F}_k$ and hence in every linear group G containing $\mathbf{F}_k \times \mathbf{F}_k$ (e.g. $G = SL_2 \times SL_2$ or $G = SL_4$). Notice that by taking groups Γ' with a given (interesting) behaviour of the isoperimetric (filling) function $F_0A(\ell)$ (see §5) one obtains Γ with the distortion of the same order of growth as $F_0A(\ell)$, for $\ell \to \infty$. This Γ is practically never free as with every element (id, w) it contains (w, id) which commutes with (id, w) (this was pointed out to me by Misha Kapovich) and it is unclear when Γ can be made non-discrete in $G \supset \Gamma$ by an arbitrarily small deformation. (An interesting case here is where Γ' is a standard group with the unsolvable word problem.)

More questions. Which Lie groups G contain strongly distorted (free) discrete subgroups? The picture seems clear for $G = SL_2\mathbf{R}$ (no such subgroup) but already $G = SL_2\mathbf{C}$ and $G = SL_3\mathbf{R}$ look mysterious. (Of course, $SL_3\mathbf{R}$ is more likely to contain a strongly distorted Γ than $SL_2\mathbf{C}$.)

3.K. Further examples of strongly distorted subgroups. We have already met the group $\Gamma = \{a, b \mid a^b = a^2\}$ (where $a^b \underset{\mathrm{def}}{=} bab^{-1}$) and observed that a is an algebraic parabolic. In fact the minimal word length of a^{2^n} (in the letters a and b) obviously satisfies $|a^{2^n}| \leq 2n+1$. Moreover, since $(a^m)^b = a^{2m}$ for all $m = 1, 2, \ldots$, the length function $\ell(m) = |a^m|$ satisfies $\ell(2m) \leq \ell(m) + 2$, which implies $\ell(m) \leq \text{const} \log m$ for all m.

3.K$_1$. Next, we take $\Gamma = \{a, b, c \mid a^b = a^2,\ b^c = b^2\}$. Now the function $\ell(m) = |a^m|$ obviously satisfies

$$\ell(m) \leq 4n + 3 \text{ for } m = 2^{2^n} ,$$

which makes the distortion *double exponential*. (Notice, that the relation

$$\ell(m) = |a^m| \lesssim \log\log m$$

cannot hold for all m as this would give us a more than exponential number of points in the balls in Γ.)

3.K$_2$. Let $\Gamma = \{a, b, c \mid a^b = a^2,\ a^c = b\}$. Since $a^{(a^n)^c} = a^{2^n}$, the length function $\ell(m) = |a^m|$ satisfies

$$\ell(2^n) \leq 2\ell(n) + 5 ,$$

which makes the distortion (of the cyclic subgroup $\{a^i\} \subset \Gamma$) to be more than multiexponential. (Notice that all action takes place in the subgroup $\Gamma' = \{a, c \mid a^{a^c} = a^2\}$ which was brought into geometric focus by S. Gersten along with the groups $\Gamma^{(i)} = \{a, c \mid a^{w_i} = a^2\}$ for $w_1 = a^c$, $w_2 = a^{a^c}$, etc., see [Ger]$_{1,3,4,6}$.)

3.K$_3$. Non-recursive distortion. Take a positive integer M and two finite sets of 4-tuples of positive integers $(i_\nu, j_\nu, k_\nu, \ell_\nu)$ and $(i'_\nu, j'_\nu, k'_\nu, \ell'_\nu)$, $\nu = 1, \ldots, n$. Then consider the group Γ generated by x, y, z, t_ν, s_ν, $\nu = 1, \ldots, n$ with the relations

$$t_\nu x^{M^2} t_\nu^{-1} = x^M,\ t_\nu x^{i_\nu} z y^{j_\nu} t_\nu^{-1} = x^{k_\nu} z y^{\ell_\nu}, t_\nu y^M t_\nu^{-1} = y^{M^2},\ \nu = 1, \ldots, n$$

and

$$s_\nu x^M s_\nu^{-1} = x^{M^2},\ s_\nu x^{i'_\nu} z y^{j'_\nu} s_\nu^{-1} = x^{k'_\nu} z y^{\ell'_\nu},\ s_\nu y^{M^2} s_\nu^{-1} = y^M,\ \nu = 1, \ldots, n ,$$

and take the subgroup $\Gamma_0 \subset \Gamma$ generated by z, t_ν and s_ν. One knows (see p.p. 49–52 in [Stil]) that for a certain choice of our M, i_ν, j_ν, etc. the subgroup Γ_0 is free and the *Magnus problem* for Γ_0 (of inclusion of $\gamma \in \Gamma$ to Γ_0) is unsolvable. Moreover, there are infinitely many positive integers i, j, such that $x^i z y^j \in \Gamma_0$ and the word length of these elements in Γ_0 cannot be majorized by any recursive function. Thus $\Gamma_0 \subset \Gamma$ has non-recursively large distortion.

3.K$'_3$. Let us augment the above Γ in order to have an *Abelian* strongly distorted subgroup. For this purpose we take a faithful action A of the above free group Γ_0 on $\mathbb{Z} \times \mathbb{Z}$, and add to Γ two extra generators a and b with the relations $[a, b] = 1$ and

$$\begin{aligned} zaz^{-1} &= A_Z(a) , & zbz^{-1} &= A_z(b) , \\ t_\nu a t_\nu^{-1} &= A_{t_\nu}(a) , & t_\nu b t_\nu^{-1} &= A_{t_\nu}(b) , \\ s_\nu a \nu_\nu^{-1} &= A_{s_\nu}(a) , & s_\nu b s_\nu^{-1} &= A_{s_\nu}(b) . \end{aligned}$$

Then, clearly, $\mathbb{Z} \times \mathbb{Z}$ is (at least) as strongly distorted in the augmented Γ as Γ_0 in Γ. Moreover, the cyclic subgroups of $\mathbb{Z} \times \mathbb{Z}$ are distorted as badly as all of $\mathbb{Z} \times \mathbb{Z}$.

3.K$''_3$. Rips' example. According to I. Rips (see [Rip]) there is a *hyperbolic small cancellation group* Γ with a *finitely generated* normal subgroup $\Gamma_0 \subset \Gamma$, such that the word problem for Γ/Γ_0 is unsolvable (this was pointed out to me by Z. Sela). Then, obviously, the distortion of $\Gamma_0 \subset \Gamma$ is non-recursively large.

Remark. It might be possible to modify Rips' example in order to have such a Γ appearing as the fundamental group of a closed manifold V with $K < 0$. (May be one should allow singular metrics with

$K < 0$.) But it is hard to imagine such a $\Gamma = \pi_1(V)$ for closed manifolds V with *constant* negative curvature. (But look at Michailova's groups!)

3.K_4. Realization problem. Let $\ell(i)$, $i \in \mathbb{Z}$, be a symmetric subadditive function, i.e. $\ell(i) = \ell(-i)$ and $\ell(i+j) \leq \ell(i) + \ell(j)$. When can one find an embedding of $\mathbb{Z}$ into a finitely presented (or, at least, finitely generated) group Γ, such that the word length in $\Gamma \supset \mathbb{Z}$ restricts to our ℓ on $\mathbb{Z}$? (Of course, the equality $\text{length}_\Gamma \mid \mathbb{Z} = \ell$ is too much to ask. A more realistic demand is some kind of equivalence of $\text{length}_\Gamma \mid \mathbb{Z}$ to a given ℓ.) For example, may one realize in some Γ a function $\ell(i) \sim i^\alpha$ for an *irrational* α ?

Remark. The function $l(i) = \text{length}_\Gamma(i)$ for a badly distorted embedding $\mathbb{Z} \subset \Gamma$ looks similar to a Kolmogorov's complexity function, which is the length of the shortest formula in a given formal language expressing the number i. As (one knows) there are groups with prescribed degrees of (non)-solvability of the word problem, one expects that the Kolmogorov complexities for various languages can be approximately (in a suitable sense) realized by embeddings of $\mathbb{Z}$ into finitely presented groups.

3.L. Distortion in free groups. Let Γ be the free group on k-generators and $\Gamma_0 \subset \Gamma$ a finitely generated subgroup. *Then Γ_0 has bounded distortion in Γ and, moreover, there is a Γ_0-equivariant Lipschitz retraction* $\Gamma \to \Gamma_0$. To see this we make Γ the fundamental group of a finite graph V and let $\widetilde{V} \to V$ be the covering graph corresponding to Γ_0. Then $\widetilde{V}$ (obviously) contains a unique *minimal* connected subgraph, the *core* V^* of $\widetilde{V}$, such that the inclusion $V^* \subset \widetilde{V}$ induces an isomorphism on π_1. This V^*, clearly, is a *finite* graph whose complement $\widetilde{V} - V^*$ (corona) consists of finitely many infinite trees growing from certain vertices of V^*, see Fig. 7.

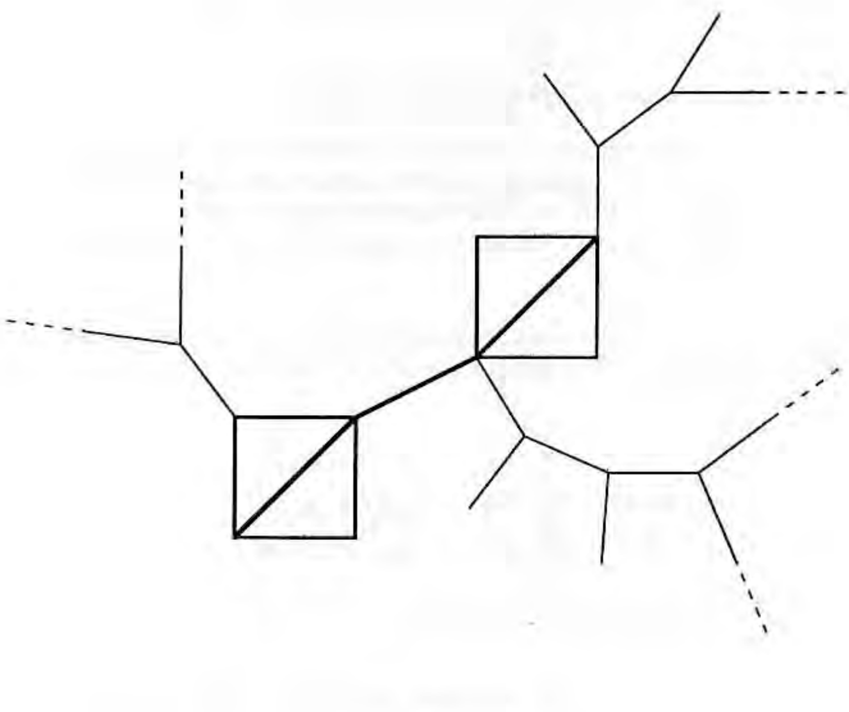

Figure 7

Then one may retract $\widetilde{V} \to V^*$ by sending each tree to its "root" in V^*. Q.E.D.

3.L′. *Remarks.* (a) Notice, that if a *free* group or a Cartesian product Γ_0 of such groups has bounded distortion in an ambient group Γ, then there always exists a Lipschitz retraction $\Gamma \to \Gamma_0$ but the Γ_0-invariance is not insured by the general principles.

(b) The above retraction property for subgroups in the free groups (obviously) generalizes to subgroups in the surface groups as everybody (in the theory of Fuchsian groups) knows.

3.L″. Let us give an estimate on the (bounded) distortion of $\Gamma_0 \subset \Gamma$ in terms of the word lengths of the generators of Γ_0. We denote by L the sum of their lengths and we claim that

$$\text{length}_0 \gamma \leq C^L \text{length}\, \gamma \ , \ \gamma \in \Gamma_0 \ , \tag{$*$}$$

where length_0 denotes the word length of elements in Γ_0 with respect to the given generators in Γ_0 and where C is some constant depending on k, the number of generators of Γ. Let $V = V(\Gamma)$ be the standard graph (consisting of k loops joint at a single vertex) presenting Γ and let $V_0 = V(\Gamma_0)$ be the corresponding graph (consisting of the loops corresponding to the generators of Γ_0) for Γ_0. The inclusion $\Gamma_0 \subset \Gamma$ and the resulting presentation of the generators of Γ_0 by some words in Γ, induces a map $V_0 \to V$ which, in turn, induces a subdivison of V_0, say V_0', for which our map is a *graph morphism*, i.e. every edge of V_0' goes onto an edge in V by a homeomorphism or it goes to a single point. Then, this morphism, call it $f : V_0' \to V_0$, can be decomposed (by a trivial argument) into morphisms of very special types,

$$V_0' \xrightarrow{f_0} V_1' \xrightarrow{f_1} V_2' \to \ldots \to V_k' \xrightarrow{f_k} V$$

where f_k is *locally injective* (i.e. injective on a neighbourhood of each vertex in V_k') and where every f_i, $i < k$, is a surjective map which is "almost injective" in the following sense: V_i' is obtained from V_i' by one of the following two elementary operations,

(1) identifying two edges issuing from the same vertex,

(2) collapsing a loop consisting of a single edge to the (unique) vertex on this loop.

In both cases we take the quotient map $V_i' \to V_{i+1}'$ for f_i.

Obvious lemma. *For every simple loop ℓ in V_{i+1}' and a vertex $v \in \ell$ (for a base point) there exists a loop $\widetilde{\ell}$ in V_i' with a vertex (base point) $\widetilde{v} \in \widetilde{\ell}$, such that $f_i(\widetilde{v}) = v$ and the loop $f_i : \widetilde{\ell} \to V_{i+1}'$ is homotopic with the base v to the loop ℓ, and such that*

$$\text{length}\, \widetilde{\ell} \leq 3 \text{ length}\, \ell \ . \tag{$+$}$$

Remark. The inequality (+) becomes an equality for the map sketched in Fig. 8.

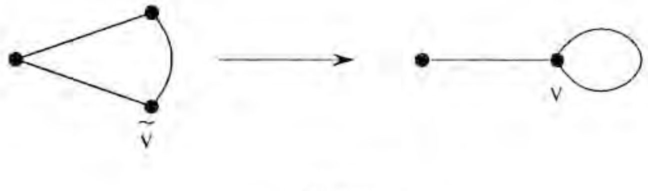

Figure 8

It follows that the composed map $V_0' \to V_k'$ contracts the lengths by at most 3^k, while the (locally injective!) map f_k does not contract at all. Then (∗) easily follows.

Isodiametric remark. The above argument is similar to that of Gersten and Cohen (see $[\text{Ger}]_6$) concerning the *isodiametric* functions of (non-free!) groups Γ, where one uses *Nielsen moves* in a free group rather than a (similar) elementary operation on graphs ((1) and (2) above).

Our inequality $(*)$ shows (together with the discussion in $[\text{Ger}]_6$) that the isoperimetric and isodiametric functions of every finitely generated group Γ are related by the following inequality slightly sharpening that in $[\text{Ger}]_6$,

$$\text{Isop } \leq \text{expexp}\, C(\text{Isodiam})$$

for some constant $C = C_\Gamma$. (In the notations of our §5 one should write

$$F_0A(\ell) \leq \text{ expexp}\left(CF_0D(\ell)\right) ,$$

where F_0A refers to "filling area" and F_0D to "filling diameter".)

Remarks on $(*)$ for non-free groups. Our inequality $(*)$ measuring the distortion of $\Gamma_0 \subset \Gamma$ in terms of the length of given generators of Γ_0 makes sense any time when the distortion of Γ_0 in Γ is known, a priori, to be bounded. (We suggest to the reader to work out the cases $\Gamma = \mathbb{Z}^n$ and Γ equal the Heisenberg group where Γ_0 does not contain the center. Also one may ponder on free products of such groups.) An especially interesting case is $\Gamma_0 = \Gamma$ where we ask the following

Question. Fix a finite generating set for Γ and let $\lambda(\ell)$ be the minimal integer λ (depending on $\ell = 1, 2, \ldots,$) such that for every system of (new) generators of Γ consisting of words of total length $\leq \ell$ each element in the original generating set can be decomposed into the product of at most λ new generators. What we want to know is the asymptotic behavior of $\lambda(\ell)$, $\ell \to \infty$, for specific groups Γ. (We have seen that $\lambda(\ell) \lesssim \exp \ell$ for free groups and it is obvious with Euclidean algorithm that $\lambda(\ell) \sim \ell$ for $\Gamma = \mathbb{Z}$.)

(a) $\Gamma' = \{a,b,c, \mid a^b = a^2,\ b^c = b^2\}$; here the (infinite cyclic) subgroup $\Gamma = \{a^i\}$ has double exponential distortion, as the powers a^{p_j} for $p_j = 2^{2^j}$ grow linearly in Γ'. It follows that $\overline{R}_0(r) \gtrsim \exp r$ and it is easy to show that, in fact, $\overline{R}_0(r)$ grows exponentially. (To do this, one first shows that for $\Gamma \subset \Gamma'_0 = \{a,b \mid a^b = a^2\}$ the function $\overline{R}_0$ grows linearly).

(b) Let us give an example of a quadratic function $\overline{R}_0(r)$. For this we take

$$\Gamma' = \{a,b,c,d \mid [a,b] = c, [a,c] = [b,c] = 1,\ d^c = c^2\}$$

and $\Gamma = \{d^i\}$. Here the distortion is $\sim \exp r^2$ and $\overline{R}_0(r)$ grows as r^2.

(c) We have constructed in §3. (cyclic) subgroups with non-recursively fast growing distortion and then we have $\overline{R}_0(r)$ of similar growth.

4.B. Remarks on the geometry and topology of Γ'-balls in $\Gamma \subset \Gamma'$. The connectedness radius $\overline{R}_0$ of Γ does not tell us the whole story about the $\mathrm{dist}_{\Gamma'}$-balls $B'(r) \subset \Gamma$, not even about the connectedness features of these balls. An obvious additional invariant to look at is the *number* of the connected component of $B'(r)$ on a fixed large scale. (In the examples of 4.A_3. this number could be easily seen to be $\gtrsim r$, but I have not looked further). The above number of components is, of course, a function of r. In fact, one should look at the homology group $H_0(B'(r))$ as it goes to $H_0(B'(R))$ for $R \geq r$. Then one has a function in *two* variables r and R, namely the rank of the inclusion homomorphism $H_0(B'(r)) \to H_0(B'(R))$, and a similar invariant (refining $\overline{R}_k(r)$ is defined for all $k = 0,\ 1,\ 2 \ldots$).

4.B_1. The case $\Gamma = \mathbb{Z} \subset \Gamma'$. Here the ball $B'(r)$ looks roughly as a union of disjoint intervals. The distribution of the lengths of these intervals (i.e. the "connected components" in $B'(r) \subset \mathbb{Z}$) as well as in the complement $\mathbb{Z} - B'(r)$ appears an interesting object to look at, but I have not worked out any serious example.

In order to get an overall geometric view on $B'(r) \subset \mathbb{Z}$ one may rescale such a ball (or the intersection of $B'(r)$ with an interval $[m,n] \subset B'(r)$ for certain $m = m(r)$ and $n = n(r)$) to the unit size and go to the limit for $r \to \infty$. Then one may hope to have in the limit a kind of a Cantor set in $[0,1]$ featuring the asymptotic geometry of $B(r)$. Unfortunately this does not work in those cases, where the gaps in $B(r)$ (or in $B(r) \cap [m,n]$) are much smaller than $\mathrm{Diam}\, B(r)$, as the limit becomes equal to the whole interval $[0\,1]$. Then one should try something more sophisticated than the linear rescaling in order to catch the large-scale geometry of $B(r)$.

A geometric approach to $B'(r)$ may be fruitful for $\mathbb{Z} = \Gamma \subset \Gamma'$ with relatively small (say, at most double exponential) distortion but for the large distortion the naive geometry of $B'(r) \subset \mathbb{Z}$ for the ordinary metric in $\mathbb{Z}$ appears rather irrelevant. To make our point clear we look at the integers as they appear to a logician operating with a fixed formal language $\mathcal{L}$. Now the ball $B_{\mathcal{L}}(r) \subset \mathbb{Z}$ consists of the integers representable by formulas of length $\leq r$ in $\mathcal{L}$ (compare 3.K_4.). Such a ball, for every large r, is highly disconnected, and looks as a Cantor set rather than a solid array of numbers from 0 to n which appears in the idealized (and illogical) counting process 1, 2, 3, ..., n. (Try to fill in the dots by numbers for $n = (2^{10!})\,!$). At the first glance this ball has certain pronounced geometric features. For example, it looks very much geometrically the same near each of its points because the Kolmogorov complexity is essentially sub-additive (if we use the addition sign in our language). But then we realize that the same is true for the set $\{\log i\}$, $i \in B(r)$, and moreover, for $\{f(i)\}$ for a wide class of (recursive) functions f. In other words the essential features of $B(r)$ survive very general transformations (corresponding, for example, to changes of the language) which completely destroy naive geometric patterns. This suggests an approach to the geometry of $B_{\mathcal{L}}(r)$ (and the balls in the subgroups with large distortion) in the spirit of the computational complexity theory.

4.C. Large $\overline{\mathbf{R}}_1$. Here we are concerned with finitely generated groups (Γ, word metric) and we want to show that the contractibility radius $\overline{R}_1(r)$ (= the minimal R such that every loop in the r-ball of the universal covering of the 2-polyhedron presenting Γ is contractible in the concentric R-ball) may display essentially the same features as $R_0(r)$ for the induced metric $\mathrm{dist}_{\Gamma'}|\Gamma$ for $\Gamma' \supset \Gamma$. For example, the following simple proposition suggests a variety of examples of groups with non-trivial (i.e. faster than linear) function $\overline{R}_1(r)$.

4.C$_1$. Gluing property. *Take two copies of* $(\Gamma', \Gamma \subset \Gamma')$ *and let* $\Gamma_1 = \Gamma' *_\Gamma \Gamma'$ *be the amalgamated product. Then the function* $\overline{R}_1(r)$ *for* Γ_1 *is minorized by* $\overline{R}_0(r)$ *for* $(\Gamma, \mathrm{dist}_{\Gamma'}|\Gamma)$.

Proof. The polyhedron P_1 presenting Γ_1 is obtained by gluing two copies of P' across P presenting Γ and then the universal covering $\widetilde{P}_1$ is glued of infinitely many copies of $\widetilde{P}'$. Then we observe that if a space X_1 is obtained by gluing two copies of X' across $X \subset X'$, then the fundamental group $\pi_1(X_1)$ comes from $\pi_1(X)$ and the relative π_1 of the pair (X', X), which is, in turn, related to π_0 of (the balls in) X. (This is the usual additivity of Van Kampen-Mayer-Vietoris). In particular $\overline{R}_0$ for $(X, \mathrm{dist}_{X'} X)$ minorizes $\overline{R}_1$ for X_1. We leave the details to the reader, who may gain some geometric insight by looking at the following

Example. Take $\mathbb{R}^2$ for X' and let $X \subset \mathbb{R}^2$ be the graph of the function $y = \rho(x)\sin x$ for a fast growing positive function $\rho(x)$. Then the intersections of X with the r-balls around $0 \in \mathbb{R}^2$ are highly disconnected and when we glue two copies of $\mathbb{R}^2$ along X the resulting space X_1 has highly non-simply connected balls around a fixed point in X_1.

4.C$'_1$. Remarks. (a) The gluing property extends to the general case of $\Gamma' *_\Gamma \Gamma''$ (and X' glued to $X'' \neq X'$ across X) if one properly takes into account the discrepancy between the metrics dist$'$ and dist$''$ on Γ (or on X).

(b) The 1-dimensional homology enjoys the same gluing property as π_1 and we can minorize not only the connectivity radius of Γ_1 (concerning π_1 of the balls) but also the *acyclicity radius* (measuring H_1, see 4.C$'_2$.) by $\overline{R}_0$ for $\Gamma \subset \Gamma'$.

(c) Our gluing property is similar to those proven by S. Gersten for the isoperimetric and isodiametric functions.

4.C$_2$. $\overline{\mathbf{R}}_1$ for aspherical 2-polyhedra (Gersten lemma). Let $\widetilde{P}$ be an acyclic (e.g. aspherical) 2-dimensional polyhedron (which appears in our applications as the universal cover of some P presenting a group Γ). Suppose we have a disk $\mathcal{D}$ embedded into $\widetilde{P}$ with the boundary circle S in P. We claim (this is a slight modification of an argument in [Ger]$_1$) that *S does not bound any other* disk *$\mathcal{D}'$ mapped into P with*

$$\mathrm{diam}\,\mathcal{D}' \leq \mathrm{diam}\,\mathcal{D}\ , \tag{$*$}$$

where "diam" means the "diameter" of the image in P. Indeed we can think of the disks in P as 2-chains and take the difference $\mathcal{C} = \mathcal{D} - \mathcal{D}'$. Since $\partial\mathcal{D} = \partial\mathcal{D}' = S$, this $\mathcal{C}$ is a 2-cycle in $\widetilde{P}$ and since $\widetilde{P}$ is acyclic of dimension 2 the cycle C is identically zero. It follows that $\mathcal{D}' \supset \mathcal{D}$, because $\mathcal{D}$ is *embedded* into $\widetilde{P}$ and so equals the support of the chain it represents.

We shall use the above when we have a sequence of circles $S_i = \partial\mathcal{D}_i$, such that $R_i = \mathrm{Diam}\,\mathcal{D}_i$ is much greater than $r_i = \mathrm{diam}\,S_i$, $r_i \to \infty$, which gives us a non-trivial bound

$$\overline{R}_1(r_i) \gtrsim R_i\ .$$

4.C$'_2$. *Remarks.* (a) The above discussion becomes more logical and the conclusion more general if we work directly with *chains* in $\widetilde{P}$ rather than with disks and relate the diameters of the *supports* of our chains by the inequality $(*)$. In this case our final conclusion is a lower bound on the *acyclicity radius* of $\widetilde{P}$, denoted $H\overline{R}_1(r)$, and defined as the minimal R for which the inclusion homology homomorphism $H_1(B(r)) \to H_1(B(R))$ vanishes.

(b) In order to make full use of *disks* (not just corresponding chains) one should (following Gersten) allow *Cockcroft 2-complexes*, where the Hurewicz homomorphism from π_2 to H_2 vanishes. Important examples of groups $\Gamma = \pi_1(P)$ with $\widetilde{P}$ Cockcroft are the fundamental groups of oriented 3-manifolds (see [Ger]$_{1,2,3,4,6}$).

4.C$_3$. *Examples.* (a) We have met in 4.A. various examples of $\Gamma \subset \Gamma'$ with large $\overline{R}_0$. Now we amalgamate and obtain groups Γ^1 with large $\overline{R}_1$. (It seems that in many concrete cases the minoration of $\overline{R}_1$ by $\overline{R}_0$ is essentially sharp and thus we may obtain a rather precise evaluation of $\overline{R}_1$ for some Γ_1). In particular, we may have Γ_1 where $\overline{R}_1(r)$ grows faster than any recursive function.

(a$'$) *Remark.* It seems, the *standard* groups Γ with non-solvable word problem have $\overline{R}_1(r)$ growing faster than recursive. Yet it is unclear if *every* group with unsolvable word problem has such $\overline{R}_1$. One can imagine a Γ, where the balls $\widetilde{B}(r)$, $r = 1, 2 \ldots$, in the covering $\widetilde{P}$ of P presenting Γ (i.e. $\pi_1(P) = \Gamma$) are all simply connected, but the decision problem

$$\pi_1(B(r)) \underset{?}{=} 1 \; , \; r = 1, \; 2, \; 3, \ldots ,$$

is unsolvable. It is even conceivable, that for certain Γ *every* finite polyhedron (up to some reasonable equivalence relation) appears in a ball like shape inside $\widetilde{P}$. (Such a possibility was apparently overlooked in the discussion on p. 43 in [Gro]$_{10}$). On the other hand, if the word problem is solvable in Γ, then $\overline{R}_1(r)$ is subrecursive because it can be (obviously) majorized by the isodiametric function (see [Ger]$_6$, and §9. in [E-C-H-P-T].

(b) Let us evaluate $\overline{R}_1$ for the group $\{a, b, c \mid a^b = a^2, \; b^c = b^2\}$. We already know, that the relation $a^b = a^2$ can be (roughly) represented by a (large) semi-disk in the hyperbolic plane, with the diameter B and the arc $A \sim \exp B$. Then we take the second disk, now with the diameter C and the arc B. Gluing them together along B we get a new disk, say $\mathcal{D}$, with the boundary consisting of two arcs of the lengths C and $A \sim \exp\exp C$, see Fig. 9 below.

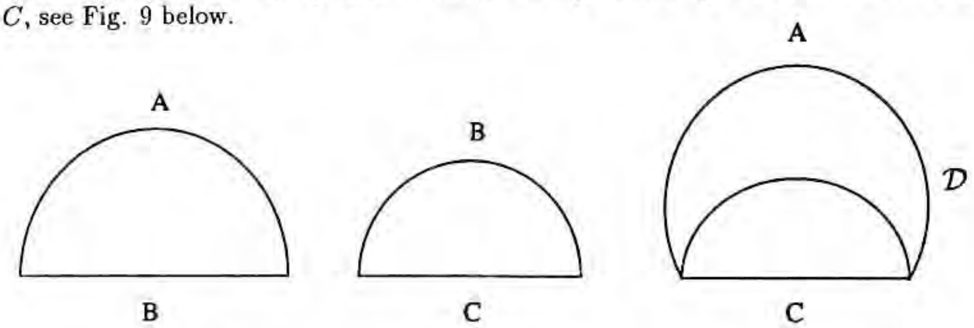

Figure 9

Then we take two copies of $\mathcal{D}$ and glue them across the A-arc. Thus we get a topological disk $\mathcal{D}_1$ bounded by a $2C$-circle, see Fig. 10.

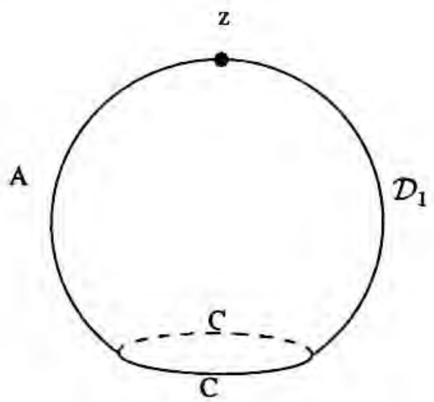

Figure 10

It is not hard to see that the distance from the $2C$-circle to the center z of the A-arc in $\mathcal{D}_1$ is about $\exp C$. On the other hand every disk $\mathcal{D}_1 = \mathcal{D}_1(C), C > 0$, quasi-isometrically embeds into the universal covering $\widetilde{P}$ of the polyhedron P presenting our Γ (this is easy to show, compare [Ger]$_6$). Since the polyhedron $\widetilde{P}$ is contractible, it has $H_2(\widetilde{P}) = 0$ and so the boundary circle $S \subset \widetilde{P}$ of the image of $\mathcal{D}_1$ in $\widetilde{P}$, which we still denote $\mathcal{D}_1 \subset \widetilde{P}$, does not bound in the complement $\widetilde{P} - \{z\}$. (If S were a boundary of some chain $\mathcal{C}$ in $\widetilde{P} - \{z\}$, we would have a non-trivial cycle in $\widetilde{P}$, namely $\mathcal{C} - \mathcal{D}_1$, see 4.C$_2$. and [Ger]$_{1,6}$). Thus we see that $\overline{R}_1(r) \gtrsim \exp r$ and then it is not hard to see that, in fact, $\overline{R}_1(r)$ grows exponentially for our Γ. This slightly improves the lower exponential estimate by J. Gersten for the isodiametric function of Γ, (see [Ger]$_6$); recall that $\overline{R}_1$ minorizes the isodiametric function.

(c) The above discussion easily generalizes to the groups

$$\Gamma_k = \{a_0, a_1, a_2, \ldots, a_k \mid a_0^{a_1} = a_0^2,\ a_1^{a_2} = a_1^2,\ \ldots, a_{k-1}^{a_i} = a_{k-1}^2\}\ .$$

Thus one conludes that $\overline{R}_1(r)$ for Γ_k grows as $\underbrace{\exp\exp\ldots\exp}_{k-1} r$.

(c$'$) Take $\Gamma = \{a_0, b \mid a_0^{a_0^b} = a_0^2\}$ and observe (following Gersten, see [Ger]$_{1,6}$), that Γ contains

$$\Gamma_\infty = \{a_i,\ i \in \mathbb{Z} \mid a_i^{a_{i+1}} = a_i^2\}\ ,$$

as a normal subgroup for b acting on Γ_∞ by

$$a_i^b = a_{i+1},\ i \in \mathbb{Z}\ .$$

Then we take the (obvious) 2-polyhedron P_∞ corresponding to the above presentation of Γ_∞ with the 1-skeleton indicated in Fig. 11.

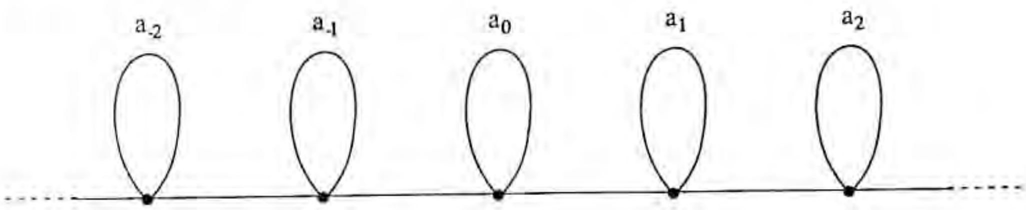

Figure 11

This P_∞ is freely acted upon by $\mathbb{Z}$ and the quotient polyhedron $P = P_\infty/\mathbb{Z}$ presents Γ. It follows that $\overline{R}_1$ for Γ grows faster than $\overline{R}_1$ for each of $\Gamma_k \subset \Gamma_\infty$ and so $\overline{R}_1$ for Γ grows faster than any iterated exponential function. (In fact, the same lower bound is valid for the acyclicity radius $H\overline{R}_1(r)$ of Γ).

4.D. Questions concerning $\overline{\mathbf{R}}_\mathbf{k}(\mathbf{r})$ for $\mathbf{k} \geq 2$. There is little doubt that there exist groups Γ with arbitrarily fast growing $\overline{R}_k$ for every given $k \geq 2$, but I have not worked out any specific example. The first class of groups to look at are $\Gamma = \pi_1(Q)$ for compact 3-dimensional aspherical polyhedra Q. One obtains many such Q by starting, say with $\Gamma_0 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, and then applying the HNN-extension to Abelian subgroups of rank two in Γ_0 thus obtaining $\Gamma_1 \supset \Gamma_0$. Next, HNN applies to Abelian subgroups in Γ_1 which gives us $\Gamma_2 \supset \Gamma_1$ and so on. One wonders if among such groups $\Gamma_i \supset \Gamma_{i-1} \supset \ldots \supset \Gamma_0$ some have large $\overline{R}_3(r)$.

There is an essential difference between the case $k = 1$ and $k \geq 2$: every finitely presented group is 1-connected on the large scale but it is not necessarily k-connected for $k \geq 2$ and so the invariants $\overline{R}_k$, $k \geq 2$, may be not even defined for certain Γ. In this case several modifications of $\overline{R}_k(r)$ are possible. For example, one may restrict to those k-spheres (in the r-ball of some thickening of Γ) which are *a priori* contractible in a suitable fixed thickening of Γ. A more general approach is to define the following function in two variables, $\overline{R}_k(r, d)$ the minimal radius of the ball in *some* d-thickening where the sphere in question becomes contractible. (Recall that the number d of a thickening $X \supset \Gamma$ refers to the Hausdorff distance from X to Γ).

§5. Filling invariants of closed curves: isoperimetric and isodiametric functions etc.; area filling in nilpotent and solvable groups; filling length, filling radius and Morse landscape; filling in k-dimensional cycles for $k \geq 2$; filling on the large scale and volume distortion of subspaces.

Given a circle S in a Riemannian manifold X (this S may have double points) there is a variety of invariants characterizing the optimal (i.e. minimal) size of surfaces in X *filling in* S i.e. having S for their boundary. Here are some of them.

I. *Filling area*, denoted Fill Area S. This is the infimal area of compact surfaces (or better to say of 2-chains D in X, filling in S. (Here one should specify the implied coefficient field. Usually we speak of chains with integral coefficients).

II. *Filling area of genus* g, denoted Fill_g Area S, where the infimum of area is taken over the surfaces *of genus* g filling in S (If we insist on $\mathbb{Z}$-coefficient these surfaces must be orientable.). Notice, that here and in future "surface" means a surface which is Lipschitz mapped into X, where the map is by no means required to be one-to-one.

A particularly important special case is that of Fill_0 Area S where the implied surface is a *disk*.

III. *Filling diameter* Fill_0 Diam S. This is the infimal diameter of disks D with some Riemannian metrics for which there are 1-Lipschitz (i.e. contracting) maps $D \to X$ sending the boundary onto S with degree 1.

IV. *Filling Radius* Fill Rad S. This is the minimal number R, such that S bounds (i.e. homologous to zero) in the R-neighbourhood $U_R(S) \subset X$.

V. *Filling length* Fill_0 Leng S. This is the minimal number L such that S can be contracted to a point by a homotopy of closed curves of length $\leq L$.

VI. *Filling Span* Fill Span. This is the minimal $\lambda > 0$ such that there exists a λ-Lipschitz map of the *unit* disk into X such that the boundary of the disk goes onto S with degree one.

Remark. One can continue this list indefinitely. In fact, every monotone invariant defined in the space of metrics on the disk (and on a surface of genus $g > 0$) gives us, via 1-Lipschitz maps to X, an invariant of S. (We have chosen the above five for the sheer geometric beauty; also they proved useful in certain situations).

Now, with every filling invariant, say $F(S)$ we associate the "filling function" $F(\ell)$, $\ell \in \mathbb{R}_+$, which is the supremum of $F(S)$ over all curves S in X of length $\leq \ell$. For the above five invariants we use the abbreviations $FA(\ell)$, $F_gA(\ell)$, $F_0D(\ell)$, $FR(\ell)$, $F_0L(\ell)$, and $FS(\ell)$. In what follows we focus our attention on infinite groups Γ with X appearing as some thickening of Γ. In fact we may use as earlier a simply connected 2-polyhedron $\widetilde{P}$ (for X) where Γ acts freely and cocompactly (Never mind that $\widetilde{P}$ is not quite a manifold.).

5.A. Isoperimetric (Dehn) functions FA and $\mathbf{F_0A}$. Our $F_0A(\ell)$ is what Gersten calls *Dehn* function. He gives a comprehensive account on F_0A (see $[\mathrm{Ger}]_6$ in this volume). The few comments we add below are stimulated by the questions put out to me by Gersten during the meeting.

5.$\mathbf{A_1}$. Upper bounds on $\mathbf{F_0A}$: *General remarks.* To have such a bound we need, in general, a specific construction of a surface of "small" area filling in $S \subset X$. In rare cases (e.g. if $K(X) \leq 0$) one can also use the variational approach, i.e. take some minimal surface M with boundary S and then establish an upper bound on Area (M) by analyzing the geometry of M in the "infinitesimal neighbourhood" of M in X, where the minimality of M can be effectively used. A somewhat similar situation arizes in the case where $\dim X = 2$ and $H_2(X) = 0$. Here every S^1 bounds a unique surface (or rather a chain, see 4.C_2.) and the only problem is to evaluate the area of this surface. This has been done by Gersten (see $[\mathrm{Ger}]_6$ and references therein) for a variety of groups Γ with aspherical (and Cockcroft) presentations.

5.$\mathbf{A_2}$. Filling area in nilpotent groups. Let X be a simply connected nilpotent Lie group. We already know that $F_0A(\ell) \lesssim \ell^\alpha$ in this case because of the "polynomial similarity" between X and $\mathbb{R}^n$, $n = \dim X$

(see 3.B_3.; notice that the polynomial bound on F_0A for nilpotent groups was originally established by Gersten in a more algebraic vein). Recall, that the polynomial similarity comes along with the exponential map of $\mathbb{R}^n = \operatorname{Lie} X$ to X and so the actual filling of a given $S \subset X$ is obtained by taking the cone of one-parameter subgroups over S. Here, to control the area, we need id $\in S$ (which we can assume) and then the filling cone is made out of the segments of one-parameter subgroups terminating on S. It is obvious that the area of this cone is bounded by const$(\ell^\alpha + 1)$ for ℓ = length S and some number $\alpha \leq \alpha_0(n)$ which can be explicitely evaluated in terms of the Lie algebra of X. (We suggest to the reader to actually make this evaluation. We do not do it ourselves since this does not, in general, lead to the best bound on F_0A).

5.A'_2. Filling in homogeneous groups. A nilpotent group X is called *homogeneous* if the Lie algebra L of X admits a *grading*. In this case there exists a semi-simple automorphism $a : L \hookleftarrow$ where all eigenvalues have absolute values < 1. It follows that X admits a 1-parameter group of automorphisms $\mathcal{A}_t : X \hookleftarrow$, $t \in \mathbb{R}^\times$, whose Lipschitz constants for some left invariant metric on X are ≤ 1 for $t \in]0,1]$. (Notice that this $\mathcal{A}_t$ is a combing in the sense of Gersten).

Remark. The above X admits a geodesic left invariant metric which is *scaled* by $\mathcal{A}_t$, i.e.

$$\operatorname{dist}(\mathcal{A}_t(x), \mathcal{A}_t(y)) = t \operatorname{dist}(x,y) .$$

Such a metric is not, in general, Riemannian but is, what we call, a *Carnot-Caratheodory* metric.

Examples. The Heisenberg groups H_{2n+1} are homogeneous for all $n = 1, 2 \ldots$. The free nilpotent groups are also homogeneous.

Let us indicate a very simple proof of the filling bound $F_0A(\ell) \lesssim \ell^\alpha$ for homogeneous nilpotent groups.

Every $S \subset X$ can be scaled to the unit lengths with an appropriate $t \in \mathbb{R}^\times$, i.e. length $\mathcal{A}_t(S) = 1$, and then $\mathcal{A}_t(S)$ is filled in by a disk D_1 of a certain area Ar_1. Then $\mathcal{A}_t^{-1}(D_1)$ fills in S and Area $\mathcal{A}_t^{-1}(D_1) \leq t^{-2\Lambda} Ar_1$, where Λ is the maximal eigenvalue of $\mathcal{A}_t$ (acting on the Lie algebra). On the other hand the length ℓ of S and the minimal eigenvalue λ of $\mathcal{A}_t$ are related by the (obvious) inequality

$$\ell \geq t^{-\lambda} .$$

Thus the area of the disk $D = \mathcal{A}_t^{-1}(D_1)$ filling in S satisfies

$$\operatorname{Area} D \leq Ar_1 \ell^{2\Lambda/\lambda} .$$

Finally we observe, that if the Lie algebra L of X admits a grading of degree d,

$$L = L_1 \oplus L_2 \oplus \ldots \oplus L_d ,$$

then one can (obviously) make $\mathcal{A}_t$ with $\Lambda \leq d\lambda$ and have Area $D \leq Ar_1 \ell^{2d}$.

5.A''_2. *Remarks.* (a) The above filling bound is by no means sharp and it also has the disadvantage of being restricted to homogeneous groups. What appeals here is the simplicity of the argument which works for the filling problem in all dimensions (compare 5.D.) and which extends to certain non-homogeneous situations. For example, if X is a horosphere in a complete manifold with pinched negative curvature K, namely $-\Lambda^2 \leq K \leq -\lambda^2$, then every circle S in X fills in a disk D in X with Area $D \leq C(\text{length } S)^{2\Lambda/\lambda}$ for some universal constant $C(\leq 100)$. (In fact, the same applies, in general, to stable leaves of Anosov systems).

(b) Let us improve the above ℓ^{2d}-bound by ℓ^{d+1}. To do this we assume $S \ni$ id and we take the cone over S built of the $\mathcal{A}_t$-semi-orbits for $t \leq 1$ starting from S. We estimate the length of such a semi-orbit $\{\mathcal{A}_t(s)\}$, $t \in [0,1]$, $s \in X$, (where $\mathcal{A}_0(s) = \text{id}$) by the same argument we used for Area D, which yields

$$\operatorname{length}\{\mathcal{A}_t(s)\} \leq C(\operatorname{dist}(\text{id}, s))^d$$

for some $C \geq 0$ and all $s \in X$ within distance ≥ 1 from id $\in X$. Then we see that the area of the filling cone (over S) is bounded by const(length $S)^{d+1}$ for length $S \geq 1$. This result is sharp in certain cases, e.g. for free nilpotent groups of degree d (see $[\text{Ger}]_6$ and 5.B_2.), but it can be improved for many nilpotent groups X. We shall indicate such an improvement in 5.A_4. based on the analytic techniques presented in the next section.

5.A_3. Filling in curves in Carnot-Caratheodory spaces. Let X be a smooth Riemannian manifold and $T_1 \subset T(X)$ be a subbundle of the tangent bundle. The $C-C$ metric in X associated to these data is defined as the infimum of the Riemannian lengths of the curves in X *tangent to* T_1 between pairs of points in X. We assume below (to exclude degeneracy), that every two points x_1 and x_2 in X can be joined by a curve tangent to T_1. Then the above definition gives us indeed a metric on X, as the $C-C$ distance between x_1 and x_2 is $< \infty$.

Now we address the following (filling) problem. Given a curve S of length ≤ 1 in X tangent to T_1, does there exist a disk D in X filling in S which is everywhere tangent to T_1 and such that $\operatorname{Area} D \leq C\,(\operatorname{length} S)^2$ for some C independent of S ? We shall restrict ourselves to the special case of the problem (which is good enough for our applications) where the Riemannian manifold X is isometric to $\mathbb{R}^n$ and S lies in the unit ball in $\mathbb{R}^n = X$ around the origin. Now the answer should depend on T_1 only.

First of all we must decide if there is any non-trivial disk at all tangent to T_1 in X. (Here and below "disk in X" refers to a smooth or a Lipschitz map of the unit disk into X and "tangent to T_1" means that the differential of the map sends the tangent bundle of the disk to T_1). It is convenient to think of our disk (at a regular point where the rank of the implied map equals 2) as the graph of a (local) map $f : \mathbb{R}^2 \to \mathbb{R}^{n-2}$ and represent the tangency condition by a system of m partial differential equations for $m = 2\operatorname{codim} T_1$. Then one can show that for *generic* T_1 with $2\operatorname{codim} T_1 > n-2$ (where the system is overdetermined) there is no non-nontrivial (i.e. with a regular point) disk at all (see p. 166 in $[\text{Gro}]_{12}$). If $2\operatorname{codim} T_1 = n-2$ and the (tangency) P.D.E. system is *determined*, regular disks usually exist but there is no way to prescribe the boundary condition $\partial D = S$. The only case where one may hope to solve the boundary problem $\partial D = S$ is where $2\operatorname{codim} T_1 < n-2$ and so our system is *underdetermined*. This system (for maps $f : \mathbb{R}^2 \to \mathbb{R}^{n-2}$) can be written in the form

$$\mathcal{D}_1 f = g \tag{$*$}$$

where $\mathcal{D}_1$ is a certain non-linear differential operator sending $(n-2)$-tuples of functions on $\mathbb{R}^2$ to m-tuples for $m = 2\operatorname{codim} T_1$.

Now we state a result which easily follows from the Nash implicit function theorem proven in $[\text{Gro}]_{12}$.

5.A'_3. *Proposition. If the above operator $\mathcal{D}_1$ is infinitesimally invertible (in the sense of* $[\text{Gro}]_{12}$*) then every smooth curve S in X tangent to T_1 bounds a (smoothly mapped) disk in X.*

Remark. The infinitesimal invertibility is a certain non-degeneracy condition on $\mathcal{D}_1$ which may be expressed by non-vanishing of certain determinants built out of the (high order) derivatives of the "coefficients" of $\mathcal{D}_1$ (see $[\text{Gro}]_{12}$).

The above proposition (and the proof by the techniques of P.D.R.) extends to families of curves in X and then it yields the following

5.A''_3. Filling Corollary. *If $\mathcal{D}_1$ is infinitesimally invertible then every closed curve $S \subset X = \mathbb{R}^n$ of length ≤ 1 lying in the unit ball and tangent to T_1 bounds a disk D tangent to T_1, such that*

$$\operatorname{Area} D \leq C\,(\operatorname{length} S)^2$$

for some $C > 0$ independent of S (but depending on T_1).

Idea of the proof. It is easy with the above remark to have our filling D with $\operatorname{Area} D \leq C(S)$ where C is bounded on every *compact family* of curves S in X tangent to T_1.

Lemma. There exists a compact family of curves tangent to T_1, say S', such that for every our S there exists an $S' \in S'$ which meets S at 6 points such that the resulting 6 arcs in S have lengths equal $1/6$

length S and the corresponding arcs in S' have the length at most $(\frac{1}{6}+\varepsilon)$ length S for a fixed small $\varepsilon > 0$ (e.g. $\varepsilon = 0.01$).

Idea of the proof. Divide S by 6 points into 6 arcs of equal length and join the adjacent points by the shortest segments tangent to T', see Fig. 12.

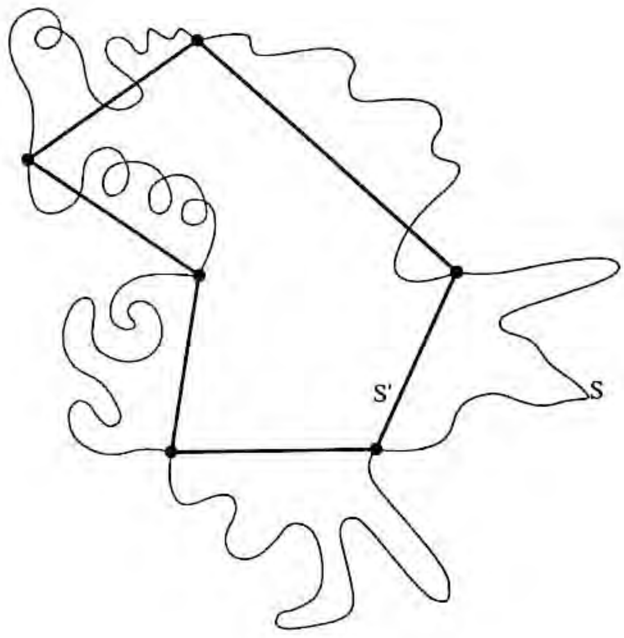

Figure 12

These segments form a curve S' with $\varepsilon = 0$ which can then be smoothed with a slight increase of ε. Q.E.D.

Now we fill in S by taking the best (i.e. the minimal among those which are tangent to T_1) fillings of S' and of the 6 circles bounded by the pairs of arcs of S and S'. Thus we see that the filling area function $\mathrm{Ar}(\ell)$ satisfies

$$\mathrm{Ar}(\ell) \leq C'\ell^2 + 6\mathrm{Ar}\Big(\big(\frac{1}{3}+\varepsilon\big)\ell\Big), \qquad (**)$$

for C' depending on S'. We take ε, such that $\Delta = 6\left(\frac{1}{3}+\varepsilon\right)^2 < 1$ and reiterate $(**)$ for $\left(\frac{1}{3}+\varepsilon\right)\ell$, $\left(\frac{1}{3}+\varepsilon\right)^2\ell$, etc. Then we obtain

$$\mathrm{Ar}(\ell) \leq C'\ell^2 + \Delta C'\ell^2 + \Delta^2 C'\ell^2 + \ldots \leq C''\ell^2 \quad \text{for } C'' = C'/1-\Delta\,.$$

Q.E.D.

5.A$_4$. Back to homogeneous nilpotent groups X. Let $L = L_1 \oplus L_2 \oplus \ldots \oplus L_d$ be a grading of the Lie algebra of X and let $T_i \subset T(X)$ be obtained by the left transport of $L_1 \oplus \ldots \oplus L_i$.

Claim. *If the differential operator associated to T_i is infinitesimally invertible then the filling area function $F_0A(\ell)$ of X is bounded by $C\ell^{2i}$ for all $\ell \geq 1$.*

Sketch of the proof. In order to fill in a curve S in X we first approximate it by a curve S_0 tangent to T_i, such that the length and the filling area of S_0 remain close to these invariants of S. Then we rescale S_0 to the unit size, fill it in by a disk (produced in the previous section) of area $\leq$ const and then scale this disk back so that it would fill in S_0. This gives as in 5.A$'_2$. the ℓ^{2i}-bound for the filling of S and consequently of S.

5.A$'_4$. *Examples.* (a) Let X be the Heisenberg group. Then L is graded as follows, $L = L_1 \oplus L_2$, where L_2 is the center and hence $\dim L_2 = 1$ and $\dim L_1 = 2n$ for $2n+1 = \dim X$. The subbundle $T_1 \subset T(X)$ of codimension one gives the (standard) *contact structure* to X and the corresponding operator is infinitesimally invertible (as is easy to check up). Thus, we recapture the *quadratic* bound on $F_0A(\ell)$ claimed (according to Gersten, see [Ger]$_6$) by Thurston. (I do not know if Thurston's proof follows the same lines).

(b) Let X be the $(4n+3)$-dimensional nilpotent group (with 3-dimensional center) which serves as a horosphere in the quaternionic hyperbolic space. Here we have the grading $L_1 \oplus L_2$ with $\dim L_1 = 4n$ and $\dim L_2 = 3$ and the corresponding operator is (by an easy argument) infinitesimally invertible for $n \geq 2$. Thus, the filling area function is *quadratic* in this case as well.

5.A$''_4$. An improvement for $d \geq 3$. Fix two independent tangent fields ∂_1 and ∂_2 in the unit disk D and look at the system of differential equations whose solutions give us the maps $D \to X$ sending the first field to a subbundle $T_1 \subset T(X)$ and ∂_2 to a larger subbundle $T_2 \supset T_1$. If $T_1 = T_2$ this is what we had met in 5.A$_3$. and what was done there can be extended to the present more general case. Then we apply the issuing analytic result to the nilpotent groups X with $L = L_1 \oplus \ldots \oplus L_d$ where T_1 comes from $L_1 \oplus \ldots \oplus L_{i_1}$ and T_2 from $L_1 \oplus \ldots \oplus L_{i_2}$ for $i_2 \geq i_1$ and thus prove the following

5.A$'''_4$. Generalization. *If the associated differential operator is infinitesimally invertible, then*

$$F_0A(\ell) \leq C\ell^{i_1+i_2},\ \ell \geq 1 .$$

Example. Take $i_1 = 1$ and $i_2 = d$, and assume that L_1 generates L as a Lie algebra. (One can always achieve this by modifying the grading.) Then one knows (see [Gro]$_{12}$) that the operator in question (responsible for the tangency of ∂_1 to T_1) is infinitesimally invertible and so $F_0A(\ell) \leq C\ell^{d+1}$ (as has been already shown in 5.A$_3$.)).

Remarks. (a) I believe the inequality provided by 5.A$'''_4$. is sharp.

(b) To make 5.A$'''_4$. practical one should work out a simple condition for the infinitesimal invertibility of our operators. (In general, checking infinitesimal invertibility is a mess of multilinear algebra but in the special case of nilpotent groups one hopes that there is a palatable criterion).

5.A$_5$. Area filling in non-homogeneous nilpotent Lie groups. Let L be a nilpotent Lie algebra and $C_i \subset L$ be the linear subspace generated by the commutators of length $\geq i$. We observe that $L = C_1 \supset C_2 \supset \ldots \supset C_d$ and that $[C_i, C_j] \subset C_{i+j}$. Then we split L agreeably with this filtration, $L = L_1 \oplus L_2 \oplus \ldots \oplus L_d$, such that $L_i \oplus \ldots \oplus L_d = C_i$. Observe that the linear subspace $L_i \subset C_i$ generates C_i as the Lie algebra for every $i = 1, \ldots d$. Fix a basis ℓ_j in L, such that the first j_1 vectors for $j_1 = \dim L_1$ lie in L_1, the following j_2 vectors for $j_2 = \dim L_2$ lie in L_2, etc. Then consider the linear operator $\alpha_t : L \to L$ which consists in multiplying the vectors in L_1 by t, the vectors in L_2 by t^2 and so on up to d where the vectors in L_d are multiplied by t^d.

Observation (compare [Gro]$_1$). *The structure constants of our Lie algebra in the basis* $\{\ell^t_j\} \underset{\text{def}}{=} \{\alpha_t \ell_j\}$ *remain bounded as* $t \to +\infty$.

This is obvious. In fact the structure constants, say $e^t_{jk\ell}$ converge for $t \to \infty$ to the structure constants of another Lie algebra L^∞ for which the linear splitting $L_1 \oplus L_2 \oplus \ldots \oplus L_d$ becomes a grading, i.e. $[L_i, L_j]^\infty \subset L_{i+j}$.

Now, let X and X^∞ be the Lie groups corresponding to L and L^∞ and let $T_i \subset T(X)$ and $T^\infty_i \subset T(X^\infty)$ be the subbundles corresponding to the subspaces $L_1 \oplus \ldots \oplus L_i$ for each $i = 1, \ldots, d$. If we denote by X^t

the space (group) X with the left frame corresponding to $\{\ell_j^t\}$, then X^t converges (in an obvious sense) to X^∞ with a certain frame. In other words we replace the rescaling used in 5.A_2'. by changes of frames and in the limit our rescaling takes us from X to X^∞.

Generalization of generalization. Let us recall the set-up of 5.A_4'''. and look at the P.D.E. systems associated to pairs of subbundles $T_{i_1} \subset T_{i_2} \subset T(X)$ and $T_{i_1}^\infty \subset T_{i_2}^\infty \subset T(X^\infty)$. Since the subbundles T_i appear as perturbations of T_i^∞ (since T_i^∞ are limits of T_i) the infinitesimal invertibility for (the operator related to) the pair $(T_{i_1}^\infty, T_{i_2}^\infty)$ implies that for (T_{i_1}, T_{i_2}). Moreover, this implies a uniform (in an obvious sense) infinitesimal invertibility of our operators in the frames $\{\ell_j^t\}$ for $t \to \infty$. As a conclusion we obtain as in 5.A_4'''. the following

5.A_5'. Upper bound on filling area. *If the operator associated to $(T_{i_1}^\infty, T_{i_2}^\infty)$ is infinitesimally invertible, then*

$$F_0A(\ell) \leq \mathrm{const}\, \ell^{i_1+i_2}, \text{ for all } \ell \geq 1 .$$

Corollary (Gersten conjecture). *For all nilpotent Lie groups*

$$F_0A(\ell) \leq \mathrm{const}\, \ell^{d+1}, \ \ell \geq 1 .$$

Proof. Since L_1^∞ generates L^∞ as a Lie algebra our operator is infinitesimally invertible.

Conjecture. Suppose, the codimension c_1 and c_2 of the above subbundles T_{i_1} and T_{i_2} satisfies

$$c_1 + c_2 < n - 2, \text{ for } n = \dim X . \tag{+}$$

Then I expect the relevant operator is infinitesimally invertible (at least for generic L with given c_1 and c_2) and consequently

$$F_0A(\ell) \leq \mathrm{const}\, \ell^{i_1+i_2}, \ \ell \geq 1 .$$

This is motivated by the observation that our P.D.E.-system (with $n-2$ unknown functions and $c_1 + c_2$ equation) is undetermined for $c_1 + c_2 < n - 2$ and the infinitesimal invertibility is a generic phenomenon for undetermined systems (see $[\mathrm{Gro}]_{12}$).

5.A_6. Filling area in lattices of rank 1. Let Γ be a non-cocompact lattice in a simple Lie group of $\mathbb{R}$-rank one. In other words Γ discretely acts on a symmetric space X of negative curvature such that the quotient space is non compact but yet has finite volume.

Claim. *Γ satisfies the quadratic isoperimetric inequality* (i.e. $F_0A(\ell) \lesssim \ell^2$) *unless X is (isometric to) one of the following three spaces,*

(1) *X is the 4-dimensional compact hyperbolic space (here and below* $\dim X = \dim_{\mathbb{R}} X$),

(2) X is the *8-dimensional quaternionic space,*

(3) X is the *hyperbolic Cayley plane (of dimension* 16).

Proof. To simplify the terminology, we assume Γ acts freely and then $V = X/\Gamma$ is a manifold with cusps which can be cut away by horospheres. Thus we obtain a compact manifold $V_0 \subset V$ with concave boundary. Each component of the boundary is finitely covered by a nil-manifold which is, in turn, infinitely covered by a nilpotent Lie group corresponding to the implied horosphere. We know, (see 5.A_4'.), that each of these groups satisfies the quadratic isoperimetric inequality and since Γ is *hyperbolic relative* the subgroups on the boundary of V_0, (see $[\mathrm{Gro}]_{14}$), Γ also satisfies a quadratic inequality as a simple argument shows.

5.A_6'. *Remarks.* (a) The above argument applies to Γ acting on (non-symmetric) spaces X with pinched negative curvature,

$$-\infty < -C^2 \leq K(X) \leq -c^2 < 0 .$$

Here again each component of the boundary of V_0 is covered by a nilmanifold (whose degree of nilpotency is bounded by the ratio C/c, see $[\text{Gro}]_{4,21}$) and the isoperimetric function of Γ is bounded by those of the nilpotent groups of the boundary. Therefore, the isoperimetric function (filling area) of Γ satisfies

$$F_0A(\ell) \lesssim \ell^\alpha ,$$

(where $\alpha \leq 1 + (\log_2(C/c) + 1)$).

(b) Since the universal covering $\widetilde{V}_0 \subset \widetilde{V} = X$ has *concave* boundary it admits a Lipschitz retraction on each component of the boundary. It follows, that the isoperimetric function (as well as any other filling function) for Γ cannot be better (i.e. smaller) than that for the nilpotent groups at the boundary. For example, in the exceptional case (1) of our rank one claim the boundary is the 3-dimensional Heisenberg group where the isoperimetry is (known to be, see $[\text{Ger}]_6$ and $5.B_2$.) *cubical* and so $F_0A(\ell) \sim \ell^3$ for the lattices Γ in $SU(2,1)$ as well. (Probably, the other two exceptions also have $F_0A(\ell)$ cubical).

(c) Since V_0 has negative curvature "many" minimal surfaces in V_0 have

$$\text{Area} \lesssim \text{length (boundary)}$$

which suggests the *linear isoperimetric inequality for* Γ *on the average*. Let us explain what it means. Given a group Γ with a fixed generating set we denote by $\mathcal{L}_\ell$ the set of all trivial words in Γ of length $\leq \ell$. We think of $\mathcal{L}_\ell$ as a finite probability space where all atoms have equal weights and then the filling area becomes a random variable whose distribution we want to understand as $\ell \to \infty$. In particular, we may ask if the expectation $E_\ell = E(\text{Fill}_0 \text{Area})$ on $\mathcal{L}_\ell$ is *subasymptotic* to the *maximum* of Fill_0 Area on $\mathcal{L}_\ell$, which is just another name for $F_0L(\ell)$. (Subasymptotic means $E_\ell/F_0L(\ell) \to 0$ for $\ell \to \infty$). In the simplest case where $\Gamma = \mathbb{Z}^n$ the standard recurrency properties of the random walk show that here indeed E_ℓ is subasymptotic to $F_0A(\ell)$ (and a similar conclusion, probably, remains valid for the nilpotent groups. This looks easy as the random walk in the nilpotent groups is well understood but I did not check the matter for the lack of time). Then, as earlier, the isoperimetric bound extends from (the nilpotent groups at) the boundary of V_0 to V_0 thus yielding a *subquadratic* isoperimetric inequality on the average for our lattices Γ acting on X at least in the case of X of *constant* negative curvature (where the horospheres are Euclidean spaces and the implied nilpotent groups are Abelian).

Notice, that the subasymptotic behaviour of E_ℓ (relative to $F_0A(\ell)$) for our lattices Γ may have another source independent of the geometry of the boundary of V_0. Namely, a random closed curve S of length ℓ in $\widetilde{V}_0$ is unlikely to stay for a long time close to a single component of the boundary. If so, the filling area of S will be *significantly* smaller than that for the curves in the boundary.

(c′) The above "random" considerations become more transparent if, instead of lattices Γ, we look at free products of Abelian (and/or nilpotent) groups.

(d) The effect of relative hyperbolicity on F_0A may be also seen in certain manifolds V_0 with *non-positive* curvature and concave boundary. A typical instance of that is provided by lattices Γ of $\mathbb{Q}$-rank one in 3.G. There, the boundary consists of several components covered by solvable Lie groups and the function $F_0A(\ell)$ for Γ is bounded by those for these solvable groups. Such bounds can be established by looking at minimal surfaces in $\widetilde{V}_0$ (which may touch the boundary of $\widetilde{V}_0$).

$5.A_7$. Filling in general Lie groups and lattices. We know (see §3.) that most of Lie groups (and lattices) are exponentially Lipschitz retracts in $X = SL(N)/SO(N)$ which is a space of non-positive curvature and so satisfy the quadratic isoperimetric inequality. Moreover, some (minimal) filling disk of a curve S of length ℓ in X (obviously) lies in the R-neighbourhood of S for $R \leq \ell/2\pi$ (which is better than a mere linear bound on the filling radius, see $[\text{Gro}]_{10}$ for an extensive discussion). It follows, that exponentially Lipschitz retracts in X satisfy the exponential inequality

$$F_0A(\ell) \lesssim \exp C\ell .$$

In particular we obtain this inequality for the connected subgroups in $SL(N)$ (and then for *all* Lie groups by a simple additional argument) and for the lattices (cocompact and non-cocompact) in the semi-simple Lie group.

Remark. The above argument applies to all filling invariants. In particular one sees this way that *every closed curve* S *in a Lie group* or in *a (thickening of a) lattice bounds* a λ*-Lipschitz image of the unit disk for* $\lambda \sim \exp C$ length S. (Here, as usual, one maps the disks into some thickening of the group in question). In other words, the filling span $F_0S(\ell)$ (which bounds the rest of the filling invariants) grows at most exponentially. This follows (via the exponentially Lipschitz retraction) from the (well-known) *linear* bound on $F_0S(\ell)$ for simply connected spaces X with $K(X) \leq 0$.

5.A₈. Thurston's assertion for Fill₀ Area in SLₙZ. According to [Ger]₆ Thurston proved that the groups $SL_n\mathbb{Z}$ for $n \geq 4$ satisfy the quadratic isoperimetric inequality, i.e. $F_0A(\ell) \sim \ell^2$ and one expects a polynomial bound for the k-dimensional filling functions for $k = 2,\ 3, \ldots,\ n-2$. But for $k = n-1$ the filling function is exponential (see [E-C-H-P-T]).

5.A₉. Quadratic filling inequalities in solvable groups. Solvable Lie groups of high $\mathbb{R}$-rank are expected to satisfy a polynomial isoperimetric inequality. Look, for example, at the semi-direct product $\mathbb{R}^n \rtimes \mathbb{R}^{n-1}$ for the diagonal action of $\mathbb{R}^{n-1} = (\mathbb{R}_+^\times)^{n-1}$ on $\mathbb{R}^n$ with determinant one. It is easy to see with the arguments in (f) of 2.B. and in 5.A$_3''$., that $F_0A(\ell) \sim \ell^2$ for $n \geq 3$. (This agrees with Thurston's claim for $SL_n\mathbb{Z}$, $n \geq 4$, and therefore must be known to the authors of [E-C-H-P-T]).

5.B. Lower bounds on the filling area. Take a smooth closed curve S in a manifold X which is known to bound in X and let us try to bound from below the area of a surface M in X with $\partial M = X$. Here is a simple situation where one can do it.

5.B₁. Fibration inequality. Suppose X is fibered over $\mathbb{R}$ by a smooth function $f : X \to \mathbb{R}$ whose gradient is everywhere ≥ 1. Let $X_t = f^{-1}(t)$, $t \in \mathbb{R}$, and $S_t = S \cap X_t$. Since the intersection is transversal for almost all t, our S_t is finite for a.a.t. and can be thought of as a 0-dimensional cycle in X_t (over $\mathbb{Z}$ if we choose to orient S and over $\mathbb{Z}_2$ in any case). Denote by $F\ell(t)$ the infimum of the length of 1-chains C in X_t with $\partial C = S_t$ and observe that every surface M with $\partial M = S$ has

$$\operatorname{length} M \cap X_t \geq F\ell(t)\ .$$

This integrates, via the *coarea inequality* (see [Gro]₁₀ all about that) to

$$\operatorname{Fill\,Area} S \geq \int_{-\infty}^{+\infty} F\ell(t)dt\ . \qquad (*)$$

Remarks. (a) The above inequality $(*)$ becomes interesting when the hypersurfaces X_t are strongly distorted in X and $F\ell(t)$ is significantly greater than the distances between the points in S_t measured in X. We shall see this presently.

(b) The inequality $(*)$ obviously generalizes to the k-dimensional isoperimetric (filling) problem and yields a lower bound on the corresponding function $\operatorname{Fill\,Vol}_k$ in terms of $\operatorname{Fill\,Vol}_{k'}$, (X_t), $t \in T$, where T may be now of dimension k'' in the interval $1 \leq k'' \leq k-1$ and $k' = k - 1 - k''$.

5.B₂. Nilpotent groups. The simplest case is where X is the 3-dimensional Heisenberg Lie group fibered over $\mathbb{R}$ by some non-trivial homomorphism $f : X \to \mathbb{R}$. The fiber is $\mathbb{R}^2$ which is, itself, fibered by the lines "parallel" to the center of X. These central lines are, as we know, quadratically distorted in X: a segment of length $\ell > 1$ on such a line has its ends within distance $\sim \sqrt{\ell}$ in X. Yet, these lines are undistorted in our fibers $X_t = \mathbb{R}^2$, which makes X_t rather distorted in X.

Now we take another homomorphism, say $g : X \to \mathbb{R}$, independent from f and look at $Y_0 = g^{-1}(0) \subset X$ which is also $\mathbb{R}^2$ sliced into central lines. We take a closed rectangular curve $\square$ in Y_0 whose one pair of opposite sides is parallel to the center and has length ℓ' while the normal sides have length ℓ. Then we replace the ℓ'-sides by the minimal geodesic segments in $X \supset Y_0$ and call the resulting curve in X by $\square_\ell$. We choose both length ℓ' and ℓ large (eventually they go to infinity) and such that $\ell' = \varepsilon\ell^2$ for a small fixed $\varepsilon > 0$. Then the sides of $\square_\ell$ corresponding to the ℓ'-sides of $\square'$ have length $\ell'' \sim \varepsilon\ell$, see Fig. 13 below.

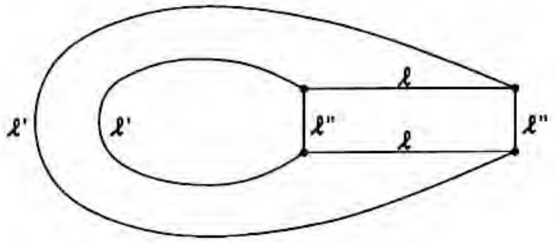

Figure 13

Now, we intersect $\square_\ell$ with X_t and we observe that the pair of ℓ-sides meets X_t at two points within distance ℓ' in X_t for t running over some segment in $\mathbf{R}$ of length $\sim \ell$. Moreover, by making this segment ε-smaller we arrive at the situation where the ℓ''-sides are not met by X_t at all for t in this smaller interval. Then $F\ell(t) = \ell'$ and

$$\text{Fill Area}\,\square_\ell \gtrsim \ell\ell' \approx \ell\,(\ell'')^2 \approx \varepsilon^2\,(\text{length}\,\square)^3 \ .$$

Thus we obtained *Gersten's inequality* $FA(\ell) \gtrsim \ell^3$ for the Heisenberg group. (Notice that we allow here all surfaces or chains M, not only the disks, filling in S and so our bound applies to $FA(\ell)$ which is a priori $\leq F_0A(\ell)$).

Let us extract the essentials from the above argument. Let X be a simply connected nilpotent Lie group and X_0 and Y_0 connected subgroups, such that

(a) $\operatorname{codim} X_0 = 1$.

(b) $\dim Y_0 = 2$.

(c) $\dim (Z_0 = X_0 \cap Y_0) = 1$.

(d) the group $Z_0 = X_0 \cap Y_0$ has polynomial distortion of degree at least d in X. A (necessary and) sufficient condition for that is

$$Z_0 \subset \underbrace{[[X,X,]X]\ldots]}_{d} \ .$$

(e) the distortion of Z_0 in X_0 is polynomial of degree at most $e \leq d$. For this it is sufficient *not to have* Z_0 in the $(e+1)$-th commutator subgroup.

$$\underbrace{\Big[[X_0,X_0],\ldots\Big]}_{e+1} \ .$$

Then our argument yields the bound

$$FA(\ell) \gtrsim \ell^\alpha \qquad (*)$$

for $\alpha = d/e + 1$.

Example. *Let X have nilpotency degree two such that*

(i) *the center Z of X equals $[X,X]$.*

(ii) $k = \dim Z > \frac{(m-1)(m-2)}{2}$ for $m = \dim(X/Z)$.

Then $FA(\ell) \gtrsim \ell^3$.

Proof. Under the above assumptions for every subgroup $X_0 \subset X$ of codimension one containing Z, there exists a 1-parameter subgroup $Z_0 \subset Z$, which is not contained in $[X_0, X_0]$, because

$$\dim Z > \frac{(m-1)(m-2)}{2} = \dim \wedge^2 (X_0/Z) .$$

We conclude the proof by generating Y_0 by Z_0 and any $y \in X - X_0$.

5.B$_3$. Solvable groups. First, let X be a 3-dimensional simply connected Lie group split into the semi-direct product, $X = \mathbb{R}^2 \rtimes \mathbb{R}$ where $\mathbb{R}$ acts on $\mathbb{R}^2$ by $(x, y) \mapsto (e^{\lambda_1 t}x, e^{\lambda_2 t}y)$. (Compare (e) in 2.B. where the notations are slightly different).

Claim. *If $\lambda_1 > 0$ and $\lambda_2 < 0$ then the isoperimetric* (i.e. *filling area) function $FA(\ell)$ grows exponentially.*

Proof. We constructed in (e) of 2.B. certain curves $\square'_d \subset X$ (see Fig. 5 in 2.B.), which we now intersect with $X_t = \mathbb{R}^2 \times t$ and apply the above fibration inequality. This gives the desired lower bound on Fill Area$\square'_d$ since the fibers X_t are strictly exponentially distorted in X (see 3.C.$_1$). Q.E.D.

Remarks. (a) The lower exponential bound for $F_0A(\ell)$ in the case $\lambda_1 = -\lambda_2$ is due to Gersten (see [Ger]$_4$).

(b) If $\lambda_1, \lambda_2 > 0$, then X is hyperbolic and $FA(\ell) \sim \ell$.

Generalization. *Let $X = N \rtimes \mathbb{R}$, where N is a simply connected nilpotent group which contains an isomorphic copy of $\mathbb{R}^2$ invariant under the action of $\mathbb{R}$ such that this action on $\mathbb{R}^2$ has two eigenvalues $\lambda_1 > 0$ and $\lambda_2 < 0$. Then $FA(\ell) \sim \exp \ell$ for this group X.*

The proof is clear at this stage. One should only keep in mind that the distortion of $\mathbb{R}^2$ in N is (at most) polynomial since N is nilpotent.

Further generalization. *Let again $X = N \rtimes \mathbb{R}$ and the action of $\mathbb{R}$ have two eigenvectors (in the Lie algebra of N), say x and y such that the corresponding eigenvalues satisfy $\lambda_1 > 0$ and $\lambda_2 < 0$. Then $FA(\ell) \sim \exp \ell$.*

Idea of the proof. If $[x, y] = 0$ we are in the situation we had earlier. To grasp the idea of the general case we assume $z = [x, y]$ lies in the center of (the nilpotent group) N. If the action of $\mathbb{R}$ on z has non-zero eigenvalue, then again we can pass to the previous case. So, the key new situation is where N is the Heisenberg group with the standard basis x, y, z in the Lie algebra and $\mathbb{R}$ acts by

$$(x, y, z) \mapsto (e^{\lambda t}x,\ e^{-\lambda t}y,\ z)$$

for some $\lambda \neq 0$.

Now consider the following hexagon with a selfintersection point in the (x, y)-plane.

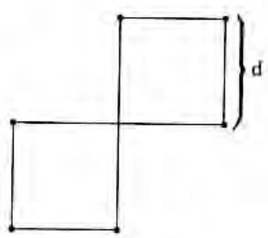

Figure 14

Since this hexagon bounds zero algebraic area in $\mathbb{R}^2$ it lifts to a hexagonal curve $\bigcirc_d$ (which is mapped to $\mathbb{R}^2$ identified with N/center). We modify this curve as earlier by replacing the six edges in N by the shortest segments in $X \supset N$. The resulting curve $\bigcirc'_d$ in X has length about $\log d$ while its filling area is bounded from below by something like d by our previous argument.

5.B$_4$. Whitney duality. We return to the general situation of a Riemannian manifold X with a closed curve $S \subset X$ whose filling area we want to bound from below. We assume $H_2(X) = 0$ and take an exterior closed 2-form ω on X. Then we observe that the integral of ω over a surface M filling in S does not depend on M

$$\int_M \omega = \int_{M'} \omega$$

for oriented surfaces ($\mathbb{Z}$-chains) having $\partial M = \partial M' = S$. (If $H_2(X) \neq 0$ the above equality remains valid if ω is exact.). Now, suppose the pointwise norm of ω on X is bounded by a constant $c \geq 0$. Then, obviously,

$$\text{Fill Area } S \geq c^{-1} \Big| \int_M \omega \Big| \tag{+}$$

where one can use any M one wishes with $\partial M = S$.

Remarks. (a) For all its triviality the inequality (+) may be quite useful as we shall see presently.

(b) A deeper aspect of the relation between the area and integrals of forms is revealed by Whitney's duality which claims that (+) is sharp (see [G-L-P], [Gro]$_{10}$). Namely, there always exists a form ω (depending on X and S) for which Fill Area $\geq c^{-1} |\int_M \omega|$, where Fill Area is defined with $\mathbb{R}$-chains filling in S. (This means we fill in the k-multiple of S by a minimal surface M_k and then go to the limit $\lim k^{-1}$ Area M_k for $k \to \infty$.). We do not (know how to) use the full power of Whitney duality but only the simplest instances of that which are clear as they stand without any explicit reference to Whitney's theorem.

5.B$'_4$. Application of (+) to Lie groups. Let $X = \mathbb{R}^2 \rtimes \mathbb{R}$ be our old friend solvable group. Now we require $\lambda_1 = -\lambda_2 \neq 0$ and we want to give another proof of the *lower exponential bound for the filling area* of the curves $\square'_d$ in X. First, we invoke the projection $p : X \to \mathbb{R}^2 = \mathbb{R}^2 \times 0 \subset X$ corresponding to the splitting of X and we denote by ω the pull-back of the area form on $\mathbb{R}^2$. This form is (pointwise) bounded on X since $\lambda_1 + \lambda_2 = 0$. Newt we construct an M filling in $\square'_d$ as follows,

$$M = M_1 + M_2$$

where M_1 is the cylinder of the projection $p : \square'_d \to \mathbb{R}^2 \subset X$ and M_2 is the square in $\mathbb{R}^2$ bounded by $\square_d = p(\square'_d) \subset \mathbb{R}^2$. Since the form ω vanishes on M_1 we have

$$\int_M \omega = \int_{M_2} = \text{area } M_2 \ .$$

As length $\square'_d \sim \log \text{area } M_2$ the proof follows.

Remarks. (a) Our proof reproduces a fragment of a beautiful argument used in [E-C-H-P-T] to bound from below the $(n-1)$-dimensional isoperimetric (i.e. filling) function for $SL_n\mathbb{Z}$.

(b) There is an obvious generalization of the above proof to more general semidirect products $X = \mathbb{R}^2 \rtimes X'$, where the implied action of the subgroups X' on $\mathbb{R}^2$ is unimodular and X' is simply connected. Here again we have a lower bound on Fill Area depending on the distortion of $\mathbb{R}^2 = \mathbb{R}^2 \times 0$ in X. Namely, the exponential distortion gives us the exponential lower bound (notice, that the distortion here does not have to be strict, i.e. the exponential distortion on a line of $\mathbb{R}^2$ will do), and the polynomial distortion has a similar effect.

We conclude our discussion on lower (and upper) bounds for Fill Area on a somewhat pessimistic note. The present methods lead to satisfactory results only in a few special cases even in the friendly geometric surroundings of solvable and nilpotent groups.

$5.B_5$. Growth of differential forms and cofilling inequalities. Consider the following problem which is, in a way, dual to the filling problem of curves in X by minimal surfaces. Take an exact 2-form ω on X and look for a "minimal" 1-form α, such that $d\alpha = \omega$. In order to make the word "minimal" precise we denote by $||\alpha||(R)$ the supremum of $||\alpha_x||$ for x running over the R-ball around a fixed point $x_0 \in X$. Then we define the function $\mathrm{Cof}(R)$ as the infimum of the functions $\beta(R)$ such that for every exact 2-form ω on X with $||\omega_x|| \leq 1$ for all $x \in X$ there exists a form α satisfying $d\alpha = w$ and $||\alpha||(R) \leq \beta(R)$.

Examples. (a) Let X be a complete simply connected Riemannian manifold with convex boundary. If $K(X) \leq -c < 0$, then the function $\mathrm{Cof}(R)$ is bounded. In other words every bounded closed 2-form ω on X is the differential of a *bounded* 1-form. To see this we use the geodesic cones in X from a fixed point $x_0 \in X$. Taking cones over the tangent vectors in X defines a linear operator from 2-form to 1-form: the value of the resulting 1-form on $\tau \in T(X)$ equals the integral of the 2-form over the infinitesimal triangle which is our cone from x_0 over τ. The curvature condition $K \leq -c < 0$ implies the bound on the areas of these triangles by const(length of the base) and our claim follows.

(a') *Remark.* The above property, properly formulated, is shared by all hyperbolic spaces and, in fact, characterizes these spaces.

(b) Let us relax the curvature condition to $K(X) \leq 0$. Then the above cone argument gives the bound $\mathrm{Cof}(R) \lesssim R$. Notice that this bound is sharp for $X = \mathbf{R}^2$ in the following strong sense: there exists an exact 2-form ω on $\mathbf{R}^2$ with $||\omega|| \leq 1$, namely $\omega = dx \wedge dy$, such that every α with $d\alpha = \omega$ satisfies $||\alpha||(R) \geq R$. This is seen by integration α over the circles of radius R passing through x_0.

The above example suggests a similar property for other "reasonable" spaces (e.g. simply connected Lie groups): there exists an exact 2-form ω on X with $||\omega|| \leq 1$, such that the forms α with $d\alpha = 1$ admit uniform lower bound on the R-balls,

$$||\alpha||(R) \leq \mathrm{Cof}_-(R) \qquad (+)$$

where $\mathrm{Cof}_-(R)$ is a function "similar" to $\mathrm{Cof}(R)$. The best similarity would be the asymptotic relation

$$\limsup_{R\to\infty} \mathrm{Cof}(R)/\mathrm{Cof}_-(R) \quad < \infty .$$

One also expects, in view of the Whitney duality, that for "reasonable" spaces the filling and cofilling functions are essentially the same, which means something like $FA(R) \sim R\,\mathrm{Cof}(R)$.

Now we look at the cofilling function in some Lie groups.

Examples. (a) If X is an exponentially Lipschitz retract in a space with $K \leq 0$, then the function $\mathrm{Cof}(R)$ for X grows at most exponentially (compare §3.).

(b) In the Heisenberg group the cofilling is at most quadratic. More generally, a homogeneous nilpotent group of nilpotency degree d has $\mathrm{Cof}(R) \lesssim R^d$. This is seen using the dilation (compare (b) in $5.A_2''$.), in the same way we used the geodesic cones for $K \leq 0$.

(c) Let X be a 3-dimensional solvable group split as usual $X = \mathbf{R}^2 \rtimes \mathbf{R}$. If the action of $\mathbf{R}$ on $\mathbf{R}^2$ is as in $5B_4'$., i.e. with two eigenvalues $\lambda_1 > 0$ and $\lambda_2 = -\lambda_1$, then the form ω constructed in $5B_4'$ is *essentially extremal* for the cofilling problem, i.e. every 1-form α with $d\alpha = \omega$ satisfies

$$||\alpha(R)|| \gtrsim \exp \varepsilon R$$

for a fixed $\varepsilon > 0$. Notice that this form is also defined if the action of $\mathbf{R}$ in $\mathbf{R}^2$ is unipotent and X equals the Heisenberg group. In this case our forms α are (uniformly) bounded from below by

$$\alpha(R) \gtrsim R^2 .$$

5.B$'_5$. Cofilling for invariant forms. If X is a Lie group or, more generally, X comes along with a cocompact action of a discrete group Γ, then we have a distinguished class of bounded forms ω on X, namely, the class of Γ-invariant forms. For example, the 2-form ω in the above (c) is invariant (under the solvable group X itself and, hence, under every discrete subgroup Γ). As exact Γ-invariant forms ω on X represent the cohomology classes in Γ, one may formulate the following

(Co)homological cofilling problem. Given a cohomology class $h \in H^2(\Gamma)$. Find a "smallest possible" 1-form α on X whose differential $\omega = d\alpha$ is a Γ-invariant form representing h.

Example. Let X be a Hermitian symmetric space with $K(X) \le 0$ and ω the implied Kähler form. One knows, that $\omega = d\alpha$ for some *bounded* 1-form. This makes X (and groups Γ acting on X) rather hyperbolic in-so-far as the complex geometry is concerned (see $[\mathrm{Gro}]_{20}$).

We want to know, in general, upper and lower bounds on the above α (e.g. bounds on $||\alpha||(R)$) in terms of Γ and h. Here is a simple example generalizing the above (c).

Example. Let $\Gamma_0 \subset \Gamma$ be a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$, such that a given cohomology class h in $H^2(\Gamma; \mathbb{R})$ restricts to *non-zero* on Γ_0 (where $H^2(\Gamma_0; \mathbb{R}) = \mathbb{R}$). If Γ_0 has strictly exponential distortion in Γ (i.e. $\mathrm{dist}_{\Gamma_0} \le c \log_\Gamma |\Gamma_0)$ then every 1-form α (on a smooth simply connected thickening X of Γ) whose differential $d\alpha$ defines h, must grow at least exponentially (on X). To see this we look at $\mathbb{R}^2$ in X thickening $\Gamma_0 \subset \Gamma \subset X$ and take R-disks in $\mathbb{R}^2$ in the Euclidean metric. These disks $D(R)$ have size $\sim \log R$ in X but the integral of the 2-form ω on X representing h over $D(R)$ is about R^2. Hence the integral of α (having $d\alpha = \omega$) over the boundary $\partial D(R)$ of length $\sim R$ must also be $\sim \exp R$, which makes the supremum of α (in the ball of radius $\sim \log R$) at least R. Q.E.D.

5.B$''_5$. Cohomology with a control on the growth. Given a positive *weight function* β on X one may define the L_∞-norm on forms with respect to β by $||\alpha|| = \sup_{x \in X} ||\alpha_x|| \beta^{-1}(x)$ and then study the cohomology of the de Rham complex of L_∞-forms. One can do the same thing for the L_p-norms, where the most interesting case is that of $p = 2$ and $\beta \equiv 1$. The resulting L_2-cohomology theory is quite fruitful already for hyperbolic groups where little of interest can be said for L_∞. (compare §8.).

5.C. Filling diameter, etc. We want to say just a few words about the rest of the filling invariants introduced at the beginning of §5.

Filling area of a given genus, Fill_g Area. This invariant becomes more interesting when it is generalized in order to cover the filling problem for *non-contractible* curves in V for $\pi_1(V) = \Gamma$ (see §6.).

Filling diameter Fill_0 Diam. This was introduced and extensively studied by S. Gersten (see $[\mathrm{Ger}]_6$), who, in particular, related Fill_0 Diam to Fill_0 Area. We have already mentioned the bound

$$\mathrm{Fill}_0 \,\mathrm{Area} \lesssim \exp\exp(C\, \mathrm{Fill}_0 \,\mathrm{Diam}) \; .$$

One can also bound from another side,

$$\mathrm{Fill}_0 \,\mathrm{Diam} \lesssim \mathrm{Fill}_0 \,\mathrm{Area}$$

(see $[\mathrm{Ger}]_6$) and this bound sometimes can be improved. For example, let Fill_0 Area grow polynomially, i.e. $F_0A(\ell) \lesssim \ell^\alpha$ for some $\alpha \ge 1$. Then

$$F_0 D(\ell) \lesssim \ell^{\alpha-1}, \text{ provided } \alpha \ge 2 \; .$$

This follows from the corresponding bound for Fill_0 Rad stated below.

Filling radii Fill Rad *and* Fill_0 Rad. The filling radius Fill Rad was defined homologically with an eye on higher dimensional generalizations (see $[\mathrm{Gro}]_{10}$). But we also may define Fill_0 Rad geometrically using disks in X filling in a given closed curve. Then the monotonicity argument (see 2.4. in $[\mathrm{Gro}]_{10}$) applies to disks and shows, in particular, that

$$F_0A(\ell) \lesssim \ell \Rightarrow F_0R(\ell) \lesssim \log R \text{ and } F_0A(\ell) \lesssim \ell^\alpha \Rightarrow F_0R(\ell) \leq \ell^{\alpha-1}, \alpha > 1 .$$

(Here $\mathrm{Fill}_0 \, \mathrm{Rad}$ is defined as the supremum of the radii of the disks D in X with $\partial D = S$, where $\mathrm{Rad}\ D \underset{\mathrm{def}}{=} \sup_{x \in D} \ \mathrm{dist}_D(x, \partial D)$. Thus

$$\mathrm{Fill}_0 \ \mathrm{Diam}\ S \leq \max(\mathrm{length}\, S,\ \mathrm{Fill}_0\, \mathrm{Rad}\ \ S) ,$$

which implies the bound on $\mathrm{Fill}_0 \, \mathrm{Diam}$ stated above).

Filling length Fill_0 Leng. This can be used (like Fill_0 Diam) to bound Fill_0 Area as follows. As we contract a curve S in X we may assume that the base point of S remains fixed because we have Γ acting on X in a cocompact (or better to say cobounded) fashion. The number of different curves represented by words of length L is about $\exp C\,L$; therefore, the contraction of S to a point can be achieved in at most $\exp C\,L$ steps if we have an a priori bound on the length of the curves by L. Thus

$$\mathrm{Fill}_0\, \mathrm{Area} \lesssim \exp\ C'\ \mathrm{Fill}_0\ \mathrm{Leng}\ .$$

Question. Can one bound Fill_0 Leng by const Fill_0 Diam for all groups Γ ? The solution would follow if one could solve the following purely geometric problem. Let D be a disk with some Riemannian metric. Can one contract the boundary $S = \partial D$ in D by a homotopy of curves of the length L bounded by

$$L \leq \mathrm{const}\max(\mathrm{length}\, S,\ DiamD) ,$$

for some universal "const", e.g. const $= 10^{10}$?

Modified filling length F_+L for nilpotent and solvable groups. Define the function $F_+L(\ell)$ for a space X as the minimal number L, such that every two paths p_0 and p in X of length $\leq \ell$ with the same ends x and y in X can be joined by a homotopy of paths p_t, $t \in [0,1]$ with the fixed ends x and y, such that

$$\mathrm{length}\, p_t \leq \ell \quad \text{for every } t \in [0,1] .$$

Clearly, $F_+L(\ell) \geq F_0L(\ell)$ and also $F_+L(\ell) \geq F_0D(\ell)$.

Claim. *If X is a simply connected nilpotent Lie group then $F_+L(\ell) \sim \ell$.*

Idea of the proof. We argue as in 5.A_5. and reduce the problem to the following elementary property of Carnot-Caratheodory spaces similar to (and simplier than) what we have met in 5.A.$_3$.

Let T be a smooth subbundle of the tangent bundle $T(\mathbb{R}^n)$ such that consecutive Lie brackets of the vector fields tangent to T span $T(\mathbb{R}^n)$. Consider to smooth paths p_0 and p_1 with common ends in $\mathbb{R}^n$ tangent to T and contained in the unit ball $B_0(1) \subset \mathbb{R}^n$. *Then there exists a homotopy of paths p_t in $B_0(1)$, such that*

(1) *p_t is tangent to T for every $t \in [0,1]$,*

(2) $\mathrm{length}\, p_t \leq \mathrm{const}\max(\mathrm{length}\, p_0, \mathrm{length}\, p_1)$,

where const *depends on T but not on p_0 and p_1.*

Remark. Our claim is, formally speaking, stronger than Gersten's linear isodiametric bound for nilpotent groups, but it seems Gersten's argument (see $[\mathrm{Ger}]_{4,6}$) bounds F_+L as well.

Now let X be our solvable group $\mathrm{Sol}_{n+1} = \mathbb{R}^n \rtimes \mathbb{R}$ defined with some diagonal linear action of $\mathbb{R} = \mathbb{R}_+^\times$ on $\mathbb{R}^n$. Here again we claim that $F_+L(\ell) \sim \ell$ which slightly improves upon Gersten's linear bound on $F_0D(\ell)$. To see the idea of the proof we assume $n = 2$ and the action has $\lambda_1 > 0$ and $\lambda_2 < 0$. Then $X = \mathrm{Sol}_3$ is sliced in two ways into hyperbolic planes that are $\mathrm{Sol}_2(\lambda_1)$ and $\mathrm{Sol}_2(\lambda_2)$ and every path can be approximated by a curve contained in a finite number k of slices (and built of s.h. segments as in (f) in 2.B.). Short homotopies between such curves for a fixed k_0 can be directly constructed in the class of such curves with a slightly larger k, (say $k_0' = 10k_0$) and then the general case follows by the iteration argument used in 5.A_3''. and in 5.A_5.

Conjecture. Every simply connected homogeneous Riemannian manifold X has $F_+L(\ell) \sim \ell$. (One can drop "simply connected" when one works on the large scale).

It is also tempting to conjecture that $F_+L(\ell) \sim \ell$ for the lattices Γ in all Lie groups. In fact, this can be easily proven for the lattices Γ with $\mathrm{rank}_{\mathbf{Q}}\,\Gamma = 1$ in semi-simple groups whenever the (concave !) boundary horospheres have F_+L linear (as it happens, for example, for lattices in $(SL_2\mathbf{R})^n$, such as Hilbert modular groups.).

Morse landscape of Γ. Let, as usual, X be a Riemannian manifold with a discrete cocompact action of Γ and, to simplify the picture, we assume X is contractible. We denote by $\mathcal{S}$ the space of closed curves in X (one should specify if the base point assumed fixed or not) and let $L(S)$, $S \in \mathcal{S}$ denote the length of S. Thus we have a (contractible) topological space with a function $L : \mathcal{S} \to \mathbf{R}_+$. The ordinary Morse theory tells you nothing about the critical points of L because $\mathcal{S}$ is contractible. Yet, if we assume that the filling length function $F_0L(\ell)$ grows faster than linearly, then the function L on $\mathcal{S}$ necessarily has infinitely many local minima. In fact L has arbitrarily deep basins consisting of curves which cannot be contracted without stretching them very much.

Besides local minima L may have highly stable saddle points which have the following topological origin. Denote by $\mathcal{S}_\ell \subset \mathcal{S}$ the level $L^{-1}([0,\ell])$ of L and let $R_k'(\ell)$ be the infimal value $\ell' \geq \ell$ such that the homological inclusion homomorphism $H_k(\mathcal{S}_\ell) \to H_k(\mathcal{S}_{\ell'})$ is trivial. Whenever the function $R_k'(\ell)$ grows faster than linearly we have stable "k-dimensional" mountain paths in the Morse landscape of $L : \mathcal{S} \to \mathbf{R}_+$ which bring to life (highly stable) critical points of L (represented by closed geodesics in X).

Question. Can one estimate the number of basins of given depth and/or evaluate the function $R_0'(\ell)$ for concrete groups Γ ? Of course, the same question arises for $k \geq 1$ but this appears more difficult.

One can extend the above question in (at least) two directions. First, one may take $V = X/\Gamma$, (assuming the action of Γ on X is free) and look at the Morse landscape of the length function on closed (now not only contractible) curves in V. Then one may study the similar problem for maps of higher dimensional manifolds (rather than of S^1) into V (in particular, into $\widetilde{V} = X$). If the underlying group Γ is, logically speaking, sufficiently complicated, then one expects a mountainous Morse landscape enforcing many solutions of variational problems of all kinds (compare $[\mathrm{Gro}]_5$). Yet, no specific result (except for local minima of $L(S)$) is known in this regard.

We do not expect much of the Morse landscape on closed curves in a *Lie* group X. For example, the function $R_k'(\ell)$ is, probably, bounded by $c_k\ell$ for some $c_k > 0$. (This is easy to prove for nilpotent Lie groups and also does not look hard for some solvable groups like $\mathbf{R}^n \rtimes \mathbf{R}^\times$). Yet, there may be something interesting in the asymptotics of c_k for $k \to \infty$ and also the Morse landscape may have non-stable critical points of geometric value.

Filling Span Fill_0 Span. This invariant majorizes the rest of them and so the upper bound for it is most desirable. It is well-known that $F_0S(\ell) \sim \ell$ for complete simply connected manifolds X with convex boundary and $K(X) \leq 0$. In fact, this is true whenever one has a combing in the sense of [E-C-H-P-T], e.g. for automatic groups. One can also easily see that $F_0S(\ell)$ is at most polynomial for nilpotent groups X. Moreover, if X is homogeneous of nilpotency degree d then $F_0S(\ell) \lesssim \ell^d$. Furthermore, our estimation for Fill Area based on Lipschitz controlled retractions obviously extend to Fill Span (and, in fact, to all conceivable filling invariants), which gives us, for example, the exponential bound for Lie groups and semi-simple lattices.

5.D. Filling in for submanifolds and cycles of dimension $k \geq 2$. Let us indicate the essential new features of the filling problem for $k > 1$.

(1) *Topology.* - There are many different topological types of connected k-dimensional manifolds for $k \geq 2$ and solutions to filling problems may be sensitive to that.

(2) *Geometry.* - The intrinsic geometry of a closed curve is characterized by the length but higher dimensional manifolds have infinitely many invariants.

(3) *Connectivity.* - Every finitely presented group Γ is 1-connected on the large scale which makes the 1-dimensional filling problem non-ambigious. For $k \geq 2$ one should either assume the large-scale k-connectivity of Γ or take extra precautions in the non-k-connected case.

(4) *Non-locality.* - A circle of length ℓ in X obviously has diameter $\leq \frac{1}{2}\ell$ and so one can often localize the filling process to a ball in X of some radius R depending only on ℓ. If $k \geq 2$, then the k-dimensional volume of a submanifold $\Sigma \subset X$, $\dim \Sigma = k$, by no means bounds the diameter of Σ. However, this difficulty can be often resolved by the filling techniques of $[\mathrm{Gro}]_{10}$. The idea is that Σ can be (kind of) decomposed into two pieces Σ_{thick} and Σ_{thin} where the thick part has the diameter under control and the thin part lies close to something lower dimensional which is taken care of by an induction on dimension.

Example. It is shown in [E-C-H-P-T] that every k-dimensional cycle Σ in (a thickening of) an automatic group bounds a chain C, satisfying

$$\mathrm{Vol}_{k+1} C \leq \mathrm{const}\,(\mathrm{Vol}_k \Sigma)\,(\mathrm{Diam}\ \Sigma)\ .$$

It follows from $[\mathrm{Gro}]_{10}$ that one can exclude the diameter and find a chain C, such that

$$\mathrm{Vol}_{k+1} C \leq \mathrm{const}\,(\mathrm{Vol}_k \Sigma)^{\frac{k+1}{k}}\ .$$

(According to $[\mathrm{Ger}]_2$ this was conjectured by Thursten).

(5) *Miscellanea.* - (a) It is sometimes difficult to pinpoint what goes wrong with a (filling) argument when the dimension increases. Look, for example, at our filling of circles in Carnot-Caratheodory spaces in 5.A_3. The technically difficult analytic part of the argument appealing to infinitesimal invertibility works equally well in all dimensions. But the simple looking iteration process (with hexagons) breaks down for $k \geq 2$, because there is no nice canonical subdivision of a k-dimensional manifold into smaller pieces (compare Fig. 12).

(b) One must be eventually able to determine the isoperimetric behavior of all connected Lie groups. To obtain some insight, let S be the solvable group considered in (f) of §2.B., i.e. the semi-direct product $\mathbb{R}^n \rtimes \mathbb{R}^{n-1}$ for $\mathbb{R}^{n-1} = (\mathbb{R}_+^\times)^{n-1}$ diagonally acting on $\mathbb{R}^n$ with determinant one. Here one expects the filling volume function (characterizing minimal $(k+1)$-chains filling k-cycles) to be Euclidean, for small k, namely $FV_{k+1}(\ell) \sim \ell^{\frac{k+1}{k}}$ for $k = 1, \ldots, n-2$ (We know this is true for $k = 1$, see 5.A_8.). Then, if $k = n-1$, the function $FV(\ell)$ is exponential (see final remarks in (f) in 2.B.). Finally, when $k \geq n$, one suspects $FV_{k+1}(\ell)$ grows almost linearly, i.e. $\lesssim \ell^\alpha$ for every $\alpha > 1$. (If $k+1 = 2n-1 = \dim S$, then

$FV_{k+1}(\ell)$ grows faster than linearly, because S is a unimodular amenable group. But for $n \leq k \leq 2n-3$ the filling function $FV_{k+1}(\ell)$ may grow linearly, for all we know.)

(b′) Let us justify our optimism concerning the filling volume in the dimensions above $\mathbb{R}$-rank by the following two remarks.

(b′$_1$) Let X be a symmetric space with non-positive curvature. If $k+1 \geq \operatorname{rank} X$, then $FV_{k+1}(\ell) \sim \ell$. In other words, *every k-cycle Σ in X bounds some chain C with*

$$\mathrm{Vol}_{k+1} C \leq \mathrm{const}\ \mathrm{Vol}\,\Sigma\ . \tag{$*$}$$

The construction of C is well known: choose a maximal flat $F \subset X$ (which is isometric to $\mathbb{R}^r$ for $r = \operatorname{rank} X$), normally project Σ to F and take the cylinder of this map for C (We may throw away the part of F which is not the image of the projection). The bound $(*)$ follows from the exponential decay of the volume contraction of the normal projection $p : X \to F$. This means that the norm of the differential of p acting on the ℓ-th exterior power of $T(X)$ satisfies

$$\|D_p \mid \wedge^\ell T_x\| \leq \exp -C\,\mathrm{dist}(x, F)$$

for all $x \in X$ and some positive constant $C > 0$. (Notice that the isoperimetry in our X below the rank is *Euclidean*, i.e. $FV_{k+1}(\ell) \lesssim \ell^{\frac{k+1}{k}}$) for all $k \geq 1$.)

(b_2'). Let X be a solvable group which is a semi-direct product, of the form $X = \mathbf{R}^n \rtimes \mathbf{R}^r$ for some linear action of $\mathbf{R}^r = (\mathbf{R}_+^\times)^r$ on $\mathbf{R}^n$.

Claim. If $k \geq r+1$ then *every k-cycle Σ bounds a chain C satisfying*

$$\mathrm{Vol}_{k+1} C \leq \mathrm{const}\Big((\mathrm{Vol}_k \Sigma)^\alpha + 1\Big)$$

for $\alpha = (k+1)/r - k$.

Proof. Define the *dilation* λx for $x = (y, z) \in X$, $y \in \mathbf{R}^n$ and $z \in \mathbf{R}^r$, by $\lambda x = (\lambda y, z)$ for all $\lambda > 0$. Using this dilation we want to reduce the (interesting) case $\mathrm{Vol}_k \Sigma \geq 1$ to $\mathrm{Vol}_{k+1} \Sigma \leq 1$. In the latter case Σ bounds C with $\mathrm{Vol}_{k+1} C \leq \mathrm{const}\,(\mathrm{Vol}_k \Sigma)^{\frac{k+1}{k}}$, since X has locally bounded geometry being homogeneous. Now if $\mathrm{Vol}\,\Sigma \geq 1$ we use our dilation with $\lambda = (\mathrm{Vol}\,\Sigma)^{-(r-k)}$. Then, obviously, the dilated cycle $\Sigma' = \lambda\Sigma$ has $\mathrm{Vol}_k \Sigma' \leq 1$. This bounds C' with $\mathrm{Vol}_{k+1} C' \leq \mathrm{const}$ which we dilate back to $C = \lambda^{-1} C'$. This increases the volume by at most the factor $\lambda^{-(k+1)}$ (recall, $\lambda < 1$). Thus

$$\mathrm{Vol}_{k+1} C \leq \mathrm{const}\,\lambda^{-(k+1)} = \mathrm{const}\,(\mathrm{Vol}_k \Sigma)^{\frac{k+1}{r-k}} .$$

Q.E.D.

Generalization. *Let X be a simply connected solvable Lie group and $N \subset X$ a nilpotent normal subgroup such that $X/N = \mathbf{R}^r$. Then the $(k+1)$-dimensional filling is polynomial for $k \geq r$*, i.e. $F\,\mathrm{Vol}_{k+1}(\ell) \lesssim \ell^\alpha$ *for some α depending on X* (and *bounded by* $\alpha_0 = \alpha_0(\dim X)$).

Idea of the proof. Use a fake dilation in N as in 5.A_5.

(c) The above (mostly conjectural) picture for solvable groups probably extends to non-cocompact lattices Γ in semi-simple Lie groups. For example, if $\mathrm{rank}_{\mathbf{Q}}\,\Gamma = 1$, then the isoperimetry of Γ is essentially the same as that of the boundary horospheres H (which are solvable groups). In fact, to find a good filling for a k-cycle Σ lying in the core X_0 of the symmetric space X on which Γ acts (recall that X_0/Γ is compact and the boundary of $X_0 \subset X$ is concave for $\mathrm{rank}_{\mathbf{Q}}\,\Gamma = 1$, see 3.H.) we first fill in Σ by the minimal chain C in X and then fill in the intersection $C \cap \partial X_0$ in ∂X_0, where a bound on $\mathrm{Vol}_k\,(C \cap \partial X_0)$ is achieved by normally moving ∂X_0 by distance $d \leq 1$. This argument shows that the filling volume functions for X and H are related by

$$F_X V_{k+1}(\ell) \lesssim F_H V_{k+1}(\ell) \quad \text{for } k \geq \mathrm{rank}\, X ,$$

and

$$F_X V_{k+1}(\ell) \lesssim F_H V_{k+1}(\ell^\alpha) \quad \text{for } \alpha = (k+1)/k \text{ and } k \leq \mathrm{rank}\, X .$$

(The presence of α is due to the fact that $\mathrm{Vol}_{k+1} C$ is bounded by $(\mathrm{Vol}_k \Sigma)^\alpha$ and an intersection of C with moving ∂X_0 can be bounded by $\mathrm{Vol}\,C$ and no better. But I am pretty certain one can somehow manage without any α).

If $\mathrm{rank}_{\mathbf{Q}}\,\Gamma \geq 2$ one expects similar results but the proofs appear more difficult. For example, if $\Gamma = SL_n\mathbf{Z}$ one's bet is $FV_{k+1}(\ell) \sim \ell^{\frac{k+1}{k}}$ for $k = 1, \ldots, n-3$, which generalizes Thurston's unpublished result for $k = 2$ and $n \geq 4$. Then one knows $FV_{k+1}(\ell)$ is exponential for $k = n-2$ (see [E-C-H-P-T]) and for $k \geq n-1$ the expected behaviour is $FV_{k+1}(\ell) \sim \ell$ (or at least $\lesssim \ell^\alpha$).

(d) *Remarks on* Fill Rad. The notion of the filling radius given at the beginning of §5. naturally extends to higher dimensions and it goes along with the filling volume as we have already mentioned (also see $[\mathrm{Gro}]_{10}$). Recall that for $k = 1$ (i.e. for the filling of curves) there is a modified filling radius $\mathrm{Fill}_0\,\mathrm{Rad}$ reflecting the geometry of *disks* filling in a given curve rather than surfaces of unspecified genus and the corresponding function F_0R is equivalent to F_0D measuring the filling diameter. Now, when we pass to $k \geq 2$, one may meaningfully generalize $\mathrm{Fill}_0 \mathrm{Rad}$ but this seems impossible for Fill Diam, because the diameter of a k-dimensional cycle is not bounded by its volume for $k \geq 2$.

Conjecture. *Every connected Lie group X has $FR(\ell) \sim \ell$ in every dimension k.* (Here the filling radius is defined on the large scale, i.e. in some thickening of X, as usual). Moreover, this may even be expected for a suitable $F_0R(\ell)$ as well as for a higher dimensional (and/or multiparametric version) of $F_+L(\ell)$ defined earlier. Then, with certain reserve one extends this conjecture to the lattices in Lie groups.

(e) *Filling and combing.* Combing, or Lipschitz contraction of X, fills in every Σ in X by a cone C over Σ which is the "orbit" of Σ under the contraction. If the combing is also Lipschitz in the time direction (as it happens, for example, for automatic groups, see [E-C-H-P-T]), then, by choosing the base of the cone in Σ, one can have this cone no bigger than $\Sigma \times [0, D]$ for $D = \mathrm{Diam}\,\Sigma$, as is pointed out in [E-C-H-P-T]. We have already mentioned that this cone property implies via $[\mathrm{Gro}]_{10}$ the Euclidean isoperimetric inequality in X, i.e.

$$FV_{k+1}(\ell) \sim \ell^{\frac{k+1}{k}} \quad \text{for all } k \geq 1 \; ,$$

and consequently, the linear bound on FR. Furthermore, Gersten observed (see $[\mathrm{Ger}]_{4,6}$) that the "Lipschitz in t" condition is unneeded for the linear bound on the filling diameter for *curves* (which is the same thing as filling radius in this case). Thus the Gersten combing insures $F_0R(\ell) \sim \ell$ for $k = 1$. One may hope for similar results for filling invariants of k-cycles for $k \geq 2$ and then try to construct combings for specific spaces.

Conjecture. Every connected Lie group X is Gersten combable.

Example. Let X admit a split extension $1 \to A \to X \to X_0 \to 1$ where $A = \mathbb{R}^m$ and X_0 is combable (e.g. Abelian). Then X is combable.

Proof. Fix a section $X_0 \to X$ and Lipschitz contract X to X_0 by $ax_0 \mapsto (ta)x_0$ for the usual similarity transformations $a \mapsto ta,\ t \in [0,1]$, in $A = \mathbb{R}^n$. Then compose this contraction with the combing in X_0.

(f) *Filling volume in nilpotent groups.* Let us try to guess the behavior of the filling volume function $FV_{k+1}(\ell)$ for the Heisenberg group H^{2n+1}. We grade the Lie algebra of H^{2n+1} by $L\left(H^{2n+1}\right) = \mathrm{Cntr} \oplus T_0$ and we denote by e_λ the dilation of H^{2n+1} which acts on $L\left(H^{2n+1}\right)$ by $(c,t) \mapsto (\lambda^2 c, \lambda t)$. Observe that e_λ preserves the (codimension one) subbundle $T \subset T\left(H^{2n+1}\right)$ corresponding to T_0 which defines a Carnot-Caratheodory geometry on H^{2n+1}.

There are three different cases for the filling problems of k-dimensional cycles $\Sigma \subset H^{2n+1}$ by k-chains depending on whether $k < n,\ k = n$ or $k > n$ about which we make the following

Conjecture:

(i) If $k < n$ then the filling volume has Euclidean behavior,

$$FV_{k+1}(\ell) \sim \ell^{\frac{k+1}{k}} \; .$$

(ii) If $k = n$ then the filling is harder than in the Euclidean space i.e. the filling volume function grows faster,

$$FV_{k+1}(\ell) \sim \ell^{\frac{k+2}{k}} \; ,$$

(iii) If $k > n$ then the filling is better than Euclidean

$$FV_{k+1}(\ell) \sim \ell^{\frac{k+2}{k+1}} \; ,$$

Explanation. Let $\Sigma_\lambda = e_\lambda(\Sigma_0)$ for a fixed Σ_0 and $\lambda \to \infty$. If Σ_0 is tangent to T then $\mathrm{Vol}_k \Sigma_\lambda \sim \lambda^k$; otherwise $\mathrm{Vol}_k \Sigma_\lambda \sim \lambda^{k+1}$. If $k < n$, and Σ_0 is tangent to T, we can fill in Σ_0 by a C_0 tangent to T and then Σ_λ is filled in by C_λ with $\mathrm{Vol}_{k+1} C_\lambda \sim \lambda^{k+1}$ which agrees with (i).

If $k = n$, then we still may have Σ_0 tangent to T but no $(k+1)$-dimensional chain C_0 can be tangent to T. Thus $\mathrm{Vol}\,\Sigma_\lambda \sim \lambda^k$ and $\mathrm{Vol}\,C_\lambda \sim \lambda^{k+2}$ which agrees with (ii).

Finally, if $k > n$, none of the two, neither Σ_0 nor C_0, can be tangent to T and so

$$\mathrm{Vol}\,\Sigma_\lambda \sim \lambda^{k+1}, \ \mathrm{Vol}\,C_\lambda \sim \lambda^{k+2} ,$$

which fits (iii).

Remarks. We already know the proof in the case $k = 1$ (see 5.A$'_2$.) and our argument can be probably adjusted to the general case. Also we know that the conjecture is valid for $k = 2n$ by the Pansu-Varopoulos theorem. Here we are going to indicate a proof of the lower bound for the middle case $k = n$ using the argument in 5.B$_2$. We invoke the homomorphism $h : H^{2n+1} \to \mathbb{R}^{2n} = H^{2n+1}/\mathrm{Cntr}$ and recall that $\mathbb{R}^{2n}$ carries a non-singular anti-symmetric 2-form ω whose lift $h^*(\omega)$ on H^{2n+1} integrates to a differential 1-form, α, i.e. $d\alpha = h^*(\omega)$, such that $\mathrm{Ker}\,\alpha = T$. We take a pair of transversal n-dimensional linear subspaces L_0 and L_1 in $\mathbb{R}^{2n}$ such that $\omega | L_i = 0$, $i = 0, 1$, and then we construct $\Sigma_0 \subset H^{2n+1}$ tangent to T such that the h-image of Σ_0 is contained in L_0 and such that $h^{-1}(L_1) \subset H^{2n+1}$ meets Σ_0 transversally at exactly two points. (The construction of Σ_0, which is a *Legendre* variety for the *contact* form α, is an elementary exercise, compare [Gro]$_{12}$). Now we expand by $\lambda \to \infty$, i.e. take $\Sigma_\lambda = e_\lambda(\Sigma_0)$, and bound the filling volume from below by $\sim \lambda^{k+2}$ by looking at the intersection of Σ_λ with translates of $\widetilde{L}_1 = h^{-1}(L_1)$. Then we may apply 5.B$_1$. since $\widetilde{L}_1$ is an Abelian subgroup in H^{2n+1} (as $\omega | L_1 = 0$) and it is quadratically (on λ) distorted in H^{2n+1}. (We suggest the reader would fill in the details).

As a corollary of the above "hyper-Euclidean" lower bound on FV_{k+1} we obtain that H^{2n+1} *admits no combing in the sense of* [E-C-H-P-T]. (This might be known to the authors of [E-C-H-P-T]).

The above heuristics generalize to all nilpotent Lie groups X (compare 5.A$_5$. where we discuss the case $k = 1$) but the proofs are non-existent yet. For example, in order to locate the first "non-Euclidean" function $FV_{k+1}(\ell)$ we take $T \subset T(X)$ corresponding to the splitting $L = [L, L] \oplus T_0$ of the Lie algebra L of X and let k_0 be the maximal dimension of submanifolds in X tangent to T. Then one believes that $FV_{k+1}(\ell) \sim \ell^{\frac{k+1}{k}}$ for $k < k_0$ and $FV_{k+1}(\ell) \sim \ell^{\frac{k+2}{k}}$ for $k = k_0$.

Similar results are expected for other filling invariants, for example, for the filling span function where the proof must be easier. Here, "filling span" refers to the following problem. Given an ℓ-Lipschitz map f of the unit sphere $S^k = \partial B^{k+1}$ into X, find an ℓ'-Lipschitz extension of f to the ball for the smallest possible ℓ'. Then the filling span function $F_0 S_{k+1}(\ell)$ provides (the best possible) bound $\ell' \leq F_0 S_{k+1}(\ell)$.

If the above subbundle T complementary to $[L, L]$ contains "many" $(k+1)$-dimensional submanifolds in the sense that the corresponding differential operator is infinitesimally invertible (see 5.A$_3$.) then one can use the techniques of 5.A$_3$. and show that $F_0 S_{k+1}(\ell) \sim \ell$. (Notice that the infinitesimal invertibility is a verifiable, albeit complicated, linear algebraic condition on the Lie algebra L). On the other hand, one can, probably, bound FS_{k+1} from below either by intersecting $\Sigma_0 = f(S^k) \subset X$ with submanifolds of the complementary dimension (as we did to bound from below $\mathrm{Fill\,Vol}_{k+1}$) or by a *limit argument* exploiting the fact that $\mathrm{Con}_\infty X$ is a Carnot-Caratheodory space determined by T. For example, if X is the Heisenberg group H^{2n+1}, the intersection with $h^{-1}(L_1)$ *suggests* the lower quadratic bound in the middle dimension $k = n$, i.e.

$$FS_{k+1}(\ell) \gtrsim \ell^2$$

(notice that $FS_{k+1}(\ell) \lesssim \ell^2$ for all nilpotent groups of nilpotency degree two as is seen with the combing given by the dilation), while the lower bound which (obviously) follows from $FV_{k+1}(\ell) \sim \ell^{\frac{k+2}{k}}$ is significantly weaker,

$$FS_{k+1}(\ell) \gtrsim \ell^{\frac{k+2}{k+1}} .$$

(I do not know where in the interval $\left[\frac{k+2}{k+1}, 2\right]$ the truth is located).

Now, let us explain the limit argument, assuming, to make the life easier, that we have a dilation on X. If we start with a Lipschitz image $\Sigma_0 = f(S^k) \subset X$ which is almost everywhere tangent to T and assume that the scaled map $f_\lambda = e_\lambda \circ f : S^k \to X$ extends to a λ'-Lipschitz map $g' : B^{k+1} \to X$ for $\lambda' \sim \lambda$, then the rescaled back maps $g'_\lambda = e_\lambda^{-1} \circ g$ subconverge, for $\lambda \to \infty$, to a Lipschitz extension of $f : S^k \to X$ to a map $g : B^{k+1} \to X$ which is, moreover, *Lipschitz for the Carnot-Caratheodory metric associated to* T. One knows (the proof is elementary) that such a g is almost everywhere tangent to T and one believes this is as restrictive a property as having a *smooth* $(k+1)$-manifold tangent to T. Thus one could arrive at an effective condition which would guarantee the (lower bound) relation

$$\ell^{-1} FS_{k+1}(\ell) \underset{\ell \to \infty}{\longrightarrow} \infty \ .$$

Example. *The Heisenberg group.* If a Lipschitz map $g : B^{k+1} \to H^{2n+1}$ is a.e. tangent to T, then the (contact) 1-form α defining T by $\operatorname{Ker} \alpha = T$ obviously satisfies $g^*(\alpha) = 0$ a.e. Since the pull-back operator g^* on exterior forms commutes with the exterior differential (see *Explanation* below) the differential $d\alpha$ is also pull-backed to a.e. zero form, and since the maximal isotropic subspaces of $d\alpha$ are n-dimensional, the map g has rank $\leq n$ a.e. Thus, for $X = H^{2n+1}$, we have arrived at yet another proof of the relation $\ell^{-1} FS_{n+1}(\ell) \to \infty$, for $\ell \to \infty$.

Remark. The above argument extends with a little effort to $FV_{k+1}(\ell)$, where the limit object is a *rectifiable current* which is a.e. tangent to T.

Explanation. The commuting of d with g^* is standard for C^2-maps. Then, using smoothing operators, one can approximate a Lipschitz map g by C^∞-maps g_i, such that g_i uniformly converge to g for $i \to \infty$ and $Dg_i \to Dg$ almost everywhere. Then the induced forms $g_i^*(d\alpha)$ weakly converge (as currents) to $g^*(d\alpha)$ which implies the required commuting property.

Generalization. For every subbundle $T \subset T(X)$ one defines the curvature form $\Omega : \wedge^2 T \to T(X)/T$ by $\Omega(\tau_1, \tau_2) = [\tau_1, \tau_2]/T$. One can easily show, that if $g : B \to X$ is a Lipschitz map tangent to T then $g^*(\Omega) = 0$ and then bound rank of g by the maximum of the dimensions of Ω-isotropic subspaces. We invite the reader to work out the case of the quaternionic Heisenberg group H^{4n+3} and obtain lower bounds for FS and FV.

(6) *Insufficiency of examples.* To the best of my knowledge, there are no examples of groups Γ which are k-connected on the large scale and which have "very large" filling invariants for $k \geq 2$, but I am pretty certain such examples must be plentiful. For instance, certain Γ should harbour k-dimensional cycles (lying in some thickening of Γ) whose filling volume admits no recursive bound in terms of the volume (compare $\overline{R}_k(r)$ for $k \geq 2$ in 4.D.).

5.E. Top-dimensional isoperimetric problems. There are two such problems. The first is a specialization of the above to $(k+1)$-dimensional k-connected polyhedra X where we look for the best fillings of k-dimensional cycles. The second problem is non-homological in nature. We consider bounded domains in X and want to relate their volumes to those of their topological boundaries. (Notice that the two problems are equivalent for $(k+1)$-dimensional *manifolds* X). Here we want to say a couple of words about the second problem. This can be formulated in purely group theoretic terms using the notion of the boundary $\partial_d F$ for subsets $F \subset \Gamma$ (see 0.5.A.), for some fixed d, say for $d = 1$. Namely, one considers in Γ the subsets F containing exactly N elements and defines the function $\operatorname{Is}(N)$, called the *isoperimetric profile* of Γ, as the infimum of $\operatorname{card} \partial_d F$ over all these F. Recall that the linear asymptotics $\operatorname{Is}(N) \sim N$ characterizes non-amenable groups and the degree of amenability can be measured by the decay of the ratio $\operatorname{Is}(N)/N$ for $N \to \infty$.

Examples. (a) If $\Gamma = \mathbb{Z}^n$ then

$$\operatorname{Is}(N) \sim N^{\frac{n-1}{n}} ,$$

as everybody knows.

(b) If Γ is a nilpotent group then $\operatorname{Is}(N) \sim N^{\frac{m-1}{m}}$, where m is the Hausdorff dimension of the asymptotic cone $\operatorname{Con}_\infty \Gamma$. This result was first proven geometrically by Pansu in his thesis for the 3-dimensional Heisenberg group (see [Pan]$_1$). Then Varopoulos obtained the general result by the techniques of random walks in Γ (see [Var], [V-S-C] and references therein). One also can prove this geometrically by extending the techniques from $[\mathrm{Gro}]_7$ (in the abstract dressing of $[\mathrm{Gro}]_{17}$) to Carnot-Caratheodory spaces.

(c) If Γ has exponential growth, then

$$\mathrm{Is}(N) \gtrsim N(\log N)^{-2} .$$

(See [Var], where the reader will find much more about $\mathrm{Is}(N)$).

5.F. Filling on the scale $\sim \ell$ for ℓ =length S. One knows that the linear isoperimetric inequality in X, i.e. the property $F_0A(\ell) \sim \ell$, is equivalent to a certain metric inequality (expressing δ-hyperbolicity, see $[\mathrm{Gro}]_{14}$) concerning the quadruples of points in X. Now we want to relate the polynomial isoperimetry $F_0A(\ell) \sim \ell^\alpha$ (and especially the quadratic case $F_0A(\ell) \sim \ell^2$) to metric inequalities concerning *finite* configuration of points in X.

Denote by $\overline{S}_k$ the standard regular k-gone in the plane and define *a k-gone S in X* as a map of the set of the vertices of $\overline{S}_k$ into X. (If X is a geodesic space, we can extend the map to the boundary curve $\partial\overline{S}_k$ by joining the adjacent vertices in X by minimal geodesic segments and thus obtain an actual closed curve in X). We call the *edges* or *sides* of S the pairs of points in X corresponding to the pairs of vertices in $\overline{S}_k$ joined by edges. The *length of an edge* is the distance between the corresponding points. The *length* of S is the sum of the lengths of the edges.

A partition $\sqcap$ of S is a combinatorial partition of $\overline{S}_k$ into k_μ-gones, $\mu = 1, \dots, \nu$, (where each of them should be homeomorphic to the disk as in Fig. 15 below, where the hexagon is partitioned into 5 pieces), and a map of the set of the vertices of this partition into X.

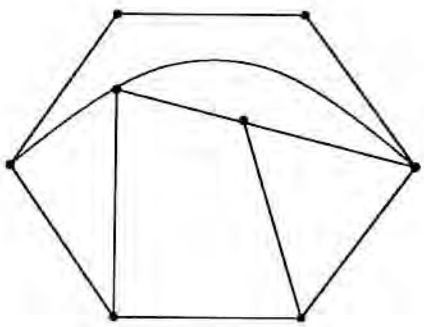

Figure 15

We denote by $\ell_1, \dots, \ell_\nu$ the lengths of the resulting μ-gons in X, $\mu = 1, \dots, \nu$ and set

$$\mathrm{Mesh}\sqcap = \max_{\mu=1,\dots,\nu} \ell_\mu .$$

Now we can state the following

Combinatorial filling problem. Find a partition $\sqcap$ of S into a given number ν of "pieces" with the minimal possible Mesh $\sqcap$.

We denote this minimal (or rather infimal, as usual) Mesh by Fill_ν Mesh(S).

Remark. There are other filling measures of S via $\sqcap$ which resemble the area somewhat better than Mesh. For example, the sum of the squared lengths, $\sum_{\mu=1}^{\nu} \ell_\mu^2$ looks the best combinatorial approximation for the area in Euclidean spaces. But we have chosen Mesh which is better suited to localize the ideas.

Now with Fill Mesh at hand we follow our route and define the *filling function* $FM(\ell|k,\nu)$ as the supremum of Fill_ν Mesh S over all k-gons of length $\leq \ell$.

Remark. Our subdivisions of S into ν "pieces" may have, a priori, arbitrarily many vertices. But in fact one can always "erase" most of the vertices without increasing ℓ_μ. Thus one may assume that the competing partitions $\sqcap$ have no more than N vertices for N bounded by something like $k+\nu$. Thus the function $FM(\ell|k,\nu)$ can be computed by looking at the N-tuples of points in X.

Corollary. *The number $FM(\ell|k,\nu)$, for every fixed triple (ℓ,k,ν), is continuous under Hausdorff limits of metric spaces. In particular, the functions $FM(\ell|k,\nu)$ for the cones $\text{Con}_\omega X$ determine the asymptotics of $FM(\ell|k,\nu)$ for X as $\ell\to\infty$ as follows: if every asymptotic cone $\text{Con}_\omega X$ has FM bounded by a function $B(\ell,k,\nu)$ independent of ω, then*

$$\limsup_{\ell\to\infty} \ell^{-1} FM(\ell|k,\nu) \leq B(\ell,k,\nu)\ .$$

(We shall see in 5.F$''_1$. that just vanishing of π_1 ($\text{Con}_\omega X$) gives a non-trivial bound on $FM(\ell/k,\nu)$).

Suppressing k. Let us define $FM(\ell|\nu)$ as $\sup_k FM(\ell|k,\nu)$ over all $k=1,\ 2,\ \dots$. The new function FM without k carries essentially the same amount of information as FM with k and it looks prettier without k. For example, it satisfies the following

Composition inequality.

$$FM\,(\ell|\nu_1\nu_2) \leq FM\Big(FM(\ell|\nu_1)|\nu_2\Big)\ . \tag{$*$}$$

This is obvious from the definitions, and this inequality takes an even more transparent form if we measure the refinement effect of subdivisions into ν pieces by $\ell^{-1}FM(\ell|\nu)$ and define

$$M(\nu) = \inf_{\ell\geq 0} \ell^{-1} FM(\ell|\nu)\ .$$

Then $(*)$ obviously implies that

$$M\,(\nu_1\nu_2) \geq M\,(\nu_1)\,M\,(\nu_2)\ . \tag{$**$}$$

5.F$_1$. Fill Mesh **and** Fill_0 Area. In what follows, we assume, to avoid irrelevant complications, that our space X has bounded local geometry. The basic example is where X is a locally finite simply connected polyhedron admitting a cocompact action of a discrete group. In this case there is the following obvious relation between Fill Mesh and Fill_0 Area. Take some $\delta>0$ and let $F_0A(\ell|\delta)$ be the minimal number ν, such that $FM(\ell|k,\nu)\leq\delta$ for all k satisfying $k\leq \ell/\delta$. Then, for every fixed δ, the function $F_0A(\ell|\delta)$ is *equivalent* to the (filling area) function $F_0A(\ell)$, (i.e. the ratio of the two is bounded by a constant as $\ell\to\infty$).

The above equivalence is not interesting by itself for our purpose as it needs $\nu\to\infty$ for $\ell\to\infty$. But it becomes useful in conjugation with the above $(**)$.

5.F$'_1$. *Let $M(\nu)>1$ for some ν and set $\alpha=\log\nu/\log M$. Then the filling area function of X is polynomial, namely,*

$$F_0A(\ell) \lesssim \ell^\alpha\ . \tag{$+$}$$

Proof. Let p be the maximal integer $\leq \log_M \ell$. Then, by (**), every S in X can be subdivided into ν^p pieces of roughly unit size and so

$$\text{Fill}_0 \text{ Area } S \lesssim \nu^p ,$$

which implies (+).

5.F$_1''$. Corollary. *Let, for every ulterfilter ω, the asymptotic cone* $\text{Con}_\omega X$ *be simply connected. Then X satisfies a polynomial isoperimetric inequality (i.e. the above (+)).*

Proof. If $\pi_1(\text{Con}_\omega X) = 0$, then for every $k = 1, \ldots$, there exist $\ell_0 > 0$, $\nu = \nu(k) = 1, \ldots$, and $\lambda = \lambda(k) < 1$, such that every k-gon in X of length $\ell \geq \ell_0$ can be subdivided into ν-pieces of length $\leq \lambda\ell$. Otherwise we would have a sequence of closed curves in X of length $\ell \to \infty$ converging to a non-contractible curve in some $\text{Con}_\omega X$. (To avoid irrelevant complications we assume throughout that X is simply connected and admits a discrete cocompact group action). This makes $M(\nu) > 1$ for large ν and the above applies.

5.F$_2$. *Open problems.* In order to complete the above discussion one should find some bounds on FM in terms of F_0A. Here is the easiest question which is still unresolved. Suppose X satisfies the quadratic isoperimetric inequality, i.e. $F_0A(\ell) \lesssim \ell^2$. Is then $M(\nu) > 1$ for some ν ? (We assume as usual, X is simply connected with bounded local geometry). It is natural to conjecture that

$$F_0A(\ell) \lesssim \ell^2 \Rightarrow \liminf_{\nu \to \infty} (\log \nu / \log M) \leq 2 .$$

This means, geometrically, that a disk D with some Riemannian metric of area A (and with the length of the boundary about $\sqrt{A}$) can be subdivided into ν pieces of perimeter $\approx A/\sqrt{\nu}$. Notice, that one may additionally assume that every ball in D of radius $r \leq \ell$ has area $\gtrsim r^2$ (see [Gro]$_{10}$). This allows one to produce coverings of D by ν balls of radii $\lesssim A/\sqrt{\nu}$ but this fails short of our aim. The above discussion suggests that a converse to 5.F$_1''$. should be true. That is, if a group Γ satisfies a quadratic (or may be even higher degree polynomial) isoperimetric inequality then $\text{Con}_\infty \Gamma$ is, conjecturally, simply connected.

5.F$_3$. The fundamental group of X as seen from infinitely far and related invariants. Let us join our observer from §2. who looks at a metric space X from an observation point positioned distance d from X. Now, for each d, the observer collects a certain finite amount of information about the loops in X of length about d and their homotopies on the scale comparable to d and then as X recedes for $d \to \infty$ this information is organized (possibly with the help of an ulterfilter) into an actual group. For example, if one regards trivial the loops with quadratic filling area, then the resulting group is (in the general case, conjecturally) just $\pi_1(\text{Con}_\infty X)$. But this is not the only possibility ! One can factor away another class of loops with a given bound for $d \to \infty$ of a specific filling invariant and thus obtain a quite different "asymptotic π_1" of X. Then one may play this game with the higher dimensional homotopy and homology. Unfortunately, at the present stage of knowledge it is unclear how to select truly interesting invariants out of the innumerable possibilities and then to evaluate them for specific spaces and groups.

5.G. Volume distortion of subspaces $X_0 \subset X$. We introduced the (ordinary) distortion of X_0 in X by comparing dist_{X_0} and dist_X on the *pairs of points* in X_0. Now, instead of pairs of points consider a space $\mathcal{S}$ of closed submanifolds (or cycles) in X_0 of a certain kind. For example $\mathcal{S}$ may consist of all (images of) maps of the circle S^1 into X_0. We specify some filling invariant for our submanifolds, for example, the filling area for circles and consider two functions on $\mathcal{S}$, that are

$$F(S) = \text{Fill Inv}_X(S)$$

and

$$F_0(S) = \text{Fill Inv}_{X_0}(S)$$

for all $S \in \mathcal{S}$. Then we define the *F-distortion* $FD(\ell)$ of X_0 in X by

$$FD(\ell) = \sup_{S \in \mathcal{S}_\ell} F(S) ,$$

where $\mathcal{S}_\ell \subset \mathcal{S}$ consists of all $S \in \mathcal{S}$ with $F_0(S) \leq \ell$. Here as usual, our attention is fixed not so much on the function $FD(\ell)$ itself but rather on the asymptotics for $\ell \to \infty$, (e.g. expressed by the equivalence $a(\ell) \sim b(\ell)$ signifying $a(\ell) \leq cb(\ell)$ and $b(\ell) \leq c\,a(\ell)$ for a fixed $c > 0$).

Warning. The above definition recaptures the distortion of §3. if we use the space $\mathcal{S}$ of the pairs of points in X_0 and use for F the length of the shortest curve between the points. However our "normalization" here differs from that in §2. as the function $FD(\ell)$ represents *the inverse function* for $\Delta\text{is}(R)$ in §3.

Example. Look again at the three dimensional solvable group $X_0 = \text{Sol}_3$ with $\lambda_1 > 0$ and $\lambda_2 < 0$ embedded into $X = H^2 \times H^2$ (where H^2 is the hyperbolic plane identified with Sol_2). We know, the ordinary distortion of this X_0 is bounded (see §3.). Now, we take $\mathcal{S}$ consisting of closed curves in X_0 and let F be the filling area (i.e. the area of the minimal surfaces filling in our curves). Then *the corresponding F-distortion is exponential*

$$FD(\ell) \sim \log \ell$$

for a fixed $c > 0$, because the filling area is quadratic in our X and exponential in X_0. (We say "exponential" rather than "logarithmic" to make clear that the distortion is quite large and to be in agreement with the terminology of §3.).

Now, for the same X and X_0 take the space of all closed surfaces (2-cycles) Σ in X_0 for $\mathcal{S}$ and let F be the filling volume.

Claim. *If $\lambda_1 \neq -\lambda_2$ then the F-distortion is bounded,* i.e. $FD(\ell) \sim \ell$. *Otherwise it is unbounded but yet quite small, say*

$$FD(\ell) \gtrsim \ell^\alpha \textit{ for every } \alpha < 1 .$$

Proof. If $\lambda_1 \neq -\lambda_2$, then the group X_0 is *not* unimodular and so the boundary Σ of every domain $D \subset X_0$ has

$$\text{Area}\,\Sigma \geq \text{const Vol}\,D ,$$

which shows $FD(\ell) \sim \ell$ in this case.

Now assume $\lambda_1 = -\lambda_2$ and so X_0 is unimodular. Since X_0 is amenable, it admits (Folner) domains D (of arbitrarily large volume) for which the ratio $\text{Vol}\,D/\text{Area}\,\partial D$ is arbitrarily large. This makes the distortion unbounded (which with our perverse convention means $FD(\ell)/\ell \to 0$ for $\ell \to \infty$) since $X = H^2 \times H^2$ satisfies *linear* isoperimetry for 3-dimensional fillings. Finally, to prove the bound $FD(\ell) \gtrsim \ell^\alpha$ we use the result by Varopoulos (see 5.E.) claiming that X_0 satisfies "almost linear" isoperimetric inequality in dimension 3.

A bound on FD. Let us indicate how to bound the distortion (i.e. bound $FD(\ell)$ from below) for certain *hypersurfaces* $X_0 \subset X$. We assume both X and X_0 are smooth Riemannian manifolds where X has (uniformly) bounded local geometry and X_0 has (uniformly) bounded local extrinsic geometry in X which amounts to a bound on the second fundamental (curvature) form of $X_0 \subset X$ and the normal injectivity radius of X_0 in X. Notice, that these assumptions are satisfied if X and X_0 are Riemannian homogeneous spaces (as for example $\text{Sol}_3 \subset H^2 \times H^2$).

Claim. *If for some k the filling volume for X and the volume distortion of X_0 in X are at most polynomial then so is the filling volume in X_0.* (Thus a lower polynomial bound on the filling volume in X_0 leads to such a bound for the volume distortion, i.e. $FD(\ell) \gtrsim \ell^\alpha$ for some $\alpha > 0$).

Sketch of the proof. Take some k-cycle $\Sigma \subset X_0$ and fill it in by a minimal k-chain C in X. Now we want to find a filling chain $C_0 \subset X_0$ whose volume is polynomially controlled by $\text{Vol}_{k+1} C$. To do this we slightly normally make X_0 to a near "parallel" hypersurface X_0', such that $\text{Vol}_k(C \cap X_0') \lesssim \text{Vol}\,C$. (Compare (c) in

(5) of 5.D.). Then the intersection $\Sigma' = C \cap X_0'$ is normally projected to X_0 where it bounds some C' of the volume polynomially controlled by $\mathrm{Vol}\,\Sigma'$. What remains to do is to join Σ and Σ' by a small chain in X_0. We use for this the normal projection to X_0 of the part of C lying between Σ and Σ'.

Questions. (a) Let X_0 be a connected subgroup in a Lie group X, such that their filling invariants in certain dimension $k+1 \geq 2$ have the same asymptotic,

$$F_X\, V_{k+1}(\ell) \sim F_{X_0}\, V_{k+1}(\ell) .$$

Is then the corresponding F-distortion of X_0 in X bounded (i.e. $FD(\ell) \sim \ell$) ?

Warning. If a (filling) function $f(\ell)$ is said to grow *exponentially* then it does not have to be necessarily equivalent to any $\exp c\ell$ in our technical sense dictating $c_1 \exp c\ell \leq f(\ell) \leq c_2 \exp c\ell$. We rather mean $\log f(\ell) \sim \ell$ which is a significantly less binding relation. (The fine equivalence $f(\ell) \sim \exp c\ell$ for our geometric functions is a subtle matter which is at no point discussed in this paper). With this in mind one should not take the above equivalence between F_X andF_{X_0} on the face value when the functions grow faster than polynomially.

5.G$_1$. Volume expansion and contraction for retractions. We saw in §3. that the distortion of $X_0 \subset X$ can be measured by Lipschitz constants (or functions depending on the distance to X_0) of retractions $X \to X_0$. First we observe, that the volume distortions provide lower bounds on these Lipschitz constants as well as the distance distortion. For example, now we have an alternative way to see that $H^2 \times H^2$ does not Lipschitz retract onto Sol_3 in there. Actually, if this Sol_3 has $\lambda_1 = -\lambda_2$ we have two obstructions, one for the area and another for the volume. The second obstruction associated to the volume can be seen yet from another angle. Recall that Sol_3 sits as a convex horosphere H_0 in $X = H^2 \times H^2$ and if we move it inward by the normal displacement the volume element exponentially decays with $t \to \infty$. Now, if we have domains $D_0 \subset H_0$ with large volume and with relatively small boundary ∂D_0, i.e. with $\mathrm{Area}\,\partial D_0/\mathrm{Vol}\,D_0 \leq \varepsilon$ for small $\varepsilon \to 0$, they move to much smaller domains, say $D_t \subset H_t$, $t < 0$, with $\mathrm{Vol}\,D_t \approx (\exp -t)\,\mathrm{Vol}\,D$. If we add such a D_t with the t-collar in X over ∂D_0, we obtain a filling of ∂D_0 by a new chain of the volume bounded by

$$(\exp -t)\,\mathrm{Vol}\,D_0 + t\;\mathrm{Area}\,\partial D_0 \leq \big((\exp -t) + t\varepsilon\big)\;\mathrm{Vol}\,D_0\;,$$

which becomes arbitrarily small compared to $\mathrm{Vol}\,D_0$ for $\varepsilon \to 0,\ t \to \infty$ if we make $t\varepsilon \to 0$.

The key point of the above discussion (besides the existence of D_0 with relatively small ∂D_0) is the exponential (3-dimensional) volume expansion of the normal projection of the horoball H^+ bounded by H_0 onto H_0. In fact we know a posteriori, that every retraction of H^+ to $H_0 = \partial H^+$ must have unbounded volume distortion, i.e. the differential of such a retraction on the exterior power $\wedge^3 T(H^+)$ must be unbounded.

It seems, in general, an interesting problem to find, for a given Riemannian homogeneous pair X and $X_0 \subset X$, a (possibly G_0-equivariant for the isometry group of the pair (X, X_0)) retraction $p : X \to X_0$ with the smallest possible norm $\|D_p| \wedge^{k+1} T_x(X)\|$ thought of as a function of $d = \mathrm{dist}(x, X_0)$. This problem (as well as the whole of the volume distortion story) extends to arbitrary finitely generated subgroups in finitely generated groups (at least if these are k-connected on the large scale) but nothing of substance is known in this generality in order to suggest a meaningful question.

On the other hand, one may try to extend our study of $H_0 = \mathrm{Sol}_3 \subset X = H^2 \times H^2$ to the horospheres in general spaces with non-positive curvature. How much are such horospheres lengthwise and/or volumewise distorted ? When they admit (Folner) exhaustions by domains D with $\mathrm{Vol}\,D/\,\mathrm{Vol}\,\partial D \to \infty$, etc.

§6. K $\leq$ 0 and semi-hyperbolicity; coning and combing; rank and geometry of flats; Tits boundary and Morse landscape at infinity; minimal surfaces and norms on homology; group action with K $\leq$ 0; semi-hyperbolic constructions and generalizations.

We start this § with a little propaganda for groups acting on spaces with $K \leq 0$ and then we explain the difficulties in globalizing the conception of non-positive curvature to a notion of semi-hyperbolicity.

6.A. Examples of K $\leq$ 0. A geodesic space has $K \leq 0$ if the distance function there is more convex than that on $\mathbb{R}^2$ in the sense of CAT (see [Gro]$_{14}$, [Gh-Ha], [C-D-P]). This includes Riemannian manifolds with the sectional curvature $K \leq 0$. Another elementary example is provided by 2-dimensional simplicial polyhedra X satisfying the $\frac{1}{6}$*-condition*: X contains no subpolyhedron isomorphic to the cone over a closed polygon with less than six edges (i.e. pentagon, quadrangle or triangle).

One may think of $K \leq 0$ as the limit case of the negative curvature condition $K < 0$ and regard (groups acting on) spaces X with $K(X) \leq 0$ as rare and exotic species compared to *hyperbolic* groups and spaces corresponding (morally rather than factually at the present state of knowledge) to $K < 0$. This point of view is justified in the local differential geometry as the curvature tensors with $K < 0$ are open and dense among those with $K \leq 0$. But on the global scale there is no (known) way of approximating $K \leq 0$ by $K < 0$, although many geometric properties of groups and spaces with $K \leq 0$ do lie on the boundary of those with $K < 0$.

This situation in geometry is vaguely similar to what we see in the combinatorial group theory where groups appear as "developments" of their presentation, which are, unlike groups themselves, *finite* objects playing the role of "local" and "infinitesimal" in geometry. Here again the non-strict negativity of combinatorial curvature expressed, for example, by the $\frac{1}{6}$-condition, may look rather special compared to $K < 0$ expressed by $1/n$, $n \geq 7$, but this point of view is by far less compelling here than in the case of the curvature tensors. For example, there are, I suspect, some natural combinatorial schemes of enumerating (finite) polyhedra, such that "$K < 0$" has infinitely smaller density of occurence than "$K \leq 0$" (never mind that in another setting the *hyperbolic groups* are the ones who turn up with probability one). In fact, there are certain (essentially local) combinatorial constructions which lead to spaces (and groups acting upon them) with $K \leq 0$ where the strict negativity is a rare exception. We mean, first of all, Davis' polyhedral realization of reflection groups, where $K < 0$ is actually impossible above a certain dimension. (See [Dav]$_{1,2}$, [Vin]$_{1,2}$, [Mou] and beware of an erroneous claim to the contrary in [Gro]$_9$).

Another construction which excludes $K < 0$ is the *Cartesian product* of spaces (and groups): even if both X_1 and X_2 have $K < 0$ we only may claim $K(X_1 \times X_2) \leq 0$ and there are serious obstructions to make the curvature strictly negative.

But what, in our opinion, brings $K \leq 0$ on a par with $K < 0$ is the following list of beautiful geometric spaces with $K \leq 0$ which are acted upon by a variety of remarkable groups

I. *Symmetric spaces X of non-compact type.* Here the strictly negative curvature $K < 0$ is relatively rare as it corresponds to the condition rank $X = 1$. An especially important example of a symmetric space is $X = SL_n\mathbb{R}/SO_n$ as every discrete linear group discretely and isometrically acts on such X. Notice that this space has rank $X = n - 1$ and so the curvature is *non-strictly* negative apart from the case of the hyperbolic plane $H^2 = SL_2\mathbb{R}/SO_2$.

II. *Euclidean buildings X of Bruhat-Tits.* These are combinatorial counterparts to symmetric spaces. For $\dim X = 1$ they are just certain trees (and hence hyperbolic) but for $\dim X \geq 2$ they have *non-strictly* negative curvature. The groups which act here are, for example, subgroups in $SL_n\mathbb{Q}_p$ and in the linear groups over more general fields with non-Archimedian norms.

III. *Infinite dimensional symmetric spaces and buildings.* The standard constructions of classical symmetric spaces (and, probably, of buildings) can be extended (usually in several ways) to the infinite dimension. For example, the space $X_n = SL_n\mathbb{R}/SO(n)$ becomes

$$X_\infty = GL_\infty/O(\infty) \times \mathbb{R}^\times ,$$

where GL_∞ is the group of invertible bounded operators on the Hilbert space $\mathbb{R}^\infty$, and $O(\infty)$ consists of the orthogonal (unitary) operators, while $\mathbb{R}^\times$ acts by multiplication by scalars. One can identify this X_∞ with projectivized space of Hilbert norms on $\mathbb{R}^\infty$ equivalent to the Hilbert structure norm.

The space X_∞, besides being non-Riemannian, appears too large for many geometric purposes. For example, it has infinite rank. Our next example looks more manageable.

Let $O(p,\infty) \subset GL_\infty$ consist of the operators preserving the form $-\sum\limits_{i=1}^{p} x_i^2 + \sum_{i=p+1}^{\infty} x_i^2$ (where the structure form on $\mathbb{R}^\infty$ is $\sum\limits_{i=1}^{\infty} x_i^2$). The corresponding Riemannian symmetric space

$$X = O(p,\infty)/O(p) \times O(\infty)$$

has $K \leq 0$ and rank $= p$. If $p = 1$ this is the infinite dimensional hyperbolic space H^∞ of constant negative curvature and for $p \geq 2$ our X has non-strictly negative curvature.

The spaces like this X look as cute and sexy as their finite dimensional brothers and sisters but they have been for years shamefully neglected by geometers and algebraists alike. The questions which interest us most (in this paper) about such spaces concern discrete isometry groups Γ acting on them where the word "discrete" requires an explanation. Namely, we do not care much for actions keeping fixed some point $x_0 \in X$, as these come from unitary representation of groups. So, the very least we require of Γ is to have *unbounded orbits* and the strongest discreteness condition is *boundness of distortion* for Γ embedded in X as an orbit. In fact we can require even more in this regard, the *existence* of a *large-scale Lipschitz retraction of* X *onto* Γ identified with some orbit in X. Notice that in both definitions we assume Γ is finitely generated and we refer to a word metric in Γ when we speak of "distortion", "Lipschitz" etc. Another important notion is that of *strict unboundness* of an action where the intersection of every orbit with each bounded subset in X is finite (or precompact if we allow *locally compact* groups Γ acting on X).

Examples. (a) Let Γ be an amalgamated product $\Gamma_1 *_\Delta \Gamma_2$, where Δ has index at least three in Γ_1 and Γ_2. Then Γ admits an unbounded action on H^∞ which is obtained by "gluing together" the (bounded) actions of Γ_1 and Γ_2 on H^∞ corresponding to the regular representations of Γ_1 and Γ_2. (We suggest to the reader to work out this construction by himself).

(b) Every tree T can be embedded into H^∞, such that the automorphism group of T strictly coboundedly acts on H^∞.

(c) *Conjecture.* For every $p = 1, 2, \ldots$, there exists a discrete (according to one of the above definitions) finitely generated group Γ acting on X_{p+1} corresponding to $O(p+1,\infty)$, which, however, admits no strictly unbounded action on X_p. In fact, a lattice Γ in a simple finite dimensional Lie group of $\mathbb{R}$-rank $\geq p+1$, probably, cannot discretely act on X_p or on other "reasonable" infinite dimensional symmetric spaces of rank $\leq p$. (It is obvious that a lattice Γ of rank $\geq p+1$ cannot have any action on X_p with *bounded distortion* since X_p admits no $\mathbb{R}^{n+1}$ inside itself with bounded distortion).

The above conjecture is justified by what we know about the case of $X_1 = H^\infty$ where there is a variety of restrictions on Γ. For example, using harmonic maps into X_1 and Bochner-Siu-Sampson formula for $K_{\mathbb{C}} < 0$ (see [Gro]$_{21}$, [Gr-Sch]), one can show that if Γ is a cocompact irreducible discrete isometry group of a symmetric Hermitian space of rank ≥ 2, then every isometric action of Γ on $X_1 = H^\infty$ has bounded orbits. This is similar to Margulis' superrigidity theorem.

IV. *The spaces of metrics and subgroups in* Diff. Let V be a compact smooth manifold. Then the space of the Riemannian metrics on V with a fixed volume element Ω can be made into an infinite dimensional Riemannian space with $K \leq 0$ with an isometric action of Diff(V,Ω). This, we hope, should aid us in the study of Diff and subgroups in Diff but nothing concrete is known in this direction (compare [Gr-D'A]). If we do not like Ω, we may use the space of *conformal structures* on V instead of the Riemannian metrics. On the other hand if we allow an invariant symplectic form ω we may restrict to the metrics agreable with ω which constitute an infinite dimensional *Hermitian* space with $K \leq 0$. All this looks nice and pretty but no consequence has followed so far. (It may be not quite just to say so because there is a sequence of remarkable papers by Zimmer starting from [Zim] , concerning Diff-action of lattices, where the negative sign of the curvature in the space of metrics plays an important role though it never comes into the open).

6.B. Large-scale effects of $K \leq 0$. Let X be a simply connected geodesic space with $K \leq 0$. The basic feature of X is the uniqueness of the geodesic segment between every two points x_1 and x_2 in X. We agree to parametrize the segment by $[0,1]$ proportional to length and write $tx + (1-t)y$ for the points on this segment corresponding to $t \in [0,1]$. The map defined by the *convex combination* $(x, y, t) \mapsto tx + (1-t)y$ is denoted

$$CC : X \times X \times [0,1] \to X .$$

Remark. For reasonable (e.g. polyhedral) spaces the existence of a CC-like map is equivalent to contractibility. Moreover, if such a space comes along with a discrete group action, then one can find an equivariant CC-map. But if $K \leq 0$ then, as we shall see presently, the map CC enjoys extra geometric properties.

Another essential feature of $K \leq 0$ is the *convexity* of the distance function and most properties of manifolds with $K \leq 0$ (and of groups acting of such manifolds) follow (or should follow) from this convexity (see $[\mathrm{Gro}]_5$). For example, one can divide individual isometries γ of X into *elliptic, parabolic* and *quasi-hyperbolic* and easily show (e.g. see $[\mathrm{Gro}]_5$) that γ is quasi-hyperbolic if and only if the cyclic subgroup $\{\gamma^i\}$ has bounded distortion in X. In particular if Γ is a discrete isometry group whose all isometries are quasi-hyperbolic (e.g. Γ has no torsion and X/Γ is compact) then every cyclic subgroup in Γ has bounded distortion. (This property has been given different names in different papers. It is "no asymptotic torsion" in $[\mathrm{Gro}]_5$, "no algebraic parabolic" in [B-G-S], and "non-zero translation numbers" in $[\mathrm{Ge\text{-}Sh}]_4$ and [A-B]).

The convexity of dist can be formulated in terms of CC as follows. Let $z_i = t\,x_i + (1-t)y_i$, $i = 1,\ 2$. Then

$$\mathrm{dist}(z_1, z_2) \leq t\ \mathrm{dist}(x_1, x_2) + (1-t)\ \mathrm{dist}(y_1, y_2)\ . \qquad (*)$$

Bicombing. It was suggested in [A-B] that the large-distance rendition of CC, called a *bounded quasi-geodesic bicombing*, is a good candidate for *semi-hyperbolicity*. A *bicombing* is a map

$$BC : X \times X \times [0,1] \to X$$

for which there exist constants $\lambda \geq 1$ and $d \geq 0$ such that

$$\mathrm{dist}(z_1, z_2) \leq \lambda\big(t\ \mathrm{dist}(x_1, x_2) + (1-t)\ \mathrm{dist}(y_1, y_2)\big) + d \qquad (**)$$

for all $x_1,\ x_2,\ y_1,\ y_2$ in X, $t \in [0,1]$ and $z_i = tx_i + (1-t)y_i$.

A bicombing is (Lipschitz) *bounded* if

$$\mathrm{dist}\big(t_1x + (1-t_1)y,\ t_2x + (1-t_2)y\big) \leq \lambda|t_1 - t_2|\ \mathrm{dist}(x, y) = d\ , \qquad (*+)$$

and a bounded bicombing is called *quasi-geodesic* if

$$\mathrm{dist}(z_1, z_2) \geq \lambda^{-1}\big(\mathrm{dist}(x_1, x_2) - t_1\ \mathrm{dist}(x_1, y_1) - t_2\ \mathrm{dist}(x_2, y_2)\big) - d$$

where $z_i = t_ix_i + (1-t_i)y_i$. (The definitions in [A-B] are shaped slightly differently. Also observe that "bounded" for CC is obvious and the quasi-geodesic property follows from $(*)$ and the triangle inequality but it does not formally follow from $(**)$ and $(*+)$). Alonso and Bridson, and also Short, put forth a serious evidence for the fact that an equivariant b.q.-g. bicombing on X (with a group Γ acting on X) imposes on X and Γ strong conditions comparable to those associated to $K \leq 0$, see [A-B]. In particular, they claim in [A-B] that the bounded distortion property of $\{\gamma^i\}$ extends to their *semi-hyperbolic groups* Γ (i.e. groups with an Γ-invariant bicombing) by a generalization of an argument of $[\mathrm{Ge\text{-}Sh}]_4$, where this had been done for bi-atomatic groups.

Another situation where BC serves almost as well as CC for $K \leq 0$ is in taking (rough) convex combinations of equivariant maps. To be geometric, let V be a compact Riemannian manifold with equivariantly bicombable $\Gamma = \pi_1(V)$ and let us look at the Morse landscape of the space $\mathcal{M}$ of maps $f : W \to V$ for another manifold W, where we assume that $\dim W \leq k$ and V is k-connected. We claim that every connected component of $\mathcal{M}$ has only one deep basin for the Lipschitz constant $\ell = \ell(f)$. Namely, *every two ℓ-Lipschitz maps f_1 and f_2 $W \to V$ can be joined by a homotopy of ℓ'-Lipschitz maps for $\ell' \leq C\,\ell$ for some constant* $C = C(V, W)$. In fact f_1 and f_2 lift to the universal covering of $\widetilde{V}$ of V, say to maps $\widetilde{f}_i : \widetilde{W} \to \widetilde{V}$, $i = 1, 2$, where $\widetilde{W}$ the covering of $\widetilde{W}$ corresponding to the image of $\Gamma' = \pi_1(W)$ in $\Gamma = \pi_1(V)$. Then BC gives us an equivariant Lipschitz "homotopy" $\widetilde{f}_t = t\widetilde{f}_1 + (1-t)\widetilde{f}_2$ which then can be easily modified to an actual homotopy. It follows, that the conjugacy problem for homomorphisms $\Gamma' \to \Gamma$ is solvable (compare [A-B]).

6.$\mathbf{B_1}$. Iterated CC and BC and small simplices. By iterating CC (or BC) one obtains a "convex combination" of $(k+1)$-points $x_0, \ldots, x_k$ in X by first joining x_0 and x_1 by the edge $tx_0 + (1-t)x_1$, $t \in [0,1]$, then joining x_2 with the points on $\{tx_0 + (1-t)x_1\}$ etc. We write the result as $p_0x_0 + p_1x_1 + \ldots + p_kx_k$ for arbitrary $p_i \geq 0$ with $\sum\limits_{i=0}^{k} p_i$.

Warning. The above "sum" of weighted points x_i is, in general, non-commutative though it is commutative for $k = 1$.

The operation $(x_i, p_i) \mapsto \Sigma p_i x_i$ can be thought of as a map

$$CC_k : X^{k+1} \times \Delta^k \to X \ ,$$

and one similarly defines

$$BC_k : X^{k+1} \times \Delta^k \to X \ ,$$

where $\Delta^k = \left\{ p_i \geq 0 | \sum\limits_{k=0}^{k} p_i = 1 \right\}$.

The inequalities $(*)$ and $(**)$ show that every simplex spanned by some point x_i is roughly smaller than the corresponding Euclidean symplex spanned by points $x_i' \in \mathbf{R}^k$ with mutual distances comparable to those between x_i. This immediately implies (compare [Gro]$_8$) the following

Corollary. *If Γ is BC, then every cohomology class $h \in H^k(\Gamma; \mathbf{R})$ has polynomial growth.*

In fact h can be represented by a cochain c, such that

$$c(\gamma_0, \ldots, \gamma_k) \leq \text{const} \prod_{i=1}^{k} \text{dist}_\Gamma(\gamma_i, \gamma_{i-1}) \ .$$

Remark and comments. (a) *Generalizations.* This corollary remains (obviously) valid for combable groups in the sense of [E-C-H-P-T]. In fact one needs even less than that, something like "polynomially Lipschitz" combing where the cone over a subset of size d is bounded by d^α. For example, this property is satisfied by the nilpotent Lie groups with the coning by (segments between points of) one-parameter subgroups (this, in general, is not even combing in the Gersten sense, it seems). This suggests generalized combings or *conings* over a point $x_0 \in X$, which are homotopies of X shrinking X to x_0 (as $(x,t) \mapsto (1-t)x_0 + tx$) with a certain control on the Lipschitz constant of such a coning $C : X \times [0,1] \to X$ on the ball of radius R in $X \times [0,1]$ around $(X_0 \times 0)$. (For discrete spaces and groups one should take extra care to "large-scale" such a definition to have discontinuous but yet Lipschitz controlled homotopies. This is slightly more delicate here than for the usual combing where the Lipschitz constant is bounded). As we mentioned above, nilpotent Lie groups have polynomially Lipschitz conings and this extends by a proper "large-scaling" to *discrete* nilpotent groups.

The subgroups in SL_n have exponential coning arising from the geodesic coning in $X = SL_n/SO(n)$ via an exponential Lipschitz retraction (see §3.); probably it is easy to extend to all Lie groups as the typical non-linear group, the universal covering of SL_2R, is combable).

(b) *Novikov conjecture.* The above corollary is quite trivial by itself but it gains weight in view of the remarkable theorem by Connes and Moscovici claiming that *the Novikov higher signature conjecture is valid for the cohomology classes* (in arbitrary groups Γ) *with polynomial growth.* Thus *the combable in the sense of* [E-C-H-P-T] *and polynomially conable groups satisfy the Novikov conjecture.*

(b$'$) *Filling criteria for polynomial growth of cocycles.* Suppose the filling volume functions $F_0V_i(\ell)$ are (at most) polynomial in ℓ for $i = 2, \ldots, k$, which means that every (contractible) $i-1$-sphere in X of volume ℓ bounds a ball of volume $\lesssim \ell^\alpha$. Then, by induction, $(k+1)$-tuples of points in X of diameter ℓ can be consistently spanned by k-simplices of volume $\lesssim \ell^{\beta^k}$ which implies a polynomial bound on k-cocycles. This conclusion obviously holds true for filling functions which are "stronger" than $F_0V_i(\ell)$, e.g. the filling span $F_0S(\ell)$ (controlling the Lipschitz constants of maps $B^i \to X$). Moreover, one may use some filling functions which are a priori "weaker" than F_0V_i. For example, one can replace F_0V_i by the filling function FV_i concerning non-spherical fillings as this does not effect the relevant cohomological properties of the resulting cochains. On the other hand, a polynomial bound for invariants such as Fill Rad cannot give better than an exponential bound on the cocycles, but the overall picture is far from clear.

(c) *Cohomology with fast growth.* A. Connes pointed out to me the following example of a cohomology class with exponential growth. Take Γ the solvable group $\mathbb{Z}^2 \rtimes \mathbb{Z}$ for $\mathbb{Z}$ acting on $\mathbb{Z}^2$ by a hyperbolic transformation. Since $\mathbb{Z}^2$ has strictly exponential distortion (see §3.) every 2-dimensional cohomology class with non-trivial value on the 2-dimensional homology of $\mathbb{Z}^2 \subset \Gamma$ has exponential growth. Connes also conjectured that there must exist groups with arbitrarily fast growth of cohomology and indeed these can be constructed as follows. Let Γ' contain an infinite cyclic subgroup, say $\{c^i\}$, with large (e.g. non-recursive, see §3.) distortion and take the amalgamated product $\Gamma = \Gamma' *_C \mathbb{Z}^k$ where C is a cyclic subgroup which equals $\{c^i\}$ on the side of Γ' and which is anything cyclic you want in $\mathbb{Z}^k$. Then the growth of the k-dimensional cohomology in Γ is of the same order of magnitude as the distortion of $\{c^i\}$ in Γ'. (See [Ger]$_5$ for another class of examples.)

The above example cannot give us a counterexample to the Novikov conjecture, since it has 1-dimensional origin (in a sense which can be made precise) and the Novikov conjecture is known for the cohomology of 1- and 2-dimensional origin (see [C-G-M]$_1$). But a truly k-dimensional example for $k \geq 3$ seems harder to obtain (which is similar to the situation with the contractibility radius $\overline{R}_k$ for $k \geq 2$, see §4.).

(d) *Bounded cohomology.* No apparent coning construction can bound cocycles better than linearly and this is, in fact, the best one can do for H^1. Yet, there are cohomology classes in $H^k(\Gamma; \mathbb{R})$ for certain Γ and $k \geq 2$ which can be represented by *bounded* cochains giving rise to *bounded cohomology* (see [Gro]$_8$). For example, if Γ is a hyperbolic group then $H^k(\Gamma; \mathbb{R})$ is bounded for all $k \geq 2$ and a certain part of the cohomology is bounded in the groups (acting on spaces) with $K \leq 0$ (see [Gro]$_{14}$, [D-T], [Sav]).

(e) *Infinite iteration of CC and BC.* The CC-inequality $(*)$ for $K \leq 0$ behaves nicely when iterated but iterating BC may lead to unpleasant divergences. For example, if we, by induction, define

$$y_i = \frac{1}{2} y_{i-1} + \frac{1}{2} x_i, \; y'_i = \frac{1}{2} y'_{i-1} + \frac{1}{2} x_i$$

for given y_1, y'_1 and some sequence $x_1, x_2, \ldots$, then, according to $(*)$, $\operatorname{dist}(y_i, y'_i) \to 0$, while $(**)$ does not preclude $\operatorname{dist}(y_i, y'_i) \to \infty$ for $i \to \infty$. It is unclear if an arbitrary (equivariant) bicombing can be used for k-th convex combinations with a control over the size (Lipschitz constant) uniform in k.

(f) Besides the *iterated* convex combination for $K \leq 0$ there is another well-known one which has the advantage of being commutative. This $\sum_{i=0}^{k} p_i x_i$ is defined as the (unique for $K \leq 0$) minimum point x of the function $\sum_{i=0}^{k} p_i \operatorname{dist}^2(x, x_i)$. However, one does not know the exact Lipschitz bounds on the corresponding map $X^{k+1} \times \Delta^k \to X$ for general spaces X with $K \leq 0$.

Question. When is the (geodesic) convex hull of a compact subset $A \subset X$ compact ? This is useful to know for infinite dimensional spaces and also for "building-like" spaces such as $\mathrm{Con}_\infty X$ and Tits' boundaries (see 6.B_2).

6.B_2. Rank and the geometry of flats. One of the guiding principles in the asymptotic geometry of spaces X with $K \leq 0$ can be expressed, roughly, as follows. All flatness of X where $K = 0$ is confined to k-flats in X which are the subspaces isometric to $\mathbb{R}^k$. One distinguishes among them *maximal flats* which are not contained in bigger ones (sometimes "maximal" refers to the maximal dimension) and then tries to show that X is "hyperbolic transversally to maximal flats".

Warning. An obvious counter example to such philosophy is provided by a metric on $\mathbb{R}^2$ which is flat outside a disk D and has $K < 0$ inside D. To take care of this, one either has to assume that X admits an isometric action of some Γ with X/Γ compact (or bounded, if we allow $\dim X = \infty$) or one may complete X in the following way

Ultercompletion $\mathcal{X}^+$ of X. Consider all sequences $x_i \in X$ with a fixed $x_0 \in X$ (where the interesting case is $\mathrm{dist}(x_0, x_i) \to \infty$ for $i \to \infty$) and take the Hausdorff limits (and sublimits) of the pointed metric spaces. Moreover, take the ulterlimits whenever the ordinary (sub-)limits are not available (which may easily happen especially for $\dim X = \infty$). The set of this limit spaces is denoted $\mathcal{X}^+$ and various properties of X may be attributed to certain spaces $X' \in \mathcal{X}^+$.

Examples. (a) If X admits a cocompact isometric action of some Γ, then all $X' \in \mathcal{X}^+$ are isometric to X and there is no need to mention $\mathcal{X}^+$ at all.

(b) Let Y be a compact foliated space containing X as a dense leaf. Then every $X' \in \mathcal{X}^+$ is isometric to some leaf in Y. (The case of compact X/Γ corresponds to a foliation with a single compact leaf).

(c) For the above metric on $\mathbb{R}^2$ flat at infinity, the space $\mathcal{X}^+$ contains, besides X, the flat plane $\mathbb{R}^2$ and nothing else (up to isometry).

(d) If X is a finite dimensional space with bounded local geometry (e.g. if $K(X) \geq -c > -\infty$) then one can always construct (this is easy) a compact foliated space Y containing up to isometry all $X' \in \mathcal{X}^+$ as leaves. In fact, one may always think of $\mathcal{X}^+$ as of a foliated space whose leaves are metric spaces and where X appears as a dense leaf.

Seven definitions of Rank.

I. $\mathrm{Rank}\, X = \sup\limits_{\omega} \dim \mathrm{Con}_\omega X$ where ω runs over all non-principal ulterfilters.

II. $\mathrm{Rank}\, X =$ the supremum of the dimensions of the flats in X.

III. $\mathrm{Rank}\, X =$ supremum of the asymptotic dimensions of *quasi-flats* in X which are subspaces quasi-isometric to Euclidean spaces.

IV. $\mathrm{Rank}\, X = \dim \partial_T X + 1$ for the *Tits boundary* $\partial_T X$ defined in (e) below.

V. $\mathrm{Rank}\, X =$ the minimal k_0 such that X satisfies linear isoperimetric inequality from dimension k_0, i.e. $FV_{k+1} \sim \ell$ for $k \geq k_0$.

VI. $\mathrm{Rank}\, X =$ the minimal k_0 with the following property. There exists a constant $\lambda \geq 0$, such that for every $\varepsilon > 0$, X admits a λ-Lipschitz selfmap $f : X \to X$ which is parallel to the identity (i.e. $\mathrm{dist}(f, \mathrm{id}) < \infty$) and volume *$\varepsilon$-contracting* above the dimension k_0. (If X is a manifold, this means $\mathrm{Vol}\, f(Y) \leq \varepsilon \mathrm{Vol}\, Y$ for all submanifolds $Y \subset X$ of dimension $> k_0$. In the general case one should work harder to make this definition stick).

VII. $\mathrm{Rank}\, X =$ the minimal k_0 with the following property. Let $X_0 \subset X$ be a closed subset which is the union of a set of *rays* in X issuing from a fixed point $x_0 \in X$ (where a ray is an isometric copy of $\mathbb{R}_+$ in X). If $\dim X_0 \geq k_0 + 1$, then X_0 has exponential growth, which means every net $N_0 \subset X_0$ has at least $\exp c R$ points in the R-balls around x_0 for some $c > 0$ and all sufficiently large R.

Remarks and comments. (a) The idea behind all these definitions is that X behaves hyperbolically in the dimensions above rank X. For example, if $\mathrm{Rank}\, X = 1$, then X should be hyperbolic as reflected in the k-dimensional geometry for $k \geq 2$.

(b) One believes the seven definitions are (essentially) equivalent in the case where the isometry group of X is cocompact on X. In the general case one should modify the definitions by invoking the ultercompletion $\mathcal{X}^+$ of X and setting

$$\operatorname{Rank}^+ X = \sup_{X' \in \mathcal{X}^+} \operatorname{Rank} X' .$$

Now the equalities $\operatorname{Rank}_{\mathrm{I}}^+ = \ldots = \operatorname{Rank}_{\mathrm{VII}}^+$ become a realistic conjecture for all X without major pathology. (Pathological or not, all X I have looked upon so far seem to have the seven Rank^+'s equal). Notice that among the seven, some are more sensitive to the "plusification" than others. For example, $\operatorname{Rank}_{\mathrm{I}} X$ is more likely to be equal to $\operatorname{Rank}^+ X$ than $\operatorname{Rank}_{\mathrm{II}} X$.

Now we briefly discuss the possibility of equivalence between the definitions:

(c) I $\sim$ II ? If X contains a k-flat then so is $\operatorname{Con}_\omega X$ for every ω and so $\dim \operatorname{Con}_\omega X \geq k$. This shows that

$$\operatorname{Rank}_{\mathrm{I}} \geq \operatorname{Rank}_{\mathrm{II}} \text{ and } \operatorname{Rank}_{\mathrm{I}}^+ \geq \operatorname{Rank}_{\mathrm{II}}^+$$

(but there are easy examples, where $\operatorname{Rank}_{\mathrm{II}} X < \operatorname{Rank}_{\mathrm{II}}^+ X$).

The reverse inequality, $\operatorname{Rank}_{\mathrm{I}} \leq \operatorname{Rank}_{\mathrm{II}}$ has been established in §2. for symmetric spaces X and that argument extends to some infinite dimensional symmetric spaces (e.g. to $O(p,\infty)/O(p) \times O(\infty)$). Let us indicate how one may extend (the plusification of) that argument to non-symmetric spaces X. Take an oriented geodesic in X and let H_t, $t \in]-\infty, \infty[$ be the corresponding horospheres. As $t \to -\infty$, the metric in H_t decreases and we want to distinguish different "directions" in H_t according to the rate of decay of the metric. Namely, we want to partition each H_t into *horocycles* denoted $h_{t,y} \subset H_t$, such that

(i) The normal projection $p_{t_1,t_2} : H_{t_1} \to H_{t_2}$ for $t_2 \leq t_1$ agrees with the partition into horocycles, i.e. the pull-back of every horocycle in H_{t_2} is a horocycle in H_{t_1}.

(ii) The induced metric on each horocycle exponentially decays as $t \to -\infty$. Thus every horocycle h is strictly exponentially distorted in X which should make $\operatorname{Con}_\infty X$ locally disconnected in $\operatorname{Con}_\infty X$.

Notice that (ii) may be (hopefully) used to define horocycles as the maximal subset on which the metric does decay exponentially. (The decay should be discussed *on the large scale* as we do not care what happens to nearby points).

(iii) When we divide X into the horocycles , the resulting quotient space, say Y, must have $K \leq 0$. Then we may proceed by induction, once we know that $\dim \operatorname{Con}_\infty Y = \dim \operatorname{Con}_\infty X$.

Example. Let our geodesic be contained in a flat $F \subset X$ with $\dim F = \operatorname{Rank}_{\mathrm{II}} X$. Then one may expect $Y = F$, if our geodesic is *regular* in an appropriate sense. (A geodesic is definitely *irregular* if it is contained in *several* maximal flats.). The theory of regular geodesics has been developed in [B-B-E-S] and [Ban-Sch] and, with the present state of art, one can, probably, complete the above argument for smooth finite dimensional manifolds X with a cocompact Γ-action, (and the real analytic case seems somewhat easier). However, in the general case of $K \leq 0$ even the inequality $\dim \operatorname{Con}_\omega X \leq \dim X$ remains unsettled. Yet, the latter inequality may be valid in a greater generality, e.g. for finitely generated groups with a kind of *polynomially Lipschitz combing* mentioned earlier, which would include, besides $K \leq 0$, such creatures as nilpotent groups.

(d) II $\sim$ III ? It is trivial that

$$\operatorname{Rank}_{\mathrm{III}} \geq \operatorname{Rank}_{\mathrm{II}} \text{ and } \operatorname{Rank}_{\mathrm{III}}^+ \geq \operatorname{Rank}_{\mathrm{II}}^+$$

since every flat is also a quasi-flat. Here we observe that $\operatorname{Rank}_{\mathrm{II}}$ is not, a priori, quasi-isometry invariant, but this invariance is known in the case of a smooth finite dimensional X with a cocompact Γ-action as was shown in [And-Sch] by the techniques of minimal varieties. Namely, if the same group Γ isometrically and cocompactly acts on two different manifolds X_1 and X_2 with $K \leq 0$ then $\operatorname{Rank}_{\mathrm{II}} X_1 = \operatorname{Rank}_{\mathrm{II}} X_2$. I gave another proof of this result in $[\mathrm{Gro}]_{21}$ using harmonic maps and my argument can be generalized in several directions. (This is also true for the original proof in [And-Sch] but I, naturally, feel more comfortable with mine). First, one can admit singular spaces with $K \leq 0$ as the corresponding theory of harmonic maps has been worked out in [Gr-Sch]. Secondly, one can forfeit Γ and work with compact foliated spaces Y. This shows that $\mathrm{Rank}^+_{\mathrm{II}}$ (but not $\mathrm{Rank}_{\mathrm{II}}$) is quasi-isometry invariant in the category of locally compact spaces (e.g. manifolds) X with $K \le 0$ and uniformly locally bounded geometry (e.g. $K \ge -c > -\infty$). In fact I am pretty sure this works in the general case, and, moreover, proves that

$$\mathrm{Rank}^+_{\mathrm{III}} \le \mathrm{Rank}^+_{\mathrm{II}} ,$$

i.e. every quasi-flat in X can be flattened down to a flat in some $X' \in \mathcal{X}^+$, but the details need checking.

(d') An important property of flats $F \subset X$ needed for my argument in $[\mathrm{Gro}]_{21}$ is the existence of a 1-Lipschitz retraction $X \to F$. In fact, the normal geodesic projection is 1-Lipschitz. Similarly, every quasi-flat $F \to X$ admits a λ-Lipschitz retraction with λ depending on $\dim F$ and the constants behind "quasi" (compare §3.). The normal projection, however, has an important advantage of being "canonical" and thus commuting with isometries and Hausdorff limits which is not the case for the retraction on quasi-flats. Yet the ambiguity here is, so to speak, compact and it can be absorbed in the foliated language of $[\mathrm{Gro}]_{21}$.

(d'') If a flat $F \subset X$ has maximal dimension k, i.e. $\dim F = k = \mathrm{Rank}^+_{\mathrm{II}} X$ then the normal projection $X \to F$ has an extra nice property of volume contraction: if a k-dimensional submanifold $H \subset X$ projects onto the R-ball $B \subset F$ then, by a simple limit argument,

$$\mathrm{Vol}_k H \ge \lambda(R,d)\, \mathrm{Vol}_k B ,$$

where $d = \mathrm{dist}(H,F) (= \inf \mathrm{dist}(h,f))$ and $\lambda(R,d) \to \infty$ for $R, d \to \infty$.

This property has definite large-scale flavor and can be used for a *definition of maximal quasi-flats* in a semi-hyperbolic setting.

Finally, we notice that $\lambda(R,d) \sim \exp c\, d$ for symmetric spaces X (see $[\mathrm{Most}]_2$) and this may be true for all reasonable (?) X.

(e) I $\sim$ IV ? Recall that the *geodesic boundary* ∂_{geo} of a metric space X with a reference point $x_0 \in X$ is defined as the space of geodesic rays in X issuing from x_0, which are *isometric* maps $[0,\infty) \mapsto X$ sending $0 \mapsto x_0$ and where the topology in ∂_{geo} is that of the uniform convergence on (finite) subsegments $[0,t] \subset [0,\infty)$. If $K \le 0$ then $\partial_{\mathrm{geo}}(X,x_0)$ does not depend on x_0 (there are canonical homeomorphisms $\partial_{\mathrm{geo}}(X,x_0) \longleftrightarrow \partial_{\mathrm{geo}}(X,x)$ and besides the usual topology on $\partial_{\mathrm{geo}}(X)$ there is another one defined by *Tits' geometry* as follows). Let $\mathrm{Co}\,\partial_{\mathrm{geo}}$ denote the union of the geodesic rays (issuing from x_0) in X identified at x_0. Thus $\mathrm{Co}\,\partial_{\mathrm{geo}} = \{d,\delta\}$, $d \in [0,\infty)$, $\delta \in \partial_{\mathrm{geo}}$, such that $(0,\delta)$ is identified with x_0. We denote by $A_1 : \mathrm{Co}\,\partial_{\mathrm{geo}} \to X$ the tautological map and observe that

$$\mathrm{dist}\, A_1(d,\delta), x_0) = d .$$

Next we let $A_\lambda(d,\delta) = A_1(\lambda d,\delta)$, we induce the metric on $\mathrm{Co}\,\partial_{\mathrm{geo}}$ by A_λ from X and set

$$\mathrm{dist}_\lambda = \lambda^{-1} \text{ (induced metric)} .$$

Since dist is convex on X, the metrics dist_λ increase with λ and converge for $\lambda \to \infty$ to some conical metric on $\mathrm{Co}\,\partial_{\mathrm{geo}}$, denoted dist_T on $\mathrm{Co}\,\partial_{\mathrm{geo}}$. The base of this cone with the corresponding matrix is called *Tits' boundary* and denoted $\partial_T X$ (see [B-G-S]). Notice that the identity map $\partial_T X \to \partial_{\mathrm{geo}} X$ is continuous but $\partial_{\mathrm{geo}} X \to \partial_T X$ is usually not continuous. For example, if X is hyperbolic, then $\partial_T X$ is discrete.

The cone $\mathrm{Co}\,\partial_T X$ is somewhat similar to $\mathrm{Con}_\infty X$ and so we expect the two may have equal dimensions. But this is not true, in general, as is seen in the following

Example. Let X equal $\mathbb{R}^2$ with a complete rotationally symmetric metric with $K \le 0$. Then $\partial_T X$ equals the circle S^1 if the total curvature of X is finite, i.e. $\int_{\mathbb{R}^2} K(x)dx > -\infty$, and $\partial_T X$ is discrete otherwise. On the other hand, if curvature decays at most quadratically, i.e. $|K(x)| \le \mathrm{const}\, \mathrm{dist}^{-2}(x,x_0)$, then $\mathrm{Con}_\infty X$ typically contains (infinitely many copies of) the one-point completion of the universal covering of $\mathbb{R}^2$ minus the origin. Thus we may have $\dim \operatorname{Co} \partial_T = 1$ and $\dim \operatorname{Con}_\infty X = 2$. But in this case there is a flat "leaf" $X' \in \mathcal{X}^+$ which saves the equality $\operatorname{Rank}^+_{\mathrm{IV}} = 2$.

Now, back to the general X with $K \leq 0$, we observe that every k-flat in X gives rise to an Euclidean $(k-1)$-sphere in $\partial_T X$. Thus

$$\operatorname{Rank}_{\mathrm{II}} \leq \operatorname{Rank}_{\mathrm{IV}} ,$$

(but it is not immediately clear if the geometry of quasi-flats makes $\operatorname{Rank}_{\mathrm{III}} \leq \operatorname{Rank}_{\mathrm{IV}}$ as well).

It is also clear that $\operatorname{Co} \partial_T$ isometrically embeds into $\operatorname{Con}_\infty X$ and so

$$\operatorname{Rank}_{\mathrm{IV}} \leq \operatorname{Rank}_{\mathrm{I}} .$$

The major reason why $\operatorname{Rank}_{\mathrm{IV}}$ may be *strictly* smaller than $\operatorname{Rank}_{\mathrm{I}}$ can be visualized as follows. Two "infinitely close" geodesic rays become disjoint in $\operatorname{Co} \partial_T$ if they diverge faster than *linearly*. On the other hand if they diverge (no faster than) *polynomially* they remain close in $\operatorname{Con}_\infty X$. (Look at the above example and also at nilpotent groups with dilations.) Yet we still hope that the distinction between "linear" and "polynomial" disappears in the process of our plusification and $\operatorname{Rank}^+_{\mathrm{IV}} = \operatorname{Rank}^+_{\mathrm{I}}$.

Geometry of ∂_{T} and Morse landscape at infinity. Much of the geometry of ∂_T can be rendered quasi-isometry invariant by looking at families of λ-Lipschitz maps $K \to S(R) \subset X$ where K is a polyhedron of a fixed size, $S(R)$ denotes the R-sphere in X around a fixed point x_0 in X and where $\lambda \sim R$ for $R \to \infty$. (More generally, one may look at the space $M_{R,\lambda}$ of λ-Lipschitz maps f of K to the complement of the R-ball in X around x_0 and study, as in Morse theory, the homology homomorphisms $H_*(M_{R,\lambda}) \to H_*(M_{R',\lambda'})$ for $R > R'$ and $\lambda < \lambda'$, for $R \to \infty$ and $\lambda \to \infty$, where $\lambda = \lambda(f)$ is looked upon as a Morse function on the space of maps in question). An intelligent choice of K would be a compact subpolyhedron in ∂_T and then our maps $K \to S(K)$ may approximate in a certain way isometric (or at least Lipschitz) maps $K \to \partial_T$. For example, let K be the k-sphere S^k, X be a symmetric space of dimension n and let us look at m-parametric families of maps for $m = n - 1 - k$. More specifically, let the parameter space be homeomorphic to S^m. Then we have the following obvious quasi-isometry invariant characterization of $\operatorname{rank} X$,

$\operatorname{rank} X \leq k+1 \Leftrightarrow$ *for each R there exists a map $S^k \times S^m \to S(R)$ of degree one whose restriction to each k-sphere $S^k \times s$ is λ-Lipschitz for $\lambda \leq \operatorname{const} R$.*

Here is a related

Question. Let X and X_1 be n-dimensional symmetric spaces of non-compact type. When does there exist a proper Lipschitz map $f : X \to X_1$ of non-zero degree, such that

$$\operatorname{dist}_{X_1}(f(x_0), f(x)) \geq \operatorname{dist}_X(x_0, x)$$

for a fixed $x_0 \in X$ and an arbitrary x ?

(f) *On the local geometry of singular spaces with $K \leq 0$.* Many of the difficulties in comparing different Rank's can be reduced to local problems concerning the structure of singularities of spaces with $K \leq 0$ and with $K \leq 1$ (Tits' boundary has $K \leq 1$). For example, the inequality $\operatorname{Rank}_{\mathrm{I}} \leq \operatorname{Rank}^+_{\mathrm{II}}$ tells you that whenever the space $Y = \operatorname{Con}_\infty X$ has $\dim Y = k$ then Y contains subsets $Y_\varepsilon \subset Y$ of size ε which look like the Euclidean ε-balls for small $\varepsilon \to 0$. (Of course, one could avoid all trouble by building-in such a property in the definition of dimension).

In order to understand the local geometry of a singular space X at x_0 (notice that ∂_T and $\operatorname{Con}_\infty$ of a non-singular space may be highly singular) we consider the tangent cone $\operatorname{Co}(X, x_0)$ which is defined similarly to $\operatorname{Co} \partial_T$, but now we use the geodesic segments (of finite length) issuing from x_0 and take the limit of $\operatorname{dist}_\lambda$ for $\lambda \to 0$ rather than for $\lambda \to \infty$ as in (e). (There is an alternative definition with ulterlimits, but this seems unneeded for the present discussion). A point x_0 is called *regular* if $\dim \operatorname{Co}(X, x_0) = \dim X$. (Here, what we truly need is $\dim \operatorname{Co}(X, x_0) \geq \dim X$). Now, if we are able to establish the existence of a regular point, we may prove that X contains an infinitesimal almost Euclidean ball of dimension $k \leq \dim X$, by induction on k as the base of the cone $\operatorname{Co}(X, x_0)$ is a space with $K \leq 1$ of dimension $\dim \operatorname{Co}(X, x_0) - 1$.

Questions. Does every (sufficiently complete) X with $K \leq 1$ contain a regular point ? If not, what should we add to $K \leq 1$? (See [B-G-P] where a similar question is tackled for $K \geq 0$).

(g) II $\sim$ V $\sim$ VI ? Suppose X admits a self-map parallel to the identity which contracts k-dimensional volume. Then an easy argument shows that X satisfies the linear isoperimetric inequality for $\operatorname{Fill Vol}_k$, i.e. $FV_k \sim \ell$. It follows that $\operatorname{Rank}_{\mathrm{VI}} \geq \operatorname{Rank}_{\mathrm{V}}$. Another obvious relation here is

$$\operatorname{Rank}_{\mathrm{II}} \leq \operatorname{Rank}_{\mathrm{V}}$$

as k-flats have $FV_k \sim \ell^{\frac{k}{k-1}} \gtrsim \ell$. (In fact even "big pieces" of k-quasi-flats provide a lower bound $\operatorname{Fill Vol}_k$, and so $\operatorname{Rank}_{\mathrm{V}} \geq \operatorname{Rank}^{+}_{\mathrm{III}}$).

Next, if X is a symmetric space (or a product of hyperbolic spaces), a volume contracting map can be actually constructed either with the gradient flow of the horofunction associated to a *regular* geodesic or with the gradient flow of the distance function to a maximal flat. This gives the inequality

$$\operatorname{Rank}_{\mathrm{VI}} \leq \operatorname{Rank}_{\mathrm{II}}$$

for the symmetric spaces X, but one does not know if $\operatorname{Rank}_{\mathrm{VI}} \leq \operatorname{Rank}^{+}_{\mathrm{II}}$ for general X with $K \leq 0$.

(h) VII $\sim$ IV ? One expects, that (at least after plusifications) "infinitely close" geodesic rays issuing from x_0, either diverge linearly or exponentially. If we have a family of rays of dimension $> \dim \partial_T X$, some of them should exponentially diverge, which would prove the inequality

$$\operatorname{Rank}_{\mathrm{VII}} \leq \operatorname{Rank}_{\mathrm{IV}} ,$$

while the reverse inequality faces the problem similar to that (but seemingly easier than) discussed in (f). Notice, that no problem arises for symmetric spaces where we do know the equalities

$$\operatorname{Rank}_i = \operatorname{Rank}^{+}_j \quad \text{for} \quad i,\ j = \mathrm{I}, \ldots,\ \mathrm{VII}, \ldots .$$

Hyperbolic corank. Consider all geodesic hyperbolic spaces Y quasi-isometrically embedded into X (i.e. $Y \subset X$ have bounded distortion) and let

$$\operatorname{corank}_{hy} X = \sup_{Y} \dim \partial_\infty Y$$

over all such $Y \subset X$.

Examples. (a) If X is hyperbolic (i.e. $\operatorname{rank} X = 1$) then $\operatorname{corank}_{hy} X = \dim \partial_\infty X$.

(b) As we have seen in §3., the hyperbolic space $H^{k+1}_{\mathbb{R}}$ admits a quasi-isometric embedding into the Cartesian product of k copies of $H^2_{\mathbb{R}}$. Thus $\operatorname{corank}_{hy} \overset{k}{\underset{1}{\times}} H^2_{\mathbb{R}} \geq k$. In fact, a similar argument shows that the Cartesian product of k hyperbolic spaces, $X = X_1 \times X_2 \times \ldots \times X_k$ always has

$$\operatorname{corank}_{hy} X \geq \sum_{i=1}^{k} \dim \partial_\infty X_i ,$$

and one expects there is equality in this case.

(e) Let X be a symmetric space. Then

$$\operatorname{corank}_{hy} X \geq n - k ,$$

for $n = \dim X$ and $k = \operatorname{rank} X$.

Proof. The key property of X is the existence of a *regular* geodesic ray $r \subset X$ issuing from a given point $x_0 \in X$. This ray is contained in a *unique* maximal flat F (by the very definition of regularity) and nearby flats containing x_0 exponentially diverge along r. Thus if we take a small $(n-k)$-dimensional disk S transversal to F at some point $x \in r \subset F$ away from x_0, then the geodesic rays starting from x_0 and meeting S exponentially diverge forming a hyperbolic subspace Y in X with $\partial_\infty Y = S$. Q.E.D.

Probably, $\text{corank}_{hy} X = n - k$ and this may be true for all "reasonable" spaces with $K(X) \leq 0$ (and generalize in the plusified form to non-reasonable X).

Remarks. (a) The notion of corank_{hy} makes sense for all metric spaces. Here we should mention groups X with infinitely many ends which contain infinite regular trees and thus have $\text{corank}_{hy} \geq \varepsilon > 0$ (as explained in (6) below). Also, solvable Lie groups X may contain large hyperbolic subspaces Y. For example if $X = \mathbb{R}^{n-1} \rtimes \mathbb{R}$ with the implied eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$, where at least ℓ of them are non-zero and have the same sign, then $\text{corank}_{hy} X \geq \ell$ (and, probably, $= \ell$ if ℓ is the maximal number of such λ_i).

(b) There are many refinements of corank_{hy}. For instance, instead of $\dim \partial_\infty Y$, one may use a Hausdorff-type dimension expressed by the exponential rate of growth of concentric R-balls in Y. This distinguishes, for example, full fledged hyperbolic trees from something like $\mathbb{R}$ which is hyperbolic only in name.

6.B$_3$. Periodic flats. A k-flat $F \subset X$, where X is acted upon by some Γ, is called periodic if there exists $\Gamma_0 \subset \Gamma$ preserving F and such that F/Γ_0 is compact. Notice, that Γ_0 in this case is commensurable to $\mathbb{Z}^k$.

Problem. When does the existence of a k-flat $F \subset X$ imply the existence of a periodic flat $F' \subset X$? When can one approximate F by periodic flats ? Here the convergence $F_i \to F$ may mean one of two slightly different things:

(1) for every R there exist R-balls $B_i \subset F_i$ which Hausdorff converge to the R-ball $B_0(R) \subset F$ around a fixed point $x_0 \in F$.

(2) $\text{dist}_{\text{Hau}}(B_i, B_0(R)) \leq \text{const} < \infty$ for some balls B_i of radii $R_i = R \to \infty$.

The positive solution to the approximation problem (in the sense of (1)) has been known since long for Γ acting on *symmetric* spaces X with $\text{Vol}\, X/\Gamma < \infty$ (see [Most]$_2$). Recently, Bangert and Schroeder (see [Ba-Sch]) solved the existence problem for periodic flats for real analytic manifolds X with $K \leq 0$ and X/Γ compact. Yet the existence problem is still open for 2-polyhedra X with $K \leq 0$ and X/Γ compact.

The periodic flat problem can be extended to general groups Γ as follows. Let Γ contain a subset quasi-isometric to $\mathbb{Z}^k$. Does then Γ contain $\mathbb{Z}^k$? For $k = 1$ this is equivalent to the Burnside problem on periodic groups as Γ contains a quasi-isometric copy of $\mathbb{Z}$ if and only if Γ is infinite.

One can generalize further by starting with a quasi-isometric embedding of a general kind of group Γ_0 (instead of $\mathbb{Z}^k$) into Γ and then seeking a subgroup $\Gamma_0' \subset \Gamma$ which is in (a suitable sense) commensurable t Γ_0. Here is a more specific problem. Given Γ_0, find a *periodic* (i.e. pure torsion) group Γ such that Γ_0 quasi-isometrically embeds into Γ. (The solution seems within reach if Γ_0 is free or free Abelian.)

6.B$_4$. The space of flats. Let $\mathcal{F}_k$ denote the space of k-flats in X with the topology of the Hausdorff convergence on bounded subsets.

Question. Up to what extent is $\mathcal{F}_k$ quasi-isometry invariant ?

To avoid minor complications one may identify *parallel* flat (i.e. those with finite mutual Hausdorff distance) and reiterate the question for the space $\overline{\mathcal{F}}_k$ of the classes of parallel flats. Also one should keep in mind the plusification of 6.B$_1$ to avoid another class of trivial counterexamples.

The spaces $\mathcal{F}_* = \bigcup_{k=1}^{\infty} \mathcal{F}_k$ and $\overline{\mathcal{F}}_* = \bigcup_{k=1}^{\infty} \overline{\mathcal{F}}_k$ have, besides the topology, another neighborhood relation. For example, two geodesics (i.e. 1-flats) may be asymptotic in one direction, i.e. contain subrays with bounded mutual Hausdorff distance. In general, for two flats F and F' in X and given ρ one has a convex subset $\Delta_\rho \subset F \times F'$ defined by

$$\Delta_\rho = \{(x, x') \in F \times F' \mid \text{dist}_X(x, x') \leq \rho\} ,$$

whose shape is basically independent of ρ as $\rho \to \infty$. In fact Δ_ρ is essentially determined by two convex subsets $F_\rho \subset F$ and $F'_\rho \subset F$ which are $F_\rho = F \cap U_\rho(F')$ and $F'_\rho = F' \cap U_\rho(F)$ where U_ρ denote the ρ-neighborhoods.

Example. For the maximal flats in the *symmetric* space X these F_ρ are (Hausdorff equivalent to) Weyl chambers in the flats along which the flats in question are asymptotic.

Question. Suppose we have two discrete cocompact actions of a group Γ on two spaces, say X and Y, with $K \leq 0$. Does there exist a Γ-equivariant homeomorphism between $\overline{\mathcal{F}}_*(X)$ and $\overline{\mathcal{F}}_*(Y)$ preserving (in a suitable sense) the above (asymptotic) neighborhood structure ?

Subquestions. Are the geodesic boundaries $\partial_{\text{geo}}(X)$ and $\partial_{\text{geo}}(Y)$ Γ-equivariantly homeomorphic ? Are they isometric for the Tits metric ? (It was pointed out to me by somebody at this meeting that if X and Y are *singular* spaces with $K \leq 0$, then one does not even know whether $\partial_{\text{geo}}(X)$ and $\partial_{\text{geo}}(Y)$ are homeomorphic without the Γ-equivariance requirement).

The answer to the above question is, obviously, positive if periodic flats are (known to be) dense in $\mathcal{F}_*$, since periodic flats correspond to Abelian subgroups in Γ. This applies, for example, to the case where X (but not necessarily Y) is a symmetric space and was used by Mostow in his proof of the rigidity theorem for Rank $X \geq 2$. (The homeomorphism between $\mathcal{F}_1(X)$ and $\mathcal{F}_1(Y)$ as well as the density of the periodic flats (geodesics) is also well-known for the negative curvature (see [Mor], [Kli], $[\text{Gro}]_3$) and more generally for $K \leq 0$ and rank $X = 1$, but it has no such a spectacular corollary as Mostow rigidity since the asymptotic neighborhood relation on $\mathcal{F}_1$ is not much of a structure). In the general non-symmetric case, one can establish certain measurable Γ-equivariant correspondence between $\mathcal{F}_*(X)$ and $\mathcal{F}_*(Y)$ but one does not know how to turn it into an actual homeomorphism (compare $[\text{Gro}]_{21}$).

Generalization. One can (quasi)-generalize the definition of $\mathcal{F}_k$ to an arbitrary Γ by taking the space $\mathcal{M}_k$ of quasi-isometric maps $f : \mathbb{Z}^k \to \Gamma$. This space is acted upon by $\mathbb{Z}^k$ and by Γ, such that the action commute, and then one defines the space of quasi-flats of Γ by

$$Q\mathcal{F}_k = \mathcal{M}_k / \mathbb{Z}^n \ .$$

(Notice that the double action is also implicitly present in the definition of $\mathcal{F}_k$, where we use, instead of $\mathcal{M}_k$ the space $\mathcal{I}$ of isometric embeddings $\mathbb{R}^k \to X$, which is naturally acted upon by the isometry group $\text{Iso}\,\mathbb{R}^k$, containing $\mathbb{R}^k$ as a subgroup, and also by Γ. Here, there is some choice in the definition of $\mathcal{F}_k$: it may be either $\mathcal{I}/\text{Iso}\,\mathbb{R}^k$ or $\mathcal{I}/\mathbb{R}^k$, but, in fact, we are truly interested in the space $\mathcal{I}$ itself with the action of $(\text{Iso}\,\mathbb{R}^k) \times \Gamma$). The space $Q\mathcal{F}_k$, as it stands, appears too big and ugly. Yet it may admit a pretty Γ-equivariant subquotient taking the role of $\mathcal{F}_k$.

6.C. Geometry of surfaces for $K \leq 0$. First, to make the point, let V be a closed manifold with $K \leq 0$ and $f_0 : S \to V$ be a C'-smooth (or, at least, Lipschitz) map of a closed surface into V. Let us try to deform f_0 in order to diminish the induced Riemannian metric. Eventually such a deformation stops as we arrive at a *tight* Lipschitz map $f_1 : S \to V$ whose every deformation increases the induced length of some curve in S. Then, one knows, that the induced metric in S, albeit singular, has $K \leq 0$.

The above can be generalized in several ways.

(1) One may work with surfaces with boundary where the boundary is fixed under the deformation.

(2) One may have a continuous family of maps $f_0 : S \times T \to V$ which deforms to a tight family f_1.

(3) One may work upstairs on X with a discrete but possibly non-free Γ-action and apply the tightening deformation to Γ-equivariant maps of Γ-surfaces to X.

Remark. There are particular minimization (tightening) deformations commonly used in geometry, such as the *heat flow* in the space of maps, but these do not enter our play at this stage.

Examples. (a) If S is homeomorphic to S^2 then a tightening deformation may terminate only at a constant map as no metric on S^2 has $K \leq 0$. (One should make precise here the meaning of the sign of the curvature for degenerate maps but this is not a serious problem).

(b) Let S be homeomorphic to the 2-torus. Then the terminal tight f_1 is either degenerate or it lands onto a *flat* torus in V, since non-flat points tend to make the induced curvature in S negative. (An actual proof is as follows. If $f : T^2 \to V$ is tight the induced metric has $K \leq 0$ and, hence, it is flat, and the torus acts by isometries. Now, using CC in 6.B. we may average f with its isometric translates by the torus which shortens the induced metric unless it is *extrinsically* flat, which means $\widetilde{f} : \widetilde{T}^2 = \mathbb{R}^2 \to X = \widetilde{V}$ is a 2-flat).

(c) Let S be a disk. Once we make it tight with $K \leq 0$ we can use the 2-dimensional geometry of such disks. For example we obtain the sharp (isoperimetric) inequality for the filling area in V as it is known for disks, etc.

(d) Let S be a cylinder. Deforming to $K \leq 0$ shows (by an elementary geometry of cylinders with $K \leq 0$) that the two components of the boundary can be joined by a homotopy of curves whose length does not exceed that of the boundary. This implies that the conjugacy problem is solvable in $\Gamma = \pi_1(V)$.

(e) If S is a closed surface of genus ≥ 2 there is no immediate visible geometric effect on S of the condition $K \leq 0$ which would not follow from the above. However there is something one can say about the universal covering of S, as it also carries a metric with $K \leq 0$. Let us make an explicit statement to make our point clear. Let V be a closed Riemannian manifold which is diffeomorphic (and, hence, Lipschitz equivalent) to a manifold with $K \leq 0$. Then there exists a constant $C = C(V)$ with the following property. Let S be a surface with a Riemannian metric and $f_0 : S \to V$ be a 1-Lipschitz map. Then there exists a homotopy $f_t : S \to V$, such that,

(i) For every $t \in [0,1]$ the map f_t is C-Lipschitz;

(ii) The Riemannian metric g_1 induced by f_1 on S satisfies a quadratic isoperimetric inequality for all disks D immersed into S with a constant independent of the immersion, i.e.

$$\operatorname{Area} D \leq C\,(\operatorname{length} \partial D)^2$$

where the area and length are measured in g_1.

This property does not seem to follow (at least immediately) from something more elementary, e.g. the isoperimetric inequality for V or the existence of a certain bicombing.

Example. *Small cancellation groups.* Let Γ be a small cancellation group which we assume, to make life easier, without torsion, for example, 1/6-groups (i.e. satisfying the "at most 1/6" condition). Let V be the 2-polyhedron corresponding to the implied 1/6-presentation. Then, every tight surface S in V is λ-Lipschitz equivalent to a surface with $K \leq 0$ where λ depend on V but not on S. Thus the above tightening discussion applies to V (after an obvious quasification). In particular, the final statement in (e) remains valid for this V.

Let us indicate some algebraic and topological problems where the geometry of surfaces of higher genus may prove useful.

Conjecture. - Let V be a compact manifold and $h' \in H^2(V;\mathbb{R})$ a cohomology class represented by a 2-form ω' on V. Then the following two conditions are equivalent

(1) The lift $\widetilde{\omega}$ of ω to the universal covering $X = \widetilde{V}$ is the differential of a *bounded* form, i.e. $\widetilde{\omega} = d\alpha$ for $||\alpha_x|| \leq \text{const} < \infty$.

(2) The class h' is *bounded* in the sense of $[\text{Gro}]_8$, i.e. it can be represented by a singular cochain $c(\sigma)$ which is bounded by a fixed constant on all singular simplices σ in V. (This is essentially equivalent for this class to be representable by a bounded cocycle $c'(\gamma_0, \gamma_1, \gamma_2)$, see §§ 6.B$_1$., 6.C$_1$.).

Remarks. (a) Notice that the existence of a bounded α with $d\alpha = \widetilde{\omega}$ does not depend on the choice of ω representing h'.

(b) If the fundamental group $\Gamma = \pi_1(V)$ is hyperbolic, then (1) and (2) are known to be equivalent. In fact a class h' satisfies (1) and (2) if and only if it is *aspherical* (i.e. vanishes on all 2-spheres in V). Then one can easily show that the conjecture is valid if $\Gamma = \pi_1(V)$ is the Cartesian product of hyperbolic groups.

(3) It is, probably, not hard to verify the conjecture for invariant forms $\widetilde{\omega}$ on symmetric spaces X. For example, if $\widetilde{\omega}$ is the Kähler form of a *Hermitian* symmetric X, then it is $d\alpha$ for a bounded α, and also it gives a bounded cohomology class on every Γ acting on X. (This is easy to show).

(4) The above evidence in favour of the conjecture is rather limited and it would be safe to make some extra assumptions on Γ ($K \leq 0$, semi-hyperbolic etc.) but even for $K(V) \leq 0$ we are able to prove (see below) only a rather weak version of the conjecture.

(5) The conjecture makes sense for the cohomology classes $h' \in H^k(V,\mathbb{R})$ for $k \geq 3$, but the implication (1) $\Rightarrow$ (2) fails to be true. For example, if V is a closed oriented k-dimensional manifold and $h' \in H^k(V;\mathbb{R})$ is the fundamental class, then the corresponding $\widetilde{\omega}$ on $X = \widetilde{V}$ is d(bounded) if and only if the fundamental group $\Gamma = \pi_1(V)$ is *non-amenable.* Yet h hoes not have to be bounded. For example if $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_1 is an infinite amenable group then h is unbounded, no matter how big Γ_2 is. (Example: take $V = T^{k-2} \times S$, where S is a surface of genus $g \geq 2$ and T^{k-2} is the $(k-2)$-torus).

6.C$_1$. Genera of 2-cycles and norms on homology. Take a homology class $h \in H_2(V)$ and consider all closed oriented surfaces (continuously mapped) in V representing h. If we choose some numerical invariants of S, say $\mathrm{In}_1(S)$, $\mathrm{In}_2(S), \ldots$, $\mathrm{In}_p(S)$, then we can characterize h by the set of the possible values of these invariants, denoted $\mathcal{INV}(h) \subset \mathbb{R}^p$.

Examples. (a) Area *of* h. Let $p = 1$ and $\mathrm{In}_1(S) = \mathrm{Area}\, S$. Then $\mathcal{INV}(h) \subset \mathbb{R}$ is a semi-infinite interval uniquely defined by its left end called

$$\mathrm{Area}(h) \underset{\mathrm{def}}{=} \inf_S \mathrm{Area}\, S\ .$$

Notice that $\mathrm{Area}(h_1 + h_2) \leq \mathrm{Area}(h_1) + \mathrm{Area}(h_2)$ and that

$$\mathrm{Area}_{\mathbb{R}}\, h = \lim_{i\to\infty} i^{-1}\, \mathrm{Area}(ih)$$

defines a *norm* on $H_2(V;\mathbb{R}) = H_2(V) \otimes \mathbb{R}$. A slightly non-trivial point here is the following *non-vanishing* property of this norm for *compact* manifolds V,

$$\mathrm{Area}_{\mathbb{R}}(h) = 0 \Leftrightarrow h \in \ \mathrm{Tor}\, H_2(V)\ .$$

(b) *Genus of* h. Denote by $g(h) =$ genus(h) the minimum of the genera of the closed connected oriented surfaces S representing h. This is again a subadditive function

$$g(h_1 + h_2) \leq g(h_1) + g(h_2)$$

and one can pass to the limit

$$g_{\mathbb{R}}(h) = \lim_{i\to\infty} i^{-1} g(ih)\ .$$

But now this norm may vanish on some non-torsion $h \in H_2(V)$. In fact, this norm is dual to the ℓ_∞-norm on $H^2(V;\mathbb{R})$ whose finiteness distinguishes the bounded cohomology (see [Gro]$_8$). For example, if the fundamental group $\Gamma = \pi_1(V)$ is amenable then $g_{\mathbb{R}}(h) = 0$ for all $h \in H_2(V)$. Moreover, the genus may be bounded on all of H^2. For example, $g(h) \leq \mathrm{const}_T$ if the fundamental group $\Gamma = \pi_1(V)$ is nilpotent (see [Bar-Ghy]). On the other hand if Γ is hyperbolic, then $g_{\mathbb{R}}(h) \neq 0$ unless h is torsion modulo the spherical part of $H_2(V)$ (i.e. the image of the Hurewicz homeomorphism $\pi_2 \to H_2$). This can be made more precise if $K(V) < 0$. Take our surface S_0 in V representing h and then make it tight by some area decreasing homotopy. The Gauss curvature of the resulting tight surface S is majorized by $K(V)$, i.e. if $K(V)$ is bounded by $-\kappa(s)$ at the points $s \in S$, then so is $K(S)$. Thus $g(S) \geq 1 + \pi \int_S \kappa(s) ds$. Therefore, if $K(V) \leq -\kappa$ on all of V, then

$$g(h) \geq 1 + \pi\, \kappa\ \mathrm{Area}(h) \qquad (*)$$

for all non-zero $h \in H_2(V)$ (where we do not even have to assume V is compact). We shall prove below a weak version of this result for $K \leq 0$, taking into account the position of h with respect to 2-flats in V.

Remark. The notion of the genus for H_2 was introduced by Thurston for 3-manifolds V, where Thurston used *embedded* surfaces. In this situation g often behaves as a norm without the $\mathbb{R}$-stabilization of ours.

(c) *Length of h.* For a connected surface S with a metric we define length S as the infimum of the lengths of the subgraphs (i.e. 1-dimensional subcomplexes) $R \subset S$ whose complements $S - R$ are topological 2-cells. (It is not hard to see that

$$\text{length}\, S < 4\,(\text{genus}\, S)\,\text{Diam}\, S\ ,$$

but there is no bound on $\text{Diam}\, S$ in terms of length S). One also knows that length S can be bounded in terms of $\text{Area}\, S$ and the length of the shortest non-contractible curve in S, called $\text{sys}\,(S)$, by length $S \le C$ genus $(S) \cdot \text{Area}(S)\big(\text{sys}\,(S)\big)^{-1}$, for some universal constant $C \le 2$, compare $[\text{Gro}]_{10}$).

Now we define length(h) as the infimum of the lengths of closed connected oriented surfaces representing h. If $X = \widetilde{V}$ satisfies a (filling) isoperimetric inequality controlled by $F_0A(\ell)$ (see §5.) then, obviously

$$\text{Area}\, h \le F_0 A\big(\text{length}(h)\big)$$

for all $h \in H_2(V)$. But there is no ***universal*** bound on Area(h) in terms of length(h) as is implicit in the examples in $6.\text{B}_1$. of classes in $H^2(\Gamma)$ non-representable by slow growing cocycles on Γ.

(d) *Genus and the geometry of flats.* Let V have $K(V) \le 0$ and let us describe the part of $H_2(V;\mathbb{R})$ which comes from 2-flats. Denote by $F_2(V)$ the space of 2-flats in V that are maps $\mathbb{R}^2 \to V$ which lifts to isometries (i.e. to flats) in the universal covering of V. The space $F_2(V)$ comes along with a natural action of $\mathbb{R}^2$ and every invariant measure μ on $F_2(V)$ gives rise to a homology class, called the *rotation "number"* $[\mu] \in H_2\big(F_2(V)\big)$ (or foliated cycle in the terminology of [Sul]). The projection of such a $[\mu]$ to $H_2(V;\mathbb{R})$ is called *flat* homology class in V. Clearly, flat classes constitute a linear subspace in $H_2(V;\mathbb{R})$.

Observation. *If $h \in H_2(V;\mathbb{R})$ is flat then $g_{\mathbb{R}}(h) = 0$.*

This has little to do with the curvature. Any time we have a foliated k-cycle whose leaves are "sufficiently amenable" (e.g. quasi-isometric to $\mathbb{R}^k$) the ℓ_1-norm of this cycle (generalizing $g_{\mathbb{R}}$ for $k \ge 3$) vanishes. (See $[\text{Gro}]_8$ and $[\text{Gro}]_{21}$ for the foliated version of the ℓ_1-norm suggested by A. Connes. *Warning*: Do not confuse this ℓ_1 with $\ell_{p=1}$ of §8.)

We do not know if the converse is true, i.e. if every h with $g_{\mathbb{R}}(h) = 0$ is flat but we can prove a weaker result concerning the pair of the invariants (genus, Area) on H_2. The corresponding subsets $\mathcal{INV}\,(h) \subset \mathbb{R}^2$ can be adequately described by the function Area(h, g) which is the infimum of the areas of the surfaces of genus g representing h. Notice that

$$\text{Area}\, h = \inf_{g \in \mathbb{Z}_+} \text{Area}(h, g)$$

and that Area(h, g) may be significantly bigger than $\text{Area}\, h$. For example, if $\Gamma = \pi_1(V)$ is hyperbolic then Area$(h, g) = \infty$ for every fixed g and all $h \in H_2(V, \mathbb{R}) -$ (a compact subset).

Claim. *Let $h \in H_2(V)$ be non-flat (after tensoring with $\mathbb{R}$) and let $g_i \in \mathbb{Z}_+$ be a sequence with $\liminf\limits_{i\to\infty}\ i^{-1} g_i = 0$. Then $\limsup\limits_{i\to\infty}\ i^{-1}\,\text{Area}(ih, g_i) = \infty$.*

Remark. What we really would like to know is Area$(ih) \gtrsim g(ih)$ for $i \to \infty$ but we are only able to estimate the larger function Area(ih, g).

Idea of the proof. Suppose we have a sequence of surfaces S_{i_j} in V representing $i_j h$ such that

$$\lim_{i_j \to \infty}\ i_j^{-1}\,\text{genus}\,(S_{i_j}) \to 0$$

and $\text{Area}\, S_{i_j} \lesssim i_j$. We may deform S_{ij} to tight surfaces without increasing the area and so we assume S_{i_j} are tight to start with. Then the average curvature of S_{i_j} goes to zero,

$$\int\limits_{S_{i_j}} K(s)ds / \,\text{Area}\, S_{i_j} \to 0\ ,$$

and so S_{ij} are getting flatter and flatter for $i_j \to \infty$. In fact it is not hard to show that the real cycles $i_j^{-1} S_{i_j}$ converge (or at least subconverge) to a flat foliated cycle representing our h. (The details are not hard and left to the reader).

Remarks and open questions. (a) It would be interesting to describe the annihilator of the space of flat cycles in $H^2(V;\mathbb{R})$. It is obvious that every $h' \in H^2(V;\mathbb{R})$ representable by a form ω' vanishing on the 2-flats is contained in this annihilator, i.e. $\langle h', h\rangle = 0$ for all flat $h \in H_2(V;\mathbb{R})$. Another interesting subspace in $H_2(V;\mathbb{R})$ annihilating the flat cycles consists of those h' for which the lifts $\widetilde{\omega}'$ of ω' to the universal covering $X = \widetilde{V}$ are d(bounded). In fact, if $\widetilde{\omega}' = d(\alpha)$ for a bounded 1-form α on X, then the integrals of ω' over large flat R-disks in V grow at most linearly (rather than quadratically in R) which makes vanish (by a simple argument) the integral of ω' over the flat cycles. The relation between these spaces is unclear. For example, does vanishing of ω' on the flats imply that $\widetilde{\omega}'$ is d(bounded) ? Or, does this vanishing insure that h' is a bounded (for the simplicial norm) class ? (It is easy to show by integrating ω' over tight surfaces that if ω is bounded by the sectional curvature of V, i.e. $|\omega(\tau)| \leq \mathrm{const}\,|K(\tau)|$ for all orthonomal 2-frames τ in V, then h' is bounded).

(e) Can one estimate the rate of decay of $i^{-1}g(ih)$ for $i \to 0$ (in the case where this quantity does decay) ? For example, what is a geometric characterization (in our case, where $K \leq 0$) of the subgroup in $H_2(V)$ on which the genus function is bounded ? Can one effectively estimate the function $\mathrm{Area}(h, g)$? Is the function $g(h)$ recursive ? (This is obvious if $g(h) \geq \mathrm{const}\,\mathrm{Area}(h)$).

(c) Suppose the norm $g_{\mathbb{R}}$ is non-degenerate on $H_2(V;\mathbb{R})$ (i.e. $g(h) \geq \mathrm{const}\,\mathrm{Area}(h)$ for some const > 0). What is the geometry of this norm ? For example is $g_{\mathbb{R}}$ smooth, piecewise smooth or piecewise linear on $H_2(V;\mathbb{R})$? What is the arithmetic of $g_{\mathbb{R}}$? For example is the number $\lim\limits_{i\to\infty} i^{-1}g(i\,h)$ rational (or at least algebraic) for $h \in H_2(V;\mathbb{Z})$?

Generalization to $H_k(V)$ for $k \geq 2$. Here we represent homology classes h by integral singular cycles $c = \sum\limits_i n_i\sigma_i$ where σ_i are singular k-simplices which are Lipschitz (we prefer them to continuous) maps of the standard k-simplex to V. We define $\mathrm{Vol}_j(\sigma_i)$ as the sum of the j-volumes of the j-faces of σ_i and set

$$\mathrm{Vol}_j(c) = \sum_i n_i \,\mathrm{Vol}_j(\sigma_i)$$

and

$$\mathrm{Vol}_j(h) = \inf_c \mathrm{Vol}_j(c)$$

over all c representing h. Notice that Vol_0 is proportional to the genus for $k = 2$ and Vol_1 essentially agrees with length. Also observe that the totality of the invariants Vol_j carries significantly more information than these quantities taken separately. Recall, that this "totality" is represented by the set of the possible values of the $k+1$ volumes on the cycles representing h which we denote in this case by $\mathcal{VOL}(h) \subset \mathbb{R}^{k+1}$. The asymptotic geometry of this function $\mathcal{VOL}(h)$ or, equivalently of its graph

$$\mathcal{VOL} = \bigcup_{h\in H_k(V)} \mathcal{VOL}(h) \subset \mathbb{R}^{k+1} \times H_k(V)\,,$$

contains a peculiar information about the topology of V and especially about $\pi_1(V)$, but even for locally symmetric spaces V with $K(V) \leq 0$ one knows very little about this $\mathcal{VOL}$. The situation is much better understood for $K < 0$ (and for the hyperbolic groups in general) where one can straighten singular simplices σ and make them very narrow, such that $\mathrm{Vol}_j(\sigma) \leq \mathrm{const}$ for $j \geq 2$ (see $[\mathrm{Gro}]_{10}$, $[\mathrm{Gro}]_{14}$). Notice, that "straightening k-simplices" is an operation similar to the iterated convex combination or bicombing in 6.B. Even for $K \leq 0$ this straightening is not bound to be geodesic (though "geodesic" is O.K. for $K < 0$) as shown by the work of Savage (see [Sav]) who studied the *projective* bicombing on $X = SL_n/SO(n)$ for the usual embedding $X \subset P^m$, $m = \frac{n(n+1)}{2} - 1$, and proved that the top-dimensional projective simplices in X have their volumes (with respect to an SL_n-invariant volume element) bounded by a universal constant

const_n. Thus he proved that the fundamental class $h' \subset H^m(X/\Gamma;\mathbb{R})$ is bounded for every cocompact torsionless Γ acting on X.

Conjecture. Let X be a symmetric space and let r denote the maximal dimension of the totally geodesic subspaces $X' \subset X$ which isometrically split by $X' = X'' \times \mathbb{R}$. Then for every cocompact lattice Γ on X and each $k \geq r+1$, all cohomology classes in $H^i(\Gamma;\mathbb{R})$ are bounded.

Notice, that this conjecture looses any meaning if for some reason $H^i(\Gamma;\mathbb{R}) = 0$, and it should be reformulated in a better way. Here is a possibility. There exists, conjecturally, a Γ-equivariant multiconing (or combing if you like this word better)

$$C_k : X^{k+1} \times \Delta^k \to X, \quad \text{for every } k > r$$

which has the formal properties of the convex combination,

$$(x_0, \dots, x_k,\ p_0, \dots, p_k) \mapsto \sum_{i=0}^{k} x_i p_i$$

(though we do not require commutativity, compare 6.B$_1$.) and such that the image of each simplex $C_k((x_0, \dots, x_k) \times \Delta^k)$ has volume $\leq$ const for some "const" depending on X and k.

Example. Let $X = X_1 \times X_2$ where X_j have strictly negative curvature for $j = 1,\ 2$, and certain dimensions k_j. Then the above multiconing does exist starting from $k = r+1 = \max(k_1, k_2)+2$, as a simple argument (using products of simplices $\Delta_1^{k_1} \times \Delta_2^{k_2}$) shows.

There is a version of the above conjecture where r is defined as the maximal dimension of linear subspaces $T \subset T(V)$ on which the curvature (tensor) of V degenerates in the sense that there exists a unit vector $t_0 \subset T$, such that $K(t_0 \wedge t) = 0$ for all $t \in T$. With such an r one may expect the conclusion of the conjecture to hold true for all manifolds with $K \leq 0$ (and there is an obvious reformulation which applies to singular spaces). On the other hand it is not clear what should be a global quasi-isometry invariant version of the curvature condition that could extend this conjecture to semi-hyperbolic spaces. A warning comes from the following

Example. Take the 2-torus minus two open disks, multiply it by the circle S_1 and glue the two boundary components, both diffeomorphic to $S^1 \times S^1$ by a diffeomorphism switching the S^1-generators. The resulting 3-manifold V obviously has a metric with $K \leq 0$, its fundamental class in $H^3(V)$ is unbounded, (i.e. *the simplicial volume* of V is zero, see $[\text{Gro}]_{2,8}$), but the universal covering $X = \widetilde{V}$ admits no isometric (quasi-isometric ?) splitting.

6.C$_2$. The commutator norm and related invariants. Let $[\Gamma,\Gamma] \subset \Gamma$ denote the commutator subgroup and $\text{com}(\gamma)$, $\gamma \in [\Gamma,\Gamma]$ the minimal number of commutators needed to represent γ. In other words, if we take the set of commutators $\{\Gamma,\Gamma\} \subset [\Gamma,\Gamma]$ for the generating set, then $\text{com}(\gamma) = \text{dist}(\gamma, \text{id})$ in the corresponding word metric. This metric, denoted com di may be also defined as

$$\text{com di}(\gamma_1, \gamma_2) = \text{com}(\gamma_1 \gamma_2^{-1})$$

and it is biinvariant on Γ. If Γ is realized by $\pi_1(V)$ then com di equals minus the maximum of one half of the Euler characteristics of the compact connected oriented surfaces with two boundary components representing given γ_1 and γ_2 in Γ.

The question one wishes to address is that of the asymptotic geometry of $[\Gamma,\Gamma]$ with this metric com di, and in particular the growth-rate of $\text{com}(\gamma)$ for $\gamma \in [\Gamma,\Gamma]$. More specifically one wants to know the behavior $\text{com}(\gamma^i)$ for a fixed $\gamma \in [\Gamma,\Gamma]$ and $i \to \infty$. If $K(V) < 0$ (or more generally, if Γ is hyperbolic) one knows (see $[\text{Gro}]_{5,14}$) that $\text{com}(\gamma^i) \sim i$ for all non-torsion elements γ in $[\Gamma,\Gamma]$. This is seen by realizing γ by a closed geodesic in V and by looking at minimal surfaces in V spanning γ^i corresponding to the product of commutators. Thus one sees a close relation between com and the genus function on H_2. In fact, if we fix a closed curve $\gamma \in V$, one can speak of the genus on the relative homology $H_2(V,\gamma)$ in the same way we did in the absolute case. (One can reduce the relative case to the absolute one by doubling V across γ without disturbing the condition $K \leq 0$.) Unfortunately, here as in the absolute case, all what we have to offer are

Open questions. For which Γ is the function com on $[\Gamma, \Gamma]$ bounded ? This is known for nilpotent groups (see [Bar-Ghy]) and appears very unlikely for fundamental groups $\Gamma = \pi_1(V)$, where $K \leq 0$, unless Γ is virtually Abelian. In fact, if $K \leq 0$, one may expect $\lim\limits_{i \to \infty} i^{-1} \operatorname{com}(\gamma^i) > 0$ for most cases with an explicit list of exceptions. Then we want to know for which groups Γ the function $\operatorname{com}(\gamma)$ is computable on all of $[\Gamma, \Gamma]$ or at least on a given cyclic subgroup $\{\gamma^i\} \subset \Gamma$. (See [Bav] for some information.)

Relation between com *and* $\ell_1 H_2$. Consider the complex of singular ℓ_1-chains in V which are infinite combinations $\sum\limits_i r_i \sigma_i$ of singular simplices σ_i of a given dimension k (now we only care for $k = 2$) where the real numbers r_i satisfy $\sum\limits_i |r_i| < \infty$. We denote the homology of this complex by $\ell_1^\Delta H_*(V)$ and observe that there is a natural pairing between $\ell_1^\Delta H_*(V)$ and $\ell_\infty^\Delta H^*(V)$, that is the cohomology of the complex of bounded cochains.

Example. Let S^1 be the circle and h the generator of $H_1(S^1)$ represented by the usual cycle which is the loop σ generating $\pi_1(S')$. Then this σ is the boundary of an obvious telescopic 2-chain (see [Gro]$_8$). It follows that every relative class in $H_2(V, \gamma)$, represented by a surface S in V with the boundary curve γ, gives rise to a class in $\ell_1^\Delta H_2(V)$. If $H_2(V)$ is finitely generated (e.g. V is compact) and the classes h corresponding to the surfaces S with $\partial S = \gamma$ all non-zero in $\ell_1^\Delta H_2(V)$ then, clearly, $\liminf\limits_{i \to \infty} i^{-1} \operatorname{com}(ih) > 0$.

Now, in order to give a geometric criterion for non-vanishing of classes in $\ell_1^\Delta H_*$ we shall slightly modify the definition in order to make possible to integrate differential 2-forms over 2-cycles. Thus we consider *$\ell_{1,1}$-chains* $\sum\limits_i r_i \sigma_i$, where $\sum\limits_i |r_i| < \infty$ as well as

$$\sum_i |r_i| \operatorname{length}(\partial \sigma_i) < \infty$$

and

$$\sum_i |r_i| \operatorname{area}(\sigma_i) < \infty \ .$$

Example. Let V be a complete surface with $K(V) = -1$ with $\operatorname{Area} V < \infty$ and (possibly) with cusps. Then the fundamental class of V is (obviously) representable by a $\ell_{1,1}$-cycle (compare [Gro]$_8$).

Let us generalize this example by attaching a cusp C to a closed curve $\gamma \subset V$. Such a cusp C topologically is $\gamma \times [0, \infty]$ for $\gamma \times 0 = \gamma$, and C carries a complete metric of constant negative curvature and finite area. Denote by $V_\gamma \supset V$ the result of the attaching the cusp $C = C_\gamma$ to V and observe that the inclusion is the homotopy equivalence. Yet the geometry of V_γ is somewhat better than that of V because every relative class h_0 in $H_2(V, \gamma) = H_2(V_\gamma, \gamma)$ gives rise to a class $h \in \ell_{1,1}^\Delta H_2(V_\gamma)$ as is clear from the preceding discussion.

Observation. *Suppose V_γ carries a metric with $K \leq 0$ which is cuspidal on $C_\gamma \subset V_\gamma$. Then the above class, $h \in \ell_{1,1}^\Delta H_2(V_\gamma)$ does not vanish*, provided ∂h_0 is non-zero in $H_1(\gamma)$.

Proof. It suffices to exhibit a bounded cocycle h' such that $\langle h', h \rangle \neq 0$. Let ω be a non-negative and not identically zero 2-form supported in C_γ (where the curvature is constant negative) and let us define the cocycle representing h' by integrating ω over geodesic simplices Δ in V_γ. Since the integral $\int\limits_\Delta \omega$ does not exceed a constant times the total curvature of Δ (as Δ has non-positive induced curvature) it is bounded by $\operatorname{const} \pi$ and so h' is indeed bounded.

Corollary. *If $K(V) \leq 0$ and $K(V)$ is strictly negative at some point $v \in \gamma$, then* $\operatorname{com}(\gamma^i) \gtrsim i$.

One can also see with the above argument, that a manifold V with $K(V) < 0$ on a dense subset in V has *infinitely* generated groups $\ell_1^\Delta H_2$ and $\ell_\infty^\Delta H^2$ apart from a few trivial cases (which are easy to isolate). This is similar to what we know (and prove by a parallel argument) for hyperbolic groups Γ : *If Γ is non-elementary, then $\ell_1^\Delta H_2(\Gamma)$ and $\ell_\infty^\Delta H^2(\Gamma)$ are infinitely generated. Furthermore, if $\Gamma \to \Gamma'$ is a surjective homeomorphism then $\ell_1^\Delta H_2(\Gamma)$ goes onto $\ell_1^\Delta H_2(\Gamma')$.*

Generalization. Suppose we have several elements $\gamma_1, \dots \gamma_k$ in Γ represented by closed curves in V and let $\text{com}(\gamma_1, \dots, \gamma_k)$ be minus the maximum of the Euler characteristic of the compact connected oriented surfaces bounded by these curves. These higher "commutator norms" seem to carry some extra information for $k \geq 3$ but there are only a few things I know about them.

(I) Suppose $\gamma_1, \dots \gamma_k$ are (realized by) closed geodesics in a manifold V with $K \leq 0$, such that K does not vanish at some points $v_j \in \gamma_j$, $j = 1, \dots, k$. Then one can bound $\text{com}(\gamma_1^{i_1}, \dots, \gamma_k^{i_k})$ from below by const $\sum_{i=1}^{k} |i_j|$, for some const > 0. This is seen again with minimal surfaces filling-in the geodesic curves $\gamma_j^{i_j}$. In fact, one can similarly estimate com of any system of ℓ curves, say $(\gamma_1', \gamma_2', \dots, \gamma_\ell')$ where each γ_j' is a multiple of some of the curves $\gamma_1, \dots, \gamma_k$. (In other words, some γ_j may be used several times). It follows, that the *normal* subgroup in Γ normally generated by $\gamma_j^{i_j}$ for sufficiently large j is *free* and, moreover, $\gamma_j^{i_j}$ are normally freely independent provided the curves γ_j are disjoint.

Remark. The above discussion also applies to hyperbolic groups but these do not supersede the above $\Gamma = \pi_1(V)$ as we allowed K to be zero somewhere on V. In fact, the fundamental groups Γ of manifolds V with $K \leq 0$, such that $K < 0$ at some point $v \in V$ *generalize* in a certain way the hyperbolic groups rather than the other way around. (Notice, we do not need V in the above discussion to be compact or smooth. It can be rather general singular space or orbispace with $K \leq 0$). This suggests a promising direction of a true generalization of hyperbolic groups which would include these Γ. The idea may be that such a Γ is hyperbolic relative the semi-hyperbolic part of Γ concentrated on V where the curvature may vanish. Another feature of these Γ is that they are semi-hyperbolic and almost surely (in a probabilistic sense) hyperbolic. For example, if we take a (long) random closed geodesic γ in V it spends (or should spend if we make the definitions right) a definite percentage of time in the region where $K \leq -\varepsilon < 0$. Then the normal subgroup $N_\gamma \subset \Gamma$ generated by γ is free and has infinite index in Γ. Thus, if we attach the relation $\gamma = 1$ to Γ the resulting group $\Gamma' = N_\gamma$ will be, essentially, as hyperbolic as Γ, but there is no general formalism to make this precise. In particular, one runs into a technical problem if one iterates the above by taking a random $\gamma' \in \Gamma'$, then $\gamma'' \in \Gamma'' = \Gamma'/N_\gamma$, etc. (If $K(V)$ is strictly negative, attaching a random relation keeps us within the realm of hyperbolic groups, compare conformal hyperbolicity in Appendix to §8.).

To give more substance to our discussion let us recall a couple of standard examples of manifolds and singular spaces, where $K < 0$ almost everywhere.

($\square$) Cubical polyhedra with no-Δ-condition (see $[\text{Gro}]_{14}$) have $K < 0$ except for a union of subtori. In particular, Davis' reflection groups Γ fit the above discussion.

($\square\square$) Let V_0 be a complete n-dimensional manifold with constant negative curvature which has two *cusps* C_0 and C_1. If we chop away these cusps, we obtain a manifold V_0' with a boundary. The boundary consists of two components $B_0 = \partial C_0$ and $B_1 = \partial C_1$ whose lifts to the universal cover H^n of V_0 are horospheres. (The cusps C_0 and C_1 we have chopped away, were covered by the corresponding horoballs). Thus B_0 and B_1 are flat Riemannian manifolds and we now suppose they are diffeomorphic. Then we fix a linear) diffeomorphism between them, which always exists in every homotopy class, and consider the manifold V obtained from V_0' by gluing B_0 to B_1 according to the chosen diffeomorphism.

Claim. *The manifold V admits a metric with $K \leq 0$ and such that*

(i) *$K < 0$ outside the flat manifold B corresponding to $B_0 = B_1$* in V.

(ii) *A complement of a tubular neighbourhood of B in V is isometric to V_0'.*

Idea of the proof. Recall that $C_0 \subset V$ topologically splits as $C_0 = B_0 \times [0, \infty)$ and the metric (of constant negative curvature) in C_0 is $g = dt + e^{-t} g_0$ for the flat metric g_0 in B_0. Thus every "slice" $B_0 \times t$ is a flat manifold obtained by rescaling g_0. Now, our diffeomorphism $B_1 \leftrightarrow B_0$ transplants another flat metric to B_0, say g_1 coming from B_1. Then we consider the following metric g' on C_0 which interpolate between g_0 on B_0 and $e^{-t} g_1$ on $B_0 \times t$ for large t,

$$g' = dt^2 + e^t g_t \ ,$$

where $g_t = g_0$ for $t \leq 1$, $g_t = g_1$ for $t \geq t_0$ for a sufficiently large t_0 and such that $g_t = a(t)g_0 + b(t)g_1$ for $t \in [1, t_0]$ where $a(t)$ and $b(t)$ are smooth positive functions in t with small derivatives and such that

$$a(t) + b(t) = 1 \quad , \quad t \in [1, t_0] \ ,$$

$$a(1) = 1 \text{ and } a(t_0) = 0 \ .$$

As we can take t_0 as large as we want, we can make the derivatives $a'(t)$ and $b'(t)$ as small as we need and then the metric g' has its curvature arbitrarily close to that of g. Now, our (V_0, g') has two *isometric* cusps and gluing by an isometry is a trivial matter.

Remarks. (1) It is somewhat harder but yet sometimes possible to glue along a *non-linear* diffeomorphism (see [Fa-Jo]$_2$).

(2) If V_0 has variable curvature pinched between two negative constants $-\infty < -\kappa_1 \leq K(V_0) \leq -\kappa_2 < 0$, then B_1 and B_2 are (diffeomorphic to) infra-nilmanifolds and, in general, V has no metric with $K \leq 0$. Yet such V may be regarded semi-hyperbolic in a suitable sense as it carries metrics of *almost negative curvature* (in a sense we do not make precise in this paper, compare the discussion on definitions of semi-hyperbolicity later on).

On the other hand, if the cusps are isometric we may have negative curvature by gluing the cusps (rather than their boundary). Thus one can see that $\pi_1(V)$ survives attaching discs to random closed curves, but it becomes not so clear if the cusps are not isometric.

(3) Let V be an irreducible locally symmetric space of finite volume and rank ≥ 2. Then there is no infinite free normal subgroup $\Gamma' \subset \Gamma$ since by a theorem of Margulis Γ/Γ' must be finite if Γ' is infinite (see [Mar]$_5$). It follows that the lower bound on $\mathrm{com}(\gamma'_1, \dots, \gamma'_\ell)$ must fail. Suppose, to make it simpler, that the curves γ'_j are positive multiples of a single closed geodesic γ in V, i.e. $\gamma'_j = \gamma^{i_j}$. One wonders if the lower bound on $\mathrm{com}(\gamma_1^{i_j}, \dots, \gamma_1^{i_\ell})$ by const $\sum_{j=1}^{\ell} |i_j|$ remains valid when all i_j are of the same sign (or are subjected to another restriction of this type).

6.D. Group actions on simply connected spaces X with $\mathbf{K \leq 0}$. The basic geometric characteristic of an isometry γ on X is the *displacement function* $\mathrm{di}_\gamma(x) = \mathrm{dist}(x, \gamma(x))$. For example, if we take some finitely generated group Γ for X then the boundness of the function di_γ on Γ signifies that the centralizer $C_\gamma \subset \Gamma$ has finite index in Γ.

If X has $K(X) \leq 0$ then the function di_γ *is convex.* This is obvious (since $\mathrm{dist}_X(x_1, x_2)$ is convex) but very useful. It follows that each level $\mathrm{di}_\gamma^{-1}[0, d] \subset X$ is a convex subset in X and in particular the minimum set $M_\gamma \subset X$ is convex (but it may be empty).

The geometry of the displacement functions of isometries γ on X is most transparent if the group Γ constituted by these γ is cocompact on X. A significant part of this geometry has been carried over to bi-automatic groups in [Ge-Sh]$_4$ and similar results for more general bicomable groups have been announced in [A-B]. However, the following exemplary result for $K \leq 0$ obtained with the displacement geometry have not found yet a semi-hyperbolic counterpart.

Theorem (Gromoll-Wolf, Lawson-Yau, see [G-W], [L-Y]). *If X/Γ is compact and Γ splits into a Cartesian product, $\Gamma = \Gamma_1 \times \Gamma_2$ with infinite Γ_1 and Γ_2, then X isometrically splits as well, $X = X_1 \times X_2$, with unbounded X_1* and X_2.

Remarks. (a) In this theorem X should be a smooth manifold without boundary. If we allow singularities, we split not X itself but its *core* which is a certain convex Γ-equivariant subset X_0 in X.

(b) The theorem implies that if the centralizer of each $\gamma \in \Gamma$ has infinite index in Γ, then the splitting is preserved by some subgroups of finite index in Γ.

(c) This theorem, as stated, does not seem suitable for semi-hyperbolization. Here are some conjectural candidates for this role.

(1) Suppose X (and hence Γ) admits a quasi-isometric splitting $X = X_1 \times X_2$ where X_i, $i = 1, 2$, are quasi-geodesic spaces with positive asymptotic dimension. Is then Γ quasi-isometric to a Cartesian product of two *infinite* groups ?

Warning. One should not expect an actual splitting as a group Γ may act irreducibly on, for example, $H^2 \times H^2$. Yet, if such a situation arises, one may be able to identify what happens. For example, if some Γ is quasi-isometric to $H^2 \times H^2$ and yet no subgroup of finite index in Γ splits, then Γ must admit a cocompact isometric action on $H^2 \times H^2$. This particular example fits into another well-known conjecture: if a finitely generated group Γ is quasi-isometric to a symmetric space X then a subgroup $\Gamma' \subset \Gamma$ of finite index admits an isometric cocompact action on X. This is known for rank $X = 1$ (where the most difficult case is that of the hyperbolic plane H^2 for X) and the case $X = X_1 \times \ldots \times X_k$ for rank $X_i = 1$ appears easy (I checked it for products of hyperbolic spaces).

(c′) One can modify the assumption of the splitting conjectures by requiring some splitting (topological or bi-Lipschitz) of the tangent cone $\mathrm{Con}_\infty \Gamma$ where one may additionally insist on such a splitting being equivariant in a suitable sense.

6.D$_1$. Thick-thin decomposition and collapsible actions. Let $\mathrm{di}_\Gamma = \inf_{\gamma \neq \mathrm{id}} \mathrm{di}_\gamma$, consider the level

$$X_\delta = \{x \in X \mid \mathrm{di}_\Gamma(x) \leq \delta\}$$

and observe that for each point $x \in X_\delta$ there exists a non-empty finite set of elements in Γ, say $M_x(\delta) \subset \Gamma$, such that

$$\gamma \in M_x(\gamma) \Leftrightarrow \mathrm{di}_\gamma(x) = \mathrm{di}_\Gamma(x) \leq \delta \; .$$

Notice that typically $M_x(\delta)$ consists of a single element γ but as we move x in X the displacement $\mathrm{di}_\gamma(x)$ may grow and at some moment we have to switch to another element $\gamma' \in \Gamma$. Since X is connected (this is our standing tacit assumption) there must appear some point $y \in X$, where $\mathrm{di}_\gamma(y) = \mathrm{di}_{\gamma'}(y) = \mathrm{di}_\Gamma(y)$. Such points y typically form a hypersurface in X, then the points where three elements from Γ meet with the same displacement (equal di_Γ) form a codimension two subset etc.

Now, suppose the local geometry of X is bounded in the following weak sense: every ball of radius $r \leq 1$ in X contains at most N disjoint balls of radius $r/4$ for a fixed N independent of r and the center of the ball.

(Generalized and weakened) Margulis' lemma. *There exists a constant $\delta_0 = \delta_0(N) > 0$, such that if $\delta \leq \delta_0$, then, for every $x \in X$, the subgroup in Γ generated by $M_x(\delta)$ is virtually nilpotent.*

The proof easily follows from the polynomial growth theorem (see $[\mathrm{Gro}]_{6,8,10}$).

Historical remark. This lemma (in a finer form) for homogeneous manifolds is due to Zassenhaus and to Kazhdan and Margulis (see [Rag]). The latter authors introduced, with the aid of this lemma, the thick-thin decomposition of (Γ acting on) a *symmetric* space X in the course of their solution of Selberg's conjecture (see [Ka-Ma]. Some fragments of the Zassenhaus-Kazhdan-Margulis argument were repeatedly used by various authors, especially by those working on Fuchsian and Kleinian groups, unaware of the source of their ideas). Then around 1970 Margulis generalized Zassenhaus' theorem to Riemannian manifolds X with $|K(X)| \leq 1$ and $|\mathrm{Inj\,Rad}| \geq 1$ and applied that to the study of complete manifolds V with $-\infty < -\kappa_1 \leq K(X) \leq -\kappa_2 < 0$ and with finite volume. (Margulis has not written down his results, except for a few (partly erroneous) remarks in $[\mathrm{Mar}]_4$, but he gave several lectures around 1970 in Leningrad).

Definitions. (a) Pick up a positive constant $\delta \leq \delta_0(N)$ and divide X into two parts, X_{thick}, where $\mathrm{di}_\Gamma \geq \delta$ and X_{thin} where $\mathrm{di}_\Gamma \leq \delta$. The division $X = X_{\mathrm{thick}} \cup X_{\mathrm{thin}}$ is called the ***thick-thin decomposition.***

Comment. If Γ has no torsion, then $\mathrm{di}_\Gamma(x)$ on $V = X/\Gamma$ equals twice the *injectivity radius* of V at x and the thin part can be characterized by the property that its dimension on the scale $\approx \delta$ is strictly less than the topological dimension of V. In fact, the thin part can be mapped to a lower dimensional space with infra-nilmanifolds for fibers (see $[\mathrm{Gro}]_4$, [G-L-P], [Fuk], [C-F-G] where this idea is expressed with different degree of precision).

(b) All of the above motivates the following more abstract definition of a *collapsed* (or *thin*) action of Γ on X. First we define an $\mathcal{N}$*-structure* on (X,Γ) as a map which assigns to each $x \in X$ a finite subset $N_x \subset \Gamma$, such that $N_{\gamma(x)} = \gamma N_x \gamma^{-1}$, for all $\gamma \in \Gamma$.

(ii) For every $\gamma \in \Gamma$ the subset $X_\gamma \subset X$ defined by $x \in X_\gamma \Leftrightarrow \gamma \in M_x$ is *open*.

(iii) For each $x \in X$ the subgroup generated by N_x in Γ is an infinite virtually nilpotent group.

(iv) There exists a constant $d \geq 0$, such that $\mathrm{di}_\gamma(x) \leq d$ whenever $x \in X_\gamma$ (i.e. $\gamma \in M_x$).

Remark. If Γ has no torsion then the collection of the finite subsets in Γ, where each of them generates an infinite virtually nilpotent subgroup constitutes in an obvious way a simplicial complex, say $Q(\Gamma)$ where Γ acts by conjugation. Then the above $\mathcal{N}$-structure can be defined as a continuous equivariant map of V to $Q(\Gamma)$.

Example. Let Γ be a non-elementary hyperbolic group without torsion. Then it is easy to see that (X,Γ) admits no $\mathcal{N}$-structure as there is no corresponding map $X \to Q(\Gamma)$ (see §6.6. in $[\mathrm{Gro}]_{10}$ for homological and geometric criteria for the non-existence of maps $X \to Q(\Gamma)$ expressed in terms of the homotopy quotient $Q(\Gamma)/\!\!/\Gamma$).

Definition. *An action of* Γ *on* X *is called collapsible (or thin) if it admits an* $\mathcal{N}$*-structure.*

Remark. There are two aspects of the notion of collapsibility. First, this can be attributed to an abstract group Γ since for certain classes of spaces X acted upon by Γ the existence of an $\mathcal{N}$-structure depends only on Γ. For example, if Γ may act freely and cocompactly on a k-connected space X_k, for an arbitrarily large k, we can define collapsibility of Γ as that for all X_k. Another aspect of the collapse is geometrical. If, for example, X has $K(X) \leq 0$ then every collapse takes a particular geometric shape. Namely, X decomposes into convex subsets, say X_i, where each of them isometrically splits, $X_i = X_i' \times \mathbb{R}$. (Compare §6.6. in $[\mathrm{Gro}]_{10}$). Then if we look downstairs on $V = X/\Gamma$ we see this V sliced into flat tori of various dimensions as in the last example in 6.C_1.

Now, suppose V is a compact non-collapsible space with $K(V) \leq 0$ (where "collapse of V" refers to that of the universal covering $X = \widetilde{V}$ acted upon by $\Gamma = \pi_1(V)$). Then one expects the topology of V to be in certain respects non-degenerate. For example, let X be a topological n-manifold without boundary (or, may be, just a pseudo-manifold).

Conjectures. (a) If V is non-collapsible then the simplicial volume $\|V\|$ of V (i.e. the ℓ_1-norm of the fundamental homology class) does not vanish. (Notice that if V is collapsible then $\|V\| = 0$, see $[\mathrm{Gro}]_8$).

(b) Take a sequence of subgroups $\Gamma_i \subset \Gamma$ of finite indices such that $\mathrm{ind}\,\Gamma_i \underset{i\to\infty}{\longrightarrow} \infty$. (It may be safer to assume $\bigcap\limits_i \Gamma_i = \{\mathrm{id}\}$). Then the minimal number of generators in Γ_i goes to infinity for $i \to \infty$. (This is known for hyperbolic groups and also some arithmetic groups by the work of Lubotzky, see $[\mathrm{Gro}]_{14}$, [Lub]).

(c) There are only finitely many cofinite extensions $\Gamma' \supset \Gamma$ without torsion (cofinite = Γ'/Γ is finite). This is proven for lattices in semi-simple groups by combining Mostow rigidity with the non-collapsibility of such groups established in [Ka-Ma].

6.D_2. Uniflat spaces and groups. Let X be a symmetric space or a Cartesian product of several hyperbolic spaces. Then the maximal flats are rather uniformly distributed in X. For example, there is a maximal flat through each point $x \in X$. Moreover each geodesic (and even each flat) is contained in a maximal flat (of dimension = $\mathrm{Rank}\,X$). Next, if we look at $\mathrm{Con}_\infty X$ we shall see that the group of bi-Lipschitz homeomorphisms of $\mathrm{Con}_\infty X$ is transitive on the set of flats in $\mathrm{Con}_\infty X$ (which are essentially the same as flats in X). Now, if we have Γ cocompactly acting on X then this uniflatness can be attributed to Γ. In fact we can do it even better with such Γ as the (maximal) periodic flats, corresponding to the (maximal) free Abelian subgroups, are dense in the space of all flats.

One knows, that if X is a *(smooth !) Riemannian manifold with* $K \leq 0$ *where every geodesic is contained in a* k*-flat for* $k \geq 2$*, then either* X *non-trivially isometrically splits,* $X = X_1 \times X_2$*, or* X *is a symmetric space (or both) provided* X *admits a cocompact* Γ*-action.* This is the celebrated theorem of Ballmann and

Burns-Spatzier (see $[\text{Ball}]_{1,2,3}$, [B-G-S], [Bu-Sp] and [Eb-He]). Yet this is unknown for singular spaces and "small" Γ acting on X.

Classification problem. One seeks a rough classification of Γ-spaces X with $K \le 0$ modulo those with $K < 0$ in the case where Γ is sufficiently large (e.g. cocompact). A first approximation conjecture in this regard would say that every X falls into some of the following three (not quite mutually exclusive) categories.

(1) Almost hyperbolic spaces as those discussed in $6.C_2$., where the curvature is negative (or can be made such) on a substantial part of X.

(2) (X, Γ) is collapsible.

(3) X is uniflat.

Notice, that in each of the three cases one may try to analyze (X, Γ) further: In case (1) one would like to reduce all flatness of X to some kind of a core $X_0 \subset X$ (where the curvature vanishes) and which has $\dim X_0 < \dim X$. Such a core is clearly visible in the Davis' reflection spaces and the manifolds obtained by gluing together hyperbolic cusps (see ($\square$) and ($\square\square$) in $6.C_2$.) where the flatness localizes to several (flat) tori. But the following examples indicate different possibilities.

Examples. (a) Let $V_1 \vee V_2$ be two closed manifolds of positive dimensions with $K \le 0$ joint at some point. Here $\pi_1(V_1 \vee V_2) = \pi_1(V_1) * \pi_1(V_2)$ and the space $V_1 \vee V_2$ should be taken as almost hyperbolic, regardless of particular geometries of V_1 and V_2. In fact, if we take a random closed geodesic in $V_1 \vee V_2$ it will visit both V_1 and V_2 (the probability of staying in one half exponentially decays with the increase of the length of the geodesic if the probability scheme is set up correctly). Such a geodesic γ enjoys all hyperbolic features such as

(i) $\text{com}(\gamma^i) \sim i$,

(ii) the normal subgroup generated by γ is free.

Intuitively, the joint point in $V_1 \vee V_2$ carries an infinite amount of negative curvature, while the flatness "localizes" to V_1 and V_2 taken *individually*.

(b) One can generalize the above discussion to the amalgamated products of (fundamental) groups which appear when one glues V_1 and V_2 along isometric locally convex subsets $W_i \subset V_i$, $i = 1, 2$.

Next, in the collapsible case (2) one can break the space into pieces each of which is the Cartesian product of something lower dimensional by $\mathbb{R}$. Finally, the uniflat spaces may be amenable to a generalized version of the Ballmann-Burns-Spatzier theorem.

Ramified covers. "Uniflatness" and "almost hyperbolicity" may be mixed up together in a quite intricate way. What we have in mind appears when we take a ramified covering of a manifold V with $K \le 0$ where the ramification locus is some totally geodesic submanifold $\Sigma \subset V$ of codimension two in V. If the normal bundle of Σ in V is (topologically) divisible by k, then there exists a unique maximal covering of the complement $V - \Sigma$ which ramifies around Σ with order k. The metric completion of this covering, denoted $X_k = \widetilde{V}(k\Sigma)$ is simply connected and has $K \le 0$. The covering group $\Gamma_k = \pi_1(V, k\Sigma)$ isometrically acts on X_k with $X_k/\Gamma = V$. Notice that we allow $k = \infty$ where X_∞ is the metric completion of the universal covering $\widetilde{V - \Sigma}$ (and where we need no extra assumption on the normal bundle of Σ).

Examples. (a) Let $V = S \times S$ for a surface S with $K(S) < 0$ and $\Sigma \subset S \times S$ be the diagonal. Then one can construct X_k for $k = \infty$ and every finite k dividing the Euler characteristic $\chi(S)$. (If S is an open surface, then every $k > 0$ will do). The geometry of such an X_k reflects the intersection pattern of geodesics in S as the ramification unflattens the flats in V corresponding to pairs of intersecting geodesics in S.

(b) Let $V = X/\Gamma$ for the symmetric space $X = O(n,2)/O(n) \times O(2)$ and let $\Sigma \subset V$ be covered by the subspace $Y = O(n-1,2)/O(n-1) \times O(2)$ isometrically embedded to X (compare [Tol]). (The normal bundle of Σ in V is non-trivial but it can be made highly divisible by passing to a finite covering of V).

Now we observe that whenever we have a ramification around Σ, there appear geodesics which *stably* intersect Σ. In particular, if $V_k \to V$ is a finite ramified cover for $k \ge 2$ (recall, that V_∞ topologically is $V - \Sigma$), then we have plenty of closed geodesics in V_k meeting Σ. (For instance, one can easily prove for the above (a) and (b) that such closed geodesics are dense in the space of all geodesics. Probably, this is true in the general case as well). We take such a closed geodesic γ in V_k which meets Σ at finitely many points and assume that at a meeting point $x \in \gamma \subset V_k$ our γ is *not contained, even locally near x, in any 2-flat of V_k*. Notice that this condition is typically satisfied. Take, for example, such a γ in the above (a) and look at the two natural projections of γ to S. Thus we obtain two broken geodesics in S which (necessarily) meet at the breaking points corresponding to the intersection of γ with the (ramified) diagonal in $S \times S$. Then the above no-flat condition is satisfied at a breaking point if it is an *isolated* intersection point of the two projections (i.e. the two should not have a segment in common). Similarly, one can show that the "no-flat" property is generic in the above (b) as well.

Whenever we have a closed geodesic γ which at some point is not contained in a local flat, we can see this γ displaces the essential hyperbolic features, such as $\operatorname{com}\gamma^i \sim i$ etc. In particular a generic geodesic γ in the above examples (a) and (b) generates a *free* normal subgroup in the group $\Gamma_k = \pi_1(V, k\Sigma)$. This is especially interesting in the case (b) where by the Margulis theorem the original group $\Gamma = \pi_1(V)$ has no non-trivial normal subgroup of infinite index.

Despite the presence of a substantial amount of hyperbolicity at Σ in $V_k, k \geq 2$, there are too many flats in V_k to make curvature negative near Σ by perturbing the metric. Namely, for every point in V_k close to the ramification locus $\Sigma_k \subset V_k$ (lying over Σ in V) there is a flat in V_k passing through this point in the above examples (a) and (b). It follows that there is no deformation of the metric of V_k, in the class of metrics with $K \leq 0$, which is supported near Σ_k. In fact, we may expect a Mostow type rigidity theorem for the groups Γ_k in these examples. (In the case (a) it is the Gromoll-Wolf, Lawson-Yau rigidity rather than that of Mostow we expect).

Remark. The group $\Gamma_\infty = \pi_1(V - \Sigma)$ in the above example (b) is not residually finite (see [Tol]) though it is by any standard semi-hyperbolic.

6.D$_3$. Non-cocompact groups. When Γ acts on X discretely and cocompactly then the ties between X and Γ become especially close as, for example, such an X is quasi-isometric to Γ with the *word* metric. We have been assuming at many occasions in our discussion that Γ is cocompact. If we have made the cocompactness assumption so often, it was only to facilitate the exposition and not because the author adheres to the conservative (and unfortunate) point of view that only cocompact groups Γ deserve serious attention. In fact the most important groups in mathematics, such as $SL_n\mathbb{Z}$, $SL_n\mathbb{Q}$, and the (Teichmüller) mapping class group, naturally (and discretely) act on spaces with $K \leq 0$ where the action is non-cocompact. Recall that $SL_n\mathbb{Z}$ acts on $X = SL_n/SO(n)$ and the group $SL_n\mathbb{Q}$ acts on the product of the above X with X_p, $p = 2, 3, 5, 7, \ldots$, which are the affine buildings replacing X for the finite primes p. The mapping class group acts on the Teichmüller space. Notice that the Teichmüller metric itself does not quite have $K \leq 0$ though it is rather semi-hyperbolic. On the other hand the Weil-Peterson metric does have $K \leq 0$. Some people may be bothered by the fact that this metric is not geodesically complete, but this property is not essential for our discussion of $K \leq 0$. In fact the Weil-Peterson metric is convex in our sense and it looks near infinity as the universal coverings $X_\infty = \widetilde{V - \Sigma}$ we met in the previous section.

The groups in the above examples share a common feature which is similar to cocompactness but is somewhat weaker: they have finite covolume and so are equivalent to the space they act on in the measure theoretic sense. (Here we should take an extra care in the case of the mapping class group). However, even the finite covolume condition feels too restrictive for my taste. Here are some reasons.

1. A self-contained theory needs infinite covolume groups as they appear, for example, as subgroups of groups Γ which do have finite covolume. Also when we stabilize X, by enlarging dimensions (which may even become infinite in the process) we also loose finite covolume (and cocompactness) at least in the literal sense. (A typical stabilization is passing from a hyperbolic space H^n to $H^{n+1} \supset H^n$ and eventually to $H^\infty \supset H^n$).

2. There are many beautiful groups Γ which are neither cocompact nor of finite covolume appearing in the theory of Kleinian groups. (In fact this theory mainly concerned with groups of infinite covolume).

3. The finite covolume assumption is redundant in certain applications, especially in the problems surrounding the Novikov higher signature conjecture.

4. Many questions become more difficult and more interesting if we do not insist on the finite covolume.

Example. Let Γ be an *amenable* group discretely acting on X with $K(X) \leq 0$. If the action is cocompact, then Γ is *virtually Abelian* by Avez' theorem (see [Av]) and Avez' proof readily extends to groups acting with finite covolume as was shown, I recall, by some student of R. Zimmer. Notice that in [Av] and in the subsequent papers X is assumed smooth Riemannian but the singularities do not seem to interfere with the proof). On the other hand, if Γ has infinite covolume it may be significantly more general. For example, every finitely generated polycyclic group Γ may act on $X = SL_n/SO(n)$ for some $n = n(\Gamma)$. A non-polycyclic solvable example is provided by the group $\Gamma = \{x, y \mid x^y = x^2\}$ represented by the matrices $\begin{pmatrix} 2^i & x \\ 0 & 2^{-i} \end{pmatrix}$ where x is a diadic rational. This Γ is discrete in $SL_2\mathbb{R} \times SL_2\mathbb{Q}_2$ and hence, acts discretely on $H^2 \times Y$, where Y is the Bruhat-Tits building associated to $SL_2\mathbb{Q}_2$, which is the infinite regular tree with three edges at every vertex. Notice, that Γ contains an Abelian subgroup with unbounded division (by 2^i). I am not certain if this is solely due to the singular nature of trees (as I fail to-day to solve exercise (ii) on p. 91 in [B-G-S] which I suggested 10 years ago).

Open problem. Find a condition on the action of an amenable Γ on X with $K(X) \leq 0$ which would insure that Γ is virtually Abelian.

The above examples indicate that the following condition may prove useful.

Strong stability. There exist some elements $\gamma_1, \ldots, \gamma_k$ in Γ, such that the maximum of their displacements $S(x) = \max\limits_{1,\ldots,k} \mathrm{di}_{\gamma_i}x$ goes to infinity as the distance from x to a fixed orbit $\Gamma(x_0) \subset X$ goes to infinity.

Notice that if Γ is finitely generated it is enough to try a generating set for $\gamma_1, \ldots, \gamma_k$. The failure of the strong stability intuitively signifies that the action of Γ can be moved to infinity. In fact, one can make this precise by invoking the "foliation" $\mathcal{X}^+$ associated to X (see 6.B.) and then "moving to infinity" amounts to an action of another group Γ' on some $X' \in \mathcal{X}^+$ where Γ' may be equal to Γ or be even bigger. (If, for instance $\Gamma = \{\gamma^i\}$, where $\inf\limits_{x \in X} \mathrm{di}_\gamma(x) = 0$, then for the sequence $x_i \to \infty$ where $\mathrm{di}_\gamma(x_i) \to 0$ the limit group Γ' contains $\mathbb{R}$, which is definitely bigger than the original group $\{\gamma^i\} = \mathbb{Z}$).

The strong stability condition is well adapted to the theory of *harmonic maps.* Namely if Y is an arbitrary Riemannian manifold with a cocompact action of Γ, then for every strongly stable action Γ on X (with $K(X) \leq 0$) there exists a unique Γ-equivariant harmonic map $Y \to X$. (This is an easy corollary of the Eells-Sampson theory of harmonic maps (see $[\text{Ee-L}]_{1,2}$) pointed out by Donaldson, see [Don] $[\text{Cor}]_1$ and [Gr-Sch]). An interesting approach to the above amenable group problem (for subgroups of cocompact groups) via harmonic maps was suggested by M. Anderson (see $[\text{And}]_1$) but, unfortunately, the argument in $[\text{And}]_1$ is not conclusive. (Compare the free subgroup theorem stated in $[\text{Ball}]_{3,4}$).

The above problem on amenable groups reflects the general lack of information on countable groups Γ' which admit *discrete* (i.e. *uniformly unbounded* as defined in III of 6.A.) actions on simply connected spaces X with $K(X) \leq 0$. For all we know, *every* group Γ may admit such an action on some X. Even if we assume X is a Riemannian manifold and Γ has no torsion, the only detectable effect of $K(X) \leq 0$ is contractibility of X and the issuing inequality $\dim \Gamma \leq \dim X$ (where $\dim \Gamma$ refers to the length of the shortest resolvent). Moreover, we do not gain much by insisting on negative curvature $K(X) \leq -\kappa < 0$ as one can "hyperbolize" every X with $K(X) \leq 0$ by taking $X' = X \times \mathbb{R}$ with the Riemannian metric $e^t g_X + dt^2$.

Test question. Let V be a (geodesically) complete Riemannian manifold with $K(V) \leq -\kappa < 0$. Does the fundamental group $\Gamma = \pi_1(V)$ have a solvable word problem ? (The positive answer seems more feasible if we assume $|K(V)| \leq C < \infty$ and/or that every $\gamma \in \Gamma$ is *hyperbolic* which means the existence of a closed geodesic in V representing the conjugacy class of γ in Γ).

Examples and non-examples. There is a variety of infinite dimensional spaces X which nearly (but not quite) have $K \leq 0$ and where every Γ may act. Take for instance the space $\ell_\infty = \ell_\infty(\Gamma)$ of bounded functions $\Gamma \to \mathbb{R}$ with the sup-norm. The obvious action of Γ on ℓ_∞ by translations is non-discrete but the discreteness can be recovered with a suitable (unbounded Lipschitz) function on Γ, e.g. with $d(\gamma) = \mathrm{dist}(\gamma, \mathrm{id})$, as follows. Let $X = \ell_\infty + d$ that is the space of functions on Γ of the form $f + d$ for all $f \in \ell_\infty$. This X is an affine space of ℓ_∞ and thus isometric to ℓ_∞. But now the obvious action by Γ is discrete (as well as isometric) on X.

It would be infinitely more satisfactory to modify the above construction by somehow replacing ℓ_∞ by the Hilbert space ℓ_2, but this is, in principle, impossible for many groups Γ. For example, if Γ has Kazhdan's T-property (e.g. $\Gamma = SL_n\mathbb{Z}, n \geq 3$) then every isometric action of Γ on the Hilbert space is bounded (i.e. has bounded orbits). Yet every subgroup Γ in $O(n,1)$ or in $U(n,1)$, has a property opposite to T (such groups are called a-T-menable in $[\text{Gro}]_{15}$) as they admit *discrete* (i.e. strictly unbounded) isometric action on ℓ_2 (though the construction is tricker). Our T-action on ℓ_∞ generalizes to ℓ_p with a function $d(\gamma)$ on Γ, such that $d'(\gamma) = d(\gamma) - d(\gamma'\gamma)$ is in ℓ_p for all γ', but $d(\gamma)$ itself is not in ℓ_p. If $\limsup_{\gamma\to\infty} d(\gamma) = 0$, we obtain a discrete action and if $\liminf_{\gamma\to\infty} d(\gamma) = 0$ then our action is unbounded (which is less than discrete but better than nothing). In particular, every non-trivial element in the 1-dimensional ℓ_p-cohomology of Γ gives us an unbounded action on ℓ_p. Also if Γ has polynomial growth the above works for all p with $d(\gamma) = (\text{dist}(\gamma, \text{id}))^\alpha$ for a suitable $\alpha = \alpha(\Gamma, p) < 0$.

Problem. Given a group Γ (e.g. $\Gamma = SL_n\mathbb{Z}$), find those p for which Γ admits a discrete (or at least unbounded) isometric action on some space ℓ_p or on another space of comparable degree of convexity, e.g. a subquotient of L_p).

Another class of spaces X with many actions appears as follows. Take some convex cone of functions on Γ, e.g. the cone of positive functions with a given rate of growth (or decay). The base of such a cone is a convex set (in a projectivized function space) and the obvious action of Γ is isometric (and often discrete) for the *Hilbert metric* on this convex set. Unfortunately, most Hilbert metrics are rather irregular (like the space ℓ_∞) and do not fulfill the CAT requirement for $K \leq 0$.

Finally, we observe that every representation of Γ by bounded linear operators on $\mathbb{R}^\infty$ gives in an action on the "symmetric space" $X_\infty = GL_\infty/O(\infty) \times \mathbb{R}^\times$). (See III in 6.A.). This can be achieved, for example, for $\mathbb{R}^\infty$ realized by $\ell_2(\Gamma, \delta)$ for some weight function δ on Γ. If $\delta(\gamma) \to \infty$ for $\gamma \to \infty$, the resulting action of Γ on X_∞ is discrete. Unfortunately, X_∞ is as irregular as L_∞, and the smooth alternative would need the group of the operators 1 + Hilbert-Schmidt. But discrete actions on the corresponding symmetric (now Riemannian) space X_2 are harder to come by.

Question. Does every Γ admit a discrete (or at least unbounded) action on X_2 ? Can one classify such actions by further geometric properties ? (The same questions apply to the spaces X_p associated to Schatten's L_p-classes of compact operators).

6.E. Semi-hyperbolicity problems. We have made every effort to present our discussion on $K \leq 0$ in a manner suitable for semi-hyperbolization, but, of course, most of the problems for $K \leq 0$ become only more difficult for semi-hyperbolic spaces and groups. There is yet another class of problem for semi-hyperbolic objects of foundational nature (which has no counter-part for $K \leq 0$) which concerns the relation between various definitions of semi-hyperbolicity. There are two groups of such definitions: the definitions of the first group mimic the coning and combing properties of $K \leq 0$ (see 6.B.) and those of the second group appeal to the geometry of surfaces (see 6.C.). The combing approach is emphasized in [A-B] following the work by Gersten and Short on bi-automatic groups. Yet it is still unclear what is the relation between the different kinds of coning, combing and bicombing. For all we know, just the existence of an [E-C-H-P-T]-combing may be good enough for the definition of semi-hyperbolicity, as no pathology has yet emerged to light with this property. Notice, that this definition is quasi-isometry invariant (which is a definite asset but not at all a necessity). Thus the non-trivial central extensions Γ of hyperbolic groups (which are, according to an observation made independently by Gersten and Epstein, quasi-isometric to trivial extensions) should be regarded semi-hyperbolic. (Notice that according to exercise (i) on p. 91 in [B-G-S] the semi-hyperbolicity of these groups is, in a certain sense, less prominent than for the trivial extension $\Gamma = \Gamma' \times$ Center).

The definition of semi-hyperbolicity via surfaces is implicit in the theory of small cancellation groups which are semi-hyperbolic in the "surface" sense. This is a definite advantage of surfaces over combings: there is no example (to the best of my knowledge) of a group with an invariant bicombing which does not come from $K \leq 0$ and pure hyperbolicity. The only result I know of relating the geometry of surfaces with combing appears in $[\text{Ge-Sh}]_{1,2,3}$ where some small cancellation groups are proven to be automatic and hence [E-C-H-P-T]-combable.

Question. Let Γ be a group with a finite *aspherical* presentation. Does then the quadratic isoperimetric inequality for Γ imply the existence of a (Γ-invariant) bicombing, or at least some combing ?

6.E$_1$. Semi-hyperbolic constructions. Whichever definition we lay down for the foundation of the semi-hyperbolic theory, the following three classes of groups must be admitted into the semi-hyperbolic pantheon.

1. *Word hyperbolic groups.* (In what follows we concentrate on the simplest aspect of the semi-hyperbolicity corresponding to *cocompact* actions which explains "*word* hyperbolic" rather than plain "hyperbolic").

2. *Groups cocompactly acting on simply connected spaces X with $K(X) \leq 0$.* Here we may allow more general convex spaces X as in [Gro]$_5$ but it is unclear if we thus obtain a larger class of groups).

3. *Small cancellation groups.*

Next, semi-hyperbolicity must be stable under the following operations:

I. *Cartesian products and direct factors.*

II. *Free products and free factors.*

III. *Commensurability*, i.e. passing to subgroups (or overgroups) of finite index.

IV. *Convex surgery*: amalgamated products, HNN-extensions, ramified covers (this is made precise below).

V. *Hyperbolic and semi-hyperbolic products; factoring away random hyperbolic relations* (see below).

VI. *Ulterlimits* (also explained below).

Explanations. (i) *Convex surgery.* When we glue spaces with $K \leq 0$ by locally isometric maps of locally convex subsets we end up again with $K \leq 0$. For example, let $W \to V$ be a locally isometric map which sends W onto a totally geodesic subvariety (which may have double points or even be everywhere multiple as a geodesic representing γ^k, $k \geq 2$). Then we can attach the cylinder $W \times [a,b]$ to V by gluing $W \times a$ to V by this map. This does not change by itself the fundamental group, but it can be used for this purpose whenever we have two locally isometric maps α and β of W to V as we can glue $W \times [a,b]$ to V at both ends by the map α on $W \times a$ and β on $W \times b$. The fundamental group Γ_1 of the resulting space V_1 is either an HNN-extension of $\Gamma = \pi_1(V)$ (this is the case if V is connected) or an amalgamated product, which appears if V consists of two components, one receiving α and the other β.

Corollary. *Let Γ_1 and Γ_2 be groups with $K \leq 0$* (which is *here* a short way to say they *cocompactly act on simply connected spaces with $K \leq 0$*). *Then the amalgamated product* of Γ_1 *with* Γ_2 *over arbitrary infinite cyclic subgroups (or finite extensions of cyclic groups) has again $K \leq 0$.*

Proof. Apply the above to geodesics $\tilde{\gamma}_i$ representing the cyclic subgroups $C_i \subset \Gamma_i$, $i = 1,\ 2$. Here we can always rescale the metric of one of the implied spaces, say of X_1, on which Γ_1 acts, in order to have $\text{length}\,(\tilde{\gamma}_1/C_1) = \text{length}\,(\tilde{\gamma}_2/C_2)$ and so the gluing process will be compatible with isometries. (If there is no torsion we can pass to the quotient spaces $V_1 = X_1/\Gamma_1$ and $V_2 = X_2/\Gamma_2$ and glue together the closed geodescis $\gamma_1 = \tilde{\gamma}_1/C_1$ and $\gamma_2 = \tilde{\gamma}_2/C_2$).

Warning. The above works for HNN-extensions if the corresponding closed geodesics have *equal* lengths but not in general, even if we avoid the obvious pitfall of gluing together multiple closed geodesics of different multiplicity (to avoid groups like $\{\alpha, \gamma \mid \gamma^\alpha = \gamma^2\}$).

Counter-example. Take the standard flat torus $T^n = S^1 \times S^1 \times \ldots \times S^1$ and first, let $\gamma_0, \ldots, \gamma_k$ be some elements in $\Gamma_0 = \mathbb{Z}^n$ represented by geodesics of equal length. Notice that each γ_i is given by an n-tuple of integers γ_i^j, $j = 1, \ldots, n$ and the length is $\left(\sum_{j=1}^{n} (\gamma_i^j)^2\right)^{\frac{1}{2}}$, so that we may have k arbitrarily large for every $n \geq 2$. Here we can glue together these geodesics in the category $K \leq 0$ which means, algebraically, adding new generators $\alpha_1, \ldots, \alpha_k$ to Γ_0 and the relations $\gamma_i^{\alpha_i} = \gamma_{i-1}$, $i = 1, \ldots, k$. Next, let us drop the assumption that the geodesics have equal length. It still may happen that there is another flat metric in T^n for which the new chosen geodesics $\gamma_i, \ldots, \gamma_k$ have equal lengths and then again the resulting group $\overline{\Gamma}_0 = \{\Gamma_0, \alpha_i \mid \gamma_i^{\alpha_i} = \gamma_{i-1}\}$ has $K \leq 0$. But if no such flat metric exists, then $\overline{\Gamma}_0$ does not have $K \leq 0$. In fact, if $\overline{\Gamma}_0$ has $K \leq 0$ then $\Gamma_0 = \mathbb{Z}^n$ must be realized by a flat torus (in the space $\overline{V}$ with $K(\overline{V}) \leq 0$ having $\pi_1(\overline{V}) = \overline{\Gamma}_0$) where geodesics conjugate in $\overline{\Gamma}_0$ necessarily have equal length.

Semi-hyperbolic moral. The amalgamated products of semi-hyperbolic groups over cyclic (or virtually cyclic) subgroups should be semi-hyperbolic. This should be also true for HNN-extensions of Γ but only if one can guarantee the "equal length" condition. For example if there is an automorphism of finite order of Γ permuting the cyclic subgroups in question.

Let us look at the above counter-example from a semi-hyperbolic view-point. Even if the group $\overline{\Gamma}_0$ emerging after gluing γ does not have $K \leq 0$ it may, a priori, be semi-hyperbolic, though it appears rather unlikely. In any case, if we take sufficiently many γ, then $\overline{\Gamma}_0$ is definitely not semi-hyperbolic. For example, if the set $\{\gamma_i\}$ consists of all prime (i.e. non-multiple) geodesics of length $\leq 100^n$, then the subgroup $\Gamma_0 \subset \overline{\Gamma}_0$ has *exponential* distortion (as an elementary argument shows) which is incompatible with semi-hyperbolicity because of the following

Important properties. Every Abelian subgroup A in a group Γ with $K \leq 0$ is finitely generated and has bounded distortion in Γ. Furthermore, if Γ has no torsion then A is contained in a unique maximal subgroup $\tilde{A}$ with rank $\tilde{A} =$ rank A (if one allows torsion, then there are at most finitely many of such maximal $\tilde{A}$). All these properties are expected to be true for the semi-hyperbolic groups Γ (and they may be already proven by Alonso, Bridson and/or Short for groups with equivariant bicombing, as the proof in the case $K \leq 0$ entirely relies on the convexity of the displacement).

Remark. In the above example of $\overline{\Gamma}_0$ one can bypass these properties of A by showing with some extra effort that the isoperimetry in $\overline{\Gamma}_0$ is exponential (for this nonsensically large 100^n).

Specific questions. (a) Let Γ be an amalgamated product of two small cancellation groups over cyclic subgroups. Does Γ satisfy the quadratic isoperimetric inequality ? (The answer should be "yes" if the desired semi-hyperbolic theory exists).

(b) Let V be a compact connected irreducible locally symmetric space with $K \leq 0$ and rank ≥ 2. Take two closed geodesics in V of *different* lengths and glue them together. Is the resulting HNN-extended fundamental group $\overline{\Gamma}$ semi-hyperbolic ? For example, does $\overline{\Gamma}$ satisfy the quadratic isoperimetric inequality ? (Since the geometry of V is rigid in the category $K \leq 0$ one can probably show without much ado that $\overline{\Gamma}$ is not of $K \leq 0$).

Amalgamation over non-cyclic subgroups. Here there are two problems. First is to decide when a subgroup $\Gamma_1 \subset \Gamma$ is "convex" in this sense or another, (e.g. realized by π_1 of a totally geodesic submanifold). The second question is when two such isomorphic subgroups are "isometric". If we work in the category $K \leq 0$, then we know for sure that every Abelian subgroup Γ_1 is convex (and this is to be expected for the semi-hyperbolic case as well. Here "convexity" may refer to some bi(combing) or to the existence of a Γ_1-equivariant Lipschitz retraction $\Gamma \to \Gamma_1$). Also we expect that certain "rigid" non-Abelian groups Γ_1 are necessarily convex inside semi-hyperbolic groups. For example, cocompact lattices Γ_1 in simple Lie groups of $\mathbb{R}$-rank ≥ 2 may have such a rigidity (which would generalize some aspects of Margulis' superrigidity).

Strong convexity. Let us observe that certain subgroups $\Gamma_1 \subset \Gamma$ may be ***strongly convex*** which means relative hyperbolicity of $\Gamma \supset \Gamma_1$ with respect to Γ_1 (see $[\text{Gro}]_{14}$). This implies the existence of a Γ_1-invariant retraction $\Gamma \to \Gamma_1$ whose local Lipschitz constant exponentially decays as $\text{dist}(\gamma, \Gamma_1) \to \infty$, in the same fashion this happens for quasi-convex subgroups of hyperbolic groups. For example, if Γ is the fundamental group of a hyperbolic manifold (with $K = -1$) with cusps, then each cuspidal (virtually Abelian) subgroup is strongly convex in Γ.

Isometry problems. Let us indicate the cases where the isometry of two subgroups Γ_1 and Γ_2 is a priori insured.

(1) *Doubling.* Take Γ with a subgroup $\Gamma_1 \subset \Gamma$. Now we take another copy of Γ and observe that the corresponding copy of Γ_1 is "isometric" to the original. Thus the amalgamated product $\Gamma \underset{\Gamma_1}{*} \Gamma$ of the two copies of Γ over Γ_1 is possible in the category $K \leq 0$ whenever Γ_1 is convex in Γ, i.e. realized by π_1 of a locally convex subset. In particular, this works if Γ_1 is an Abelian subgroup in Γ. Then we expect such amalgamated product $\Gamma \underset{\text{Ab}}{*} \Gamma$ to exist in the semi-hyperbolic domain as well.

(2) *Internal isometries.* Let Γ be a normal subgroup of finite index in $\Gamma' \supset \Gamma$ where $K(\Gamma') \leq 0$ and let Γ_1 and Γ_2 be two subgroups in Γ which are conjugate in Γ'. Then these are isometric, and if they are convex one can form the HNN-extension $\overline{\Gamma}$ of Γ by gluing Γ_1 to Γ_2 with $K \leq 0$. Again this might work as well in the semi-hyperbolic case.

(3) *Rigid groups.* Let Γ_1 and Γ_2 be isomorphic subgroups which are rigid in the sense indicated above. (For example, there are some rigid virtually Abelian (crystallographic) groups $\Gamma_1 \supset A = \mathbb{Z}^n$. Such a group Γ_1 is rigid if the action $x \mapsto \gamma x \gamma^{-1}$ of Γ_1 on $\mathbb{R}^n = \mathbb{Z}^n \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ is irreducible). Then if we realize them by totally geodesic submanifolds V_1 and V_2 in V, these are isometric up to a scale factor and so the gluing is possible as in the case of cyclic subgroups. In particular, one can always amalgamate in the category $K \leq 0$ over rigid crystallographic subgroups but in the general semi-hyperbolic surroundings this appears more difficult but still possible.

Remark. Amalgamation over non-isometric Abelian subgroups is, in general, impossible in the category $K \leq 0$. For example, if such amalgamation between two torsionless crystallographic groups does have $K \leq 0$, then the corresponding tori must have been isometric for some choices of the flat metrics we glue together. But I do not know if such amalgamated products are still semi-hyperbolic in a certain sense.

(4) *Strongly convex subgroups.* Let Γ_1 and Γ_2 be isomorphic strongly convex Abelian or virtually Abelian subgroups in $\Gamma_1' \supset \Gamma_1$ and $\Gamma_2' \supset \Gamma_2$. Then we expect that the amalgamated product Γ between Γ_1' and Γ_2' by a given isomorphism $\Gamma_1 \leftrightarrow \Gamma_2$ is semi-hyperbolic whenever Γ_i', $i = 1, 2$ are semi-hyperbolic. This is similar to what happens to the gluing of cusps described earlier, (which suggests a possibility of certain HNN-extensions in this situation).

Remarks and questions on word hyperbolic groups. Let Γ be a word hyperbolic group and $\Gamma_1 \subset \Gamma$ a quasi-convex subgroup. Then the amalgamation $\overline{\Gamma}$ of two copies of Γ over Γ_1 is semi-hyperbolic in any conceivable sense (this is easy to prove). Yet $\overline{\Gamma}$ need not be hyperbolic as was pointed out to me by H. Casson at this meeting. For example, if Γ_1 is realized as π_1 of a totally geodesic surface S inside something with $K < 0$ and S intersects itself over a pair of closed geodesics in S, then these geodesics give rise to a copy of $\mathbb{Z} \times \mathbb{Z}$ in Γ. In fact the algebraic mechanism breeding this non-hyperbolicity is quite apparent: every pair of non-torsion elements γ_1 and γ_1' in Γ_1 which are not conjugate in Γ_1 but are conjugate by some γ in the ambient group Γ, produces a pair of independent commuting elements in $\overline{\Gamma}$. A natural *conjecture* is that the converse is also true: if the inclusion $\Gamma_1 \subset \Gamma$ induces an *injective* map on the sets of the conjugacy classes, say $[\Gamma_1] \underset{\text{inj}}{\to} [\Gamma]$, then $\overline{\Gamma}$ is hyperbolic.

Besides the non-injectivity of the map $[\Gamma_1] \to [\Gamma]$ there is an asymptotic source of non-hyperbolicity which may be a priori more general. To see this we invoke the ideal hyperbolic boundary $\partial_\infty \Gamma$ and the limit set $\partial_\infty \Gamma_1 \subset \partial_\infty \Gamma$ (which *is* the ideal boundary of Γ_1 for it is convex). Then we observe that if there is some γ in Γ outside Γ_1, such that $\gamma(\partial_\infty \Gamma_1) \subset \Gamma$ meets $\partial_\infty \Gamma_1$ (recall that Γ acts on $\partial_\infty \Gamma$), i.e.

$$\partial_\infty \Gamma_1 \cap \gamma(\partial_\infty \Gamma_1) \neq \emptyset, \gamma \in \Gamma - \Gamma_1 ,$$

then $\overline{\Gamma}$ is non-hyperbolic. In fact, if the above intersection is non-empty it necessarily contains at least *two* points as a simple argument shows. Therefore there exists a line ℓ in Γ (joining these two points) which lies within bounded distance from Γ_1 and $\gamma \Gamma_1 \gamma^{-1}$, and which gives rise (in an obvious way) to a quasi-flat, i.e. a quasi-isometric embedding $\mathbb{R}^2 \to \overline{\Gamma}$ (If ℓ is invariant under some $\gamma_1 \in \Gamma_1 \cap \gamma \Gamma_1 \gamma^{-1}$, then this quasi-flat is periodic and reduces to $\mathbb{Z} \times \mathbb{Z}$ sitting in $\overline{\Gamma}$ as we have seen earlier).

Now, this asymptotic picture can be reversed.

Asymptotic criterion for hyperbolicity of amalgamated product and HNN-extensions. *Let Γ and Γ' be word hyperbolic groups and $\Gamma_1 \subset \Gamma$ and $\Gamma_1' \subset \Gamma'$ quasi-convex mutually isomorphic subgroups where Γ_1 satisfies the asymptotic disjointness condition for every $\gamma \in \Gamma - \Gamma_1$,* i.e.

$$\partial_\infty \Gamma_1 \cap \gamma (\partial_\infty \Gamma_1) = \emptyset . \tag{$*$}$$

Then the amalgamated product of Γ with Γ' with an arbitrary isomorphism $\Gamma_1 \leftrightarrow \Gamma_1'$ is a word hyperbolic group. Furthermore, this remains valid for the HNN*-extensions of Γ with pairs of subgroups Γ_1 and Γ_1' in Γ where Γ_1 satisfies $(*)$ and where, moreover,*

$$\partial_\infty \Gamma_1 \cap \partial_\infty \Gamma_1' = \emptyset . \tag{$**$}$$

The proof relies on the elementary properties of quasi-convex sets (see $[\text{Gro}]_{14}$) and is left to the reader.

Remark. The condition (**) is not indispensable, as is seen in the Jorgensen example where $\Gamma = \Gamma_1 = \Gamma_1'$ is a surface group and where the equality $\Gamma_1 = \Gamma_1'$ is offset by the hyperbolicity of the connecting isomorphism $\Gamma_1 \hookleftarrow$. The hyperbolicity was made transparent in these examples by Bestvina-Feighn (see [Be-Fe]) whose argument may apply to more general HNN-extensions where $\partial_\infty \Gamma_1$ meets $\partial_\infty \Gamma_1'$.

If one tries to reduce the algebraic conjecture on hyperbolicity of $\overline{\Gamma}$ to the asymptotic criterion one immediately stumbles into the following

Problem. Let Γ_1 and Γ_2 be quasi-convex subgroups in a hyperbolic group Γ. Does then

$$\partial_\infty(\Gamma_1 \cap \Gamma_2) = \partial_\infty \Gamma_1 \cap \partial_\infty \Gamma_2 \ ?$$

Or, at least, does the following implication hold true ?

$$\partial_\infty \Gamma_1 \cap \partial_\infty \Gamma_2 \neq \emptyset \Rightarrow \Gamma_1 \cap \Gamma_2 \subset \mathbb{Z} \ .$$

(The latter question would have a positive answer if one knew that the existence of a quasi-flat in a semi-hyperbolic group Δ implied the existence of $\mathbb{Z} \times \mathbb{Z}$ in Δ. This makes me feel I miss something obvious around here).

Ramified coverings. Ramified covering of spaces V with $K \leq 0$ (or $K < 0$) whose ramification locus W has codimension two in V often carry metric with $K \leq 0 (K < 0)$. The simplest case is where W is totally geodesic in V but one may admit more general (singular) W (see $[\text{Gro}]_{14}$, [Ch-Da]). Also removing some (e.g. totally geodesic of codimension 2) W from V may leave the condition $K \leq 0$ (or $K < 0$) intact but one should address here the cocompactness problem. For example, if we remove a closed geodesic from a hyperbolic 3-manifold, what remains can be made *compact* with $K \leq 0$ but not with $K < 0$. (Strictly speaking, one does not truly need W of codimension two for ramified coverings but rather a locally convex subset $U \subset V$ whose boundary ∂U in V admits a nontrivial covering. Thus one obtains a wide variety of examples, especially if V is singular).

Now we want to extend ramified coverings to the hyperbolic and semi-hyperbolic categories. If Γ is a hyperbolic group and $\Gamma_1 \subset \Gamma$ is a quasi-convex subgroup one can see ramified coverings by looking at the boundary $\partial_\infty \Gamma \supset \partial_\infty \Gamma_1$. Namely, we take the Γ- orbit of $\partial_\infty \Gamma_1$ in $\partial_\infty \Gamma$, called $W_\infty \subset \partial_\infty \Gamma$, and consider some ramified covering $\widetilde{\partial}$ of $\partial_\infty \Gamma$ which ramifies over W_∞ such that the (localized at W_∞) ramification data are Γ-equivariant. For example, we may take the universal covering or the one which ramifies with given order p along W_∞. Then the action of Γ combines with that of the deck transformation group on $\widetilde{\partial}$ and gives us a certain group $\widetilde{\Gamma}$ acting on $\widetilde{\partial}$ with a natural homomorphism $\widetilde{\Gamma} \to \Gamma$. In order to make the above meaningful one should take care of the following two difficulties,

1. $\partial_\infty \Gamma$ and $\partial_\infty \Gamma_1$, may be rather ragged spaces with no good covering theory.
2. The action of Γ on $\partial_\infty \Gamma$ is by no means discrete.

Let us indicate how one may approach these problems. First let $\partial_\infty \Gamma_1$ be disjoint from $\gamma(\partial_\infty \Gamma_1)$ for all $\gamma \in \Gamma - \Gamma_1$ and *postulate* that for every finite system of elements $\gamma_1, \dots, \gamma_k$ *there exists a unique* maximal ramified covering of $\partial_\infty \Gamma_1$ which ramifies with certain order p over $\bigcup_{i=1}^{k} \gamma_i(\partial_\infty \Gamma_1)$. Then we take the direct limit of the deck transformation groups of these coverings as the subsets $\{\gamma_1, \dots, \gamma_k\}$ exhaust Γ and observe that our $\widetilde{\Gamma}$ makes sense in the limit. All this perfectly works, for example, if $\partial_\infty \Gamma$ is a topological sphere of dimension ≥ 2 and $\partial_\infty \Gamma_1$ is a topological submanifold of codimension two with (topologically) trivial normal bundle. (This does not work in the simplest case where $\partial_\infty \Gamma$ is the topological circle S^1 as we did not properly take into account the fundamental group of $\partial_\infty \Gamma$. We suggest to the reader to recapture the ramified covers of surfaces by working in this S^1).

Next, assuming we know how to define $\widetilde{\Gamma}$, we try to decide when it is hyperbolic.

Claim. *If different translates $\gamma(\partial_\infty \Gamma_1)$, $\gamma \in \Gamma$, are mutually disjoint (unless they coincide), $\partial_\infty \Gamma$ and $\partial_\infty \Gamma - \partial_\infty \Gamma_1$* are locally good *enough for the covering theory and $\pi_1(\partial_\infty \Gamma) = 0$* (just to make life easier) then the group $\widetilde{\Gamma}$, *defined with given finite ramification index on the orbit $\Gamma(\partial_\infty \Gamma_1)$, is word hyperbolic.*

This follows by a simple quasi-convexity argument which we omit.

Non-hyperbolic generalization. The above construction of $\widetilde{\Gamma}$ does not immediately extend to the semi-hyperbolic case as we lack (at least at the present moment) an invariant notion of the ideal boundary $\partial_\infty \Gamma$. But there is another route to define $\widetilde{\Gamma}$ which we shall briefly describe. First we need the notion of the *large-scale fundamental group of the complement* $\Gamma - \Gamma_1$ and then of $\Gamma - \bigcup_{i=1}^{k} \gamma_i \Gamma_1 \gamma_i^{-1}$. This may be done with the yoga of thickening of §1. As in the hyperbolic case, the situation is the friendliest if

(a) Γ is simply connected at infinity;

(b) Γ_1 is *disjoint from* $\gamma\Gamma_1\gamma^{-1}$ *at* infinity for all $\gamma \in \Gamma - \Gamma_1$. This means $\mathrm{dist}(\Gamma_1 - B(R),\ \gamma\Gamma_1\gamma^{-1} - B(R)) \to \infty$, where $B(R)$ is the ball in Γ around the identity of radius $R \to \infty$.

Then we can define the limit of the groups π_1 of $\Gamma - \bigcup_{i=1}^{k} \gamma_1 \Gamma_1 \gamma_i^{-1}$ at infinity as $\{\gamma_i\} \to \Gamma$ and eventually construct $\widetilde{\Gamma}$ with given ramification data.

So, it seems, there is a meaningful general theory of ramified coverings $\widetilde{\Gamma}$ of Γ with a given ramification locus (at the Γ-orbit of a subgroup Γ_1 or a finite collection of subgroups) but the details and examples are to be worked out yet. But even when the general theory goes through we are still in need of a useful criterion for $\widetilde{\Gamma}$ to be semi-hyperbolic. For the moment, the only such criterion I see (besides the geometry with $K \leq 0$) appeals to certain relative hyperbolicity of Γ_1 in Γ.

Test questions. Let Γ be a cocompact lattice in a simple Lie group G of $\mathbb{R}$ − rank ≥ 2 such that the corresponding symmetric space has no totally geodesic submanifold of codimension two (e.g. $G = SL_n\mathbb{R}$ $n \geq 4$). Are there subgroups $\Gamma_1 \subset \Gamma$ with ramified coverings $\widetilde{\Gamma}$ around them ? Can such $\widetilde{\Gamma}$ be semi-hyperbolic?

Hyperbolic and semi-hyperbolic products. If a group Γ is hyperbolic relative to a subgroup $\Gamma_1 \subset \Gamma$ (or a finite collection of subgroups) then hyperbolicity or semi-hyperbolicity of Γ_1 implies the corresponding property of Γ, and a similar situation appears in certain cases of *relative semi-hyperbolicity*.

Example. Let X be a metric space which is hyperbolic or semi-hyperbolic in some sense (e.g. $K(X) \leq 0$) and let $X_0 \subset X$ be a subset bounded by a disjoint union of concave hypersurfaces (i.e. $X - X_0$ is a union of disjoint convex subsets). Then, if each component of the boundary is semi-hyperbolic, then so is X_0. To make this precise let us stick to some bicombing definition of semi-hyperbolicity. Thus we assume every two points in X are joint by a "path" reasonably depending on the end points. Now, if the end-points lie in X_0, the "path" may go outside to the complement $X - X_0$, but then it crosses some component of the boundary ∂X_0 at two points (as the "convexity" of ∂X_0 should be understood relative to the bicombing). Then, as ∂X_0 is semi-hyperbolic, these two points are joint by a path in ∂X_0, and this equally applies to all pairs of crossing points in different components of ∂X_0. Thus we obtain paths between pairs of points in X_0 whose dependence of the end points is as good (Lipschitz) as that in X and in ∂X_0, provided there is a certain coherence between the paths in X and ∂X_0 needed to avoid the following unfortunate situation: some path in X is accidentally contained in ∂X_0 but the bicombing of ∂X_0 assigns to the same end points (contained in ∂X_0) an entirely different path in ∂X_0 lying quite far from the X-path. This difficulty does not appear if X is hyperbolic relative to $X - X_0$, but it is not clear how to achieve the coherence in the semi-hyperbolic situation without making very stringent assumptions. To see what may happen, we specialize the above discussion to the case where X is geodesically complete manifold with $K(X) \leq 0$ acted upon by Γ such that X_0 is Γ-invariant and X_0/Γ is compact. We assume furthermore, that each "boundary subgroup" of Γ, i.e. the subgroup sending a component of ∂X_0 into itself is Abelian. We have already met such situation for $K(X) = -1$, where these components were horospheres with *flat* induced metrics. Now, the *induced* Riemannian metrics may be non-flat (and then the internal curvature of X_0 is not ≤ 0). Yet we expect X_0 is semi-hyperbolic in every conceivable sense. We know already that X_0 is as good, isoperimetrically speaking, as the Euclidean space (see §5.) but now we are also concerned with (bi)combing. If $K(X) \leq -\kappa < 0$ then we are in a relative hyperbolic situation where the existence of Γ-invariant bicombing on X_0 is immediate with the above construction, but it is unclear how to prove this in general.

Factoring away random hyperbolic geodesics. We have already met on several occasions with the following situation. Given a manifold V with $K(V) \leq 0$ and a "long random" geodesic which spends a "sufficient" time in the region in V where $K \le -\varepsilon < 0$. Then the normal subgroup generated by γ is free and the factorgroup $\overline{\Gamma}$ of $\Gamma = \pi_1(V)$ obtained with the relation $[\gamma] = 1$ is semi-hyperbolic in the sense that the tight surfaces S in the space $\overline{V}$ (obtained by attaching the disk to V along γ) are λ-Lipschitz to those with $K \le 0$ for λ depending on $\overline{V}$ but not on S. Notice that the actual condition on γ which leads to this conclusion becomes clear in the course of the proof as we look at tight surfaces in $\overline{V}$. (This condition contains, as an ingredient, a geometric version of the combinatorial $\frac{1}{n}$-condition for a suitably large n expressing the idea that no two pieces of γ follow each other closely for a significant amount of time. It takes a certain effort to show that this is satisfied by generic long geodesics in V in order to arrive at our "random" formulation, compare $[\mathrm{Gro}]_{14}$). Also notice that many aspects of the passage from Γ to $\overline{\Gamma}$ remain unclear. What is, for example, the most general condition on γ where the surfaces work ? (We know that γ may be hyperbolic without a single bit of actual negative curvature as happens, for example, for ramified coverings of unflat manifolds). Then we do not know how to prove (or disprove) that $\overline{\Gamma}$ has a good enough (bi)combing to be promoted a full member to the semi-hyperbolic ranks.

Now, let us look at a particular instance of $\overline{\Gamma}$ where the above problems are happily resolved. We take the free product $\Gamma = \Gamma_1 * \Gamma_2$ of two semi-hyperbolic groups Γ_1 and Γ_2 and we want to show that adding a generic relation (or finited collection of these) to Γ does not disturb semi-hyperbolicity. We start with the geometrically transparent situations where $\Gamma = \pi_1(V_i)$, $i = 1, 2$, for $K(V_i) \le 0$ and then we produce a *non-compact* (unlike V_i) space V with $\pi_1(V) = \Gamma$ having strictly negative curvature $K(V) \le -1$. This V has two cusps corresponding to V_i so that the hyperbolicity of Γ relative to Γ_i becomes geometrically visible. Here is

The construction of V. Take $V_i' = V_i \times \mathbf{R}_+$ with the Riemannian metric $g_i' = e^{-t}g + dt^2$, $i = 1, 2$ and observe that $K(g_i) \le 0$ implies $K(g_i') \le -1$ as a simple computation (or an argument) shows. Both V_i' have convex "boundaries" $V_i \times 0$ and one can join them by a bridge between boundary points, so that the joint space $\overline{V}$ also has $K \le -1$ with compact convex boundary and which is isometric, sufficiently far away from this boundary to the disjoint union of V_1' and V_2'. If one cuts away the infinity, i.e. takes $V_t \subset V$, that is V minus $V_1 \times [t, \infty) \cup V_2 \times [t, \infty)$ for some (large) $t > 0$ one obtains a space with concave boundary (and so not having $K < 0$ near this boundary in the CAT sense unless the intrinsic curvature of this boundary has $K < 0$, compare $[\mathrm{Gro}]_5$). Now, we realize the relation we want to add to Γ by a closed geodesic γ in V and we take V_t with a sufficiently large t so that $\gamma \subset V_t$. Now, we know perfectly well when $\overline{V}$ is hyperbolic, i.e. the universal covering of $\overline{V}$ is hyperbolic in the sense of $[\mathrm{Gro}]_{14}$, since V has strictly negative curvature. If this is the case we end-up in (almost) the same situation we started with: the group $\overline{\Gamma}$ is hyperbolic relative to Γ_1 and Γ_2 and so, by our earlier discussion, it inherits all semi-hyperbolic features from Γ_1 and Γ_2. (The situation is not quite the same as now we have hyperbolicity rather than $K \le -\varepsilon < 0$ which is, a priori, a weaker condition.) All of the above equally applies to finite collections of relations and leads to the following

Conclusion. *Let $\overline{\Gamma}$ be obtained from $\Gamma = \Gamma_1 * \Gamma_2$ by adding a system of relations satisfying a sufficiently strong small cancellation (in the broad sense of* $[\mathrm{Gro}]_{14}$*) condition. If the groups Γ_1 and Γ_2 have $K \le 0$, then $\overline{\Gamma}$ is semi-hyperbolic (in any conceivable sense).*

Remarks. (a) This obviously generalizes to free products of several (≥ 2) groups and also to certain amalgamated products and HNN-extensions.

(b) The next logical generalization would be that where Γ_i are semi-hyperbolic. Unfortunately, the disk attaching procedure in the hyperbolic framework needs the construction of the geodesic flow which has been realized so far only for *word* hyperbolic groups (see $[\mathrm{Gro}]_{14}$). However, I feel certain this should work in the present case. Moreover, one may, probably, add to $\Gamma_1 * \Gamma_2$ sequences of random relations of increasing length as in the word hyperbolic case.

Example. Let Γ_1 and Γ_2 be cocompact lattices in simple Lie groups of $\mathbf{R}$-rank ≥ 2. Then the groups $\overline{\Gamma}$ obtained from $\Gamma_1 * \Gamma_2$ with small cancellation relations are semi-hyperbolic (but have no apparent metrics with $K \le 0$) and may serve as a test ground for the semi-hyperbolic theory. Notice, that the starting groups Γ_i are very special (T-property, often congruence subgroup property, no normal subgroup of infinite index etc.) and one wonders how this shows up in $\overline{\Gamma}$.

Scaling and ulterlimits. If X with a given metric dist has $K \le 0$ then so does $(X, \lambda\,\mathrm{dist})$ for all $\lambda > 0$, in particular for those ≤ 1. So, when X is simply connected, it is no surprise that the asymptotic cone

$\mathrm{Con}_\omega X$ also has $K \leq 0$ whichever ulterfilter ω is used for Con_ω. This is obvious, since the CAT-inequality defining $K \leq 0$ appeals to the geometry of quadruples of points in X and is scale invariant. In particular, $K(X) \leq 0$ makes $\mathrm{Con}_\infty X$ contractible. The situation is somewhat different for the to-day's definitions of semi-hyperbolicity as these impose retrictions on k-tuples of points in X with unbounded k. Yet, as these conditions reduce (in the known definitions) to the Lipschitz properties of certain maps, there is no problem of passing to ulterlimits. For example, the existence of an [E-C-H-P-T]-combing implies that for $\mathrm{Con}_\omega X$ as well as the contractibility of $\mathrm{Con}_\omega X$ (but this is far from truth for Gersten's combing). On the other hand the passage of semi-hyperbolic properties from $\mathrm{Con}_\infty X$ back to X is more difficult (at least for the moment). In particular, there is no working definition of semi-hyperbolicity of the following form: *there is a number k , such that every k-tuple of points in X ....*

Example. Say X is (λ, k)-semi-hyperbolic if every k-tuple of points is λ-bi-Lipschitz equivalent to a k-tuple of points in some space with $K \leq 0$ (in the CAT-sense). Now, assume this is satisfied for some λ, say $\lambda = 100$ and a very large $k = k(\lambda)$, e.g. $k = \exp\exp(100!)$. Does then X look semi-hyperbolic provided it is a geodesic space? It seems the geodesic condition is not enough and one should modify the above definition by bringing it to the form: *for every k-tuple of points in X there exists a k′-tuple such that ...* Also one may try to adjust such a definition to a given isometry group Γ acting on X.

Stability and localisation of $K \leq 0$ and semi-hyperbolicity. The geometry of $K \leq 0$ may become virtually unrecognizable if we apply a λ-bi-Lipschitz homeomorphism with large λ. Yet if $\lambda = 1 + \varepsilon$ for small $\varepsilon > 0$, many qualitative features remain intact. Similarly, a small perturbation of basic geometric features of $K \leq 0$ remains "semi-hyperbolic" in some definite geometric sense.

Examples. (a) Suppose that the convexity condition on the distance function is replaced by

$$\mathrm{dist}(z_1, z_2) \leq \lambda\big(t\,\mathrm{dist}(x_1, x_2) + (1-t)\,\mathrm{dist}(y_1, y_2)\big) + d\ ,$$

(compare 6.B) where z_i, $i = 1,\ 2$, denote the points on *some* minimal geodesic segments between x_i and y_i dividing the segments in the proportion $t : (1-t)$ (we do not assume such segments are unique). Then if $d = 0$ and $\lambda = (1+\varepsilon)\,\mathrm{dist}(x_1, x_2)$ for a small $\varepsilon > 0$, e.g. $\varepsilon = 0.1$, it is easy to recapture basic qualitative features of $K \leq 0$. For example, one can fill in every geodesic simplex using the subdivisions as in Fig. 16.

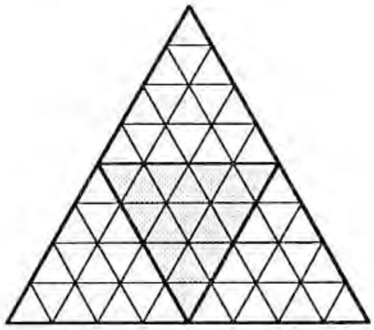

Figure 16

Every act of subdivision diminuishes the size of simplices by the factor $2 - \varepsilon$ which does make them smaller for $\varepsilon < 1$ and forces the process eventually to converge. Thus one proves that the space X is contractible. (If we allow $d > 0$ the same remains valid on the large scale. For example, $\mathrm{Con}_\omega X$ remains contractible).

(b) Suppose that every closed curve of length ℓ in X bounds a disk of area $\leq (1+\varepsilon)2\pi\ell^2$. Is then X contractible ? (It seems easy to prove $\pi_2(X) = 0$ by looking at a minimal sphere in X with the usual conformal parameter).

(c) It would be tempting to try to find for a given (semi-hyperbolic) group Γ the best (i.e. smallest) ε so that Γ admits a Γ-invariant metric which is ε-close to $K \leq 0$ in one sense or another. But any specific question seems premature at the present state of art as we are not able so far to distinguish between $K \leq 0$ and semi-hyperbolicity and/or to find examples with small ε which yet cannot be made zero.

Stability of the flow of chambers. This is another kind of stability which appears in (many) uniflat spaces X with $K \leq 1$. To keep it simple, let X be a Cartesian product of two simply connected spaces X_1 and X_2 with $K \leq -\varepsilon < 0$. Then the 2-flats in X are products of geodesics in these spaces. Now we take a quasi-flat i.e. a λ-bi-Lipschitz (on the large scale, but this is immaterial) embedding $f : \mathbb{R}^2 \to X$. If $\lambda = 1 + \varepsilon$ for a small positive ε, e.g. $\varepsilon = 0.1$ as earlier, then there is a (necessarily unique) flat asymptotic to our quasi-flat, but this is not so for large ε. To see this we observe that the Tits boundary of X is the graph whose vertex set is the disjoint union of $\partial_\infty X_1$ and $\partial_\infty X_2$ and the edges are $[x_1, x_2]$ for all $x_1 \in \partial_\infty X_1$ and $x_2 \in \partial_\infty X_2$. There is an obvious $1-1$ correspondence between the flats in X and simple circuits (cycles) in this graph consisting of exactly four edges. On the other hand, the geodesic cone in X over every simple $2m$-circuit is a quasi-flat which is far from any flat if $2m \geq 6$. Finally one knows since long that every quasi-flat with $\lambda = 1 + \varepsilon$ where ε is small is shadowed by an actual flat. In fact the foliation of the space of flats, (i.e. isometric maps $\mathbb{R}^2 \to X$) into $\mathbb{R}^2$-orbits is stable (e.g. see $[\text{Gro}]_{19}$).

Remark. The above discussion extends to the classical *uniflat spaces* which are Cartesian products of symmetric spaces and those with $K \leq -\varepsilon < 0$. Yet the geometry of quasi-flats still needs clarification. Namely one suspects that in reasonable spaces (e.g. sufficiently uniflat or sufficiently homogeneous) *quasi-flats of maximal dimension are asymptotically conical.* Let us indicate some arguments in favour of this conjecture. First let us look at the plane $P(= \mathbb{R}^2)$ topologically embedded into a 2-dimension polyhedron Y with $K \leq 0$, where we assume the isometry group of Y is cocompact. Suppose that our P with the (piecewise linear) metric is quasi-isometric to the flat $\mathbb{R}^2$.

Claim. *The induced metric in P is flat outside a compact subset.*

Proof. If a topological plane P with a metric with $K \leq 0$ is bi-Lipschitz to the flat $\mathbb{R}^2$, then $\int_P K(p)dp < \infty$, since the area $A(R)$ of the R-ball $B(R)$ in P is related to the total curvature of this ball

$$K(R) = \int_{B(R)} K(p)dp$$

by

$$A(R) \sim \int_0^K \int_0^K 1 + \int_0^R K(R)dR \,.$$

Thus

$$\int_P K = \infty \Rightarrow A(R)/R^2 \underset{R\to\infty}{\to} \infty \,.$$

Now, at every point $p \in P \subset Y$, where it is not flat there is a definite (non-zero) amount of integral negative curvature, as the metric in Y is (so we assumed) piece-wise linear. This proves our claim.

Now, for a 2-dimensional quasi-flat Q in a general X, we pass to $\text{Con}_\omega X$ and obtain some $P \subset \text{Con}_\omega X$ which is bi-Lipschitz to $\mathbb{R}^2$. If our asymptotic cone is "polyhedral", e.g. the product of two $\mathbb{R}$-trees as for the above $X = X_1 \times X_2$ or a 2-dimensional Euclidean $\mathbb{R}$- building (as it should be for a symmetric space X of rank 2), then P is asymptotically flat and hence conical at infinity in $\text{Con}_\omega X$. This indicates that Q is asymptotically conical in X in a certain weak sense and we expect there is an actual geodesic cone in X within bounded Hausdorff distance from Q as suggested by the above mentioned stability of the foliation of flats. (In fact, the *proof* of this stability may do the job for our Q).

Remarks. (a) Our discussion extends with a minor effort to k-dimensional quasi-flats Q for $k \geq 2$ (and corresponding polyhedral P bi-Lipschitz to $\mathbb{R}^k$).

(b) What we have just shown indicates that maximal quasi-flats Q in (a reasonable) X are divided in *countably many* classes according to the total amount of curvature, where we find on the bottom quasi-flats

Q asymptotic to flats. It would be nice to clarify the picture in the simplest case of 2-polyhedra with $K \le 0$ and next for the small cancellation groups.

Localization and recognition of semi-hyperbolicity. The property of a group Γ being δ-hyperbolic is, in principle, recognizable in terms of an arbitrary presentation of Γ, because hyperbolicity is a self-reproducing property: if a ball of a fixed radius in Γ is sufficiently hyperbolic then larger balls are even more hyperbolic. (This is similar to strict monotonicity of functions $f(t)$ locally recognizable by $f'(t) \ge \delta > 0$). But the condition $K \le 0$ is not like that at all ! For example the round two sphere of huge radius looks everywhere locally as if $K = 0$, but on the global scale it develops very differently from $\mathbb{R}^2$. This leads to the difficulty of recognition of semi-hyperbolicity in groups. Even in the friendliest environment of small cancellation groups (or even those given by 2-polyhedra with $K \le 0$), we are lost if we come across a non small cancellation presentation of such a group in a non-hyperbolic case. Yet there is some hope that a local recognition of (some) semi-hyperbolicity is possible, because, intuitively, there are two mutually exclusive modes of growth (or divergence) in a space with $K \le 0$, which are *linear* and *exponential*. The linear growth is confined to the maximal (quasi)-flats and these (quasi)-flats exponentially diverges (more or less like geodesic rays for $K \le -\varepsilon < 0$). Now, as we know, the exponential divergence is a self-reproducing phenomenon and so the difficulty resides where the growth is linear. But there, despite the above spherical counter-example there is a localizable geometric criterion, namely the polynomial growth criterion for nilpotency (which includes flatness corresponding to Abelian groups). Unfortunately the proof of this theorem in $[\text{Gro}]_6$ is very non-effective and recognizability follows by a non-constructive compactness argument.

There is another form of the localization-recognition problem, where we have a metric on Γ which is ε-close to $K \le 0$ on a large ball and we ask if this makes all of Γ close to $K \le 0$. This is similar to the corresponding geometric problem of studying spaces V of *almost negative* curvature, which means $K \le +\varepsilon$, for a small ε depending on some geometric characteristic of V (compare $[\text{Gro}]_4$, [Fu-Ya]). Here is a typical problem of this type geometers are unable to settle yet. Let X_0 be an irreducible symmetric space of rank ≥ 2 (and $K \le 0$, as usual). Consider a *compact* Riemannian manifold V where every unit ball is $(1+\varepsilon)$-bi-Lipschitz to such a ball in X_0. Does there exist a universal upper bound $\varepsilon_0 = \varepsilon_0(X_0) > 0$ on ε, such that $\varepsilon < \varepsilon_0$ would imply, for example, that V is an aspherical space ?

Generalized semi-hyperbolicity. I believe the *true* notion of semi-hyperbolicity must include nilpotent groups. For example, nilpotent groups with dilations are selfsimilar (for suitable Carnot-Caratheodory metrics) and thus should be regarded as *flat* objects. Then every nilpotent group Γ has a unique asymptotic cone $\text{Con}_\infty \Gamma$ which is a *contractible* space as it should be for $K \le 0$. In fact, the *Lipschitz contractibility* of $\text{Con}_\infty X$ may turn out a viable candidate for a definition of semi-hyperbolicity as it agrees with (bi)combings of X satisfying polinomial bounds on the Lipschitz constants. Various operations over semi-hyperbolic groups discussed in this section naturally extend to the more general nilpotent situation. For example, every (non-cocompact) lattice in a simple Lie group of $\mathbb{R}$-rank one is semi-hyperbolic in the nilpotent sense. But what we do not have for nilpotent groups is anything resembling uniflat spaces. The only examples, where *flat* = *Abelian* may be replaced by *flat* = *nilpotent* appear as central extensions of the flat = Abelian semi-hyperbolic spaces and alike. It seems the nilpotent groups lack some (reflection) symmetry inherent in the Abelian groups. But one should prove yet that no true nil-uniflatness exists.

Finally we may ask ourselves if one can (and should) work out even more general notions of semi-hyperbolicity in order to include polycyclic groups and/or non-cocompact lattices in Lie groups. These groups do have some remarkable geometric features and one may be tempted to scrutinize them by axiomatizing. For example, many (if not all) polycyclic groups X become hyperbolic after the following stabilization $X \rightsquigarrow X' = X \times \mathbb{R}^n$ with $g_{X'} = e^t g_X + \sum_{i=1}^{n} e^{\lambda_i t} dt_i^2$. (The simplest instance of that, $g \rightsquigarrow e^t g + dt^2$, turns $K \le 0$ on X to $K \le -\varepsilon < 0$ on $X' = X \times \mathbb{R}$, as we have seen earlier).

§7. Hyperbolic groups and K < 0; hierarchy in the hyperbolic ranks; pinching and conformal geometry at infinity; round trees and strongly branched polyhedra; Kähler and anti-Kähler groups; uniform embeddings and a-T-menability.

7.A. Hyperbolic Zoo. We start our discussion by enlisting the basic known examples of word hyperbolic groups.

I. *One-dimensional groups* Γ. These are the word hyperbolic groups whose hyperbolic boundary $\partial_\infty\Gamma$ has topological dimension zero. One can easily show that $\operatorname{Hypdim}\Gamma = 1 \Leftrightarrow \Gamma$ is virtually free, (i.e. Γ contains a free group of finite index), where

$$\operatorname{Hypdim}\Gamma \underset{\text{def}}{=} \dim \partial_\infty\Gamma + 1 \ .$$

Remark. One could use here the asymptotic dimension as, obviously,

$$\operatorname{Hypdim}\Gamma = 1 \Leftrightarrow \operatorname{Asdim}\Gamma = 1 \Leftrightarrow \operatorname{Asdim}_+\Gamma = 1 \ ,$$

for word hyperbolic groups Γ. In fact, a theorem by Bestvina-Mess (see [Be-Me]) implies that $\operatorname{Hypdim}\Gamma \geq \operatorname{Asdim}\Gamma$ for all word hyperbolic groups Γ. Furthermore, if the boundary $\partial_\infty\Gamma$ is *locally contractible*, one can easily prove the opposite inequality by mapping (thickenings of Γ) to a (geodesic) cone over $\partial_\infty\Gamma$, but the general case remains open for the moment.

II. *Two-dimensional groups.* Here there are several important subclasses.

$\mathrm{II}_{K<0}$. *Fundamental groups of compact 2-dimensional polyhedra and orbihedra with negative curvature.* Notice that 2-dimensional polyhedra with $K < 0$ are readily available. For example, every polyhedron P built of i-gons with $i \geq 7$ has negative curvature provided every two gons meet in P over a common face (i.e. an edge or a vertex, but not over several edges).

$\mathrm{II}_{1/7}$. *Small cancellation groups.* Here we allow any kind of a small cancellation condition (not only 1/7) which insures hyperbolicity. Notice that already 1/7-groups form a huge class. Namely, if we fix the number $k \geq 2$ of generators and look at all presentations with ℓ-relations given by words of length λ which is large compared to ℓ, then a majority of such presentations will be 1/7. (See $[\mathrm{Gro}]_{14}$, [Cham] and §9.)

$\mathrm{II}_{\mathrm{seq}}$. *Sequentially random groups.* Fix k generators for $k \geq 2$ and consider sequences of words $w_1, \dots, w_\ell$, where the length of w_i fast grows with i, e.g. $\operatorname{length} w_i \sim \lambda^i$ for a large λ. Then again a dominating majority of the groups given by such presentations are word hyperbolic of dimension two (see $[\mathrm{Gro}]_{14}$, [Cham]). Here, (unlike the small cancellation case) we may have sequences of word hyperbolic groups $\Gamma_\ell = \{\gamma_1, \dots, \gamma_k \mid w_1, \dots, w_\ell\}$, $\ell = 1, 2, \dots$, which give us in the limit interesting groups Γ_∞ of infinite type.

III. *Polyhedral groups with* $K \leq 0$. Let P be a compact polyhedron of dimension n with piecewise linear (i.e. Euclidean) metric of non-positive curvature (i.e. CAT(0)). Then there is a simple geometric criterion for hyperbolicity of the universal covering $\widetilde{P}$ of P and hence of the fundamental group $\Gamma = \pi_1(P)$.

No flat criterion. *Γ is word hyperbolic if and only if there is no isometric embedding* $\mathbb{R}_2 \to \widetilde{P}$.

There is also a local version of this criterion which we use here to *define* $K < 0$. Namely, we say that the *curvature vanishes at some face* F of P (our P is built of convex Euclidean polyhedra or something like that) if there is an isometric map of a domain $D \subset \mathbb{R}^2$ into P, say $I : D \to P$, such that the pull-back $I^{-1}(F) \subset D$ is non-empty and is strictly contained in D (i.e. does not approach the boundary of D). Then we say $K(P) < 0$ if $K(P) \leq 0$ and the curvature vanishes at no face.

Remarks. (a) The above definition is similar to the standard one-layer small cancellation condition for 2-polyhedra (e.g. see 4.7.A. in $[\mathrm{Gro}]_{14}$), except that we presuppose here that $K \leq 0$. It would be nice to work out a multi-dimensional one-layer condition which would yield hyperbolicity (or semi-hyperbolicity) without any assumption on the curvature.

(b) One might generalize the above definitions by allowing metrics which have constant curvature $\kappa_F \leq 0$ on every face F of P (our piecewise linear metrics have $\kappa_F = 0$ for all $F \subset P$). This may seem at first sight a considerably larger class including, for example, compact manifolds with constant negative curvature (as these can be subdivided into convex cells). But probably, every metric with $\kappa_F \leq 0$ can be deformed to the one with $\kappa_F = 0$ without disturbing the property $K \leq 0$. (One may need first to subdivide P).

(c) The n-dimensional polyhedra with $K \leq 0$ generalize, formally speaking, those of dimension 2 defined in $\mathrm{II}_{K<0}$. But the experience shows that the case $n = 2$ is rather exceptional. Namely, there are, in a certain sense, far more 2-dimensional polyhedra with $K < 0$ than for $n \geq 3$, as the known combinatorial construction of higher dimensional polyhedra with $K < 0$ arrives by a few very restricted routes, such as the *hyperbolization* (see 3.4 in $[\mathrm{Gro}]_{14}$) and ramification of Davis' reflection polyhedra, where the simplest example is the flat n-torus ramified over a collection of mutually orthogonal subtori of codimension two (see $[\mathrm{Gro}]_5$). Notice that there may be some additional constructions in the low dimension like 3 and 4. For example, one can build a lot of $K < 0$ in dimension 3 out of dodecahedra.

One common feature of the known higher dimensional constructions is that they lead to groups Γ which *do not* satisfy Kazhdan's property T. In fact, the groups of hyperbolized polyhedra decompose into amalgamated products and the reflection groups are *a-T-menable* (see [B-J-S]) according to the following definition (compare $[\mathrm{Gro}]_{15}$).

A-T-menability. A locally compact group G is called a-T-menable if it admits a *proper* isometric action on the Hilbert space $\mathbb{R}^\infty$, where "proper" means that for every bounded subset $B \subset \mathbb{R}^\infty$ the subset $\Delta_B \subset G$ consisting of those $g \in G$ for which the translate $g(B)$ meets B is precompact.

This property is opposite to the Kazhdan T-property which requires every isometric action of G on $\mathbb{R}^\infty$ to have a fixed point. Thus, if G is both T and a-T-menable, then it is compact.

Remark on $\mathbb{R}^\infty \sim H^\infty_{\mathbb{R},\mathbb{C}}$. One can replace $\mathbb{R}^\infty$ in the above discussion either by the real or the complex infinite dimensional hyperbolic space as the three spaces $\mathbb{R}^\infty$, $H^\infty_{\mathbb{R}}$ and $H^\infty_{\mathbb{C}}$ are equivalent in the following sense: each of these spaces, say X, can be embedded into another one, say Y, such that

(a) the map $X \to Y$ is Lipschitz and the induced metric on X is uniformly bounded from below by dist_X, i.e.

$$\mathrm{dist}_Y(x_1, x_2) \geq \alpha\big(\mathrm{dist}_X(x_1, x_2)\big)$$

for a function $\alpha(d)$ satisfying $\alpha(d) \to \infty$ for $d \to \infty$.

(b) Every isometry of X extends to an isometry of Y; moreover, there is a homomorphism $\mathrm{Iso}\, X \to \mathrm{Iso}\, Y$ which makes our map $X \to Y$ equivariant.

Proof. The required embedding of $\mathbb{R}^\infty$ to $H^\infty_{\mathbb{R}}$ is the one which identifies $\mathbb{R}^\infty$ with a horosphere in $H^\infty_{\mathbb{R}}$. Also the embedding $H^\infty_{\mathbb{R}} \to H^\infty_{\mathbb{C}}$ is obvious as $H^\infty_{\mathbb{C}}$ appears as a complexification of $H^\infty_{\mathbb{R}}$. Then one constructs the embeddings $H^\infty_{\mathbb{R}}$, $H^\infty_{\mathbb{C}} \to \mathbb{R}^\infty$ by prescribing the ambient metric in terms of the hyperbolic metric by

$$\mathrm{dist}_{\mathbb{R}^\infty} = (\mathrm{dist}_H)^{\frac{1}{2}} .$$

This $\mathrm{dist}_{\mathbb{R}^\infty}$ is Hilbertian by [Fa-Ha]. The remaining embeddings $H^\infty_{\mathbb{C}} \to H^\infty_{\mathbb{R}}$ and $\mathbb{R}^\infty \to H^\infty_{\mathbb{C}}$ are obtained by composing the ones we have already constructed. Q.E.D.

Note that our embedding $H^\infty_{\mathbb{C}} \to H^\infty_{\mathbb{R}}$ is rather degenerate as the isometries of $H^\infty_{\mathbb{C}}$ extended to $H^\infty_{\mathbb{R}}$ fix a point at the ideal boundary of $H^\infty_{\mathbb{R}}$. We shall see later on that this is in the nature of things.

Questions. Are polyhedral groups Γ with $K < 0$ a-T-menable or at least non-T ? (May be this becomes true if one manages to isolate and rule out the low-dimensional phenomena). If the answer turns out negative one may seek interesting homomorphisms of such Γ to $O(k, \infty)$ for $k > 2$. (Recall that $O(1, \infty)$ is the isometry group of $H^\infty_{\mathbb{R}}$).

IV. *Hyperbolic lattices.* These are the lattices in the Lie groups Γ of $\mathbb{R}$-rank one that are:

$O(n, 1)$ acting on $H^n_{\mathbb{R}}$,

$U(n, 1)$ acting on $H^{2n}_{\mathbb{C}}$ (where $2n$ is the real dimension),

$Sp(n, 1)$ acting on $H^{4n}_{\mathbb{H}}$,

and F_4^{-20} acting on the hyperbolic Cayley plane $H^{16}_{\mathbf{Ca}}$

(such a lattice Γ is hyperbolic if and only if G/Γ is compact but the essential properties of non-cocompact lattices are the same as for cocompact ones).

The basic common feature of the cocompact lattices $\Gamma \subset G$ is the large size of the quasi-isometry group $\mathrm{Q\,iso}\,\Gamma = \mathrm{Q\,iso}\,G = \mathrm{Q\,iso}\,X$ for the underlying hyperbolic symmetric space X. Namely this group is transitive on the ideal boundary $\partial\Gamma = \partial G = \partial X$; moreover, it contains a compact subgroup which is transitive on this boundary. (Notice that the one-dimensional hyperbolic groups have equally large Q iso but no other example seems to be known. Possibly, there are highly symmetric hyperbolic 2-polyhedra with large Q iso, compare $[\mathrm{Ben}]_{1-3}$, [Ba-Br], [Hag]). Probably, this symmetry is characteristic for lattices and in any case there must be many quasi-isometry characterizations for lattices because every group quasi-isometric to a lattice in G (and hence to G) is (by now known to be) *commensurable* to a lattice in G. (The most difficult case of $G = O(2,1)$ has been recently settled by Casson and independently by Gabai). One also believes that lattices can be characterized by a presence of rather weak invariant geometric structures. Here is a standing

Conjecture. Let the ideal boundary of a word hyperbolic group Γ admit a Γ-equivariant structure of a C^2-smooth manifold (which necessarily must be a sphere). Then Γ is a lattice. (Notice that one cannot relax here C^2 to C^1, see [H-P].)

As we go from $O(n,1)$-lattices to (those in) $U(n,1)$ and then to $Sp(n,1)$ and F_4^{-20} there is a strong feeling of an ascent. For example the $U(n,1)$ lattices are more scarce than those in $O(n,1)$ but they are tighter knit than $O(n,1)$-lattices. And the lattices in $Sp(n,1)$ and F_4^{-20} are even more scarce and tight. Here are specific properties giving this feeling.

(i) *Surgery.* One can cut and paste manifolds of constant negative curvature along totally geodesic hypersurfaces and derive new hyperbolic lattices (and non-lattices) from given ones. Thus one can "inter-breed" arithmetic lattices and produce *non-arithmetic lattices* in $O(n,1)$ for all $n \geq 2$. Also one can produce a variety of interesting polyhedral groups (see III above) starting from $O(n,1)$-lattices (but the full extent of the hyperbolic surgery is yet not revealed, compare [Gr-Pia], [Gr-Th]).

The "surgery" becomes quite limited when one turns to $U(n,1)$-lattices, though the complex hyperplanes (of real codimension 2) can be used in a nontrivial way. This was shown by Mostow who found, among other things, non-arithmetic (complex reflection) lattices in $U(2,1)$ and $U(3,1)$. (It is unknown if there are non-arithmetic lattices in $U(n,1)$ for $n \geq 4$). But when we finally turn to $Sp(n,1)$, $n \geq 2$, and F_4^{-20} no (apparent) surgery is available, as all totally geodesic submanifolds have codimension ≥ 4, and one knows that all lattices in these groups are arithmetic.

(ii) *Quasi-isometries.* The quasi-isometry group of $O(n,1)$ is the largest possible as it consists of all (quasi-conformal) transformations of the sphere $S^{n-1} = \partial_\infty O(n,1)$. Next, for $U(n,1)$ this group consists of the transformations of S^{2n-1} preserving the standard contact structure on S^{2n-1}. This is still an infinite dimensional group albeit a much smaller one than the group of all transformations. But for G equal $Sp(n,1)$, $n \geq 2$, and F_4^{-20} the quasi-isometry group acting on $\partial_\infty G$ equals the group G itself by a theorem of Pansu (see $[\mathrm{Pan}]_4$). (There is a more general conjecture, stated in a letter of Margulis to Prasad about 15 years ago, claiming that every quasi-isometry of a symmetric space or an Euclidean building is parallel to an isometry, apart from the cases derived from $O(n,1)$, $U(n,1)$ and automorphisms of trees).

Notice that (i) and (ii) indicate that the internal symmetry of a lattice decreases as we ascend from $O(n,1)$ to $Sp(n,1)$ via $U(n,1)$. The following fundamental properties of these lattices Γ demonstrate a decrease of the *external* symmetry as well as a certain increase of a kind of density of Γ which can be thought of as a properly measured excess of the number of relations Γ over the number of generators.

(iii) T *and* a-T. The group $O(n,1)$ and $U(n,1)$, and hence their lattices, are a-T-amenable while $Sp(n,1)$ and F_4 (along with their lattices) are Kazhdan T. (The a-T-menability follows from the earlier Remark on $\mathbb{R}^\infty \sim H^\infty_{\mathbb{R},\mathbb{C}}$. The T-property for these groups is due to Kostant, see [H-V]). This imposes strong restrictions on homomorphisms between such groups as

Every homomorphism of a T-group to an a-T-menable one has finite (or precompact if we allow non-discrete locally compact groups) image.

This is obvious with our definitions. So we see that there are no homomorphisms with unbounded images from Sp and F_4^{-20}-lattices to $U(n,1)$ (and hence to $O(n,1)$) and we shall see soon that there are very few homomorphisms from U-lattices to $O(n,1)$ (including $n = \infty$).

Remark on $P_{K<0}$. The polyhedral groups with $K<0$ seem to fit on the bottom of this hierarchy lattices just next to the O-lattices (though only very exceptional polyhedral groups are isomorphic to O-lattices). Here are specific

Conjectures. Every homomorphism of an Sp- or F_4^{-20}-lattice to a polyhedral group Γ has finite image. In fact every measureable Γ-valued cocycle on such a lattice is expected to be equivalent to a bounded one. Also every homomorphism of a U-lattice into Γ must factor through a subgroup in Γ commensurable to a surface group. (These conjectures probably can be approached via harmonic maps of pertinent locally symmetric spaces and foliations into P as suggested by the results for maps into buildings, see [Gr-Sch]).

(iv) *Superrigidity.* Consider a group Γ discretely and isometrically acting on a Riemannian manifold X and the same Γ acting isometrically (but not necessarily discretely) on another Riemannian manifold, say Y. If Y is contractible (as it is in our story) then there exists a continuous Γ-equivariant map $f_0 : X \to Y$ which is unique up to equivariant homotopy.

Question. What is the best geometric representative f_1 in the Γ-equivariant homotopy class of f_0 ?

Of course one may expect a meaningful answer only if X and Y are rather special manifolds. The following theorem provides an answer.

Superrigidity Theorem. *Let X be a symmetric space and Γ a discrete isometry group with* $\mathrm{Vol}(X/\Gamma) < \infty$, *such that no subgroup $\Gamma' \subset \Gamma$ of finite index admits a splitting $\Gamma' = \Gamma_1' \times \Gamma_2''$, where Γ_1' is a $U(n,1)$ or $O(n,1)$-lattice for $n \geq 1$. Let Y be another symmetric space with an isometric Γ action and $f_0 : X \to Y$ an equivariant map. Then f_0 is Γ-equivariantly homotopic to a unique geodesic map f_1.* (Recall, a map $X \to Y$ is *geodesic* if its graph in $X \times Y$ is a totally geodesic submanifold).

This theorem for rank $X \geq 2$ is proven by Margulis using ergodic theory and the remaining case where $X = H_{\mathbf{H}}^{4n}$ or $H_{\mathbf{Ca}}^{16}$ is due to Corlette who relies in his proof on the theory of harmonic maps. (The mixed cases of split Γ easily follow from these two.)

Remarks on generalizations. The above theorem generalizes to Y (and sometimes X) being Euclidean buildings (and products of buildings and symmetric spaces, see $[\mathrm{Mar}]_5$, [Gr-Sch]). Also there are generalizations to some cases where Y is an *infinite dimensional* symmetric space, such as $Y = O(k,\infty)/O(k) \times O(\infty)$. In fact, the theory of harmonic maps works whenever the target space Y has $K(Y) \leq 0$ even if $\dim Y = \infty$. Let us make a precise statement in the case where X/Γ is compact and Y is a smooth Riemannian manifold of finite or infinite dimension with an isometric action of Γ. Ideally, we would like to homotope a given map $f_0 : X \to Y$ to a *harmonic map* $f : X \to Y$ by a homotopy of Γ-equivariant maps, where a map f is called harmonic (in the present framework) if it minimizes the *energy* that is $\int\limits_{X/\Gamma} \|Df\|^2\, dx$. (This integral makes sense as the norm of the differential is Γ-invariant). But, this may be impossible even if $\dim Y < \infty$. Take for example a parabolic action of $\Gamma = \mathbb{Z}$ on $Y = H^n$ which fixes a single point on the ideal boundary $\partial_\infty H^n$ and let $X = \mathbb{R}$ with the obvious action of $\mathbb{Z}$. Then the energy minimization process tries to move every Γ-equivariant map $\mathbb{R} \to H^n$ to this fixed point which does not belong to H^n and so the process diverges. Yet we can recover the convergence if we allow a modification of the space Y in the course of minimization (compare "plusification" in 6.B.).

(Ulter)limits and weak containments between Γ-spaces. Let (Y_i, y_i) be a sequence of pointed metric spaces with isometric actions of a fixed group Γ. One can define, using an ulterfilter, a certain huge limit Γ-space but this limit will appear in our discussion only as a figure of speach. We say that a Γ-space Y' is *Γ-contained in a sublimit* of Y_i if the following condition is satisfied.

$(*)$ For every finite subset $Z \subset Y$ there exist injective maps $\mu_{i_j} : Z \to Y_{i_j}$ for $i_j \to \infty$, such that

(i) $\mathrm{dist}(z, y_{i_j}) \leq \mathrm{const}$, for all $z \in Z$ and $j = 1, 2, \dots$.

(ii) The metrics on Z induced from Y_{i_j} converge for $j \to \infty$ to the metric $\mathrm{dist}_{Y'} | Z$.

(iii) The (partially defined) action of Γ on the image of Z in Y_{i_j} converges to the action in Y'. That is, if $z' = \gamma(z) \in Z$ for $z \in Z$ and $\gamma \in \Gamma$, then

$$\mathrm{dist}\Big(\mu_{i_j}(z'),\ \gamma\big(\mu_{i_j}(z)\big)\Big) \underset{j\to\infty}{\longrightarrow} 0\,.$$

Next, we say that a Γ-space Y' is *weakly Γ-contained* in Y if there exists a sequence $y_i \in Y$, such that Y' is Γ-contained in a sublimit of the sequence (Y, y_i). (This agrees with the corresponding notion in the theory of unitary representation as it is used in Kazhdan's definition of T : a group Γ satisfies T if whenever an isometric action of Γ on the Hilbert sphere $S^\infty \subset \mathbb{R}^\infty$ weakly contains the trivial action of Γ on the one-point space, the action has a fixed point in S^∞).

Now we return to $f_0 : X \to Y$ and reformulate the basic existence theorem due to Eells and Sampson in our terms (compare $[\text{Gro}]_{21}$, [Gr-Sch]).

If $K(Y) \leq 0$ then there exists a Riemannian manifold Y' with a Γ-action which is weakly Γ-contained in Y and which receives a Γ-equivariant harmonic map $X \to Y'$. Furthermore, if Y is strongly complete (see below) then one can choose Y' isometric to a totally geodesic subspace in Y.

Definition. A metric space Y is *strongly complete* if every separable metric space Y' which is weakly contained in Y admits an isometric embedding into Y (where "weakly contained" means that for every finite subset $Z \subset Y'$ there exist embeddings $\mu_i : Z \to Y$, such that the induced metrics converge to $\text{dist}_{Y'}|Z$ as $i \to \infty$).

Example. Finite dimensional homogeneous Riemannian manifolds are (obviously) strongly complete. What is more interesting is the strong completeness of the "classical" infinite dimensional symmetric spaces, such as $\mathbb{R}^\infty$, S^∞, H^∞, $O(k,\infty)/O(k) \times O(\infty)$ etc.

We conclude our brief discussion on the use of harmonic maps for proving rigidity by recalling that the final step, harmonic $\Longrightarrow$ geodesic, is achieved with *Bochner formulas* which are completely unsensitive to the dimension of the ambient space Y and that such formulas are known for almost all (some people may know them by now for all) symmetric spaces X which contain no $H^n_{\mathbb{R}}$ and $H^{2n}_{\mathbb{C}}$-factors (compare the superrigidity theorem). In these cases the existence of an action of Γ on a symmetric space Y (of finite or infinite dimension) implies (at the very least) that either this action weakly contains the trivial action or Y contains a totally geodesic subspace isometric to a factor of X (or to X itself if X is irreducible). Further discussion on rigidity and related matters mentioned in this section appears in [Gr-Pa].

On the failure of superrigidity for $O(n,1)$ and $U(n,1)$. It is easy to make up examples of cocompact lattices $\Gamma \subset O(n,1)$ and $\Gamma' \subset O(n+1,1)$, such that Γ embeds onto a quasi-convex subgroup in Γ' which is not totally geodesic. In fact one can realize such an example by a map between compact manifolds of constant negative curvature, say $f : V \to V'$, such that f is geodesic away from a closed totally geodesic hypersurface $V_0 \subset V$ and f does have a corner along V_0. The presence of such a corner for $\dim V \geq 2$ makes it impossible to homotope f to a geodesic map, since the limit set of $\Gamma = \pi_1(V)$ in $S^n = \partial_\infty \Gamma'$ is a *non-smooth* sphere $S^{n-1} \subset S^n$ with singularities coming from corners. (To produce a corned embedding start with some V' having totally geodesic hypersurfaces V_1 and V_2 meeting across V_0 and glue V out of half of V_1 and V_2 as schematically shown on Fig. 17.

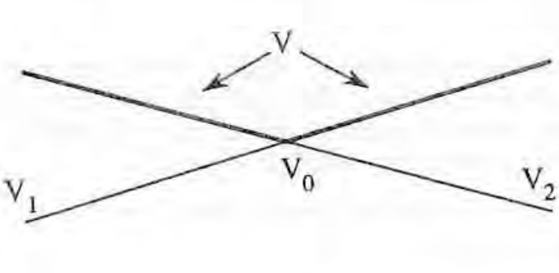

Figure 17

Apparently, superrigidity may also fail for $H_{\mathbb{C}}^{2n}$ but not so drastically as for $H_{\mathbb{R}}^{n}$. In fact, if in the Superrigidity theorem we take $X = H_{\mathbb{C}}^{n}$ then the conclusion remains valid in the weaker form where "geodesic" is replaced by "pluriharmonic" (for the natural complex structure on $H_{\mathbb{C}}^{2n}$).

A comparison between $Sp(n,1)$-lattices and those of rank ≥ 2. As far as the superrigidity, arithmeticity and the T-property are concerned the lattices in $Sp(n,1)$ and F_4^{-20} behave as in the case of rank ≥ 2. But yet for rank ≥ 2 the lattices are infinitely more rigid. For example, a theorem by Margulis asserts that every lattice Γ in a simple Lie group of $\mathbb{R}$-rank > 2 is *almost simple*. Thus it behaves as an overcooled liquid: add an extra relation to Γ and it crystallizes to a finite group. But the lattices in $Sp(n,1)$ and F_4 are after all hyperbolic groups which are soft and amorphous creatures (by the group theoretic standards) even though they may look rigid and unyielding from the representation theoretic viewpoint.

On the increase of rigidity with dimension. It seems that the lattices of every classical series, such as Γ in $O(n,1)$ etc. grow with n in the hierarchy of rigidity. The only definite result in this direction I am aware of appears in [Co-Haa] where the $Sp(n,1)$-groups are shown to satisfy stronger and stronger versions of the T-property as n increases.

V. *Quasi-convex discrete groups.* Let Γ be a discrete isometry group of a symmetric space X with $K(X) \leq 0$ such that $\operatorname{Vol} X/\Gamma = \infty$. There are several notions which express the idea of convexity of Γ such as

(i) every Γ-orbit $\Gamma(x) \subset X$ is *geodesically quasi-convex*: $\operatorname{dist}(tx + (1-t)\gamma(x), \Gamma(x)) \leq c < \infty$, for all $\gamma \in \Gamma$ and $t \in [0,1]$ (where c may depend on $x \in X$);

(ii) there exists a large-scale *Lipschitz Γ-equivariant retraction* $X \to \Gamma(x)$;

(iii) the orbit map $\gamma \mapsto \gamma(x)$ is a *quasi-isometric embedding* of Γ to X.

If furthermore, Γ is word hyperbolic as an abstract group (which is of primary interest to us in this section) then we may require that the limit set of Γ in the ideal geodesic boundary $S^{n-1} = \partial_{\text{geo}} X$, $n = \dim X$, equals the ideal hyperbolic boundary $\partial_\infty \Gamma$. (Notice that S^{n-1} is a topological sphere and moreover, it has a natural Lipschitz structure invariant under $\operatorname{Iso} X \supset \Gamma$, but if $\operatorname{rank} X \geq 2$ it may lack a Γ-invariant smooth structure, as is seen, for example, on $S^2 = \partial_{\text{geo}}(H^2 \times \mathbb{R})$ which has no invariant C^1-structure). The simplest examples of quasi-convex subgroups come from lattices acting on totally geodesic subspaces in X and sometimes one can "glue" such subgroups together by so-called *combination theorems*. Yet in all these examples (as far as I can see) the dimension of Γ (understood in the asymptotic or in the hyperbolic sense) does not exceed the maximum m of dimensions of proper totally geodesic subspaces in X.

Question. Is always $\dim \Gamma \leq m$? If not what is the best bound on $\dim \Gamma$ in terms of X ?

Remarks and examples. (a) *Non-convex groups.* If we do not assume any kind of convexity we may have $\dim \Gamma = n - 1$, where Γ fixes a horofunction on X. Yet such examples can be probably suppressed by some very mild restrictions on Γ more general than any kind of convexity.

(b) *Reflection groups.* Consider a convex polyhedral cone in the space $\mathbb{R}^{p+q}$ with the scalar product $\langle x, y\rangle = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$ and let $\Gamma \subset O(p,q)$ be the group generated by the reflections around the codimension one faces with respect to our scalar product. Under certain assumptions on the angles between the faces this Γ is discrete. It may be semi-hyperbolic of arbitrarily large dimension. For example every Coxeter group Γ comes this way (see $[\text{Vin}]_1$, [Mous]). However, there are no examples of hyperbolic reflection groups of large (e.g. ≥ 30) dimension (see $[\text{Vin}]_2$ and [Mous] on this matter) and one does not know for most groups $O(p,q)$ what is the maximal possible dimension $d = d(p,q)$ a discrete reflection group $\Gamma \subset O(p,q)$ may have.

(c) *A bound on* $\operatorname{codim} \Gamma$ *by* $\operatorname{rank} X$. Suppose Γ is hyperbolic and $\partial_\infty \Gamma$ embeds into $\partial_{\text{geo}} X$ (e.g. Γ has geodesically quasi-convex orbits in X). Recall that $\partial_{\text{geo}} X$ is partitioned into simplices (chambers) corresponding to the Weyl chambers (and their faces) in the maximal flats in X. Since the action of $\Gamma \subset \operatorname{Iso} X$ is isometric with respect to the Tits metric, which is compatible with the topology on each simplex of this partition, the limit set $\partial_\infty \Gamma \subset \partial_{\text{geo}} X$ must be "transversal" to these simplices. For example, the fixed points in $\partial_\infty \Gamma$ of non-torsion elements $\gamma \in \Gamma$ are *isolated* points in $\partial_\infty \Gamma$ endowed with the Tits metric. It follows that

$$\operatorname{codim} \Gamma \underset{\text{def}}{=} \dim X - \dim \Gamma \geq \operatorname{rank} X - 1 \; .$$

(d) *Convex cocompact groups of isometries of hyperbolic symmetric spaces.* If X has rank one then all possible notions of (quasi-)convexity for Γ coincide. In fact, if Γ has subexponential distortion in X (i.e. in $\operatorname{Iso} X$) then there exists a geodesically convex Γ-invariant subset $X_0 \subset X$ such that X_0/Γ is compact, which is expressed by saying Γ is *convex cocompact.* (Thus "cocompact" is a special case of "convex cocompact" and *not* the other way around). There is a general feeling that if $\dim X$ is large and/or $\operatorname{codim}\Gamma$ is small, then Γ is, in a certain sense, piecewise geodesic. One may conjecture, for example, that X_0 contains a totally geodesic subspace $Y \subset X$, such that $\dim Y \to \infty$, provided $\dim X \to \infty$ while $\operatorname{codim}\Gamma$ stays bounded. (Here $\operatorname{codim}\Gamma$ may be interpreted as the topological codimension of the limit set $\partial_\infty\Gamma \subset \partial_\infty X$).

The convex groups Γ of low codimension (and their limit sets) gain additional structure when we pass from $X = H^n_{\mathbb{R}}$ to $H^{2n}_{\mathbb{C}}$ and then to $H^{4n}_{\mathbb{H}}$ and $H^{16}_{\mathbb{C}a}$.

(d′) *Complex hyperbolic groups* Γ. First we observe, following Corlette (see [Cor]$_2$) that $\partial_\infty\Gamma \subset \partial_\infty H^{2n}_{\mathbb{C}}$, for $n \geq 2$, must have codimension at least two. Indeed, assume Γ has no torsion (or pass to such subgroup of finite index in Γ) and look at the complex manifold $V_0 = X_0/\Gamma$. This V_0 has strictly convex, and hence strictly pseudoconvex (or $\mathbb{C}$-convex) boundary and, therefore, it homotopy retracts, by a theorem of Grauert, to a union of a (unique) maximal compact complex subvariety $W_0 \subset \operatorname{Int} V_0$ and a (non-unique) polyhedral subset $K \subset V_0$ having $\dim K \leq n$. Thus X_0 admits a Γ-equivariant retraction onto a subset of codimension ≥ 2 which immediately implies $\operatorname{asdim}\Gamma \leq 2n-2$ and (with a little thought) $\dim \partial_\infty\Gamma \leq 2n-3$. In fact, whenever $\dim\Gamma > n$, there is a compact complex subvariety W_0 in (the interior of) $V_0 = X_0/\Gamma$ having $\dim_{\mathbb{R}} V_0 = \dim\Gamma$. In particular $\dim\Gamma$ is necessarily *even* if it is $\geq n+1$. (Also notice that, according to Grauert, one can blow down W_0 to a single point, such that the resulting complex (!) space $\overline{V}_0$ is Stein.) This agrees with our general philosophy that low codimensional Γ are similar in most respects (such as rigidity) to lattices. Yet, in the simplest case where W_0 is a *divisor* in V_0, we do not know when each irreducible component of W_0 (of codimension 2) is totally geodesic in V_0. (This may be expected for $n \geq 3$ and further reinforced by extra conditions such as (i) Γ is contained in a *lattice* Γ' acting on X, (ii) W_0 is a smooth variety whose extrinsic curvature in V_0 is bounded by a small constant ε).

Remarks on rigidity. (d$'_1$) Suppose W_0 *is* a totally geodesic submanifold in V_0 and assume there is no K-part in V_0 (which can be insured by a purely algebraic assumption on Γ, for example, by requiring that $\partial_\infty\Gamma$ is a topological sphere). Then we claim that the only possible deformations of $\Gamma \subset \operatorname{Iso} X$ come from "rotations" of X around the totally geodesic submanifold $X_0 = \widetilde{V}_0 \subset X$. This amounts to saying that if we deform Γ to a (also convex) subgroup $\Gamma' \subset \operatorname{Iso} X$, then the corresponding $W'_0 \subset V'_0$ remains *geodesic.* To see this let first W_0 and W'_0 be complex *curves* (which is, in truth, not allowed by our assumption $\dim\Gamma > n = \dim_{\mathbb{C}} X$). Then, by the elementary Kähler geometry, the area of such a curve W is an integer multiple of a fixed (normalization) constant and, hence, constant under deformations. The total Gauss curvature of W is obtained by integrating over W certain expressions of the form $-dw - \kappa(w)dw$, where $\kappa(w) \geq 0$ and $\kappa(w) > 0$ at each point $w \in W$, where W is *not* totally geodesic. Hence $\kappa(w) = 0$ in our case as the total Gauss curvature does not change under deformations. Now, we come back to reality, where $\dim_{\mathbb{C}} W_0 \geq 2$. Here we know that W'_0 is biholomorphic to W_0 as the (unique) harmonic map $W_0 \to V'_0$ realizing a given isomorphism $\Gamma \to \Gamma'$ is necessarily holomorphic by the Siu rigidity theorem (see [Gr-Pa]). Then the above integration argument applies to the tautological one-dimensional complex foliation of the space of totally geodesic complex curves in W_0 (compare [Mok]).

(d'_2) The Mostow rigidity theorem for cocompact lattices in $\operatorname{Iso} X$ claims that every two such mutually isomorphic lattices, say Γ and Γ', can be joined by biholomorphic maps between $V = X/\Gamma$ and $V' = X/\Gamma'$. One can view this from another angle by taking the diagonal action of $\Gamma \simeq \Gamma'$ on $X \times X$ for Γ acting on the first factor and Γ' on the second one, and then by looking on the graph of our map $V \leftrightarrow V'$ lifted to $V^{\square} = (X \times X)/\Gamma$. The Mostow theorem becomes equivalent to the existence of compact complex submanifolds in $V^{\square}$ or, better to say (especially if Γ has torsion), a Γ-invariant complex submanifold $Y \subset X \times X$ with Y/Γ compact. This leads to the following

Problem. Let Z (for $X \times X$) be a symmetric Hermitian space and Γ a discrete group of isometries satisfying some quasi-convexity condition. When does Z contain a Γ-invariant complex subspace $Y \subset Z$ with a "small" (e.g. compact or quasi-projective algebraic) quotient space Y/Γ ? When, on the contrary, is Y/Γ Stein ? When is it a holomorphically convex space ? etc.

Notice that the approach via the Grauert pseudo-convexity theory only works for $\dim \Gamma > \dim_{\mathbb{C}} Z$ and does apply, for example, to the above $Z = X \times X$ of the Mostow rigidity theorem (which suggests an augmentation of Grauert by Siu. Also one may try complex k-convexity).

(d'') *Convex cocompact groups* Γ *acting on* $H^{4n}_{\mathbb{H}}$ and $H^{16}_{\mathbf{Ca}}$. We use the usual normalization of the metric in our space $X = H^{4n}_{\mathbb{H}}$ or $H^{16}_{\mathbf{Ca}}$ such that the ball of radius R has the following volume growth, $\operatorname{Vol} B(R) \sim \exp \delta R$ where δ is the Hausdorff dimension of $\partial_\infty X$ with the (standard) Carnot-Caratheodory-Mostow metric (for which the action of $\operatorname{Iso} X$ on $\partial_\infty X$ is conformal, see [Gro-Pa]), that is $\delta = 4n + 2$ for the quaternionic hyperbolic spaces $H^{4n}_{\mathbb{H}}$ and $\delta = 22(= (16-1) + (8-1))$ for $H^{16}_{\mathbf{Ca}}$. If Γ is a discrete group of isometries of X with $\operatorname{Vol} X/\Gamma < \infty$ then the number of elements of every Γ-orbit in B, call it $N_\Gamma(R)$, grows roughly as $B(R)$, i.e. as $\exp \delta R$, but if Γ is of infinite covolume $N_\Gamma(R)$ may grow slower. Then one defines $\delta(\Gamma)$ as the infimum of those d for which $N_\Gamma(R) \lesssim \exp dR$. If Γ is a convex cocompact group then this $\delta(\Gamma)$ equals to the Hausdorff dimension of $\partial_\infty \Gamma \subset \partial_\infty X$ with respect to the CCM- metric by a theorem of Sullivan (originally proven for $H^n_{\mathbb{R}}$ and extended to other hyperbolic spaces by Corlette). The following beautiful theorem of Corlette shows that the size of Γ acting on $H^{4n}_{\mathbb{H}}$ and $H^{16}_{\mathbf{Ca}}$ is bounded geometrically as well as topologically.

Theorem (see [Cor]$_2$). *If* Γ *is a discrete group of infinite covolume then* $\delta(\Gamma) \le 4n$ *(i.e. "codim"* $G \ge 2$*) for* $H^{4N}_{\mathbb{H}}$ *and* $\delta(\Gamma) \le 16$ *(i.e. "codim"* ≥ 6*) for* $H^{16}_{\mathbf{Ca}}$.

Corlette's proof relies on the representation theory and especially on a quantitative version of the T-property for the groups $Sp(n,1)$ and F_4^{-20} due to Kostant. (Notice, we have slightly cheated with the definition of $\delta(\Gamma)$ but this hardly matters). Observe that a weak qualitative version of Corlette's theorem is valid (and obvious) for all symmetric spaces X where the group $\operatorname{Iso} X$ is T (i.e. X has no factors isometric to $\mathbb{R}^n$, $H^n_{\mathbb{R}}$ or $H^{2n}_{\mathbb{C}}$). Namely, one can bound Γ of infinite covolume by giving a non-trivial *lower* bound on the volumes of the concentric R-balls in the quotient space. We have already mentioned that, by Kazhdan's observation, these balls $B(X/\Gamma, R)$ cannot have subexponential growth, but it is equally clear that

$$\operatorname{Vol} B(X/\Gamma, R) \exp -\varepsilon \; R \underset{R\to\infty}{\to} \infty ,$$

for some $\varepsilon = \varepsilon(X) > 0$. (The reader who wants to know more about this ε and $\delta(\Gamma)$ should consult an expert in the representation theory.)

Bounds on $\dim \Gamma$. If Γ is convex cocompact, we may try to estimate $\operatorname{codim}(\partial_\infty \Gamma \subset \partial_\infty X)$ in terms of $\delta(X) - \delta(\Gamma)$ that is the Hausdorff codimension for CCM-metric. It is immediate (as observed in [Cor]$_2$) that

$$\operatorname{codim} \partial_\infty \Gamma \ge CCM \operatorname{codim} \partial_\infty \Gamma = \frac{1}{2}\big(\delta(X) - \delta(\Gamma)\big) .$$

(The constant $1/2$ appears because $\partial_\infty \Gamma$ is modelled on a nilpotent group of nilpotency degree 2). On the other hand, if we have a *smooth* submanifold $M \subset \partial_\infty X$, of sufficiently large dimension, then $\operatorname{codim} M = CCM \operatorname{codim} M$ where "large dimension" is such that M is necessarily transversal at a generic point to the subbundle $T_1 \subset T(\partial_\infty X)$, responsible for the CCM-metric. For example, if $X = H^{2n}_{\mathbb{C}}$, then $\partial_\infty X$ is modelled by the Heisenberg group, $\operatorname{codim} T_1 = 1$, and M *cannot* be everywhere tangent to S for $\dim M \ge n$ which makes "large" in this case. This "large" can be also computed by simple linear algebra for $H^{4n}_{\mathbb{H}}$ and $H^{16}_{\mathbf{Ca}}$. (A fast evaluation I have made in my head says that $\dim M \ge 2n+2$ is "large" for $H^{4n}_{\mathbb{H}}$ and $\dim M \ge 11$ is "large" for $H^{16}_{\mathbf{Ca}}$). On the basis of such "smooth" considerations Pansu suggested that Corlette's theorem for $H^{4n}_{\mathbb{H}}$ would yield the connectedness of the complement $\partial_\infty H^{4n}_{\mathbb{H}} - \partial_\infty \Gamma$ privately conjectured by Corlette prior to his proof in [Cor]$_2$. Let us spell out an explicit conjecture underlying Pansu's suggestion.

Conjecture. Let Y be a smooth Carnot-Caratheodory space, consider all smooth m-dimensional submanifolds in Y and denote by $\delta_m = \delta_m(Y)$ the infimum of their CC-Hausdorff dimensions. (Notice that $\delta_m(Y)$ is a relatively easy computable invariant of Y). Then every compact subset in Y of topological dimension $\ge m$ has CC-Hausdorff dimension $\ge \delta_m$.

Remarks. (i) One may need some extra regularity condition on Y, such as real analyticity or a certain homogeneity (equisingularity) to avoid possible pathologies.

(ii) The number $\delta_m(Y)$ seems closely related to the isoperimetric (filling) exponents for m-dimensional filling inequalities for Y (compare 5.A_2 - A_5.) and to the infδim-invariants appearing in 7.C_1.

(d''') *Foliated way.* The space of all isometric copies of $H^{2n}_{\mathbb{C}}$ in $H^{4n}_{\mathbb{H}}$ is tautologically foliated into these very $H^{2n}_{\mathbb{C}}$ and a complex analysis on the resulting foliation can be used for the study of $H^{4n}_{\mathbb{H}}$ (see $[\mathrm{Gro}]_{21}$). If one could prove a Grauert type theorem for holomorphic foliations with leaves having pseudoconvex boundaries (which appears realistic and not too difficult), one would show for compact coconvex Γ acting on $H^{4n}_{\mathbb{H}}$ and $H^{16}_{\mathbf{Ca}}$ that all their topology above middle dimension comes from totally geodesic subspaces. Namely, the quotient manifold $V = X/\Gamma$ (here $X = H^{4n}_{\mathbb{H}}$ or $H^{16}_{\mathbf{Ca}}$ and Γ has no torsion) should admit a homotopy retraction onto a union $W \cup K$, where W is a finite union of compact (immersed) totally geodesic submanifolds and K is a compact subset with $\dim K \leq \frac{1}{2}\dim X$. This seems a plausible conjecture; it also generalizes to other (non-Hermitian) symmetric spaces which contain Hermitian subspaces. Moreover, under especially favourable circumstances, one should be able to reduce dimension of K further to $\dim K \leq \frac{1}{2}\dim X - 1$.

Example: stability of geodesic submanifolds (compare $[\mathrm{Gro}]_{19}$). Let our $V = X/\Gamma$ contain a closed submanifold $W_0 \subset V$ which is *ε-geodesic* (which means the exterior curvature of W_0 in V is $\leq \varepsilon$) for a small $\varepsilon > 0$. The question is whether W_0 can be perturbed to a totally geodesic submanifold. If X is Hermitian symmetric, $\dim W_0 > \frac{1}{2}\dim X$ and W_0 has a small strictly convex (or at least strictly pseudoconvex) neighborhoood, then, by the Grauert theorem, W_0 can be approximated by a complex subvariety W which, for small enough ε, can be seen to be non-singular and C^2-approximating W_0. Yet it is not clear when this W is totally geodesic. Furthermore, if X is non-Hermitian but something like $H^{4n}_{\mathbb{H}}$, then the geodesic approximation of W_0 is probably possible via the foliated way. (It may be worthwhile to reiterate in this framework the abstract stability problem raised in $[\mathrm{Gro}]_{19}$: when can an almost locally symmetric Riemannian metric on a closed manifold be deformed to a locally symmetric one ?)

VI. *Underlattices and overlattices. An underlattice* $\overline{\Gamma}$ is a group which admits an epimorphism $\Gamma \to \overline{\Gamma}$ of a lattice $\Gamma \subset G$ for a simple Lie group G of $\mathbb{R}$-rank 1. (We could enlarge the class by allowing products of such Γ's). The least interesting of these $\overline{\Gamma}$ are those where the homomorphism $\Gamma \to \overline{\Gamma}$ factors through a free group. This indeed may happen as there are (hyperbolic) $O(n,1)$-lattices of a given dimension n which admit an epimorphism onto $\mathbb{F}_k$ for a given k. On the other hand some underlattices are very close to lattices. Here are standard examples.

(i) *Collapsing cusps.* Let V be a locally symmetric space of negative curvature with cusps whose fundamental group is our Γ. Recall, that V decomposes

$$V = V_0 \cup C_1 \cup \ldots \cup C_k ,$$

where V_0 is a compact manifold with (concave) boundary and C_i, $i = 1, \ldots, k$ are the cusps. Each cusp lifts to a horoball in the universal covering X of V and C_i is unique up to a choice of a horosphere in a given family of concentric horospheres. Topologically $C_i = \partial C_i \times \mathbb{R}_+$ where the boundary ∂C_i is an infranilmanifold whose fundamental group Γ_i injects into $\Gamma = \pi_1(V)$ and the boundary ∂V_0 is the disjoint union $\partial V_0 = \bigcup_{i=1}^{k} \partial C_i$. Now we attach topological cones to all components of the boundary of V_0. The resulting singular space, say $\overline{V}_0$, has $\pi_1(\overline{V}_0) = \overline{\Gamma} = \{\Gamma \mid \Gamma_i = 1\}$.

Claim. *Suppose the metric in V is normalized such that $K(V) \leq -1$. Then if every cusp C_i is sufficiently thick, that is if the injectivity radius of V at each point $v \in \partial C_i \subset V$ satisfies* Inj $\mathrm{Rad}_v \geq \rho$, *for a sufficiently large ρ, then $\overline{V}_0$ admits a singular metric with $K < 0$. In particular, $\overline{\Gamma}$ is a hyperbolic group of dimension $n = \dim V$.*

This is explained somewhere in $[\mathrm{Gro}]_{14}$. Notice that a sufficient bound for ρ is $\rho \geq 2$.

Next, even if the cusps are not thick, we can perform a similar procedure by first looking at regular finite coverings of ∂C_i corresponding to normal subgroups $\widetilde{\Gamma}_i \subset \Gamma_i$ of finite indices. If these coverings have the injectivity radii at ∂C_i at least ρ, then the space $\overline{V}_0$ admits a structure of an orbispace with $K < 0$ where the orbistructure is localised at the vertices of the cones, $c_i \in \overline{V}_0$, $i = 1, \ldots, k$, with the ramification groups $\Gamma_i/\widetilde{\Gamma}_i$. The corresponding group we obtain is $\overline{\Gamma} = \{\Gamma | \widetilde{\Gamma}_i = 1\}$ which is hyperbolic (with torsion) of dimension n.

A similar operation may be performed with compact totally geodesic submanifolds in V and, more generally, with locally convex subsets $W_0 \subset V$. Namely, if *the normal injectivity radius of* W_0 is $\geq \rho$, i.e. if the ρ-neighborhood $W_\rho \supset W_0$ homotopy retracts on W_0, then the space $\overline{V}$ obtained by removing W_ρ and coning the boundary of $V_\rho = V - W_\rho$ (which equals the boundary of W_ρ), admits a metric with $K < 0$ and the fundamental group of this space is hyperbolic of dimension $n = \dim V$, provided $\rho \geq 1$ (and where V assumed compact, i.e. without cusps). Furthermore, if ρ is small, then $K < 0$ may be achieved with an orbifold structure at the vertex of the cone corresponding to a (sufficiently large) finite covering of the boundary ∂V_ρ. (If W_0 is a totally geodesic submanifold of *codimension two*, then $\pi_1(\partial W_\rho) \neq \pi_1(W_\rho) = \pi_1(W_0)$ and one does not have to "uncover" the extra loop in $\partial V_\rho = \partial W_\rho$ to have $K < 0$).

If one is content with $\overline{\Gamma}$ being hyperbolic without insisting on $K < 0$, one can cone certain totally geodesic submanifolds $W \subset V$ which may have very small normal injectivity radius and yet have the group $\overline{\Gamma} = \{\Gamma | \Gamma_W = 1\}$ hyperbolic, where $\Gamma_W \subset \Gamma$ denotes the union of the images of the fundamental groups of the connected components of W in $\Gamma = \pi_1(V)$. (Our coning applies separately to each connected component of W, if we do not want extra "free" generators in $\overline{\Gamma}$).

Let us formulate a "small cancellation" condition which insures the hyperbolicity of $\overline{\Gamma}$. We assume as earlier $K \leq -1$ and we fix a certain small constant ε, say $\varepsilon = 1$, and say W ε-approaches itself along an R-ball B if there exist two different (totally geodesic) submanifolds Y and Y' in X which cover certain components of W in V (it may be the same component, though we assume $Y \neq Y'$) and an R-ball $\widetilde{B} \subset Y$ which projects to B in W and such that $\mathrm{dist}(b, Y_2) \leq \varepsilon$ for all $b \in B$. (We do not assume $\dim Y = \dim Y'$).

Small cancellation condition. If W ε-approaches itself along an R-ball $B \subset W$ then the injectivity radius of W (with the Riemannian metric induced from V) at the center of B is at least kR for some constant $k \geq 1$.

Claim. *For every* $\varepsilon > 0$ *there is* $k = k(\varepsilon)$ (e.g. *for* $\varepsilon = 1$ *one may take* $k = 7$) *such that if* W *satisfies the* $k(\varepsilon)$*-condition for a fixed* $\varepsilon > 0$, *then the group* $\overline{\Gamma}$ *is word hyperbolic and has dimension* $n = \dim V$.

In fact, by looking at minimal filling disks in $\overline{V}$ one sees that $\overline{\Gamma}$ satisfies the linear isoperimetric inequality and that there is no non-trivial normal relation in $\Gamma_W \subset \Gamma$.

Remarks (a). The above claim, properly modified, applies to all hyperbolic groups Γ, though one should be careful about torsion, (especially 2-torsion), which was overlooked in $[\mathrm{Gro}]_{14}$ and put straight in $[\mathrm{Delz}]_{1,2}$.

(b) The small cancellation condition we imposed on W seems unduly restrictive especially if $\dim W \geq 2$.

Our constructions of interesting factor-groups $\overline{\Gamma}$ of lattices Γ depend on the presence of special convex subgroups in Γ associated to totally geodesic submanifolds (and/or cusps).

Questions. What are other possible factor groups of lattices Γ ? For example, do $Sp(n,1)$ and F_4^{-20}-lattices Γ admit infinite (hyperbolic) factorgroups $\overline{\Gamma}$ with $\dim \overline{\Gamma} < \dim \Gamma$, say with $\dim \overline{\Gamma} = 2$? Are there (hyperbolic) factor groups $\overline{\Gamma}$ of cocompact $U(n,1)$-lattices with $\dim \overline{\Gamma} = \dim \Gamma - k$, where k is an *odd* positive integer ?

Here one should be aware of the following slightly troublesome factorgroups of Γ. Take a generic system of elements $\gamma_1, \dots \gamma_k$ in Γ and add arbitrary relations $w_1, \dots, w_\ell$ in γ_i. The resulting group $\overline{\Gamma} = \{\Gamma | w_j = 1\}$ sits between Γ and $\overline{\overline{\Gamma}} = \{\Gamma | \gamma_i = 1\}$. A variation of this construction may be applied to free products $\Gamma * \Gamma'$ (see $[\mathrm{Gro}]_{14}$) which gives us $\overline{\Gamma}_*$ which is a factorgroup of Γ as well as of Γ' and has $\dim \overline{\Gamma}_* = \max(\dim \Gamma, \dim \Gamma', 2)$. But in some intuitive sense such $\overline{\Gamma}^*$ is *reducible* (see below) and contains a piece corresponding to Γ' which may have dimension $> \dim \Gamma$ (if $\dim \Gamma' > \dim \Gamma$). Then one feels uncomfortable regarding $\overline{\Gamma}^*$ as an undergroup of Γ but one may try to isolate the part of (the ideal boundary of) $\overline{\Gamma}^*$ belonging to Γ whose dimension does not exceed that of (the ideal boundary of) Γ unless Γ is virtually free.

On reducibility. In order to define reducibility one needs a topological characterization of the spaces appearing as ideal boundaries of groups which would include the idea that such a space has "no boundary". Then a space Δ "without boundary" is called *reducible* if it contains a proper closed subset Δ_0 of full dimension which also has "no boundary". A particularly strong reducibility appears if Δ contains a (closed) subset Δ' of codimension ≥ 2 such that removing Δ' from Δ changes the connectivity of Δ, e.g. makes a connected space disconnected. This strong reducibility can be defined also for non-hyperbolic groups Γ by using the large-scale topology of Γ instead of $\partial_\infty\Gamma$.

It would be interesting to find an algebraic criterion for reducibility.

Test Questions. Let Γ be a hyperbolic group with $\dim \partial_\infty\Gamma \geq 2$ and suppose there are two points $\delta_1, \delta_2 \in \partial_\infty\Gamma$, such that the complement $\partial_\infty\Gamma - \{\delta_1, \delta_2\}$ is disconnected. Does there exist a non-torsion element $\gamma \in \Gamma$ whose two fixed points in $\partial_\infty\Gamma$ also disconnect $\partial_\infty\Gamma$? If so, when does Γ decompose into an amalgamated product along a virtually cyclic subgroup containing γ ? What happens if $\partial_\infty\Gamma$ of dimension ≥ 3 can be made disconnected by a topological circle in $\partial_\infty\Gamma$?

T-property for underlattices. Since the $Sp(n,1)$- and F_4^{-20}-lattices are T so are their factor groups. This was pointed out to me by D. Kazhdan who observed that this property, together with "generic hyperbolic factorization", yields uncountably many isomorphism classes of finitely generated T-groups, contrary to what was earlier conjectured by A. Connes.

Overlattices. These are the groups $\widetilde{\Gamma}$ coming along with epimorphisms into lattices, $\widetilde{\Gamma} \to \Gamma$, similar to ramified coverings (see 6.E_1.). In fact we do have ramified coverings for $O(n,1)$ and $U(n,1)$-lattices as these may have convex subgroups of codimension two, e.g. associated with totally geodesic submanifolds of codimension two. On the other hand, in view of our earlier discussion on convex groups in $Sp(n,1)$ and F_4^{-20}, no such subgroup is expected in $Sp(n,1)$ and F_4^{-20}-lattices and so they should not have any overgroup $\widetilde{\Gamma}$ even remotedly resembling ramified coverings. Here is a specific

Conjecture. Let Γ be a cocompact lattice in $Sp(n,1)$, $n \geq 2$, and $\widetilde{\Gamma}$ a hyperbolic group whose boundary is homeomorphic to $S^{4n-1} = \partial_\infty\Gamma$ and let $\widetilde{\Gamma} \to \Gamma$ be a surjective homomorphism which induces an isomorphism on the top dimensional cohomology, i.e. $H^{4n}(\Gamma; \mathbf{R}) \xrightarrow{\sim} H^{4n}(\widetilde{\Gamma}; \mathbf{R})$. Then the kernel of the homomorphism $\widetilde{\Gamma} \to \Gamma$ must be at least 2-dimensional, i.e. some finitely generated subgroup in this kernel must be not virtually free.

Remark. The hyperbolization procedure (see $[\mathrm{Gro}]_{14}$) gives us $\widetilde{\Gamma} \to \Gamma$ where, however, dim(kernel) $= \dim\Gamma - 1$.

We have more or less exhausted our resources of hyperbolic groups with many questions remaining open. For example, for a given n we want to have (or show it does not exist) a word hyperbolic group Γ of dimension n having (at least some of) the following properties.

(1) $\partial_\infty\Gamma$ is homeomorphic to S^{n-1} and/or Γ appears as the fundamental group of a closed n-dimensional aspherical manifold (or pseudomanifold) V. (It would be too much to ask for a metric with $K < 0$ on V).

(2) Γ has no (quasi-convex) subgroup Γ' of infinite index with $\operatorname{codim}\Gamma' \leq k$ for a given $k \geq n-2$. (The strongest requirement would be asking that every infinite subgroup of infinite index in Γ is free).

(3) Γ satisfies Kazhdan's T-property. (Notice that all known irreducible hyperbolic T-groups are even dimensional).

(4) Γ has no proper subgroup of finite index. (Or, on the contrary, Γ is residually finite).

(5) Γ receives no quasi-isometric embedding of the hyperbolic space H^k for a given $k \geq 2$.

(6) Γ admits no quasi-isometric embedding into a given symmetric space X, e.g. into $H^{4N}_{\mathbf{H}}$ for a given (finite or infinite) N.

We can continue this list by imposing more and more conditions on Γ but this only adds to our frustration as we cannot prove anything.

7.B. Hyperbolic or $\mathbf{K < 0}$? The basic foundational question is as follows:

Does every word hyperbolic group admit a discrete cocompact action on some metric geodesic space X with $K(X) < 0$?

It seems we do no loose (or gain) much by allowing non locally compact infinite dimensional spaces X where "cocompact" should be replaced by "cobounded" and $K < 0$ must be forced to strictness by requiring $K \leq \varepsilon < 0$.

A more general question concerns discrete *non-cocompact* actions of groups Γ on hyperbolic spaces X where one wishes to replace such an X by a quasi-isometric space Y of negative curvature with a discrete isometric action of the same group Γ (and where the implied quasi-isometry $X \leftrightarrow Y$ is better to be Γ-equivariant).

It took me about ten years finally to accept the failure in resolving these questions and admit "hyperbolicity" as a permanent definition replacing "coarse hyperbolicity" whose lifespan I thought would be measured by the time needed to prove "hyperbolicity" = "$K < 0$". (Compare [Gro]$_5$.)

The first case where the ($K < 0$)-question remains open concerns small cancellation groups Γ. One still hopes to prove that such a Γ admits a discrete cocompact isometric action on a 2-polyhedron X with $K \le 0$, where $K \le 0$ could be strengthened to $K < 0$ for Γ hyperbolic. Yet one is unable to prove even the weak version where one would admit an infinite dimensional space X with $K(X) \le 0$.

The second foundational question is more geometric. *Can every Γ-space X with $K < 0$ be equivariantly smoothed ?*

Here "smoothing" means including X into a smooth Riemannian manifold X_0 of negative curvature with convex boundary where we prefer $\dim X_0 < \infty$ but would be content with an infinite dimensional X if there is nothing better in the offer.

Next we want to include X_0 as a convex subset into a geodesically complete Γ-manifold $X_+ \supset X_0$ whose curvature is pinched between two negative constants. Then the ideal boundary $\partial_\infty X$ (and $\partial_\infty \Gamma$ for X/Γ compact) Γ-equivariantly embeds into $S^{n-1} = \partial_\infty X_+$, $n = \dim X_+$. This gives us a certain smoothness of $\partial_\infty X$ with the Γ-action as the action of Γ on $\partial_\infty X_+$ (besides being hyperbolic) is Hölder for some smooth structure on the sphere S^{n-1}, where the Hölder exponent depends only on the pinching. (If $-c_1 \le K \le -c_2 < 0$, then every $\gamma : \partial_\infty X_+ \hookleftarrow$ is C^α for $\alpha \ge c_2/c_1$; moreover, if $c_1 < 4c_2$, then one can take "C^1-smooth" for C^α; here one should be aware that the composition of two C^α-maps is, in general, no better than C^{α^2}.)

Remark. One can show that every word hyperbolic group Γ acts on some sphere S^{n-1}. This is the easiest if Γ has no torsion and thus representable as $\pi_1(P)$ for some finite (Rips') polyhedron P. This P can be thickened to a compact manifold with boundary $V_0 \supset P$ and then V_0 can be "completed" by attaching the cylinder to the boundary $B_0 = \partial V_0$, where the cylinder $B_0 \times \mathbb{R}_+$, for $B_0 = B_0 \times 0$ is endowed with the Riemannian metric $e^t g_0 + dt^2$. The universal covering X_+ of $V_+ = V \cup B_0 \times \mathbb{R}_+$ is obviously hyperbolic and one can show that $\partial_\infty X_+$ is homeomorphic to S^{n-1}, provided the dimension of V_0 is sufficiently large (e.g. $\dim V_0 \ge 2\dim P + 3$). However, the proof (I have in mind using Cannon's double suspension theorem) does not (?) automatically provide a Hölder control on the implied homeomorphism between $\partial_\infty X_+$ and the standard sphere, where "Hölder" here refers to the metric g_β on $\partial_\infty X_+$ obtained as follows. Let g denote some Γ-equivariant Riemannian metric on X_+ which is $e^t g_0 + dt^2$ outside $X_0 = \widetilde{V}_0 \subset X$ and take the conformal metric $g_\beta = \varphi g$ for $\varphi(x) = \exp\big(-\beta \operatorname{dist}(x, x_0)\big)$ for a fixed point $x_0 \in X_+$ and some sufficiently small $\beta > 0$. Then the metric completion of (X_+, g_β) is homeomorphic, according to Floyd to $X_+ \cup \partial_\infty X_+$ (see [Gr-Pa] and Appendix to §8.) and so g_ρ induces a metric on $\partial_\infty X_+$ also denoted g_β. (If the metric g on X is negatively pinched, then $(\partial_\infty X_+, g_\beta)$ is Hölder equivalent, by the obvious map between $\partial_\infty X$ and the unit tangent sphere S^{n-1} at $x_0 \in X$, where the implied Hölder constant depends on the pinching and β).

The above discussion suggests the following

Question. Take some class C of transformations of the sphere S^{n-1} distinguished by local regularity properties, e.g. the class of C^α-Hölder maps. For which word hyperbolic groups Γ can one find a smooth structure on $\partial_\infty X \supset \partial_\infty \Gamma$ (we assume $\dim X = n$), such that all $\gamma : \partial_\infty X_+ \hookleftarrow$ will belong to C ? (The definition of C should make sense with respect to any smooth structure).

Some answers will be given in the next section.

7.C. Around the pinching problem. The *pinching constant* of Γ may be defined as the minimal (or infimal) $\epsilon \ge 0$ such that Γ admits a discrete isometric action on some geodesically complete Riemannian manifold X of a given (or, on the contrary, unspecified) dimension with negative curvature pinched by

$-(1+\varepsilon)^2 \le K(X) \le -1$. (Nothing would change if we said $-(1+\varepsilon)^2\kappa \le K(X) \le -\kappa$ for some $\kappa > 0$ as we could always rescale the metric in X). Here one badly needs at least one such X in order to raise the discussion from the ground. Then, to keep in the air, one should have some means of showing for certain Γ that ε *cannot* be made arbitrarily small.

In what follows we shall define several asymptotic invariants of general hyperbolic groups (and spaces) reflecting the following property of pinched manifolds X. We assume here that X is simply connected with $K < 0$ and we do not mind convex boundary. We consider concentric spheres $S(R)$ in X around a fixed point in X and denote by $p_r : S(R+r) \to S(R)$ the normal projection for some fixed R. These spheres are smooth hypersurfaces (possibly, with boundaries) and the maps p_r are smooth. If $K(X) < -1$ then p_r exponentially (in r) contract the lengths of smooth curves $C \subset S(R+r)$, where the contraction is measured by the number $\lambda_r(C) \underset{\text{def}}{=} \operatorname{length} p_r(C)/\operatorname{length} C$. Now, if the curvature is pinched between -1 and $-1-\varepsilon$, then a standard estimate reads,

$$\lambda_r(C) \le \left(\lambda_r(C')\right)^{1+\varepsilon} \tag{$*$}$$

for every pair of smooth curves C and C' in $S(R+r)$. For example, if the curvature is constant (i.e. $\varepsilon = 0$) then $\lambda_r(C) = \lambda_r(C')$ for all curves C and C'.

The inequality $(*)$ yields similar relations for the *volume contraction* for submanifolds $M \subset S(R+r)$ of dimension $k = 1, 2, \dots, n-1 = \dim S(R+r)$, that is

$$\wedge^k(M) = \wedge^k_r(M) \underset{\text{def}}{=} \operatorname{Vol}_k p_r(M)/\operatorname{Vol}_k M \ ,$$

(so that $\wedge^1_r$ equals λ_r of the previous definition for curves). For example the top-dimension contraction $\wedge^{n-1}$ of every open subset $M \subset S(R+r)$ is bounded by

$$\wedge^{n-1}(M) \le \left(\wedge^1(C)\right)^{\alpha} \tag{$+$}$$

where C is an arbitrary curve in $S(R+r)$ and $\alpha = (n-1)(1+\varepsilon)$; also there is the lower bound

$$\wedge^{n-1}(M) \ge \left(\wedge^1(C)\right)^{\beta} \tag{$-$}$$

for $\beta = (n-1)/(1+\varepsilon)$. Furthermore, one has stronger inequalities if one considers all smooth curves C in M and takes the supremum and infimum of $\wedge^1(C)$ over these curves. Then one has

$$\wedge^{n-1}(M) \le \sup_{C \subset M} \left(\wedge^1(C)\right)^{\alpha'} \ , \tag{$+*$}$$

for $\alpha' = 1 + (n-2)(1+\varepsilon)$ and

$$\wedge^{n-1}(M) \ge \inf_{C \subset M} \left(\wedge^1(C)\right)^{\beta'} \tag{$-*$}$$

for $\beta' = 1 + (n-2)/(1+\varepsilon)$. Finally we state the following generalized and refined version of $(+*)$. We take a smooth submanifold $M \subset S(R+r)$ of dimension $k+\ell$ and let $\mathcal{N}$ be a family of smooth k-dimensional submanifolds $N \subset M$ satisfying the following condition: each neighborhood in M contains some $N \in \mathcal{N}$. Then

$$\wedge^{k+\ell}(M) \le \sup_{N \in \mathcal{N}} \left(\wedge^k(N)\right)^{\alpha} \tag{$++$}$$

for $\alpha = 1 + \ell k^{-1}(1+\varepsilon)$.

It is also worth recalling at this stage that pinching controls (via $(++)$) filling inequalities in horospheres $S \subset X$. For example, every k-dimensional cycle M has

$$\operatorname{Fill}\operatorname{Vol}_{k+1} M \lesssim (\operatorname{Vol} M)^{\gamma}$$

$$\text{for} \quad \gamma = (1+\varepsilon)(k+1)k^{-1}$$

and

$$\text{Fill Rad}\, M \lesssim (\text{Vol}\, M)^{\rho}$$

for $\rho = (1+\varepsilon)/k$ (compare §5), where the filling volume and radius are measured in S (and *not* in $X > S$).

Before starting our definitions for general hyperbolic spaces let us look at an example where inequality (++) becomes sharp.

Computation in $H^{2n}_{\mathbb{C}}$. The metric here is 4-pinched, i.e. it is pinched with $\varepsilon = 1$. The spheres $S(R) \subset H^{2n}_{\mathbb{C}}$ carry codimension one tangent subbundles $T_1 \subset T(S(R))$ which are invariant under normal projections. If two curves C and C' in $S(R+r)$ are simultaneously tangent or simultaneously transversal to this subbundle then $\lambda_r(C) \sim \lambda_r(C')$ for large r. But if C is tangent to T_1 and C' is transversal to it then $\lambda_r(C') \sim (\lambda_r(C))^2$ for $r \to \infty$. The inequality (++) may be sharp *only* if $\ell = 1$ and $k \leq n-1$. For this we need $M \subset S(R+r)$ *transversal* to T_1 and such that $T_1 \cap T(M)$ is an *integrable* subbundle in $T(M)$. Then we take the set of integral varieties (local leaves) of $T_1 \cap T(M)$ for $\mathcal{N}$ and observe that $\wedge^{k+1}(M) \sim (\wedge^k(N))^{\frac{k+2}{k}}$, for all $N \in \mathcal{N}$ which amounts to equality in (++) with $\varepsilon = 1$. (Notice that the integrability of $T_1 \cap T(M)$ insures the condition on $\mathcal{N}$ required for the validity of (++)).

On the other hand if we look at submanifolds N in $S(R+r)$ of dimension $k \geq n$ and M of dimension $k+\ell$, then we observe that

$$\wedge^{k+\ell}(M) \sim (\wedge^k(N))^{\frac{k+\ell+1}{k+1}}$$

which is in agreement with α in (++) even for $\varepsilon = 0$.

7.C$_1$. Quasi-isometric description of α. We attempt now to define a constant similar to the exponent α in (++) which we want to be a quasi-isometry invariant of X. Eventually, we should be able to evaluate such a constant for concrete spaces X (e.g. for $H^{2n}_{\mathbb{C}}$) in order to show that X is *not* quasi-isometric to a space with $(1+\varepsilon)^2$-pinched curvature for certain ε depending on our (quasi-isometrically defined) constant.

We assume here that X is a geodesic hyperbolic space and with each subset $A \subset \partial_\infty X$ we associate the geodesic cone $CA \subset \partial_\infty X$ from a fixed point $x_0 \in X$ and a separated net in CA denoted $C^0A \subset CA$. Then we can speak of the growth of CA by looking at the number card $(C^0A \cap B(R))$ for the concentric R-balls in X around x_0 as $R \to \infty$. This number is, in fact, finite if A is compact (or if $\partial_\infty X$ is compact which is the case for most of our applications) and $\text{gro}(A;R) \underset{\text{def}}{=} \text{card}(C^0A \cap B(R))$ grows exponentially with R. The rate of this growth is rather unsensitive to the choice of the net C^0A, as the cardinality changes by a multiplicative constant in most cases as we change C^0A inside CA. Also the shape of CA does not change much if we replace it by the geodesic cone over A with respect to another metric g' in X which is quasi-isometric to the original one. However, the *balls* $B'(R) \subset X$ with respect to g' may be quite different from $B(R)$ which makes all the difference for $\text{card}(C^0A \cap B(R))$. For example, by just rescaling the metric we may go to $B'(R) = B(aR)$ and, accordingly, the growth function $\text{gro}(A;R)$ may switch from $\exp \lambda R$ to $\exp a\lambda R$. However, if we compare two growth functions $\text{gro}(A;R)$ and $\text{gro}(B;R)$, the *relative exponent* of the growth, say $b(R) = b(A{:}B;R)$ defined by the equation

$$\text{gro}(A;R) = (\text{gro}(B;R))^{b(R)}$$

is essentially invariant under rescaling of the metric in X and we want to extract from $b(R)$ a number which will be a quasi-isometric invariant of X. The idea of removing the dependence of $b(R) = b_g(R)$ on a particular metric g consists in taking an appropriate supremum (or infimum) over all metrics quasi-isometric to a given one. This idea was brought to this context by Pansu (see [Pan]$_5$) from the conformal geometry, where it is commonly used. In fact the invariants we are going to construct are similar (and sometimes identical) to Pansu's conformal dimension, though our presentation follows a more geometric route suggested by pinching of hyperbolic cusps (see [Gro]$_{4,21}$).

Definition of $\inf\delta\text{im}(\mathcal{A}{:}\mathcal{B})$. Here $\mathcal{A}$ denotes a family of subsets $A \subset \partial_\infty X$ and $\mathcal{B}$ is another such family. Our $\inf\delta\text{im}$ is supposed to measure something like the infimum of the relative dimensions $\inf \dim A/\dim B$

over $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The most important case is where $\mathcal{B}$ consists of a single set, e.g. $\partial_\infty X$ itself and then we just write $\inf\delta\mathrm{im}\,\mathcal{A}$ (instead of $\inf\delta\mathrm{im}\,\mathcal{A}{:}\{\partial_\infty X\}$).

We shall measure the growth of subsets, say $A \subset \partial_\infty X$, by taking cones CA and nets $C^0 A$ with respect to a fixed metric. These will be uniformly dense and separated throughout the discussion (say, we always take 1-separated 2-nets). Then we say that a metric dist$'$ on X in our quasi-isometry class is *$\mathcal{B}$-finite* if the corresponding distance function $d'(x) = \mathrm{dist}'(x, x_0)$ satisfies

$$\sum_{C^0 B} \exp -d'(x) \leq \mathrm{const} < \infty \ , B \in \mathcal{B} \ ,$$

for some const = const($\mathcal{B}$). (This means, roughly speaking, that

$$\mathrm{gro}(B; R) \lesssim \exp d'R$$

for all $B \in \mathcal{B}$). Then we define $\inf\delta\,\mathrm{im}(\mathcal{A}{:}\mathcal{B})$ as the infimum of $\delta > 0$ with the following property: for every $\mathcal{B}$-finite metric on X there exists a subset $A \in \mathcal{A}$ such that

$$\sum_{C^0 A} \exp -\delta d'(x) < \infty \ .$$

It is obvious, our $\inf\delta\mathrm{im}$ is a quasi-isometry invariant.

If $\mathcal{B}$ consists of a single compact subset B in $\partial_\infty X$ of dimension m (in examples, $B = \partial_\infty X$) and $\mathcal{A}_k$ consists of all k-dimensional subsets in B, then one may expect that

$$\inf\delta\mathrm{im}(\mathcal{A}_k{:}\mathcal{B}) \leq k/m \ ,$$

this appears easy for $k = 1$. In fact, $\inf\delta\mathrm{im}\,\mathcal{A}_1$ for $\mathcal{B} = \{\partial_\infty X\}$ is, probably, related in most cases to Pansu's conformal dimension by

$$\inf\delta\mathrm{im}\,\mathcal{A}_1 = (\mathrm{conf}\dim \partial_\infty X)^{-1} \ ,$$

(compare 8.C$_2$.).

Besides serving as a quasi-isometry invariant, our $\inf\delta\mathrm{im}$ may provide an obstruction for the existence of a quasi-isometric embedding $X \to Y$, where the space Y does not have to be necessarily hyperbolic. For example, let Y be a simply connected space with $K(Y) \leq 0$ and A and B two subsets in the geodesic boundary $\partial_{\mathrm{geo}} Y$. We take as earlier nets in the geodesic cones over A and B, called $C^0 A$ and $C^0 B$ in Y and define $\delta\mathrm{im}_Y(A{:}B)$, roughly, as the "ratio" of the exponents of their growth. More precisely, we consider pairs of number α, β, such that the sums

$$\sum_{C^0 A} \exp -\alpha d(x) \quad \text{and} \quad \sum_{C^0 B} \exp -\beta d(x)$$

converge, where $d(x) = \mathrm{dist}_Y(x_0, x)$, and then take the infimum of the ratios α/β for $\delta\mathrm{im}_Y(A{:}B)$. Notice, this is *not* a quasi-isometric invariant of Y. The most important case here is where $B = \partial_{\mathrm{geo}} X$ and then we write $\delta\mathrm{im}_Y\, A$ for $\delta\mathrm{im}_Y(A{:}B)$. Similarly one defines $\delta\mathrm{im}_Y(\mathcal{A}{:}\mathcal{B})$ for families of subsets in $\partial_{\mathrm{geo}} Y$.

Now, let $X \to Y$ be a quasi-isometric embedding which extends to a topological embedding $\partial_\infty X \hookrightarrow \partial_{\mathrm{geo}} Y$ where X is assumed hyperbolic. (This extension condition is certainly quite restrictive. Yet it is satisfied if the image of the embedding is quasi-convex Y which means that the geodesic cone in Y from a fixed point $y \in Y$ over $X \subset Y$ is contained in a ρ-neighborhood of X for some $\rho < \infty$). Then

$$\inf\delta\mathrm{im}\,\mathcal{A}_k \geq \delta\mathrm{im}_Y\, \mathcal{A}_k^Y$$

where $\mathcal{A}_k$ denotes the set of all compact k-dimensional subsets in $\partial_\infty X$ and $\mathcal{A}_k^Y$ is such a family in $\partial_{\mathrm{geo}} Y$.

Remarks (a) In order to use this estimate one needs, besides an upper bound on $\inf \delta\mathrm{im}\,\mathcal{A}_k$ (these appear in the examples following this discussion), a non-trivial lower bound on $\delta\mathrm{im}_Y\,\mathcal{A}_k^Y$. Such a lower bound is, in fact, possible whenever $k \geq \mathrm{Rank}\,X$ for a suitable definition of Rank (compare 6.B.) and it is especially easy for somewhat larger k. For example, let Y be a Riemannian manifold with $K \leq 0$ such that for every $(k+1)$-dimensional linear subspace $T \subset T(Y)$ every $(k+1)$-dimensional submanifold $Y_0 \subset Y$ having $T_{y_0}(Y_0) = T$ and geodesic at y_0 (where y_0 is the base-point of T) satisfies

$$\mathrm{Ricci}\,T \underset{\mathrm{def}}{=} \mathrm{Ricci}_{y_0}(Y_0) \leq -\delta^2$$

for a fixed $\delta > 0$, while the Ricci curvature of X is bounded everywhere from below by $-\Delta^2$. Then, clearly, $\delta\mathrm{im}_Y\,\mathcal{A}_k^Y \geq \delta/\Delta$.

(b) It is desirable to render the above discussion quasi-isometry invariant. For this we have to exhibit a quasi-isometry invariant class of subsets Z (replacing the geodesic cones) such that every Z in this class admits a lower bound on the growth, i.e. the intersections of Z with concentric R-balls in Y should satisfy

$$\mathrm{card}_\varepsilon\big(Z \cap B(R)\big) \gtrsim \exp \alpha R$$

for all $\varepsilon > 0$ and an $\alpha > 0$ independent on Z in a given class (but depending on the class), where $\mathrm{card}_\varepsilon$ denotes the minimal number of ε-balls needed to cover the set in question. Here are two candidates for such classes of subsets.

(1). *Families of maximal quasi-flats.* Take $Z_0 = \mathbb{R}^k \times A$ for some compact connected metric space A and let the sets Z in our class be the images of the continuous maps $\varphi : Z_0 \to Y$, such that

(i) φ is λ-quasi-isometric on each $\mathbb{R}^k \times a$, $a \in A$. (In other words $\varphi(\mathbb{R}^k \times a)$ is a quasi-flat in Y where "quasi" is uniform in a).

(ii) The Hausdorff distance between the images $\varphi(\mathbb{R}^k \times a_1)$ and $\varphi(\mathbb{R}^k \times a_2)$ is infinite whenever $a_1 \neq a_2$. (One may require more by giving some bound on the rate of divergence between the quasi-flats $\varphi(\mathbb{R}^k \times a_1)$ and $\varphi(\mathbb{R}^k \times a_2)$).

Question. Suppose Y is a "nice" semihyperbolic space of rank k (e.g. Y is a symmetric space or a product of k hyperbolic spaces). Is then

$$\mathrm{card}_\varepsilon\big(Z \cap B(R)\big) \gtrsim \exp \alpha R$$

for some α independent of Z ?

(2). *Quasi-minimal subvarieties.* Let $Z \subset Y$ be a $(k+1)$-dimensional subvariety (cycle) which *infinitely spreads* (compare $[\mathrm{Gro}]_{21}$) in the sense that

$$\mathrm{Fill\,Rad}\,\partial\big(Z \cap B(R)\big) \underset{R \to \infty}{\to} \infty \ .$$

(One could alternatively require that Z has asymptotic dimension $k+1$. Another possibility would be to insist on a lower bound on Fill Rad, say

$$\liminf_{R \to \infty} R^{-1}\,\mathrm{Fill\,Rad}\big(Z \cap B(R)\big) > 0.)$$

Next, we say that Z is *strongly quasi-minimal* if there exists a large-scale λ-Lipschitz retraction $Y \to Z$. Notice that "minimal" does not imply "strongly quasi-minimal").

Question. Let again Y be a "nice" semi-hyperbolic of rank $\leq k$. Is then

$$\mathrm{card}_\varepsilon\big(Z \cap B(R)\big) \gtrsim \exp \alpha R\ ?$$

Remarks. (a) The condition rank $\leq k$ implies for "nice" Y the linear isoperimetric inequality for filling in of k-dimensional cycles. This immediately gives a universal bound on the growth of honestly *minimal* infinite cycles Z, but for quasi-minimal cycles the growth-rate α may, a priori, depend on the implied quasi-minimality (Lipschitz) constant λ. What we want to prove is, in fact, that α does *not* depend on λ.

(b) There are other families of subsets $Z \subset Y$ where the above discussion applies. For example, one may use (1) and (2) simultaneously and look at quasi-minimal cycles Z which decomposes into unions of maximal quasi-flats.

Now let us evaluate $\inf \delta\mathrm{im}$ for specific hyperbolic spaces X.

Examples. (a) *Real hyperbolic space* $H^n_{\mathbb{R}}$. Here the boundary $\partial_\infty H^n_R$ equals the standard sphere S^{n-1}. Take $\mathcal{A}_k$ a family of k-dimensional subsets in S^{n-1} containing the smooth k-dimensional submanifolds (e.g. the family of all k-dimensional subsets). It is obvious that $\inf \delta\mathrm{im}\,\mathcal{A}_k \geq k/n-1$. Then, in order to prove that, in fact, $\inf \delta\mathrm{im}\,\mathcal{A}_k = k/n-1$, we observe that the summation over nets can be replaced by the integration over cones with respect to the hyperbolic metric. Thus, we start with a distance function $d'(x)$, such that

$$\int_{H^n_{\mathbb{R}}} (\exp -d'(x))\,dx < \infty \tag{*}$$

and we have to show that there exists a smooth k-dimensional submanifold $A \subset S^{n-1}$, such that

$$\int_{CA} (\exp -\delta d'(x))\,dx < \infty \tag{**}$$

for every $\delta > k/n-1$. To do that we take some probability measure $d\mu$ on $\mathcal{A}$ supported on the set of smooth k-dimensional submanifolds $A \subset S^{n-1}$, such that the push-forward of $dad\mu$ to S^{n-1} has continuous density with respect to the spherical measure. (For example, we may take some neighborhood $U \subset S^{n-1}$ which smoothly splits by $A \times M$ and take a smooth measure on M for $d\mu$). Then we express (*) and (**) in polar coordinates in $H^n_{\mathbb{R}}$ minus a fixed (large) ball,

$$\int_{H^n_{\mathbb{R}}} (\exp -d'(x))\,dx \leq \int_{S^{n-1}\times[1,\infty)} (\exp -d'(s,t))\,(\exp(n-1)t)\,ds\,dt\ .$$

We replace ds by (the push-forward of) $dad\mu$ and see that

$$\int_{S^{n-1}\times[1,\infty)} \exp\big((n-1)t - d'\big)\,dad\mu dt < \infty\ ,$$

and by Hölder inequality, for every $\delta > \frac{k}{n-1}$, we have

$$\int_{S^{n-1}\times[1,\infty)} \exp(kt - \delta d')\,dad\mu dt < \infty\ .$$

Then we observe that the measure $\exp(kt)dadt$ is equivalent to the Riemannian measure dx on the cone CA and so we see that

$$\int_{\mathcal{A}_k} d\mu \int_{CA} \exp(-\delta d')\,dx < \infty\ .$$

Thus for some $A \subset \mathcal{A}_k$ the interior integral is $< \infty$ as well. Q.E.D.

Question. Let Γ be a word hyperbolic group whose boundary is homeomorphic to S^{n-1} and such that the family $\mathcal{A}_k$ of the compact k-dimensional subsets has $\inf \delta\mathrm{im}\,\mathcal{A}_k = k/n$ for a given $k = 1, \ldots, n-2$ (or for every $k = 1, \ldots, n-2$). Is then Γ commensurable to a lattice in $O(n,1)$? (We know, this commensurability is equivalent to the existence of a quasi-isometry $\Gamma \leftrightarrow H^n$).

Remark. The answer is "no" in complete generality without assuming $\partial_\infty \Gamma$ is homeomorphic to S^{n-1}. For example, if we glue together k copies of the 2-torus minus a disk over the boundary circle, the fundamental group Γ_k of the resulting space $(\Gamma_k = \mathbb{F}_2 *_{\mathbb{Z} \ldots \mathbb{Z}} \ldots * \mathbb{F}_2)$ has (by an easy argument) $\inf \delta\mathrm{im}\, \mathcal{A}_1 = 1$ for all $k \geq 2$. Yet it is not a surface group unless $k = 2$.

(b) *Complex hyperbolic space* $H^{2n}_{\mathbb{C}}$. Here the boundary is S^{2n-1} with a codimension one subbundle $T_1 \subset T(S^{2n-1})$. The hyperbolic measure of $H^{2n}_{\mathbb{C}}$ in polar coordinates is (equivalent to) $\exp(2nt)dsdt$. In fact for every smooth k-dimensional submanifold $A \subset S^{2n-1}$, transversal to T_1, the measure on the cone CA is (equivalent to) $(\exp(k+1)t)dadt$ (instead of $\exp(kt)dadt$ for $H^n_{\mathbb{R}}$), but if A is everywhere tangent to T_1 then this measure is $\exp(kt)dadt$. Notice that such tangent submanifolds A exist for $k \leq n-1$ and moreover, there (obviously) exists a measure $d\mu$ on the family of all such A with $\dim A = k \leq n-1$ whose push-forward to S^{2n-1} has continuous density with respect to the spherical measure. Then the computation for $H^n_{\mathbb{R}}$ extends to $H^{2n}_{\mathbb{C}}$ and shows that

$$\inf \delta\mathrm{im}\, \mathcal{A}_k = k/2n$$

for the family $\mathcal{A}$ of all k-dimensional subsets in S^{2n-1} where $k \leq n-1$. Notice that this dimension is $k/2n-1$ for $H^{2n}_{\mathbb{R}}$ which shows, in particular, that $H^{2n}_{\mathbb{C}}$ is not quasi-isometric to $H^{2n}_{\mathbb{R}}$. (This is an old result by Mostow.)

Case $k \geq n$. If $\mathcal{A}$ consists of a single compact subset A whose Hausdorff dimension for the Carnot-Caratheodory metric is $\geq k+1$, then $\inf \delta\mathrm{im}\{A\} \geq (2n/k+1) - \varepsilon$ for every $\varepsilon > 0$. Unfortunately, we do not know yet if the k-dimensional compact subsets for $k \geq n$ have $CCH \dim \geq k+1$ (compare d″ in V of 7.A.).

(c) Let $X = [0,1]^{n-1} \times [1,\infty]$ with the metric $\sum_{i=1}^{n-1} (\exp \lambda_i t)^2 ds_i^2 + dt^2$ for some $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1}$. Here we claim that the family $\mathcal{A}$ of the k-dimensional subsets in $\partial_\infty X = [0,1]^{n-1}$ has

$$\inf \delta\mathrm{im}\, \mathcal{A} = \sum_{i=1}^{k} \lambda_i \Big/ \sum_{i=1}^{n-1} \lambda_i \ .$$

The upper bound on $\inf \delta\mathrm{im}\, \mathcal{A}$ is obtained with the splitting $\partial_\infty X = A \times M$ corresponding to the coordinates $s_1, \ldots, s_k$ and $s_{k+1}, \ldots, s_{n-1}$ and with the measure $d\mu = ds_{k+1} \ldots ds_{n-1}$ on M. The lower bound is achieved by a simple inductive argument mimicing the standard proof of the classical inequality

$$\text{Haus}\dim \geq \text{top}\dim \ .$$

Corollary. *The numbers* $\overline{\lambda}_i = \lambda_i \Big/ \sum_{j=1}^{n-1} \lambda_j$ *can be recovered from the quasi-isometry type of* X.

This was earlier proven by Pansu (see $[\text{Pan}]_{6,7,8}$) by using 1-dimensional L_p-cohomology of X. In fact Pansu looks at the solvable group $\mathbb{R}^{n-1} \ltimes \mathbb{R}$ with the metric $\sum_i (\exp \lambda_i t)^2 ds_i^2 + dt^2$, where our computation can also be applied.

(d) *Pinched manifolds.* Let X be a Riemannian manifold with $-(1+\varepsilon)^2 \leq K(X) \leq -1$. Then it is easy to see that the family $\mathcal{A}_k$ of the compact k-dimensional subsets in $\partial_\infty X$ has

$$\inf \delta\mathrm{im}\, \mathcal{A}_k \geq k\big((1+\varepsilon)(n-1)\big)^{-1} \qquad (+)$$

(essentially, because the topological dimension of A is bounded by the Hausdorff dimension).

(d′) *Remark.* The inequality (+) is, strictly speaking, sharp. One may have a metric which has $K = -1$ in one open geodesic cone in X and $K = -(1+\varepsilon)^2$ in another such cone. Yet, there is something ridiculous about the inequality (+) as it does not become an equality for $k = n-1 = \dim \partial_\infty X$. However I do not know how to modify (+) (by adjusting definitions or by imposing some restrictions on X) in order to have $k\big((1+\varepsilon)(n-1-k)+k\big)^{-1}$ on the right hand side.

7.C$_2$. Further pinching invariants. Let us indicate other possible characteristics measuring the asymmetry of the (exponential) growth of hyperbolic spaces X. First, we observe that our examples do not reveal anything about $\inf\delta\mathrm{im}(\mathcal{A}{:}\mathcal{B})$ for general $\mathcal{B}$. For example, if $X = H^{2n}_{\mathbb{C}}$ and $\mathcal{A} = \mathcal{A}_k$ while $\mathcal{B}$ constitutes a sufficiently large family of ℓ-dimensional subsets, we expect that $\inf\delta\mathrm{im}(\mathcal{A}{:}\mathcal{B}) \le k/\ell + 1$ for $k < n$ and $\ell \ge n$ (where a "sufficiently large" $\mathcal{B}$ may be, for example, the set of the levels of a continuous map $S^{2n-1} \to \mathbb{R}^{2n-1-\ell}$). This, for $k = n-1$ and $\ell = n$, together with the (d')-version of the above (+) with $\ell = n$ in place of $n-1$, would show that $H^{2n}_{\mathbb{C}}$ cannot be $(1+\varepsilon)^2$-pinched for $\varepsilon < 1$.

Our $\inf\delta\mathrm{im}$ measures, in essence, the discrepancy in the rate of growth of geodesic cones $C \subset X$ of different dimensions. One may try to use other invariants of the cones C besides the growth. For example, one may intersect C with the concentric spheres in $S(R) \subset X$ (or, better, with the bands $S(R_1, R_2)$ between $S(R_1)$ and $S(R_2)$ for $R_2 = (1+\varepsilon)R_1$) and look at the growth of $\mathrm{Diam}\big(C \cap S(R)\big)$ for $R \to \infty$, or at similar higher dimensional invariants measuring some spread of $S \cap S(R)$. (An instance of such a spread is the *k-width* of $C \cap S(R)$, see [Gro]$_{16}$.)

The difference of growth of X in different dimensions can be also measured with the filling radii for k-dimensional cycles $Z_k \subset X$. One knows that $\mathrm{Fill\,Rad}\, Z_k \lesssim \rho_k \log \mathrm{Vol}\, Z_k$ and the exp of the ratios $\log \rho_k \,/\, \log \rho_\ell$ for the "best" constants, ρ_k and ρ_ℓ, have the same flavour as $\delta\mathrm{im}\,(\mathcal{A}_k{:}\mathcal{A}_\ell)$.

Another possibility is suggested by looking at the isoperimetric exponents in the horospheres $H \subset X$. In such H the filling volume is (usually) polynomial, $\mathrm{Fill\,Vol}_{k+1} \lesssim (\mathrm{Vol}_k)^{\alpha_k}$ where the (non) pinching is responsible for the deviation of α_k from $(k+1)/k$.

The isoperimetric exponents α_k in horospheres and in families of concentric spheres $S(R) \subset X$ bring along the idea of functional isoperimetric invariants in $S(R)$ and X, such as various *Poincaré-Sobolev exponents* in the inequalities comparing L_p and L_q norms of functions (and form) and their differentials. Ultimately, we turn to the de Rham complex $\Omega^0 \to \Omega^1 \to \ldots$ of forms on X and study its chain homotopy invariants with respect to L_p-topologies for various p in different degrees (see §8). The advantage of such an approach is the automatic quasi-isometric invariance, so that there is no need to take infima (or suprema) over the metrics on X in a given quasi-isometry class (albeit these infima and suprema are implicit in the L_p-cohomology formalism).

$\mathrm{Inf}\,\delta\mathrm{im}$ *for partitions.* Let $A_b, b \in B$ be a family of closed subsets in $\partial_\infty X$. Then one may compare the exponential growth-rate of the cones $CA_b \subset X$ with the divergence *between* the cones CA_b and $CA_{b'}$ for $b \ne b'$, where the distance between CA_b and $CA_{b'}$ may be measured in the Hausdorff sense within the R-ball $B(R)$ or (if A_b and $A_{b'}$ do not intersect) as the infimum of the distances between the points in $CA_b - B(R)$ and $CA_{b'} - B(R)$. Further invariants of this kind are associated with more complicated combinatorial arrangements of subsets, such as a sequence of finer and finer partitions or systems of subsets having the intersection pattern similar to that of 2-planes in $\mathbb{R}^3$.

Making $\inf\delta\mathrm{im}$ *(and similar invariants)* Γ*-equivariant.* Suppose we have a discrete group Γ operating in X and let us redefine $\inf\delta\mathrm{im}$ by taking only Γ*-equivariant* metrics dist' on X. We call this $\inf\delta\mathrm{im}_\Gamma$ and observe that

$$\inf\delta\mathrm{im}_\Gamma \le \inf\delta\mathrm{im}\ .$$

Practically, nothing else is known about this $\inf\delta\mathrm{im}_\Gamma$ except that an invariant of similar nature (implicitly) appears in the non-pinching examples in §4 of [Gr-Th]. Namely, suppose that we have a Γ-invariant metric with $K \le -1$. Then, for a given $c > 0$, we take the maximal number $p = p(\Gamma, c)$ such that there exists a free subgroup $\mathbb{F}_p \subset \Gamma$, where the displacements of some point $x_0 \in X$ satisfy

$$\mathrm{dist}\big(x_0, \varphi_i(x_0)\big) \le c\ ,$$

for the generators $\varphi_1, \ldots, \varphi_p$ of $\mathbf{F}_p$. It follows that the ball of radius R in X has

$$\mathrm{Vol}\, B(R) \gtrsim \exp R p c^{-1} . \qquad (*)$$

The point is that there are groups Γ for which p becomes arbitrarily large while c is kept bounded. Namely, this is the case for the groups Γ_i, $i = 1, 2, \ldots$, obtained by amalgamating i copies of a fixed group Γ over a subgroup $\Gamma' \subset \Gamma$, such that a certain k-dimensional homology class h in $H_k(\Gamma')$ vanishes in $H_k(\Gamma)$, where k should be at least two and h must not come from virtually nilpotent subgroups in Γ'. (See [Gr-Th] for a detailed discussion). It is shown in [Gr-Th] that $p(\Gamma_i, c) \geq i$ for $c = c(\Gamma_i) \leq \mathrm{const}(\Gamma, \Gamma')$ and thus the lower bound $-(1+\varepsilon)^2$ on $K(X)$ satisfies

$$n(1+\varepsilon) \gtrsim \log i$$

for $n = \dim X$, as follows from $(*)$. Notice, that the condition $K \leq -1$ appears in the proof via an upper bound on the volumes of the geodesic simplices of dimensions k and $k+1$. This suggests the following

Conjecture. The above Γ-manifolds X with $\Gamma = \Gamma_i$ satisfy

$$\min(\inf\delta\mathrm{im}_{\Gamma_i} \mathcal{A}_k,\ \inf\delta\mathrm{im}_{\Gamma_i} \mathcal{A}_{k+1}) \lesssim (\log i)^{-1}$$

for $i \to \infty$. Or, at least,

$$\inf\delta\mathrm{im}_{\Gamma_i} \mathcal{A}_1 \lesssim (\log i)^{-1} .$$

(Of course, we would like to have such inequalities for non-invariant $\inf\delta\mathrm{im}$, but this appears even more difficult).

The idea behind this conjecture is that the increase in the "branching" of Γ decreases the "relative weight" of certain subsets in $\partial_\infty\Gamma$ with respect to all of $\partial_\infty\Gamma$, as $\partial_\infty\Gamma$ increases with the "branching number" i without changing its topological dimension. We shall see later on in 7.C$_3$. that this idea becomes a reality for another class of strongly branched polyhedra.

Additional remarks and questions. (a) It follows from the above discussion that the infinite amalgamated product, say $\Gamma_\infty = \bigcup_{i=1}^{\infty} \Gamma_i$, cannot faithfully and discretely act on any simply connected finite dimensional manifold X with $-\infty < -C \leq K(X) \leq -1$.

(b) Instead of amalgamating mutually isomorphic groups one could take different $\Gamma^{(j)} \supset \Gamma'$ and make Γ^i by amalgamating these along Γ' over $j = 1, \ldots, i$. For example, one could take a sequence of compact manifolds $V^{(j)}$ of constant negative curvature having mutually isometric (or just diffeomorphic in the case $\dim V^{(j)} = 3$) boundaries (compare [Gr-Th]) and then glue $V^{(1)}, \ldots, V^{(i)}$ together across the boundaries. The fundamental group Γ_i of the resulting space V_i falls into our category if $\dim V^{(j)} \geq 3$ (see [Gr-Th]) and one may ask when, for example, the group $\Gamma_\infty = \bigcup_i \Gamma_i$ is "totally unpinched" as, we know, does happen for $V^{(1)} = V^{(2)} = \cdots$. In fact this is easy to show (by the argument of [Gr-Th]) if $\mathrm{Vol}^{(j)} \,/\, \log j \underset{i\to\infty}{\to} 0$. (If $\dim V^{(j)} \geq 4$, it is unclear if there are such sequences of manifolds $V^{(j)}$, where every two of them are mutually non-isometric). On the other hand, if the boundary $V' = \partial V^{(j)} \subset V^{(j)}$ has a sufficiently large collar, that is, if the normal exponential map is injective on $V' \times [0, \rho_i]$ for $\rho_j / \log j \underset{j\to\infty}{\to} \infty$, then the amalgamated product of $\Gamma^{(j)}$ starting from a sufficiently large (j_0) can be arranged to act on the hyperbolic space $H^{n+1}_{\mathbb{R}}$ for $n+1 = \dim V^{(i)} + 1$ (by a simple "combination" argument).

(c) Let V be a closed $n+1$-dimensional aspherical manifold and $V' \subset V$ an oriented submanifold of dimension $n-1$ representing zero class in $H_{n-1}(V)$. Then for every $k = 1, 2, \ldots$, there exists a cyclic ramified covering $\widetilde{V}_k \to V$ of order k with the ramification locus V'.

Conjecture. If $n \geq 3$ then the pinching constant of $\widetilde{V}_k$ goes to ∞ as $k \to \infty$.

Remarks. (a) It is shown in [Gr-Th], that there are at most finitely many values of k for which $\widetilde{V}_k$ may have a metric of constant negative curvature (or an arbitrary locally symmetric metric). But there is no realistic bound on the number of the exceptional k.

(b) There are cases where the fundamental group $\pi_1(\Gamma_k)$ contains the amalgamated product of k-copies of some Γ over $\Gamma' = \pi_1(V')$, and then the pinching constant of $\widetilde{V}_k$ does go to infinity (see [Gr-Th]).

(c) The pinching problem is significantly easier for the full ramified fundamental group $\sqcap \supset \pi_1\left(\widetilde{V}_k\right)$. This $\sqcap$ acts on the universal ramified cover X of V with k-th order ramification along V', such that $X/\sqcap = V$. One can show, that if X has a $\sqcap$-invariant metric g of negative curvature pinched between $-c_k$ and -1, then necessarily $-c_k \to -\infty$ for $k \to \infty$, provided $\dim V \geq 4$. This can be shown by first observing that $\mathrm{Vol}(X,g)/\sqcap \leq$ const, for some constant independent of k (this follows from $K \leq -1$), and then by giving a lower bound on the volume of some ρ-neighborhood of $V' \subset V$ (using c_k) by the argument similar to (and easier than) that in [Gro]$_2$.

(d) There are, among $\widetilde{V}_k$, (where V is a closed manifold of constant negative curvature of dimension ≥ 4) some of which admit metrics with the curvatures pinched between $-(1+\varepsilon)^2$ and -1 for arbitrarily small $\varepsilon > 0$, but yet having no metrics of constant curvature (see [Gr-Th]). One wonders if $\inf \delta\mathrm{im}\, \mathcal{A}_1$ approaches $1/n$ in these examples (for $n = \dim V - 1$) without becoming equal to $1/n$, and the same question applies to the ε-pinched manifolds V' constructed in the end of 7.D. by surgeries over submanifolds with ρ-thick collars.

7.C$_3$. Round trees and infδim for strongly branched spaces. The simplest example of what we call a *round tree* X_0 is obtained by rotating an ordinary tree $T \subset H^2_{\mathbb{R}} \subset H^3_{\mathbb{R}}$ around a geodesic, see Fig. 18 below.

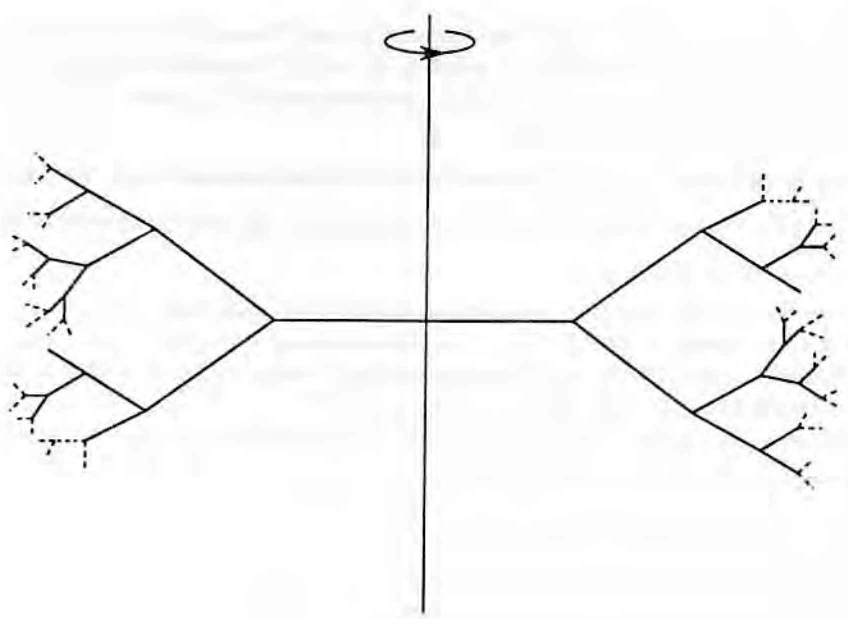

Figure 18

This $X_\circ$ sits in the hyperbolic space $H^3_{\mathbb{R}}$ where it decomposes into uncountable unions of quasi-isometric copies of $H^2_{\mathbb{R}}$, *round branches*, where every two intersect over a disk. The ideal boundary $\partial_\infty X_\circ \subset \partial_\infty H^3_{\mathbb{R}} = S^2$ is the Cartesian product of S^1 by a Cantor set K, whose points correspond to the round branches.

Abstractly speaking, a *two dimensional round tree* $X_\circ$ is a two dimensional space of negative curvature which admits an isometric action of S^1 with a single fixed point x_0, such that there is an isometrically embedded tree $T \subset X_\circ$ which intersects every S_1-orbit at exactly one point. To facilitate the exposition we assume that our metric has constant curvature on the nonsingular locus of $X_\circ$ and thus all round branches are isometric to $H^2_{\mathbb{R}}$. We assume furthermore, that T is regular with exactly $b+1$ edges at each vertex and with all edges but one having the same length δ where the fixed point $x_0 \in X_\circ$ serves of the center of the exceptional edge in T, which has length 2δ. Thus the ball $B(R)$ in X around x_0 branches at the values $R = \delta,\ 2\delta,\ 3\delta, \ldots,$ where the boundary $\partial B(k\delta)$ is the disjoint union of b^{k-1} circles of equal length. We denote this length by L_k and observe that the ratio L_{k+1}/L_k is asymptotically constant for $k \to \infty$, where this constant is denoted by a.

Observation. *The family $\mathcal{A}_1$ of the 1-dimensional subsets in $\partial_\infty X$ has*

$$\inf \delta \mathrm{im}\, \mathcal{A}_1 = (\log a)/(\log a + \log b)\ .$$

The proof here is the same as for the classical hyperbolic spaces in 7.C.

Now, one can use $X_\circ$ as a measuring rod to bound $\inf\delta\mathrm{im}\,\mathcal{A}_1$ for more general spaces, where $X_\circ$ (or at least a sector of $X_\circ$) can be quasi-isometrically embedded to.

Example. Let X be a simply connected 2-dimensional polyhedron where all 2-cells are p-gons and where the link of every vertex is the q-dimensional octahedral graph. Notice that all simple circuits in this graph have length four and each vertex has degree q. (The simplest example for $p = 3k$ is the universal ramified covering of the 2-dimensional skeleton of the q-dimensional octahedron with k-th order ramification at the center of each triangle, see $[\mathrm{Ben}]_{1,2,3}$, [Ba-Br] and [Hag] more about such polyhedra). If $p \geq 5$, this X has a metric of negative curvature invariant under the automorphisms of X. Furthermore it is not hard to see that X contains a sector of a round tree (i.e. the part $[s,s'] \times T \subset X'_\circ = S^1 \times (T_\circ - \{x_0\}) \subset X_\circ$, where the parameters a and b of the tree approximately are $a \approx p$ and $b \approx q$. (The inclusion of a sector into X is easier to see for the octahedral X rather than for similar polyhedra of this type such as tetrahedral, cubical etc.). Thus $\inf\delta\mathrm{im}\,\mathcal{A}_1 \to 0$ if $q \to \infty$ for p being kept fixed. An interesting new feature of this example is the possibility to have a cocompact lattice Γ acting on X and thus having arbitrarily small $\inf\delta\mathrm{im}\,\mathcal{A}_1$ for such groups Γ with $\dim \partial_\infty \Gamma = 1$.

Remark. (a) It seems obvious (and the proof should be straightforward) that $\inf\delta\mathrm{im}\,\mathcal{A}_1 \to 1$ if $p \to \infty$ for q being kept fixed. To make this interesting one should prove that $\inf\delta\mathrm{im}\,\mathcal{A}_1 < 1$ whenever $q \geq 3$, which appears slightly harder to do. In fact one should be able to compute $\inf\delta\mathrm{im}$ explicitly for our spaces X and, possibly, for the hyperbolic groups in general in terms of a Markov structure. (Probably, a natural framework for this is provided by the category of Markov spaces.)

(b) Let us indicate a realization of (some of) our Γ by reflection groups in $H^q_{\mathbb{R}}$. Denote by ℓ the length of an edge of the regular p-gon in $H^q_{\mathbb{R}}$ having all angles $\pi/2$ and take q mutually orthogonal geodesic segments of length ℓ in $H^q_{\mathbb{R}}$ emanating from a point $x \in H^q_{\mathbb{R}}$. Then take $\frac{q(q-1)}{2}$ plane regular p-gons in $H^q_{\mathbb{R}}$, each having an orthogonal pair among these segments for a pair of edges. Now for each edge e in every p-gon P we take the hyperplane $H^{q-1}(e) \subset H^q_{\mathbb{R}}$ containing e and perpendicular to the plane of P (there are slightly less than $pq(q-1)/2$ of these hyperplanes) and denote by $\Gamma(p,q)$ the group of isometries of $H^q_{\mathbb{R}}$ generated by reflections in the hyperplanes $H^{q-1}(e)$. This is (easily seen to be) a discrete (convex cocompact) group and the $\Gamma(p,q)$-orbit of the union of our $q(q-1)/2$ p-gons is isomorphic to X. It is clear that the Hausdorff dimension of $\partial_\infty \Gamma(p,q) \subset S^{q-1} = \partial_\infty H^q_{\mathbb{R}}$ satisfies

$$\mathrm{Hau}\dim \partial_\infty \Gamma(p,q) \geq (\inf\delta\mathrm{im}\,\mathcal{A}_1)^{-1}$$

(where $\mathcal{A}_1$ denotes the family of 1-dimensional subsets in $\partial_\infty X = \partial_\infty \Gamma(p,q)$) and, probably, this Hausdim equals $(\inf\delta\mathrm{im})^{-1}$.

(c) It would be interesting to find closed manifolds $V(\Gamma)$ of a fixed dimension n (say $n = 5$) having negative curvature and $\pi_1(V)$ containing a subgroup commensurable to Γ. (Some possibility here opens by Davis' reflection construction which leads to a semi-hyperbolic reflection group $\Gamma^+ \supset \Gamma$ cocompactly acting on $\mathbb{R}^n$ for a given $n \geq 5$, but it is not clear how to hyperbolize this Γ^+).

7.D. Kähler and a(nti)-Kähler. A remarkable and quite mysterious class of groups Γ is constituted by the fundamental groups of algebraic varieties and certain more general complex spaces, e.g. compact Kähler manifolds. Here we are concerned with *hyperbolic* groups Γ. Basic examples of these Γ are cocompact lattices in $U(n,1)$. Furthermore, if Γ is a non-cocompact lattice, the shrinking of cusps of $V = X/\Gamma$ (see 7.A.) keeps us in the complex (and in Kähler in our sense) category, so we get more examples of hyperbolic Kähler groups. Furthermore, ramifying along totally geodesic complex hypersurfaces and blowing down complex totally geodesic submanifolds in $V = X/\Gamma$ (which may have certain selfintersections) gives us further examples of hyperbolic Kähler groups. Probably, there are other manipulations over $H^{2n}_{\mathbb{C}}$ which would give us such groups (compare [Mos-Siu]) but nothing definite is known beyond this. For example, one does not (seem to) know if there are complex ramified coverings $\widetilde{V}$ of locally symmetric spaces V with $K(V) \leq 0$ having $\pi_1(\widetilde{V})$ hyperbolic where the interesting V are those covered by $H^2 \times H^2$ or by $X = O(n,2)/O(n) \times O(2)$ (compare [Tol]).

Problem. Can one characterize Kähler groups by their asymptotic geometry ? (See $[\text{Gro}]_{20}$ for the first steps in this direction).

Most interesting results concerning Kähler groups follow by the route of harmonic maps suggested by Siu in [Siu]. For example, Siu's results imply (also see [Sam]) that the groups of constant negative curvature are *a-Kähler* (i.e. anti-Kähler) in the following sense. First, we say that Γ is a *Kähler group* if it appears as a fundamental group of (a possibly singular) Kähler orbifold. In order not to get lost in definitions we only consider two cases. (1) Γ acts discretely isometrically and cocompactly on a Kähler manifold. (2) Γ is the fundamental group of a complete projective algebraic variety over $\mathbb{C}$.

Now we claim that such a Γ does not have much in common with (groups of) *constant* negative curvature. Namely, every isometric action of Γ on $H^n_{\mathbb{R}}$ (where $n = \infty$ is allowed) must be quite degenerate. We shall state here the simplest result of this type.

Theorem. *Let $h : \Gamma \to \Gamma'$ be a homomorphism of Γ into a discrete group Γ' of isometries of $H^n_{\mathbb{R}}$ where every non-identity element $\gamma' \in \Gamma$ is hyperbolic (as defined below). Then the image $h(\Gamma) \subset h(\Gamma')$ is commensurable to the fundamental group of a (real) surface (i.e. $h(\Gamma)$ is virtually free or a surface group).*

Definition. An isometry $\gamma' = X \hookleftarrow$ is called here *hyperbolic* if

$$\inf_{x \in X} \operatorname{dist}(x, \gamma'(x)) > 0 .$$

For example, if Γ' has no torsion and $H^n_{\mathbb{R}}/\Gamma$ is convex cocompact (which makes $n < \infty$) then all $\gamma' \in \Gamma'$ are hyperbolic.

Remark. The proof of the above theorem needs significantly less than $K = -1$.

In fact, the proof extends to the spaces X (in place of $H^n_{\mathbb{R}}$) where a certain curvature, denoted $K_{\mathbb{C}}X$, is strictly negative (and in the case where $\dim X = \infty$ it would be safer to assume $\inf K(X) > -\infty$). This generalization, as well as the theorem itself, is obtained by the theory of harmonic maps where the condition $K_{\mathbb{C}} < 0$ enters via the Bochner-Siu-Sampson formula (see [Sam], [Ca-To] and $[\text{Gro}]_{21}$ for more about it).

The condition $K_{\mathbb{C}} < 0$ is satisfied if $K = -1$ and then, obviously, "in a small neighborhood of $K = -1$", i.e. whenever $-(1+\varepsilon)^2 < K \leq -1$ for some universal $\epsilon > 0$. (In fact $\epsilon = 1$, as was checked up recently by Hernandez, see [Her]). Notice that, in fact, $K_{\mathbb{C}} < 0$ only needs a *local* version of the pinching condition that is

$$-(1+\varepsilon)^2 \kappa(x) \leq K_x(X) \leq -\kappa(x)$$

for some positive (non-constant!) function $\kappa(x)$ on X. Nobody knows any (global) asymptotic property of X issuing from such pinching; yet we do know that Kähler groups cannot be so pinched !

Another general result about Kähler groups which also shows that they stand much higher in the hyperbolic hierarchy than $K = -1$ and polyhedral groups with $K < 0$, is the *non-existence of essential actions on trees.* The proof is obtained with harmonic maps (see [Gr-Sch]).

Problem. Give an asymptotic characterization of hyperbolic a-Kähler groups Γ which receive few homomorphisms from Kähler groups. (In fact, it seems more logical to use Kähler foliations "mapped" into Γ, rather than groups, to define "a-Kähler". In other words, "a-Kähler" should be an invariant in the measure theoretic "thickening" of the category of groups).

Specific question . Does every homomorphism of a Kähler group to a small cancellation group (or a more general 2-dimensional group) factor through a virtual surface group ? (The harmonic maps approach may eventually work for the fundamental groups of 2-polyhedra with negative curvature but more general 2-dimensional groups require a new idea).

Examples of strongly pinched manifolds. To make use of the fact that ε-pinched manifolds are a-Kähler one needs examples of these which do not map (at least in an obvious way) to manifolds of constant curvature (as do ramified covers exhibited in [Gr-Th] which may have arbitrarily strongly pinched curvature). Such examples can be, in fact, produced with hyperbolic surgery starting from manifolds V of constant curvature $K = -1$. Namely, if γ is a closed geodesic in V with a ρ-thick collar (i.e. *regular* ρ-neigborhood) then one can attach a 2-handle to $V \times [0,1]$, such that the resulting manifold V' will have a metric with $K(V') \in [-(1+\delta)^2, -1]$ for $\delta \approx \rho^{-1}$ and with convex boundary (compare 6.A. in $[\text{Gro}]_{21}$). This also applies to finite collections of geodesics with sufficiently thick mutually disjoint collars and also generalizes to totally geodesic (and more general locally convex) submanifolds of dimension ≥ 2.

Idea of the construction. Let $W \subset V$ be a locally convex subset whose r-neighborhoods $W_r \subset V$ are also locally convex for $r \in [0, \rho]$ (which means W has ρ-collar). Then the boundary $\Sigma_\rho = \partial W_\rho \subset V$ has intrinsic curvature close to zero for large ρ and large injectivity radius. There is an almost isometric embedding of Σ_ρ to the ρ'-sphere $S(\rho')$, for $\rho' = \rho/2$ in the N-dimensional hyperbolic space $H^N_{\mathbb{R}}$ for $N = 2\dim V$, such that the relative curvature of Σ_ρ in $S(\rho')$ is small for large ρ. Then we attach to $V - W_\rho$ the hyperbolic ball $B(\rho')$ across $\Sigma(\rho) \subset S(\rho') = \partial B(\rho')$. The resulting singular space $(V - W_\rho) \cup B(\rho')$ can then be thickened to an N-dimensional manifold V' with almost constant curvature and convex boundary.

Question. Do manifolds V' obtained this way carry complete metrics with $K = -1$?

There are other natural classes of metrics (besides $K_{\mathbb{C}} < 0$ and pinching) defined by curvature conditions stronger than $K < 0$; for example, a prominent class is defined by requiring the *curvature operator* to be negative. Unfortunately we have no asymptotic interpretation of these conditions, but it would be quite useful to extend such conditions to singular spaces (with $K \leq 0$).

7.E. Uniform embeddings. Recall, that a large-scale *Lipschitz* map $f : X \to Y$ is called a *uniform embedding* (*placement in the terminology of* $[\text{Gro}]_{15}$) if the *distance compression of* f defined by

$$M_f(d) = M(d) = \inf_{(x_1,x_2)\in\Delta_d^-} \operatorname{dist} \left(f(x_1)f(x_2)\right) ,$$

$$\text{for} \quad \Delta_d^- = \{(x_1,x_2) \in X \times X \mid \operatorname{dist}(x_1,x_2) \geq d\} ,$$

satisfies

$$M(d) \to \infty \quad \text{for } d \to \infty .$$

Basic group theoretic example. A monomorphism between finitely generated groups is an uniform embedding for the word metrics in these groups.

If X comes along with a Γ-action, a uniform embedding is called *equivariant* if the pull-back to X of the metric on Y is Γ-invariant.

Example. A-T-menability (compare 7.A.). A group Γ is called a-T-menable if there is a homomorphism $\Gamma \to \operatorname{Iso}\mathbb{R}^\infty$ such that every orbit map $\gamma \mapsto \gamma(x)$, $x \in \mathbb{R}^\infty$ is a uniform embedding of Γ to $\mathbb{R}^\infty$. Such an embedding is, obviously, equivariant. Conversely, if Γ admits a Γ-equivariant uniform embedding into $\mathbb{R}^\infty$, then it is a-T-menable since the Hilbert space structure on the span of some orbit is uniquely determined by the Hilbert metric restricted to this orbit.

Question. Is every amenable group a-T-menable ? (This is easy for finitely generated nilpotent groups).

Remark about $H^\infty_{\mathbb{R}}$ and $H^\infty_{\mathbb{C}}$. One can replace $\mathbb{R}^\infty$ in the definition of a-T-menability by $H^\infty_{\mathbb{H}}$ or $H^\infty_{\mathbb{C}}$ (see 7.A.). Then one can refine this notion by imposing extra conditions on the action of Γ on $H^\infty_{\mathbb{R}}$ and $H^\infty_{\mathbb{C}}$, such as the *hyperbolicity* of the action of Γ defined in the previous section. As we know, most Kähler groups admit no hyperbolic actions on $H^\infty_{\mathbb{R}}$ though they may have such actions on $H^\infty_{\mathbb{C}}$. This shows that there is no "hyperbolic" equivariant embedding of $H^\infty_{\mathbb{C}}$ to $H^\infty_{\mathbb{R}}$. In fact every such embedding is, probably, parabolic in a certain sense (e.g. has the image in a horosphere, as the embedding indicated in 7.A.).

Problem. Let Γ be a discrete hyperbolic group of isometries of $H^\infty_{\mathbb{C}}$. Give a geometric criterion on Γ (expressing the idea that Γ is sufficiently large) that would rule out any discrete hyperbolic action of Γ on $H^\infty_{\mathbb{R}}$.

7.E$_1$. Almost equivariant embeddings. A map of a Γ-space X into a metric space Y is called *almost equivariant*, if the induced distance function, say d on $X \times X$ is *almost invariant*,

$$|d(\gamma(x_1), \gamma(x_2)) - d(x_1, x_2)| \le C < \infty$$

for all $x_1,\ x_2 \in X$ and $\gamma \in \Gamma$.

This definition is motivated by the observation by Z. Sela (see [Sel]) that the narrowness of (quasi) geodesic simplices leads to such embeddings $\Gamma \to \mathbb{R}^\infty$. For example, let X be a simply connected manifold with $-\infty < -C \le K(X) \le -1$. Fix a number $\varepsilon > 0$, consider a geodesic segment $[x_1, x_2] \subset X$ between two points and let $[x_1, x_2]_\varepsilon \in \mathbb{R}^\infty = L_2(X)$ be the characteristic function of the ε-neighborhood $U_\varepsilon([x_1, x_2]) \subset X$. Then

$$\|[x_0, x_1]_\varepsilon + [x_0, x_2]_\varepsilon - [x_1, x_2]_\varepsilon\|_{L_2} \le \delta \qquad (*)$$

for some constant $\delta = \delta(X, \varepsilon)$ and all triples of points x_0, x_1, x_2 in X.

Now we map X to $\mathbb{R}^\infty = L_2(X)$ by $x \mapsto [x_0, x]_\varepsilon$ for a fixed x_0 and observe using $(*)$ that this map is an almost equivariant uniform embedding for $\Gamma = \mathrm{Iso}(X)$.

Remark. One can introduce a sign into $(*)$ by normally projecting $U_\varepsilon([x_1, x_2])$ onto the segment $[x_1, x_2]$ and pulling back the standard 1-form dx from this segment to U_ε. Thus we obtain a map from $X \times X$ to the L_2-space of 1-forms on X, say $\mathbb{R}^\infty = L_2 \wedge^1 (X)$, denoted $(x_1, x_2) \mapsto [x_1, x_2]'_\varepsilon$ and satisfying the following (almost) cocycle relations

$$\left.\begin{aligned} [x_1, x_2]'_\varepsilon &= -[x_2, x_1]'_\varepsilon \\ \big\|[x_0, x_1]'_\varepsilon + [x_1, x_2]'_\varepsilon &+ [x_2, x_0]'_\varepsilon\big\|_{L_2} \le \delta\ . \end{aligned}\right\} \binom{*}{*}$$

This says that the *2-cocycle*

$$(x_1, x_2, x_3) \mapsto [x_0, x_1]'_\varepsilon + [x_1, x_2]'_\varepsilon + [x_2 x_0]'_\varepsilon$$

(on $X \times X \times X$) with values in $L_2 \wedge^1 (X)$ is *bounded* and thus, represents a (bounded) cohomology class h_ε in $H^2_{\text{bnd}}(\Gamma; L_2 \wedge^1 (X))$ for the group Γ of the isometries of X. If $h = 0$ then X equivariantly embeds into $L_2 \wedge^1 (X)$ which makes Γ a-T-menable. Thus, $h_\varepsilon \ne 0$ for some hyperbolic groups Γ but I do not see what happens in general.

Additional questions. What are possible uniform embeddings between classical symmetric spaces X and Y ? Namely for which X and Y does there exist one of the following three: uniform embedding $X \to Y$, uniform almost equivariant embedding $X \to Y$, uniform equivariant embedding $X \to Y$? For example, let X be a (finite dimensional) irreducible symmetric space of rank ≥ 2 and $Y = H^\infty_{\mathbb{H}}$. Then, the super-rigidity techniques (harmonic maps and/or ergodic theory in the spirit of Margulis-Zimmer) should eventually rule out *invariant* uniform embeddings $X \to Y$. If we do not impose any invariance condition, then a uniform embedding $X \to Y$ does not seem impossible (unlike the case of Y of finite dimension treated in the next section) and I dare to make no conjecture for the almost equivariant case.

7.E₂. Monotonicity of rank under uniform embeddings. We want to show that spaces Y with $K(Y) \leq 0$ having (an appropriately defined) rank ≥ 2 admit no uniform embeddings into hyperbolic spaces X of *bounded exponential growth* (see below) e.g. into word hyperbolic groups.

Compression lemma. *Let X be a hyperbolic space and $f : \mathbb{R}^2 \to X$ a uniform embedding. Then it is strictly distance compressing,* i.e.

$$\limsup_{\|a-b\|\to\infty} \operatorname{dist}_X(f(a), f(b)/\|a-b\| \to 0 , \tag{$*$}$$

where $\|a-b\| \underset{\mathrm{def}}{=} \operatorname{dist}_{\mathbb{R}^2}(a,b)$. Moreover,

$$\operatorname{dist}_X(f(a), f(b)) \leq \psi(\|a-b\|) ,$$

where the function $\psi(d)$ is determined by X, the Lipschitz constant of f and the distance compression $M_f(d)$ of f, and where $\psi(d)$ satisfies,

$$d^{-1}\psi(d) \to 0 \quad \text{for } d \to \infty .$$

Proof. Take the closed rectangular curve in $\mathbb{R}^2$ with vertices a, b, a', b', where $\|a-a'\| = \|a-b\|^{\frac{1}{2}}$ and where eventually $\|a-b\| \to \infty$. The f-image of this curve in X (see Fig. 19) bounds a surface S of area $\lesssim \|a-b\|$ (by the hyperbolicity of X). We intersect S with the boundaries of the ρ-neighborhoods of the f-image of the edge $[a,a']$ and observe that the lengths $\ell(\rho)$ of these intersections satisfy (by the coarea formula)

$$\int_0^\infty \ell(\rho) \leq \operatorname{area} S \lesssim \|a-b\| .$$

On the other hand, if $\rho \leq D = \operatorname{dist}\Big(f([a,a']), f([b,b'])\Big)$ then the curve $\ell(\rho)$ joins the f-images of the segments $[a,b]$ and $[a',b']$ which lie distance $\|a-b\|^{\frac{1}{2}}$ apart in $\mathbb{R}^2$. Therefore $\ell(\rho) \geq M_f\left(\|a-b\|^{\frac{1}{2}}\right)$ for these ρ. It follows that

$$DM_f\left(\|a-b\|^{\frac{1}{2}}\right) \leq \text{const}\, \|a-b\|$$

and so $D/\|a-b\| \to 0$ for $\|a-b\| \to \infty$. Q.E.D.

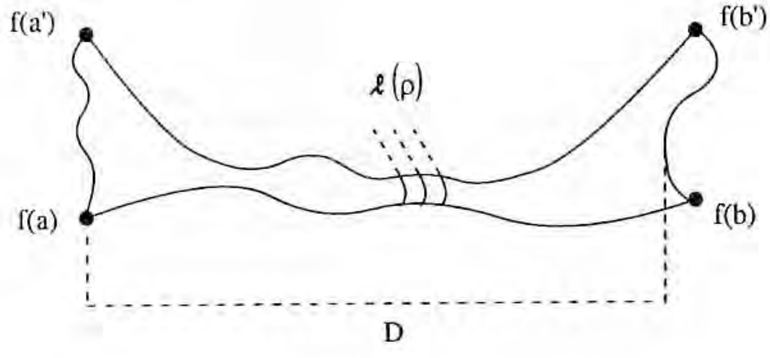

Figure 19

Now, we assume X has *bounded growth*, i.e. the concentric balls around a fixed point have

$$\operatorname{card}_\varepsilon B_X(R) \lesssim \exp \kappa R$$

where the ε-cardinality refers to the minimal number of ε-balls needed to cover $B(R)$. One can use here any positive ε and we stick to $\operatorname{card}_{\varepsilon=1}$.

Then we impose the following two conditions on Y.

(a) For every two points y and y' in Y there exists a λ-quasi-isometric embedding $\mathbb{R}^2 \to Y$ whose image contains y and y' and where λ is some constant *independent* of y and y'. (This condition expresses the idea of $\operatorname{Rank} Y \geq 2$. In fact, our present application needs significantly less than this condition.)

(b) Y has at least exponential growth, i.e. every R-ball in Y satisfies

$$\operatorname{card}_\delta B(R) \geq \operatorname{const}_\delta \exp \mu R$$

for all $\delta > 0$ and a fixed $\mu > 0$.

Claim. *There is no uniform embedding* $Y \to X$.

Proof. The compression lemma shows (with condition (a)) that every uniform embedding $f : Y \to X$ strongly contracts distances, i.e.

$$\operatorname{dist}(f(y_1) f(y_2)) \leq \psi(\operatorname{dist}(y_1, y_2))$$

for some function $\psi(d)$ satisfying

$$d^{-1}\psi(d) \to 0 \quad \text{for } d \to \infty .$$

Thus every R-ball $\subset Y$ goes into a much smaller ball (of radius $\psi(R)$) in X, and as the map is an uniform embedding it does not essentially diminuish the ε-cardinalities, i.e.

$$\operatorname{card}_{\varepsilon=1} f(B(R)) \geq \operatorname{card}_\delta B(R)$$

for some $\delta > 0$ (depending on M_f). But the function $\exp \psi(R)$ is incomparably smaller than $\exp R$ since $\psi(R)$ is by so much smaller than R, which immediately leads to the desired contradiction.

Remark. If X has bounded growth then the function ψ in the lemma can be bounded by $\psi(d) \lesssim d^\alpha$ for some $\alpha < 1$. This is seen by applying f to parallel segments $[a_i, b_i]$ in $\mathbb{R}^2$ for $i = 1, \ldots, k \approx \|a_i - b_i\|^\beta$ and $\|a_i - a_{i+1}\| \approx \|a - b\|^\gamma$ for suitable β and γ. Notice that the standard horospherical embedding of $\mathbb{R}^2 \to H^3_\mathbb{R}$ has $\psi(d) \approx \log d$ and it is claimed in 4.3.B. of [Gro]$_{15}$ that $\psi(d) \lesssim \log d$ in general. Yet, *now* I doubt this is true (though it is true for *rotationally symmetric* maps of $\mathbb{R}^2$ to $H^3_\mathbb{R}$ which the reader is invited to check up).

Modification and generalization to the situation where "Rank" $X \geq 2$. We want to modify the above non-embedding claim by somewhat relaxing the conditions on X and f while adding restrictions on Y. First, to make the idea clear, we assume Y decomposes into a Cartesian product of two geodesic metric spaces $Y = Y_1 \times Y_2$. We consider an arbitrary Lipschitz map $f : Y \to X$ and then we make the following assumption on the restriction of f to every flat square $\square_d \subset Y$, i.e. $\square_d = [y_1, y_1'] \times [y_2, y_2']$, $d = \operatorname{dist}(y_1, y_1') = \operatorname{dist}(y_2, y_2')$, where $[y_1, y_1']$ and $[y_2, y_2']$ are minimizing geodesic segments in Y_1 and Y_2 respectively.

(FV_2). *The filling area (2-volume) of the f-image of the boundary $\partial \square_d$ in X satisfies*

$$\operatorname{Fill Area} f(\partial \square_d) \leq \varphi(\operatorname{Area} \square_d = d^2)$$

for some function $\varphi(A)$ satisfying $A^{-2}\varphi(A) \to 0$ for $A \to \infty$. (Notice that this condition is satisfied for all f if the space X is hyperbolic.)

What we claim now is the following

Compression alternative. There are two possibilities:

(1) *The map f restricted to $Y_1 = Y_1 \times y_2 \subset Y = Y_1 \times Y_2$ for a given point $y_2 \in Y_2$ is strictly distance compressing at a given y_1,*

$$\operatorname{dist}(f(y_1), f(y_1')) \leq \psi(\operatorname{dist}(y_1, y_1'))$$

for some $\psi(d)$ satisfying $d^{-1}\psi(d) \underset{d\to\infty}{\longrightarrow} 0$, and all y_1' in Y_1.

(2) *There exists a sequence of subsets $C_R \subset Y$, $R = 1, 2, \ldots$, such that the projection of C_R to Y_2 covers the R-ball in Y_2 around y_2 and such that the f-image of C_R in X can be covered by* const R *balls of radii $\psi(R)$ for the above function ψ.*

An immediate consequence of this alternative is the non-existence of a uniform embedding $f : Y \to X$, whenever Y_1 and Y_2 grow exponentially while X has *bounded* (exponential) growth. In fact we get slightly more than "non uniform embedding". Namely,

For every Lipschitz map $f : Y \to X$ there exist points y_i and y_i' in Y, such that $\operatorname{dist}(y_i, y_i') \to \infty$ *while* $\operatorname{dist}(f(y_i), f(y_i'))$ *remains bounded. (If X has bounded local geometry we may even have* $\operatorname{dist}(f(y_i), f(y_i')) \to 0$.)

Proof of the alternative. If (1) is violated we find pairs of points y_1 and y_1' in Y_1 within distance $\approx R^2 \to \infty$ such that the distance between $f(y_1, y_2)$ and $f(y_1', y_2)$ in X is $\geq cR^2$. Then we take an arbitrary point y_2' in the R-ball in Y_2 around y_2 and look on the f-image of the rectangular $\square = [y_1, y_1'] \times [y_2, y_2']$. Then the boundary $f(\partial \square)$ in X can be filled in by a surface S of Area $S \leq \varphi'(\text{Area } \square)$ where

$$a^{-1}\varphi'(a) \to 0 \quad \text{for } a \to \infty\ ,$$

as easily follows from (FV_2). Next, as in the proof of the compression lemma, we intersect S with the boundaries of ρ-neighborhoods of $f(y_1 \times [y_2, y_2'])$ and thus find a point $y' \in Y_1 \times Y_2$ projecting to y_2' whose f-image lies ψ-close to $f([y_1, y_1'] \times y_2)$, meaning $\operatorname{dist}(y, f[y_1, y_1'] \times y_2) \leq \psi(\operatorname{dist}(y_2, y_2'))$. Then the union of these y' over all $y_1' \in B_{y_1}(R)$ constitutes our set C_R. Q.E.D.

Generalization to $Y = Y_1 \times Y_2 \times \ldots \times Y_k$. Here again we look at a Lipschitz map $f : Y \to X$ and assume it satisfies the following condition

(VF_k). *The filling volume of the f-image of the boundary of each d-cube $\square_d = [y_1, y_1'] \times \ldots \times [y_k, y_k']$ satisfies*

$$\operatorname{Fill Vol}_k f(\partial \square_d) \leq \varphi(\text{Vol } \square_d = d^k)$$

for $V^{-1}\varphi(V) \to 0, V \to \infty$.

Notice that (VF_k) is satisfied for all Lipschitz maps whenever X is *weakly k-hyperbolic* according to the following

Definition. A metric space X is *weakly k-hyperbolic* if the unit $(k-1)$-sphere mapped into X by a λ-Lipschitz map has

$$\operatorname{Fill Vol}_k \leq \varphi(\lambda^k)$$

for some φ satisfying $\lambda^{-1}\varphi(\lambda) \underset{\lambda\to\infty}{\longrightarrow} 0$. (The reader may work out the large-scale rendition of this definition.)

The condition (VF_k) for cubes obviously yields a similar condition for all rectangular solids and we use it for solids with edges of the lengths $R^k, R^{k-1}, \ldots, R$ for $R \to \infty$. If the R^k-edge of such a solid is not much compressed by f, we find an intersection of the filling of the f-image of our solid with the ρ-neighborhood of (one of the two) $(R^{k-1} \times \ldots \times R)$-faces whose $(k-1)$-volume is roughly φ-small (i.e. much smaller than $R^{k-1} \times R^{k-2} \times \ldots \times R$). This allows an induction on k and leads to the following

Conclusion. *Given a Lipschitz map $f : Y \to X$ satisfying (FV_k), one can find subsets $C_R \subset Y$, $R = 1, 2, \ldots$, such that the projection of C_R to some of Y_i, $i = 1, \ldots, k$, covers an R-ball around a given point, while the f-image of C_R in X can be covered by* const R^{k-1} *balls of radii $\psi(R)$ for some function ψ satisfying $R^{-1}\psi(R) \to 0$ for $R \to \infty$.*

Corollary. *If the spaces Y_i, $i = 1, \ldots, k$ have exponential growth and X is a weakly k-hyperbolic space of bounded (exponential) growth, then there is no uniform embedding.*

$$Y_1 \times Y_2 \times \ldots \times Y_k \to X \ .$$

Further generalization. We want to extend this result to more general spaces Y such, for example, as symmetric spaces Y of rank k of non-compact type.

Exponential rank. We define by induction a number, called $\exp \operatorname{rank} Y$. First we say that

$$\exp \operatorname{rank} Y \geq 1$$

if for each $\delta > 0$ every R-ball in Y has

$$\operatorname{card}_\delta B(R) \geq \operatorname{const}_\delta \exp \mu R$$

for some constants $\operatorname{const}_\delta > 0$ and $\mu > 0$. Then the inequality

$$\exp \operatorname{rank} Y \geq k$$

signifies that Y contains sufficiently many (as explained below) subspaces Y' which are λ-quasi-isometric to Cartesian products $Y' = Y'' \times I$, where I is a segement in $\mathbb{R}$ and $\exp \operatorname{rank} Y'' \geq k-1$, and moreover, such that the implied constants μ, $\operatorname{const}_\delta$ and λ are the same for all spaces in question.

Explanation. "Sufficiently many" means that there exists a number k such that every two points in Y can be joint by a connected chain (i.e. union) of *special lines*, which are $y'' \times I \subset Y' \subset Y$ for various Y' in Y and y'' in Y''.

Examples. (a) The product Y of k spaces Y_i of exponential growth has $\exp \operatorname{rank} \geq k$. A special line here is a geodesic segment contained in some subspace $y_1 \times \ldots \times y_{i-1} \times Y_i \times y_{i+1} \times \ldots \times y_k \subset Y$.

(b) Let Y be a symmetric space of noncompact type (i.e. $K(Y) \leq 0$). It uniquely splits, $Y = Y_0 \times \mathbb{R}^\ell$, where Y_0 is semi-simple, i.e. has no Euclidean factor. Then

$$\exp \operatorname{rank} Y \geq \operatorname{rank} Y_0 \ .$$

(In fact the two ranks are equal in an obvious sense.) The special lines here, in the case $Y = Y_0$, are the *supersingular geodesics* $\gamma \in Y$ where the set Y' of the geodesics parallel to γ splits by $Y' = Y'' \times \mathbb{R}$ where Y'' has rank $k-1$ and no Euclidean factor.

Now we come to the final version of our

Non-embedding theorem. *Let X be a weakly k-hyperbolic metric space of bounded growth and let $\exp \operatorname{rank} Y \geq k$. Then there is no uniform embedding $Y \to X$.*

The proof proceeds by induction on k, more or less as for $Y = Y_1 \times \ldots \times Y_k$.

Corollary. *Let X and Y be symmetric spaces of non-compact type satisfying $\operatorname{rank} X < \operatorname{rank} Y_0$, where Y_0 is the semi-simple part of Y. Then there is no uniform embedding $Y \to X$.*

Remarks. (a) In the case $\operatorname{rank} X = 1$ we may relax the assumption on $Y = Y_0 \times \mathbb{R}^\ell$ by requiring that

$$\operatorname{rank} Y \geq 2 \quad \text{and} \quad \operatorname{rank}(Y_0 \times \mathbb{R}) \geq 2$$

(see compression lemma and the subsequent discussion), but I do not know if in the general case the condition $\operatorname{rank} Y_0 \geq k$ can be replaced by

$$\operatorname{rank} Y \geq k \quad \text{and} \quad \operatorname{rank}(Y_0 \times \mathbb{R}) \geq k \ .$$

(b) There is no known geometric obstruction for uniform embeddings into infinite dimensional spaces. In particular, it is unclear whether every separable metric space can be uniformly embedded into the Hilbert space $\mathbb{R}^\infty$. On the other hand, such obstructions are known for *Lipschitz* uniform embedding. (Recall that our definition requires Lipschitz on the large scale only). For example, the space ℓ_1 admits no uniform Lipschitz embedding into $\ell_2 = \mathbb{R}^\infty$, because a 1-Lipschitz map of the n-dimensional cubical graph $\square^n$ into $\mathbb{R}^\infty$ contracts some great diagonal in $\square^k$ by factor (at least) $\sqrt{n}$, as every analyst knows.

§8. $\mathbf{L_p}$-cohomology and conformal dimensions at infinity; $\mathbf{L_2}$-Betti numbers and spectrum near zero; vanishing and non-vanishing of $\mathbf{L_pH^*(X)}$.

Let X be a *uniformly locally bounded* simplicial polyhedron, i.e. where each k-simplex has at most c_k neighbours, for some $c_k = c_k(X)$. We denote by ℓ_pC^k for $p \in [1,\infty]$, the space of p summable real cochains of degree k (which are ℓ_p-functions on the set of k-simplices) and observe that the coboundary

$$d_k : \ell_pC^k \to \ell_pC^{k+1}$$

is a *bounded* linear operator. Then we define the *non-reduced ℓ_p-cohomology* by $\operatorname{Ker} d_k / \operatorname{Im} d_{k-1}$ and the *reduced ℓ_p-cohomology* by

$$\overline{\ell_pH}^k(X) = \operatorname{Ker} d_k / \overline{\operatorname{Im} d}_{k-1} \ ,$$

where the bar over Im signifies the closure in the ℓ_p-topology. One says the ℓ_p-cohomology is *(non)-reduced* in dimension k if $\operatorname{Im} d_k$ is (not) closed.

The ℓ_p-cohomology is functorial for *uniformly proper* (in the obvious sense) simplicial maps $f : X \to Y$ as the induced homomorphisms on cochains send $\ell_pC^*(Y)$ to $\ell_pC^*(X)$. Furthermore, the induced homomorphism f^* on ℓ_p-cohomology is invariant under uniformly locally bounded simplicial homotopies as these induce chain homotopies in the category of complexes of *topological* vector spaces and *bounded* linear operators. In particular, if X and Y are *uniformly contractible*, mutually *quasi-isometric* polyhedra, then their ℓ_p-cohomology is isomorphic. (Recall that the polyhedral simplicial structures carry along natural metrics associated with them).

If X is a smooth Riemannian manifold, one can also define the L_p-cohomology of the de Rham complex of smooth L_p-forms with differentials in L_p. If X has uniformly locally bounded geometry there is a uniformly locally bounded triangulation of X, where all k-simplices are uniformly bi-Lipschitz to the standard k-simplex. Then the usual proof of the de Rham theorem shows that the complexes of ℓ_p-cochains and L_p-forms are mutually chain homotopy equivalent and, hence,

$$L_p\text{- cohomology } = \ell_p\text{- cohomology} \ ,$$

and the isomorphism between L_p and ℓ_p commutes with every isometry group Γ of X preserving the triangulation.

Two essential new features of ℓ_p-cohomology are:

(i) If X is a connected infinite complex then, (obviously)

$$\ell_pH^0(X) = 0 \text{ for } 1 \le p < \infty \ .$$

(ii) Typically, if $\overline{\ell_pH}^k(X) \neq 0$ for certain k and p, then this cohomology is infinitely dimensional. For example, *if X admits an isometry $\varphi : X \to X$ with unbounded orbits, then*

$$\overline{\ell_pH}^k(X) \neq 0 \Rightarrow \dim \overline{\ell_pH}^k(X) = \infty \ ,$$

provided $p \neq 1, \infty$.

Linear proof: ℓ_q-homology. Using the complex of ℓ_q-chains $\partial : \ell_qC_* \leftarrow$ one defines ℓ_q-homology and observes that $\overline{\ell_2H}_k$ is dual to $\overline{\ell_pH}^k$ for $\frac{1}{p} + \frac{1}{q} = 1$. Thus every non-zero $h \in \ell_pH^k$ has non-zero pairing with some $h' \in \overline{\ell_qH}_k$ and if we apply the iterates of φ to h' we see that

$$\langle h, \varphi^i_*(h') \rangle \underset{i\to\infty}{\longrightarrow} 0 \ .$$

Hence, the span of $(\varphi^i)^*(h)$ in ℓ_pH^k is infinite dimensional.

Non-linear proof: p-harmonic cochains. For every cohomology class h there exists a unique closed cochain $c \in \ell_pC^k$ (closed means $dc = 0$) which minimizes the ℓ_p-norm in the class h of c. Such a c satisfies a certain non-linear equation, written $\Delta_p c \leq 0$, where Δ_p denotes the gradient of the ℓ_p-norm restricted to the space of the cochains cohomologous to (i.e. belonging to) h. Since the ℓ_p-norm is strictly convex for $p \neq 1, \infty$, the equation has a unique solution c; moreover, if we have some c' with $\|\Delta_p c'\|_{\ell_p} \leq \varepsilon \|c'\|_{\ell_p}$ then the extremal c lies ε'-close to c' with $\varepsilon' \to 0$ for $\varepsilon \to 0$.

Now we realize a non-zero h by our extremal c, take $c_1 = c$, $c_2 = (\varphi^{i_1})^* c, c_3 = (\varphi^{i_2})^* (c), \dots$, for a fast growing sequence of numbers $0 \ll i_1 \ll i_2 \ll \dots$. Then every non-zero linear combination $\sigma = \sum_{i=1}^{k} a_i c_i$ is a non-zero almost Δ_p-harmonic cochain, i.e. $\|\Delta_p \sigma\|_{\ell_\epsilon} \leq \varepsilon \|\sigma\|_{\epsilon}$, which implies σ is non-cohomologous to zero.

Corollary. *Let the above X be uniformly contractible and φ be a translation (parallel to the identity) i.e. $\sup_{x \in X} \operatorname{dist}(x, \varphi(x)) < \infty$. Then $\overline{\ell_p H}^* = 0$ for $p \neq 1, \infty$.*

Proof. The extra conditions on X and φ show that φ^* is an isomorphism on $\ell_p H^*$.

Example to the proof. *Let $X = X' \times \mathbb{R}$, where X' is arbitrary. Then $\overline{\ell_p H}^* = 0$ for $p \neq 0, \infty$. In particular, $l_p H^*(\mathbb{R}^n) = 0$ for $p \neq 0, \infty$ and $n \geq 1$.*

Remark. The n-dimensional ℓ_1- cohomology of $\mathbb{R}^n$ equals (by an easy argument) to the cohomology with the compact supports and this equals $\mathbb{R}$. Also the n-dimensional ℓ_∞-cohomology of $\mathbb{R}^n$ is non-zero and, moreover, infinite dimensional.

First non-trivial examples. (a) Let X be the infinite regular tree with three edges at every vertex. Then $\ell_q H_1 \neq 0$ for all $s \neq 1$, where a non-trivial ℓ_q-cycle is indicated on Fig. 20.

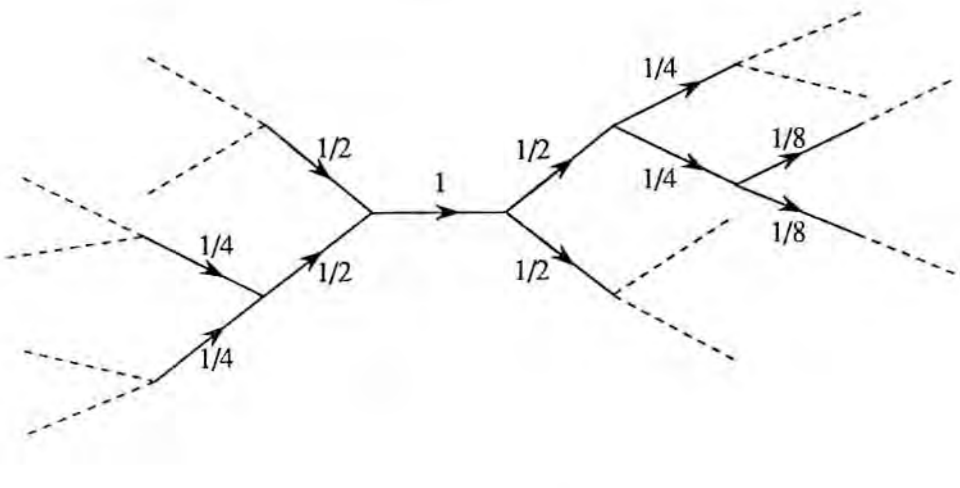

Figure 20

This cycle has non-zero pairing with every non-zero 1-cochain c supported on a single edge. It follows that the groups $\overline{\ell_p H}^1$ are infinite dimensional for this tree.

(b) Quasi-isometrically embed X into the hyperbolic plane $H^2_{\mathbb{R}}$ as a tree with geodesic edges and normally project $H^2_{\mathbb{R}}$ onto an edge. Then our c, made into a smooth 1-form supported on this edge, lifts to a closed 1-form on $H^2_{\mathbb{R}}$ which is, clearly, L_p for every $p > 1$ (compare Fig. 22 in 8.C$_2$.). Thus

$$L_p H^1(H^2_{\mathbb{R}}) \neq 0 \quad \text{for } p \neq 1, \infty .$$

(c) Generalize to $H^n_{\mathbb{R}}$ for $n \geq 2$ and observe that the pull-back form on $H^n_{\mathbb{R}}$ is in L_p for every $p > n-1$, and it non-trivially pairs with the ℓ_q-cycle represented by the tree $X \subset H^n_{\mathbb{R}}$. Thus $\overline{L_pH}^1(H^n_{\mathbb{R}}) \neq 0$ for $p > n-1$.

(d) Replace $H^n_{\mathbb{R}}$ by an arbitrary contractible n-dimensional manifold Y with negative curvature pinched between -1 and $-(1+\varepsilon)^2$. Then the above construction shows that

$$\overline{L_pH}^1(Y) \neq 0 \quad \text{for } p > (1+\varepsilon)(n-1) .$$

This result is due to Pansu (see $[\text{Pan}]_{6,7}$) who also shows that $\ell_pH_1(Y) = 0$ for $p < (n-1)/(1+\varepsilon)$.

(e) Take a conformal map φ ("Poincaré model") of $H^n_{\mathbb{R}}$ onto the unit ball $B \subset \mathbb{R}^n$ and observe that conformal maps preserve the L_p-norm on k-forms for $kp = n$. Thus, bounded k-forms ω on B pull back to L_p-forms on $H^n_{\mathbb{R}}$ for $p = n/k$. This applies, in particular, to the closed form $\omega = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k$ on $B \subset \mathbb{R}^n$ which is non-trivial in $\overline{L_pH}^k(H^n_{\mathbb{R}})$, $p = n/k$. Indeed, take the complementary form $\omega' = dx_{k+1} \wedge \ldots \wedge dx_n$ and observe that the (cup) product of the two pullbacks, i.e.

$$\varphi^*(\omega) \wedge \varphi^*(\omega') = \varphi^*(dx_1 \wedge \ldots \wedge dx_n)$$

is non-zero in $\overline{L_1H}^n(H^n_{\mathbb{R}})$ because it integrates over $H^n_{\mathbb{R}}$ to

$$\int_B dx_1 \wedge \ldots \wedge dx_n = \operatorname{Vol} B \neq 0 .$$

It follows that both forms $\varphi^*(\omega)$ and $\varphi^*(\omega')$ represent non-trivial cohomology classes in $\overline{L_{n/k}H}^k(H^n_{\mathbb{R}})$ and $\overline{L_{n/n-k}H}^{n-k}(H^n_{\mathbb{R}})$ respectively. In particular, if $n = 2m$, then the m-dimensional L_2-cohomology of $H^n_{\mathbb{R}}$ is non-zero. (One can show that the L_2-cohomology of $H^n_{\mathbb{R}}$ is multiplicatively generated by the 1-dimensional L_n-cohomology. One also knows, that the k-dimensional reduced L_2-cohomology of $H^n_{\mathbb{R}}$ vanishes for all $k \neq n/2$ and this remains true for all n-dimensional symmetric spaces of non-compact type).

(e) *Maps with p-summable gradients.* Let an n-dimensional manifold X admit a smooth map $\varphi : X \to \mathbb{R}^n$ such that

(i) $\int_X \|D\varphi\|^p \leq \infty$ for some $p \geq 1$,

(ii) $\int_X \|\wedge^n \varphi\| \leq \infty$, where $\wedge^n$ denotes (the action of) the differential $D\varphi$ on the n-th exterior power of $T(X)$, (this is, in plain words, the Jacobian of φ).

(iii) $\int_X \varphi^*(dx_1 \wedge \ldots dx_n) \neq 0$, (which makes sense in view of (ii)).

Then, clearly, $\overline{L_qH}^k(X) \neq 0$ for $q \geq p/k$.

More generally, if we replace (i) by

$$(i)_k \int_X \| \wedge^k \varphi\|^q < \infty \text{ for some } k \leq n/2 \text{ and } q \geq 1 ,$$

we conclude that

$$(i)_\ell \int_X \|\wedge^\ell \varphi\|^r < \infty \text{ for } \ell = n-k \text{ and } r = qk/\ell$$

and that $\overline{L_pH}^k(X) \neq 0$, provided $q^{-1} + \ell/qk \leq 1$. In particular, if $k = n/2$ and $q \geq 2$, then the k-dimensional L_2-cohomology of X is non-zero. We shall look later on such maps $X \to B \subset \mathbb{R}^n$ for manifolds X with pinched negative curvature and for symmetric spaces of non-compact type, as we shall continue this geometric discussion in section 8.C_2. but first we want to present a remarkable special feature of the L_2-cohomology.

8.A. Betti numbers made ℓ_2. Suppose, our polyhedron X comes along with a discrete action of a group Γ such that X/Γ is compact. Then, following Von Neumann and Atiyah (see [At]), one can define (see below) for each k a certain *real number* called the *Von Neumann dimension* or *the ℓ_2-Betti number*.

$$h^k(X:\Gamma) = \dim_\Gamma \ell_2 H^k(X)$$

which is an invariant of the Hilbert space $\ell_2 H^k(X)$ with the unitary Γ-action induced by the action of Γ on X and which expresses the intuitive idea of the ordinary dimension divided by the cardinality of Γ.

If X is k-acyclic, then h^k depends only on Γ and is denoted $h^k(\Gamma)$. Notice that for $k \geq 2$ a k-acyclic cocompact Γ-space does not always exist, but $h^k(\Gamma)$ is defined anyway, though it may be infinite. (See $[\text{Ch-Gr}]_{2,3}$, where the reader finds details and references.)

Definition of $h^k(X{:}\Gamma)$. Let $\ell_2 C^k$ denote the space of ℓ_2-cochains on X and $\ell_2\mathcal{H}^k \subset \ell_2 C^k$ denote the orthogonal complement of the image of the coboundary operator in its kernel,

$$\ell_2\mathcal{H}^k = \ker d_k \ominus \operatorname{Im} d_{k-1} .$$

Clearly, $\ell_2\mathcal{H}^k$ is canonically isomorphic to $\overline{\ell_2 H}^k(X)$. (The cochains c in $\ell_2\mathcal{H}^k$ are called *harmonic* as they satisfy the combinatorial Laplace equation $(d^*d + dd^*)(c) = 0$). We denote by P^k the orthogonal projection $\ell_2 C^k(X) \to \ell_2\mathcal{H}^k$ and observe that the operator P_k (being a bounded operator on ℓ_2) is given by a unique *kernel* which is a function $\sqcap^k(\sigma, \sigma')$ on the pairs of oriented k-simplices in X, such that

$$\left(P^k(c(\sigma))\right)(\sigma') = \sum_\sigma \sqcap^k(\sigma,\sigma')c(\sigma)$$

and where $\sqcap$ is skew-symmetric for the changes of orientations of σ and σ'. Then we observe that the function $tr_k(\sigma) = \sqcap^k(\sigma,\sigma)$ is Γ-invariant on the set S_k of the non-oriented k-simplices of X. We denote by Γ_σ the subgroup in Γ consisting of the transformations mapping σ into itself and observe that the function

$$\operatorname{Ind}\Gamma_\sigma = \operatorname{card}\Gamma/\Gamma_\sigma \quad \text{on } S_k$$

is Γ-invariant and thus is defined on the (finite !) set of Γ-orbits, denoted $\overline{S_k} = S_k/\Gamma$, where the above function tr is defined as well. Now we set

$$\operatorname{trace}_\Gamma P^k = \sum_{\overline{\sigma}\in\overline{S}_k} \left(\operatorname{Ind}\Gamma_{\overline{\sigma}}\right)^{-1} tr_k(\overline{\sigma}) ,$$

and define

$$h^k(X:\Gamma) = \dim_\Gamma \ell_2 H^k(X) = \dim_\Gamma \ell_2\mathcal{H}^k = \operatorname{trace}_\Gamma P^k .$$

(Notice, that $\operatorname{Ind}_{\overline{\sigma}}$ disappears from the formulae if the action of Γ on X is free).

Here are some basic properties of the numbers h^k which indicate their close similarity to the ordinary Betti numbers.

(1) $h^k(X:\Gamma) = 0 \Leftrightarrow \overline{\ell_2 H}^k(X) = 0$. In particular, if X is contractible, the vanishing (or non-vanishing) of $h^k(\Gamma) = h^k(X:\Gamma)$ is a quasi-isometry invariant of Γ.

(2) $h^0(\Gamma) = (\operatorname{card}\Gamma)^{-1}$. In particular, $h^0(\Gamma) = 0$ if and only if Γ is infinite.

(3) *Künneth formula.* $h^k(\Gamma_1 \times \Gamma_2) = \sum_{i=0}^{k} h^i(\Gamma_1)h^{k-i}(\Gamma_2)$ (with the convention $0 \cdot \infty = 0$ if some of h^i happens to be infinite). For example, if Γ_1 and Γ_2 are infinite then $h^1(\Gamma_1 \times \Gamma_2) = 0$.

(4) Define the ℓ_2- *Euler characteristic* by

$$\chi_{\ell_2}(X:\Gamma) = \sum_{i=0}^{\dim X} (-1)^i h^i(X:\Gamma) .$$

Then define the *orbi-Euler characteristic* of X/Γ by

$$\chi_{\text{orb}}(X/\Gamma) = \sum_{\overline{\sigma}} (-1)^{\dim \overline{\sigma}} (\operatorname{Ind} \Gamma_{\overline{\sigma}})^{-1} ,$$

where the summations take place over all $\overline{\sigma} \in \overline{S} = \bigcup_{k=0}^{\dim X} \overline{S}_k$ for $\overline{S}_k = S_k/\Gamma$ and where S_k denotes the set of the k-simplices in X. (If the action of Γ on X is free, χ_{orb} equals the usual Euler characteristic $\chi(X/\Gamma)$).

The following fundamental relation easily follows from the basic formal properties of the Von Neumann dimension (which we did not explain) but it was brought into focus relatively recently by Atiyah in the framework of his more general (and more profound) L_2-index theorem (see [At]).

Euler-Poincaré-Atiyah formula (see [Gr-Ch]).

$$\chi_{\ell_2} = \chi_{\text{orb}} . \qquad (*)$$

This formula is especially useful where X is a connected infinite (to make $h^0 = 0$) 2-dimension polyhedron. Then $(*)$ reduces to the relation

$$h^2(X:\Gamma) - h^1(X:\Gamma) = \chi_{\text{orb}}(X/\Gamma)$$

where χ_{orb} is an easily computable (rational) number. Now we can see, that if $\chi_{\text{orb}} < 0$, then $\overline{\ell_2 H}^1(X:\Gamma) \neq 0$ and if $\chi_{\text{orb}} > 0$, then $\overline{\ell_2 H}^2(X:\Gamma) \neq 0$.

(4)′ *Example.* Let Γ be a small cancellation group with no torsion given by s generators and r relations. Then $\chi_{\text{orb}} = r + 1 - s$ and therefore

$$s > r + 1 \Rightarrow \overline{\ell_2 H}^1(\Gamma) \neq 0$$

and

$$r > s - 1 \Rightarrow \overline{\ell_2 H}^2(\Gamma) \neq 0 .$$

(Here $\overline{\ell_2 H}^*(\Gamma)$ refers to $\overline{\ell_2 H}^*(X)$ for the universal covering X of the (aspherical !) polyhedron V presenting Γ.)

Notice that there is no direct proof of the above non-vanishing properties for $\ell_2 H^*(\Gamma)$, nor such properties are known for $\ell_p H^*$, where $p \neq 2$.

(4)″ Let us generalize (4)′ by allowing torsion. Namely, let

$$\Gamma = \{\gamma_1, \dots, \gamma_s \mid (w_1)^{e_1} = 1, \dots, (w_2)^{e_r} = 1\} .$$

Then, if the presentation satisfies a small cancellation condition and, hence, $V = X/\Gamma$ for a contractible 2-polyhedron X with the action of Γ having fixed points of orders e_i on the 2-cells corresponding to the relation $(w_i)^{e_i} = 1$, then

$$\chi_{\text{orb}} = 1 - s + \sum_{i=1} e_i^{-1} .$$

In particular,

$$s > 1 + \sum_{i=1} e_i^{-1} \Rightarrow \overline{\ell_2 H}^1(\Gamma) \neq 0 .$$

Notice that the ordinary first Betti number can be easily zero for such a group Γ.

Question. Can one actually compute h^1 and h^2 in the examples (4)′ and (4)″ ? For instance, let Γ be a small cancellation group with two generators and a single relation. Can one ever have $\overline{\ell_2 H}^2(\Gamma) \neq 0$?

8.A$_1$. Vanishing theorems for h^i. The only known mean for computing the ℓ_2-Betti numbers consists in combining the Euler-Poincaré-Atiyah formula with some vanishing theorems, where one may use in the process some simple linear (homological) algebra, such as the Künneth formula, spectral sequences, Mayer-Vietoriis, etc. Here is the list of known vanishing theorems.

(A) ***Amenable goups.*** *Every amenable group* Γ *has* $h^k(\Gamma) = 0$ for $k > 0$. *Furthermore, if* Γ *admits an infinite amenable normal subgroup then* $h^k(\Gamma) = 0$ *for all* $k = 0,\ 1, \ldots,$ (see [Ch-Gr]$_3$).

(A$_1$) ***Corollary.*** *Generalized Gottlieb-Stallings-Rosset theorem). Suppose* Γ *admits a free discrete cocompact action on a contractible polyhedron and there exists an infinite normal amenable* (may be, infinitely generated) *subgroup in* Γ. *Then the ordinary Euler characteristic* $\chi(\Gamma) = \sum_i (-1)^i b^i(\Gamma)$ *vanishes.*

(A$_2$) *Remark.* Let X be a contractible polyhedron which admits cocompact action of an infinite *amenable* group Γ. Then

$$\overline{\ell_p H}^k(X) = 0 \quad \text{for } 1 < p < \infty \quad \text{and } k = 0,\ 1, \ldots .$$

This can be shown by the geometric argument used in [Gro]$_8$ in the case $p = 2$. Probably, all of (A) generalizes to $p \in]1, \infty[$.

(B) ***Symmetric spaces.*** *If* X *is a symmetric space of non-compact type then the reduced* L_2*-cohomology vanishes everywhere except for the middle dimension* (as follows from Harish-Chandra's theory of discrete series). *In fact, it vanishes in the middle dimension if and only if the Euler-Poincaré-Chern-Weil form* Ω *on* X *is zero.*

This leads to the similar property of (not necessarily cocompact) lattices in $\operatorname{Iso} X$ by the following

Theorem (see [Ch-Gr]$_{1,2}$). *Let* X *be a complete contractible Riemannian manifold with uniformly bounded local geometry* (e.g. *having* $|K(X)| \leq \text{const} < \infty$ *and* $\operatorname{Inj}\operatorname{Rad}(X) \geq \varepsilon > 0$) *and* Γ *be a discrete faithful isometry group acting on* X *with* $\operatorname{Vol} X/\Gamma < \infty$. *Then the* L_2*-Betti numbers of* X *(which can be defined and which are finite in the present case) are equal to those of* Γ,

$$h^k(\Gamma) = \dim_\Gamma L_2 H^k(X) < \infty .$$

Furthermore, if X *is* Γ*-equivariantly homotopic to a* Γ*-space* Y *where* Y/Γ *is compact* (e.g. X is a real analytic Riemannian manifold with $K(X) \leq 0$), *then*

$$\chi(\Gamma) \underset{\text{def}}{=} \chi_{\text{orb}}(Y/\Gamma) = \chi_{L_2}(X : \Gamma) \underset{\text{def}}{=} \sum_{i=0}^{\dim X} (-1)^i \dim_\Gamma L_2 H^i(X) .$$

Corollary. *Let* Γ_2 *and a* Γ_2 *be lattices acting on some symmetric spaces* X_1 *and* X_2 *such that the Euler form* Ω_1 *on* X_1 *is zero while* Ω_2 *on* X_2 *is non-zero. Then* Γ_1 *is not quasi-isometric to* Γ_2.

Remark. One believes that Γ_1 is not quasi-isometric to Γ_2 unless X_1 is quasi-isometric to X_2. (A problem here appears when Γ_1 and Γ_2 are non-cocompact. The known techniques are sufficient for the implication that X_1 *is quasi-isometric to* $X_2 \Rightarrow \operatorname{Iso} X_1$ *is isomorphic to* $\operatorname{Iso} X_2$, which is not very useful for non-cocompact Γ_i, $i = 1,\ 2$, as these are not quasi-isometric to X_i.)

(C) ***Kähler manifolds.*** *Let* X *be a geodesically complete Kähler manifold with uniformly bounded local geometry. Then* $\overline{L_2 H}^1(X) = 0$ *unless* X *admits a proper holomorphic map onto a Riemann surface* Y *such that the implied map is* Γ*-equivariant for* $\Gamma = \operatorname{Iso} X$ *and some homomorphism* $\Gamma \to \operatorname{Aut}_{\mathbb{C}} Y$.

Notice that most X do not admit proper holomorphic maps to Y. For example, the existence of such a map necessarily makes $H_{n-2}(X) \neq 0$ for $n = \dim_{\mathbb{R}} X$.

The above vanishing theorem extends to $L_2 H^k$, $k \geq 2$ if X is hyperbolic and $H^2(X) = 0$. Moreover, *If the Kähler form* ω *on* X *is the differential of a bounded 1-form on* X, *then*

$$L_2 H^k(X) = 0 \quad \text{for } k \neq \dim_{\mathbb{C}} X .$$

Remarks. (a) The proof of these vanishing theorems uses the L_2-Hodge theory (see $[\text{Gro}]_{18}$) and does not extend (at the first sight) to the L_p-cohomology for $p \neq 2$.

(b) The middle dimensional L_2-cohomology is known to be reduced and non-zero under the above assumption $\omega = d(\text{bounded})$. This is proven in $[\text{Gro}]_{20}$ with a suitable L_2-index theorem.

(C_1) *Quaternionic manifolds.* Let X be a geodesically complete locally symmetric space modelled on (i.e. locally isometric to) $H^{4n}_{\mathbf{H}}$. This space comes along with a canonical 4-form Ω. This Ω has the same essential features as a Kähler form (pointed out to me by D. Toledo and P. Pansu with the reference to p. 419 in [Bess]) which allow an extension of the above Kähler discussion to X. (This was also observed by K. Corlette). Thus, *if* $\Omega = d(\text{bounded})$ *then* $L_2h^k(X) = 0$ *for* $k \neq 2n$ *and* $\overline{L_2H}^{2n}(X) \neq 0$.

(D) *T-groups.* If a group Γ is Kazhdan's T, then $h^1(\Gamma) = 0$ (as is well known and easy to prove). In particular, every factorgroup Γ of a quaternionic lattice has $h^1(\Gamma) = 0$ (though it may have $h^2(\Gamma) \neq 0$ as we shall see later on). On the other hand, we recall that there exists a cocompact lattice $\Gamma_0 \subset O(n,1)$, for every $n \geq 2$, which admits the free group F_2 as a factor group and so $h^1(\Gamma_0) = 0$ does not automatically yield the same for factor groups.

(E) ***Pinched manifolds*** (see [Don-Xa]). *If X is a simply connected, geodesically complete manifold with a sufficiently pinched negative curvature, namely*

$$-1 \leq K(X) \leq -c^2 < 1 - \left(\frac{n-2}{n-1}\right)^2 , \qquad (*)$$

then $L_2H^k(X) = 0$ *unless*

$$k = n/2 \quad \text{for } n = \dim X \textit{ even}$$

or

$$k = \frac{n \pm 1}{2} \quad \text{for } n \textit{ odd} .$$

Example. Look at the fundamental groups Γ of ramified coverings of closed manifolds of constant negative curvature along totally geodesic submanifolds of codimension 2. Some of these coverings do admit pinched metrics as in $(*)$ (see [Gr-Th]) and thus $\ell_2H^k(\Gamma) = 0$ for $\left|k - \frac{n}{2}\right| > 1$.

Remark. The proof of Donnelly-Xavier in [Don-Xa] uses polar coordinates in X and generalizes to L_p-cohomology. In fact, using polar coordinates one can efficiently study the L_p-cohomology as a function space on the sphere at infinity (see $[\text{Str}]_1$, $[\text{Pan}]_6$). An important result here is due to Pansu shows for $-1 \leq K(X) \leq -c^2 < 0$ that the k-dimensional L_p-cohomology is *reduced* for

$$p < 1 + \frac{c\,(n-k)}{k-1} ,$$

(where "reduced" means that the image of the exterior differential is *closed* in the space of k-forms). The techniques employed by Pansu may eventually prove useful in settling the best pinching constant problem for the spaces quasi-isometric to $H^{2n}_{\mathbf{C}}$, $H^{4n}_{\mathbf{H}}$ and $H^{16}_{\mathbf{Ca}}$.

Question. Let V be a closed manifold of constant negative curvature and let $X \to V$ be an infinite regular (Galois) covering, where the fundamental group $\pi_1(X) \subset \pi_1(V)$ is normally generated by a closed geodesic γ in V with a sufficiently thick ρ-collar (compare the end of 7.D.). Does then the k-dimensional L_2-cohomology of X vanish for $k \neq \frac{n}{2}, \frac{n\pm1}{2}$? (We know this does happen for locally symmetric spaces modelled on $H^{2n}_{\mathbf{C}}$, $H^{4n}_{\mathbf{H}}$ and $H^{16}_{\mathbf{Ca}}$).

(F) *Weakly branched polyhedra.* Say that an n-dimensional polyhedron X is *weakly branched* if every open n-simplex is contained in an open connected subset $X' \subset X$ such that X^1 is an *infinite* subpolyhedron in X which is also a *pseudo-manifold,* i.e. every $(n-1)$-simplex in X' has exactly two adjacent n-simplices. It is obvious, that every n-dimensional ℓ_p-cycle in X for $p \neq \infty$ is zero. Hence,

$$\overline{\ell_pH}^n(X) = 0 \quad \text{for } p \neq 1, \infty .$$

Example. Let P be a finite connected 2-dimenional polyhedron and $\Sigma \subset P$ be the subcomplex consisting of the edges having $k \neq 2$ adjacent 2-faces. Suppose each connected component P_0 in the complement $P - \Sigma$ has infinite image of $\pi_1(P_0) \to \pi_1(P)$. Then the universal covering $X = \widetilde{P}$ has $\overline{\ell_2 H}^2(X) = 0$ and so

$$h^1(X : \pi_1(P)) = 1 - \chi(P) .$$

In particular, the fundamental group of a connected surface has

$$h^1(\pi_1(P)) = 1 - \chi(P) \quad \text{and } h^k(\pi_1(P)) = 0 \quad \text{for } k \geq 2 .$$

(G) *Vanishing conjectures.* It was suggested some 15 years ago by I. Singer that the fundamental groups Γ of closed manifolds V with non-positive curvature must have $h^k(\Gamma) = 0$ for $k \neq \dim V/2$. (This would imply that $\chi(V) \geq 0$ for $\dim V = 4m$ and $\chi(V) \leq 0$ for $\dim V = 4m+2$, which remains an open question at the present day). In fact, for all we know, this may be true for the closed aspherical manifolds V with *no* assumption on the curvature and also for the hyperbolic groups Γ having $\partial_\infty \Gamma$ homeomorphic to the sphere S^{n-1}. On the other hand, one cannot exclude a counterexample among (strongly pinched) ramified coverings of closed $(2k+1)$-dimensional manifolds of constant negative curvature. Recall that $H_{\mathbb{R}}^{2k+1}$ has non-zero, *non-reduced* cohomology $L_2 H^{k+1}$ which may lead to $\overline{L_2 H}^{k+1} \neq 0$ for some ramified coverings of $H_{\mathbb{R}}^{2k+1}$. In fact, non-zero $L_2 H^2(H_{\mathbb{R}}^3)$ does breed non-zero $\overline{L_2 H}^2$ (and hence, $\overline{L_2 H}^1$) for perturbations of the constant curvature metric, such as $g_\varepsilon = dt^2 + e^t dx_1^2 + e^{(1+\varepsilon)t} dx_2^2$, as was shown by M. Anderson (see [And]$_2$).

An important source of groups Γ with $h^k(\Gamma) = 0$ comes from direct products $\Gamma_0 \times \Gamma_1$, where Γ_0 and Γ_1 are infinite (and thus have $h^0 = 0$) and where $h^i(\Gamma_0) = 0$ for $i = 0, 1, \ldots, k-1$. One wonders whether this remains true for semidirect products under certain finiteness assumptions. (These are needed, as is seen in the example of $\mathsf{F}_2 = \mathsf{F}_\infty \rtimes \mathsf{F}_1$). Here is a specific

Conjecture. Let a compact aspherical manifold V be fibered over the circle. Then the group $\Gamma = \pi_1(V)$ has $h^k(\Gamma) = 0$ for all $k = 0, 1, \ldots$.

Interesting examples where this is unknown are provided by ramified coverings of the 5-torus many of which fiber over S^1 and have π_1 hyperbolic.

8.A$_2$. Examples of computation of $h^k(\Gamma)$. (1) If $h^k(\Gamma) = 0$ for all k but $k = k_0$, then $h^{k_0}(\Gamma) = \pm\chi(\Gamma)$. Thus one computes h^k for (compact and non-compact) lattices in the semisimple Lie groups.

(2) *Underlattices.* Let $\overline{\Gamma}$ be a hyperbolic factorgroup of a hyperbolic lattice Γ. Then $\overline{\Gamma}$ acts on a locally symmetric manifold (or orbifold, if we allow torsion), X (covering V with $\pi_1(V) = \Gamma$). If X is modelled on $H_{\mathbb{C}}^{2n}$ (or $H_{\mathbb{H}}^{4n}$) and if the canonical Kähler form of degree 2 (or 4 if we live over $\mathbb{H}$) is exact, then the hyperbolicity of $\overline{\Gamma}$ implies that this form is d(bounded) and so $L_2 H^k(X) = 0$ unless $k = n$ (or $k = 2n$ for X modelled by $H_{\mathbb{H}}^{4n}$). Furthermore, assume that the group $\pi_1(X)$, which equals the kernel of the homomorphism $\Gamma \to \overline{\Gamma}$, is freely normally generated by the fundamental group of a closed totally geodesic submanifold W in the (compact locally symmetric) manifold V presenting Γ as $\pi_1(V)$ (e.g. $\ker = \pi_1(X)$ is normally generated in $\pi_1(V) \supset \pi_1(X)$ by a single closed geodesic in V), such that the restriction of every closed L_2-form on X to each lift of W to X is exact on W. (This is automatic for $\dim W < \frac{1}{2}\dim V$). Then

$$h^k(\overline{\Gamma}) = b^{k-1}(W), \quad \text{for } k < \frac{1}{2}\dim X ,$$

and

$$h^k(\overline{\Gamma}) = (-1)^k \chi(V) + b^{k-1}(W), \quad \text{for } k = \frac{1}{2}\dim X ,$$

where b^{k-1} denote the real Betti numbers of W for $k \geq 2$ with the convention $b^{-1} = b^0 = 0$. (This follows by the L_2-Mayer-Vietoris theorem, see [Ch-Gr]$_3$).

Now, let us concentrate on the (*complex* hyperbolic) case where W is a *totally real* oriented submanifold in V having all connected components of dimension n. Then if n is *even*, the lifts of the components of W

to X are independent in the n-dimensional L_2-homology of X, since each component of W has non-zero selfintersection number (equal to $\chi(W)$). It follows, that $\chi(V) = h^n(\Gamma) \geq b^n(W)$ (= the number ν of the components of W). (This can be improved with the Hodge decomposition in X and/or the L_2-signature theorem). Furthermore,

$$h^n(\overline{\Gamma}) = \chi(V) - \nu + b^{n-1}(W)$$

(where $b^{n-1}(W) = b^1(W)$ by the Poincaré duality. Notice that ν does not appear in the previous formula for $h^{k=n}(\overline{\Gamma})$, since the lift of W to X *there* is assumed L_2-homologous to zero).

Questions. (a) What happens for n odd ?

(b) Let V be a compact locally symmetric space modelled on $H^{2n}_{\mathbb{C}}$, $n \geq 2$, and W a submanifold (*without* double points) modelled on $H^n_{\mathbb{R}} \subset H^{2n}_{\mathbb{C}}$. Is there a universal bound

$$\mathrm{Vol}\, W \leq \mathrm{const}_n \,\mathrm{Vol}\, V \ ?$$

(One knows there is some bound $\mathrm{Vol}\, W \leq \varphi(\mathrm{Vol}\, V)$ which follows from the affirmative solution by Marina Ratner of the Ragunathan-Margulis conjecture).

(3) *Amalgamated products.* A variety of examples with computable $\ell_2 H^*$ is exhibited in [Ch-Gr]$_3$ and [Gr]$_{18}$. For instance, one can realize an arbitrary *constructive* sequence of real numbers β_i, $i = 1, 2, \ldots$, with β_1 and β_2 rational, by $h^i(\Gamma)$ for some finitely presented group Γ. (One does not know if h^1 and/or h^2 are necessarily rational for finitely presented groups Γ).

A general useful relation for $h^1(\Gamma)$, where $\Gamma = \Gamma_1 \underset{\Gamma_2}{*} \Gamma_3$, reads

$$h^1(\Gamma) - h^0(\Gamma) \geq \sum_{i=1}^{3} (-1)^i \left(h^0(\Gamma_i) - h^1(\Gamma_i)\right) .$$

For example, if $\mathrm{card}(\Gamma_1/\Gamma_2) \geq 2$ and $\mathrm{card}(\Gamma_3/\Gamma_2) \geq 3$, then $h^1(\Gamma) \neq 0$. (This can be also seen geometrically by looking at the ends of Γ, see 7.C$_2$.).

8.A$_3$. L$_2$-cohomology of residually finite groups. Let $\Gamma_1 \supset \Gamma_2 \supset \ldots$ be a descending sequence of normal subgroups in Γ of finite index such that $\bigcap_{i=1}^{\infty} = \{\mathrm{id}\}$. Then the ordinary real Betti numbers b^k of Γ_i give a lower bound for $h^k(\Gamma)$ by the following inequality essentially due to Kazhdan (see [Kaz]$_2$, [Ch-Gr]$_{2,3}$)

$$h^k(\Gamma) \geq \limsup_{i \to \infty} b^k(\Gamma_i)/\,\mathrm{card}(\Gamma/\Gamma_i) \ . \tag{$*$}$$

Question. Does the equality hold, $h^k(\Gamma) = \lim b^k(\Gamma_i)/\,\mathrm{card}(\Gamma/\Gamma_i)$?

The inequality $(*)$ becomes more transparent if $\Gamma = \pi_1(V)$ where V is a compact manifold and $\overline{\Gamma}_i = \Gamma/\Gamma_i$ appear as the Galois groups of regular coverings $\widetilde{V}_i \to V$. Then the L_2-cohomology of the limit covering space $\widetilde{V}_\infty$ acted upon by $\overline{\Gamma}_\infty = \Gamma/\bigcap_{i=1}^{\infty} \Gamma_i$ satisfies

$$h^k(\widetilde{V}_\infty : \overline{\Gamma}_\infty) \underset{\mathrm{def}}{=} \dim_{\overline{\Gamma}_\infty} L_2 H^k(\widetilde{V}_\infty) \geq \limsup_{i \to \infty} \dim_{\overline{\Gamma}_i} L_2 H^k(\widetilde{V}_i) = h^k(\widetilde{V}_i : \overline{\Gamma}_i) \,. \tag{$**$}$$

Notice that $(**)$ makes sense and is valid for an arbitrary descending sequence of normal subgroups, possibly of infinite index, and that if $\overline{\Gamma}_i$ is finite then $L_2 H^k\left(\widetilde{V}_i\right) = H^k(\widetilde{V}_i; \mathbb{R})$ and $\dim_{\overline{\Gamma}_i} L_2 H^k(\widetilde{V}_i) = \dim H^k(\widetilde{V}_i)/\,\mathrm{card}\,\overline{\Gamma}_i$. The formula $(**)$ suggests that there is a certain convergence of Hilbert spaces $L_2 H^k(\widetilde{V}_i)$ to $L_2 H^k(\widetilde{V}_\infty)$, which is indeed there as explained in [Ch-Gr]$_{2,3}$.

8. A_4. Several open questions concerning L_2-cohomology. Let Γ discretely and cocompactly act on a contractible polyhedron X. We already know that vanishing or non-vanishing of $h^k(\Gamma) = h^k(X : \Gamma)$ is a quasi-isometry invariant of Γ (and X). On the other hand, h^k itself is not such an invariant. For example, if $\Gamma' \subset \Gamma$ is a subgroup of finite index d then $h^k(\Gamma') = dh^k(\Gamma)$.

Is the ratio $h^k(\Gamma)/h^\ell(\Gamma)$ a quasi-isometry invariant for all k and ℓ ?

There is a closely related question which does not directly involve L_2-cohomology and which applies to the case where X is a contractible manifold and the action is cocompact and free. In this case there are two basic numerical invariants of the manifold $V = X/\Gamma$: *the Euler characteristic* $\chi(V)$ and *the signature* $\sigma(V)$ (which is defined if $\dim V = 4m$ and which has an L_2-meaning on X by the Atiyah L_2-signature theorem).

Is the ratio $\sigma(V)/\chi(V)$ a quasi-isometry invariant of Γ ? Is the sign of $\chi(V)$ (or at least, vanishing versus nonvanishing of $\chi(V)$) a quasi-isometry invariant ? Do the following implications hold

$$||V|| = 0 \Rightarrow \chi(V) = 0 \Rightarrow \sigma(V) = 0 \ ? \qquad (+)$$

(Where $||V||$ denotes the simplicial volume of V, i.e. the simplicial ℓ_1-norm of the fundamental homology class of V, see [Gro]$_8$).

In fact, in all available examples,

$$||V|| = 0 \Rightarrow L_2H^*(X) = 0 \ ,$$

but one does not know if this is a general rule. Also, one may expect a quantitative version of (+), i.e. $|\sigma(V)| \leq \chi(V)$ for 4-dimensional aspherical manifolds V (this would follow from the vanishing of $L_2H^k(X)$ for $k \neq \frac{1}{2}\dim X$) and $|\chi(V)| \leq \mathrm{const}_n ||V||$.

Our final geometric question is related to the old conjecture claiming the vanishing of $\chi(V)$ for compact affine flat manifolds.

Does the L_2-cohomology of the universal covering of a compact affine manifold vanish ?

This is not even known for complete affine manifolds where the vanishing of $\chi(V)$ is proven by Kostant and Sullivan.

Comments on the quasi-isometry questions. The basic examples of non-commensurable quasi-isometric groups are provided by cocompact lattices Γ in a fixed locally compact group G. One knows that every two such lattices, say Γ_1 and Γ_2 in G, have proportional L_2-Betti numbers. In fact, this remains valid if one (or both) of $\Gamma_i, i = 1,\ 2$ is non-cocompact and so Γ_1 may be not even quasi-isometric to Γ_2. Another interesting example due to Gersten of quasi-isometric Γ_1 and Γ_2 is where Γ_1 is a non-trivial central extension of a hyperbolic group and Γ_2 the trivial extension, e.g. Γ_1 equals the fundamental group of the unit tangent bundle of a closed surface S while $\Gamma_2 = \pi_1(S) \times \mathbb{Z}$. However, both Γ_1 and Γ_2 here have zero ℓ_2-Betti numbers and so this is a not a very convincing illustration of L_2-ideas.

Finally, let Γ_m be the free product of the free group on m generators with an infinite group Γ_0, i.e. $\Gamma_m = \mathsf{F}_m * \Gamma_0$. Here $h^1(\Gamma_m) = m + h^1(\Gamma_0)$ and $h^k(\Gamma_m) = h^k(\Gamma_0)$ for $k \geq 2$. What is unknown is whether Γ_m is quasi-isometric to Γ_n for $n > m \geq 2$. (The quasi-isometry between Γ_m and Γ_n would immediately follow if we knew that F_m and F_n were bi-Lipschitz equivalent). One can make up similar "almost counter-examples" with free products amalgamated over finite subgroups of different orders and so it seems unlikely that h^k/h^ℓ is a quasi-isometry invariant without some extra "irreducibility" assumption on the groups in question. Unfortunately, such an assumption may eventually imply that $\Gamma_1$and Γ_2 are cocompact in some ambient G which would make the quasi-isometry conjectures look silly.

Recall that the success of the definition of h^k is due to the possibility to sum (or integrate) Γ-invariant functions on certain Γ-spaces Σ where Γ acts with finitely many orbits (or, at least, with finite covolume). The true question behind the quasi-isometry invariance problem for h^k/h^ℓ is that of classification of possible summations over Σ where Γ-invariance is replaced by a weakest possible condition. Let us give an example to explain what we want to say.

Integration over foliations. To simplify the terminology we switch from Σ to (Riemannian) manifolds X. Then we take a foliation $\mathcal{X}$ whose leaves are Riemannian manifolds and let $\mathcal{X}$ carry an ergodic transversal measure. Now, we can integrate functions over $\mathcal{X}$ and this integration can be thought of (this I learned from A. Connes) as an averaging (or summation) over a generic leaf X of $\mathcal{X}$. In fact, the L_2-cohomology theory was extended by Connes to the foliated framework, which can be used to prove the following invariance of h^k/h^ℓ under *measurable quasi-isometries* between groups.

Definition. We say that Γ_i-spaces X_i, $i = 1,\ 2$ are mutually measurably quasi-isometric if there exist finite measure spaces Y_i with measure preserving Γ_i-actions and a measurable bijection

$$X_1 \times Y_1 \leftrightarrow X_2 \times Y_2\ ,$$

such that

(a) for almost all $y \in Y_1$ the leaf $X_1 \times y$ quasi-isometrically goes to some leaf $X_2 \times y' \in X_2 \times Y_2$ and the resulting correspondence between Y_1 and Y_2 is an isomorphism between these measure spaces.

(b) Almost every Γ_1-orbit in $X_1 \times Y_1$ for the diagonal action of Γ_1 on $X_1 \times Y_2$ goes to a Γ_2-orbit in $X_2 \times Y_2$. (Notice that this is possible only if the actions of Γ_i on Y_i are mutually orbit equivalent.)

Claim. Let $\Gamma_i, i = 1,\ 2$ act isometrically, discretely and cocompactly on contractible manifolds X_i. If X_i is measurably quasi-isometric to X_2 then the groups Γ_1 and Γ_2 have proportional ℓ_2-Betti numbers.

Idea of the proof. - By the above mentioned Connes' L_2-cohomology theory one has well defined L_2-Betti numbers of $(X_i \times Y_i)/\Gamma_i$ which satisfy

$$\mu_1\, h^k(\Gamma_1) = h^k\big((X_1 \times Y_1)/\Gamma_1\big) = h^k\big((X_2 \times Y_2)/\Gamma_2\big) = \mu_2 h^k(\Gamma_2)$$

where μ_i denotes the full measure of Y_i.

Remarks. (a) An equivalent (and more intuitive) form of the above definition consists of requiring the existence of a suitable measure on the space of quasi-isometries $X_1 \leftrightarrow X_2$ invariant under the natural actions of Γ_1 and Γ_2.

(b) The above claim generalizes what we already know for *cocompact* lattices in G but does not quite cover the case of non-cocompact lattices.

Computability and rationality. Let Γ act discretely and cocompactly on a locally finite polyhedron X. Are the ℓ_2-Betti numbers $h^k(X : \Gamma)$ effectively computable in a suitable sense ? Are they necessarily rational (or at least constructive) numbers ? (Rationality is an old question going back to Atiyah). Are there, at least, effective criteria for vanishing, non-vanishing and localizing h^k in a given interval ? Can, for example, the ℓ_2-Betti numbers of a hyperbolic group Γ be effectively computed in terms of a Markov structure on Γ ?

Let us give a more specific form to the computability question. Suppose, for every R, there is a harmonic k-cochain $a = a(R)$ on the R-ball in X around a fixed point which has the value at a central simplex equal one and whose ℓ_2-norm is bounded by a fixed constant C independent of R. Then, as $R \to \infty$, these cochains $a(R)$ subconverge to a non-zero harmonic ℓ_2-cochain on X thus making $h^k \neq 0$. The question is whether the existence of $a(R_0)$ for some large but effectively computable $R_0 = R_0(C)$ implies the existence of $a(R)$ for all R.

Notice that there is a partial effectiveness for bounding h^k. Namely, if we find an R_0, such that every harmonic k-cochain on $B(R_0)$ satisfies $|a(\sigma)| \leq \varepsilon \|a\|_{\ell_2}$ for all simplices σ at (i.e. adjacent to) the center of the ball, then we conclude that $h^k(X : \Gamma) \leq C\varepsilon$ for some explicit constant $C > 0$. (Unfortunately, we do not know, a priori, how large R_0 should be to insure a given ε).

ℓ_2-Betti numbers for generic groups Γ. Suppose Γ is given by $p \geq 2$ generators and q relations. Generically, if the lengths of the relations are sufficiently large, compared to q, this is a hyperbolic (and sometimes small cancellation) group. In particular, there are only two non-zero numbers $h^1(\Gamma)$ and $h^2(\Gamma)$. These satisfy $h^2 - h^1 = q - p + 1$ as well as $h^1 \leq p$ and $h^2 \leq q$. That is all we know. Also it seems not impossible that $h^1 = 0$ if q is significantly bigger than p and, conversely, $h^2 = 0$ if p is significantly bigger than q. (Here one would feel safer by restricting to relations of controlled length). But the intermediate domain looks quite dark. For example, may one have both $h^1(\Gamma)$ and $h^2(\Gamma)$ non-zero for a generic Γ ? Are there generic groups with both h^1 and h^2 zero ? (These should have $p = q + 1$).

To be specific let Γ be given by two generators and one relation. We have already mentioned the problem of finding a single such Γ with $h^1 \neq 0$. On the other hand, the only groups Γ where $h^1 = 0$ is insured are those of the form $\{a, b | a^b = a^i\}$, as they are solvable. The groups $\{a, b | (a^i)^b = a^i\}$ are also likely to have $h^1 = 0$, but for generic Γ we conjecture $h^1 \neq 0$.

We conclude by several non-generic questions about $\overline{\ell_2 H}^1$.

Let $1 \to \Gamma_1 \to \Gamma \to \Gamma_2 \to 1$, where Γ_1 and Γ_2 are infinite groups of finite type (e.g. fundamental groups of finite aspherical polyhedra). Is then $\overline{\ell_2 H}^1(\Gamma) = 0$? (We have stated this problem earlier for $\Gamma_2 = \mathbb{Z}$).

Let Γ' be obtained from Γ by adding a single relation. Is then the ℓ_2-Betti number $h^1(\Gamma')$ bounded from below by $h^1(\Gamma') \geq h^1(\Gamma) - 1$? Let

$$\Gamma = \left\{\gamma_1, \ldots, \gamma_r \mid (r_1)^{d_1} = 1, \ldots (r_q)^{d_s} = 1\right\}$$

where each d_j, $j = 1, \ldots, s$, is of the form $d_j = p^{k_j}$ for a fixed prime p. Is then $h^1(\Gamma) \geq r - 1 - \sum_{j=1}^{s} d_j^{-1}$? (This is so in the generic case).

8.A$_5$. Spectral density near zero and ℓ_2-acyclicity. The reduced ℓ_2-cohomology is a chain homotopy invariant of a complex of ℓ_2-cochains in the category of Hilbert spaces and *bounded* operators but this is not the only invariant. For example, one may have a (short) complex $0 \to \mathbb{R}^\infty \underset{\delta}{\to} \mathbb{R}^\infty \to 0$ where δ has dense image without being onto, say, by mapping each vector $(x_1, x_2 \ldots, x_i, \ldots)$ to $(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_i x_i, \ldots)$ where λ_i are non-zero numbers converging to zero. Here the rate of convergence $\lambda_i \to 0$ serves as a chain homotopy invariant of the above complex. (This "numerical" information may be, probably, encoded in the *non-reduced* cohomology, which is the quotient $\mathbb{R}^\infty / \mathrm{Im}\, d$ in this case, as was suggested by P. Pansu.)

Now, take an arbitrary complex of Hilbert spaces

$$0 \to C^0 \underset{d_0}{\to} C^1 \underset{d_1}{\to} \ldots \underset{d_{n-1}}{\to} C^n \to 0 ,$$

let $\Delta = \Delta_0 \oplus \Delta_1 \oplus \ldots \oplus \Delta_n : C \hookleftarrow$, for $(C, d) = \bigoplus_i (C^i, d_i)$, be the Laplace operator, $\Delta = dd^* + d^*d$ and observe that the following $6\frac{1}{2}$ conditions are equivalent.

(1) C is acyclic, i.e. the non-reduced homology, i.e. $\mathrm{Ker}\, d / \mathrm{Im}\, d$, is zero.

(2) Δ is invertible by a bounded operator.

(3) The spectrum of Δ does not contain zero. Recall that spec Δ, by definition, is the subset in $\mathbb{C}$ consisting of those z for which $\Delta - z$ is *not* invertible by a bounded operator. Thus spec Δ is a *closed* subset and since Δ is a symmetric operator its spectrum lies in $\mathbb{R}$. For example, the spectrum of Δ for the above short complex equals $\{\lambda_1^2, \lambda_2^2, \ldots \lambda_i^2, \ldots, 0\}$.

(4) The complex (C,d) is contractible in the category of Hilbert spaces and bounded operators. This means there exists a bounded (homotopy) operator $h = \bigoplus_i h_i : C \hookleftarrow$ for $h_i : C_i \to C_{i-1}$, such that $hd + dh =$ Id.

(5) There exists a constant $\lambda > 0$, such that every ℓ_2-cochain c satisfies $\|c\|_{\ell_2} \le \lambda(\|dc\|_{\ell_2} + \|d^* c\|_{\ell_2})$ (where d^* denotes throughout this section the adjoint operator).

(5′) There exists a constant λ', such that every ℓ_2-cochain c satisfies $\|c\|_{\ell_2} \le \lambda' \|\Delta c\|_{\ell_2}$.

(6) There exists a constant μ, such that every d-closed cochain c (i.e. $c \in \operatorname{Ker} d$) is of the form db, where $\|b\|_{\ell_2} \le \mu \|c\|_{\ell_2}$.

Definition. A simplicial polyhedron X is called *ℓ_2-acyclic* if the associated complex of ℓ^2-cochains is acyclic, i.e. satisfies (one of the) above conditions (1) - (6).

Question. Do (non-empty) ℓ_2-acyclic polyhedra exist ?

Probably, if we do not insist on $\dim X < \infty$ or on X being uniformly locally finite (i.e. at most k neighbours for each simplex) such ℓ_2-acyclic X may be easily constructible. (Notice that if the i-skeleton of X is not uniformly locally finite, then the operator δ may be unbounded on $(i-1)$-cochains and the above discussion needs to be made more precise). But ℓ_2-acyclic X may exist even with these precautions. In fact, I see no reason why there would not exist a 2-dimensional (uniformly) locally finite ℓ_2-acyclic polyhedron X. Yet 1-dimensional ℓ_2-acyclic polyhedra X do not exist. To see this we observe that for every X of dimension n the ordinary n-dimensional homology *inject* into n-dimensional ℓ_2-homology which is dual to ℓ_2-cohomology. Thus, if X is ℓ_2-acyclic, then $H_n(X;\mathbb{R}) = 0$. In particular, if $n = 1$, then X must be a *tree*. Now, if Δ_0 is invertible (on $\ell_2 C^0$), i.e. if $\varphi \le \lambda \Delta_0 \varphi$ for all 0-chains, then X satisfies the (top dimensional) linear isoperimetric inequality for bounded subsets in X and therefore, there exists a number a, such that for every segment $[x_1, x_2]$ in X of length $\ge a$ there is an *infinite* sub-subtree in X growing from an interior point x in $[x_1, x_2]$ which meets $[x_1, x_2]$ at no point besides x as in Fig. 21.

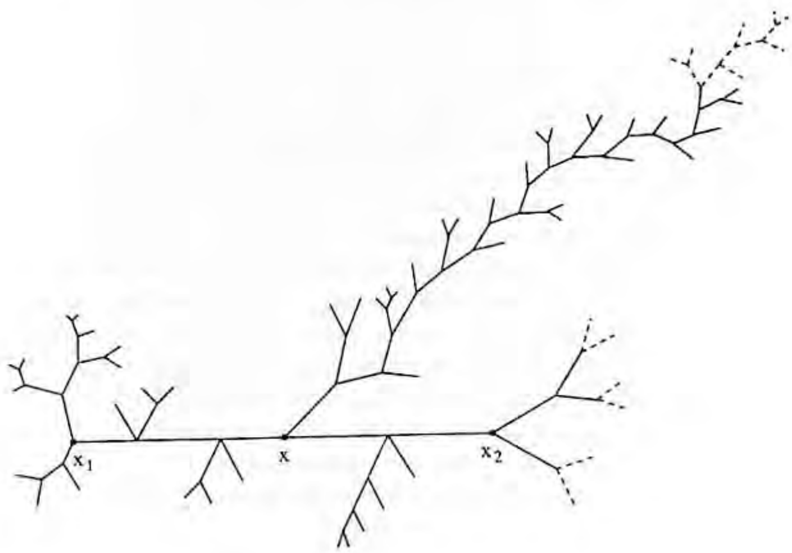

Figure 21

It follows, X contains a quasi-isometric copy of an infinite regular triadic tree Tree_3 which, as we know, supports a non-zero ℓ_2-cycle. This makes $\overline{\ell_2 H}^1$ non-zero. (Apparently the implication

$$\Delta_0 \text{ is invertible} \Rightarrow X \text{ contains a quasi-isometric copy of } \mathrm{Tree}_3$$

is valid for all polyhedra X of dimension ≥ 1).

The ℓ_2-acyclicity question seems especially interesting in the following two cases.

(A) The polyhedron X is *uniformly contractible* (i.e. there exists a function $R(r)$ such that every r-ball in X is contractible inside the concentric $R(r)$-ball).

(B) The polyhedron X admits a discrete cocompact action of a group Γ by simplicial automorphisms of X.

In fact, in the situation $(A)\cap(B)$, the ℓ_2-acyclicity becomes a property of Γ independent of a particular contractible polyhedron where Γ acts.

L_2-acyclicity. If X is a smooth Riemannian manifold one has the Laplace-de Rham operator Δ on the space of differential forms, denoted $L_2\Omega^* = \overset{\dim X}{\underset{i=0}{\oplus}} L_2\Omega^i$, and then X is called *L_2-acyclic* if the spectrum of Δ contains no zero. If X has uniformly locally bounded geometry, it can be triangulated into simplices uniformly bi-Lipschitz equivalent to standard unit simplices $\Delta^i \subset \mathbb{R}^i$, $i = 0, 1, \ldots, n$, and then, as we already know, the complexes $(L_2\Omega^*, d_{\text{exterior}})$ and $(\ell_2 C^*, d_{\text{combinatorial}})$ are naturally chain homotopy equivalent in the category of Hilbert spaces and bounded operators. (Notice, the exterior differential d as well as the Laplace-de Rham operator $dd^* + d^*d$ are unbounded but this causes no problem). Thus

$$L_2\text{- acyclicity} \quad = \quad \ell_2\text{- acyclicity} .$$

Conjecture. Let X be a contractible Riemannian manifold without boundary which admits a cocompact isometric action of a group Γ. Then X is *not* L_2-acyclic.

Motivation. It is explained in $[\text{Gro}]_{13}$ that both properties of X, contractibility + the existence of a cocompact Γ-action and the L_2-acyclicity indicate that X is *large* in a certain unspecified geometric sense. For example, many (if not all) known contractible Γ-manifolds X are *hyper-Euclidean* which means the existence of a proper Lipschitz map $X \to \mathbb{R}^n$, $n = \dim X$. (This may be true for all contractible Γ-manifolds). Then it was pointed out in $[\text{Gro}]_{13}$ that hyper-Euclidean manifolds *cannot* be L_2-acyclic for the reason which we explain here for n *even*. In this case $\mathbb{R}^n$ supports a non-trivial element of K^0 with compact support represented by some complex vector bundle F on $\mathbb{R}^n$ with a fixed trivialization at infinity. We equip this bundle with a Riemannian connection and then we pull back F to X by the composition φ_ε of a given Lipschitz map $\varphi : X \to \mathbb{R}^n$ and a scaling of $\mathbb{R}^n$ given by $y \mapsto \varepsilon y$ for a small $\varepsilon > 0$. We assume without loss of generality that φ is smooth and then the induced bundle $\varphi_\varepsilon^*(F)$ on X has ε-flat connection. Now comes the key point: the introduction of $\varphi_\varepsilon^*(F)$ brings along non-trivial topology which forces the signature operator on X twisted with $\varphi_\varepsilon^*(F)$ (or some multiple of this bundle) become *non-zero* (see [Gr-Law] where this is done for the generalized Dirac operator). On the other hand, as is pointed out in [Va-Wi], the twisted signature operator (Vafa and Witten work in [Va-Wi] with the Dirac on a compact manifold but their argument is quite general) can be viewed as an ε-perturbation of the ordinary signature operator whose spectrum is bounded from below by a positive λ if X is L_2-acyclic. Then the ε-perturbation, for ε small compared to λ, also has spectrum away from zero which is incompatible with the non-vanishing of the index. (See $[\text{Roe}]_{1-4}$, $[\text{Hur}]_{1,2}$, $[\text{Gro}]_{20}$ and $[\text{C-G-M}]_{1,2}$ for more about this kind of reasoning).

Remarks. (a) *Homologization.* The contractibility of X is not indispensable. The important thing is non-vanishing of the natural homomorphism $H_n(X/\Gamma) \to H_n(\Gamma; \mathbb{R}), n = \dim X$ (where the action is assumed orientation-preserving). In fact, starting from this point one can develop a cohomology theory based on hyper-Euclidean ideas (see $[\text{C-G-M}]_2$).

(b) *Lipschitz families.* There is a variety of generalizations of the "hyper-Euclidean" condition which amount to bringing-in families of (Lipschitz) maps $X \to \mathbb{R}^N$, $N > \dim X$. For example, one has a suitable (for Δ to be non-invertible) family whenever X admits a uniform Lipschitz embedding into some simply connected manifold Y of non-positive curvature (compare [Gr-Law], $[\text{C-G-M}]_2$).

(c) *Blow-up of boundaries.* If X is a manifold with boundary there is a natural (blown up) manifold $X^+ = X \cup \partial X \times [0,1]$ for ∂X identified with $\partial X \times 0$ and where $\partial X \times [0,\infty)$ is given the metric $e^t g_0 + dt^2$ for the original metric g_0 in ∂X. Now the notion of L_2-acyclicity may be applied to X^+. For example, if V_0 is an aspherical polyhedron, it can be first thickened to a manifold V with boundary and then the +-construction applies to the universal covering X of V. In this case the L_2-acyclicity of X^+ is, in fact, a property of the fundamental group $\Gamma = \pi_1(V_0)$, as a simple argument shows. (It is sometimes useful to try more sophisticated "blow-ups". For example, given X without boundary, take first a suitable complexification Y of X with pseudoconvex boundary, and then take (Y, Bergman metric) for Y^+. The Kähler geometry of such Y^+ may provide you with non-trivial spectral information, see [Gro]$_{20}$).

(d) *Dirac and Dolbeault.* The notion of L_2-acyclicity generalizes to other natural elliptic complexes, such as the Dolbeault complex and the Dirac operator. The Dolbeault complex very rarely is L_2-acyclic (see [Gro]$_{20}$) but Dirac may have spectrum away from zero. In fact, this is the case for manifolds X of strictly positive scalar curvature (see [Gr-Law] for all about it).

(e) *Lip-deg order.* The above discussion suggests the following order relation between complete oriented Riemannian manifolds without boundary of a fixed dimension n, $X \underset{\text{Lip}}{\geq} Y \Leftrightarrow \exists$ a proper Lipschitz map $X \to Y$ of degree 1. More generally, we write $X \underset{\text{Lip}}{\geq} dY$ if there is such a map of degree d. This relation is hopelessly complicated in the category of all Riemannian manifolds but it may be reasonable for the universal coverings of compact aspherical manifolds, where it can be attributed to the fundamental groups of these manifolds. The first unsolved problem is to decide whether every such universal covering X satisfies $X \underset{\text{Lip}}{\geq} \mathbb{R}^n$ (as was already stated in the hyper-Euclidean language).

(e′) Δ^k*-order.* Generalize the above by writing $X \underset{\wedge^k}{\geq} Y$ if there exists a smooth proper map $\varphi : X \to Y$ of degree one such that $\| \wedge^k \varphi\|(x) \leq \text{const} < \infty$ for all $x \in X$, where $\wedge^k\varphi : \wedge^k T(X) \to \wedge^k T(Y)$ denotes the k-th exterior power of the differential of φ. (Notice that the relation $X \underset{\wedge^1}{\geq} Y$ is equivalent to $X \underset{\text{Lip}}{\geq} Y$). This $\wedge^k$-comparison (as in the case Lip=$\wedge^1$) is especially interesting for the infinite (e.g. universal) coverings of closed (e.g. aspherical) manifolds though a purely group theoretic reformulation seems harder for $k \geq 2$. (Notice that the case $k = n = \dim X$ reduces to a comparison of the volumes of (the ends of) X and Y and does not appear useful from our point of view). The most prominent among the $\wedge^k$-relations is the one with $k = 2$ as $\wedge^2$-contracting maps allow pull-backs of *almost flat vector bundles.* (See [Gr-Law] where the emphasis is laid upon $\wedge^2$-contracting maps $X \to \mathbb{R}^n$ and the existence of these is called $\wedge^2$-*enlargability* of X).

(f) There is another purely topological order on the set of all closed oriented manifolds of a fixed dimension n, where $V \geq W$ signifies the existence of a continuous map $V \to W$ of degree one and $V \geq dW$ refers to a map of degree d. This order is detected by the simplicial volume $\|V\|$ as $V \geq dW \Rightarrow \|V\| \geq |d|\|W\|$ and the question arises to classify all such monotone invariants of manifolds. This appears impossibly difficult unless one imposes some extra conditions on such invariants similar to those satisfied by the simplicial volume. Recall that an essential property of the simplicial volume is a certain continuity which is best expressed by extending (following A. Connes) $\|V\|$ from closed manifolds V to *foliations* with transversal measures. Such foliations generalize groups (such as $\pi_1(V)$), where actual groups correspond to closed leaves, (and if closed leaves are dense then all leaves appear in the closure of the space of groups, while the foliation itself, as an object of a measure theoretic thickening of the category of groups, may be also thought of in this case as a limit of groups). Then one tries to determine the space of all "continuous" monotone maps from the "spaces of manifolds" to $\mathbb{R}_+$ (where one may postulate or derive from continuity other properties of such invariants mimicing those of $V \mapsto \|V\|$ such as the dependence on the group $\Gamma = \pi_1(V)$ only).

8.A$_6$. Von Neumann spectral measure near zero. Let X be, as earlier, a uniformly locally finite polyhedron and recall that by the spectral theorem the Laplace operator Δ on i-cochains is canonically isomorphic to the multiplication operator by t on $[0,\infty)$ on the L_2-space of function $f(t)$, $t \in [0,\infty)$ for some measure $\mu = \mu_\Delta$ on $[0,\infty)$. We denote by $H^k_\lambda \subset \ell_2 C^k$ the subspace of i-cochains in ℓ^2 corresponding to the functions $f(t)$ vanishing outside the segment $[0,\lambda]$ and let P^k_λ be the normal (spectral) projection of

$\ell_2 C^k$ to H^k_λ. Now we assume there is a group Γ cocompactly acting on X, define the number h^k_λ by

$$h^k_\lambda = \mathrm{trace}_\Gamma P^k_\lambda (= \dim_\Gamma H^k_\lambda) .$$

This number, for a fixed λ, does depend on a particular triangulation of X but the asymptotic behaviour of h^k_λ for $\lambda \to 0$ is a homotopy invariant of the Γ-space X. (See [Nov-Sh]$_{1,2}$, [Gr-Sh]$_{1,2}$, [Efr]). In fact, for many interesting examples $h^k_\lambda \sim \lambda^\alpha$ and then the asymptotics is expressed by the number α which carries a non-trivial homotopy theoretic information about (X, Γ).

Remarks. (a) Notice that h^k_0 equals the ℓ_2-Betti number $h^k(X : \Gamma)$. When this is non-zero, the new homotopy theoretic information lies in the asymptotics of $h^k_\lambda - h_0$ as this approaches zero.

(b) Unlike the ℓ_2-Betti numbers the asymptotics of h^k_λ for $\lambda \to 0$ may be quasi-isometry invariant for *contractible* polyhedra X. In fact, one can slightly change the definition of h^k_λ which makes it "Γ-free" as follows. The operator P^k_λ is given by a certain kernel $\sqcap^k_\lambda(\sigma, \sigma')$ (compare to the definition of h^k in 8.A.) and trace Γ is obtained by summing $\sqcap^k_\lambda(\sigma, \sigma)$ over Σ/Γ. Now, we forget about Γ and define

$$\check{h}^k_\lambda = \sup_\sigma \sqcap^k_\lambda(\sigma, \sigma) ,$$

for σ running over all k-simplices σ in X. Since $\sqcap^k_\lambda(\sigma, \sigma) \geq 0$ (as Δ is a positive operator),

$$h^k_\lambda \geq \check{h}^k_\lambda \geq \text{const } h^k_\lambda ,$$

where const equals 1/(number of Γ-orbits of k-simplices in X). Thus, we do not change the asymptotics by switching from h^k_λ to $\check{h}^k_\lambda$. (If we work with a Riemannian manifold, then $\check{h}^k_\lambda$ should be defined by averaging $\sqcap^k_\lambda(x, x)$ over a ball in X). Now, suppose we have a Lipschitz homotopy equivalence between X and Y which is given by a pair of uniformly proper simplicial maps $f : X \to Y$ and $g : Y \to X$, such that both composed maps fg and gf are *Lipschitz homotopic* to the identities $X \hookleftarrow$ and $Y \hookleftarrow$ according to the following definition.

Two Lipschitz (e.g. simplicial) maps $A \to B$ are *Lipschitz homotopic* if there exists a homotopy by a *Lipschitz* map $A \times [0, 1] \to B$ for the obvious metric in the product $A \times [0, 1]$. (Notice, that every quasi-isometry between uniformly contractible spaces with uniformly bounded local geometries can be modified into a Lipschitz homotopy equivalence). Such a homotopy equivalence induces a very special chain homotopy equivalence between the ℓ_2-cochain complexes of X and Y as every Lipschitz homotopy induces a chain homotopy in the algebra of operators induced by selfhomotopies of the pertinent spaces moving all points by uniformly bounded distance. Then it seems to follow (I did not check it 100 %) that the numbers $\check{h}^k_\lambda(X)$ and $\check{h}^k_\mu(Y)$ are related by

$$\check{h}^k_\lambda(X) \leq C\, \check{h}^k_\lambda(Y) \tag{$*$}$$

for $\lambda' = C_\lambda$, where C is a constant depending on the Lipschitz constants of our maps and homotopies, and conversely

$$\check{h}^k_\lambda(Y) \leq C \check{h}^k_{\lambda'}(X) . \tag{$**$}$$

(The inequalities ($*$) and ($**$) express exactly what we mean under the equivalence of the asymptotics of $h^k_\lambda(X)$ and $h^k_\lambda(Y)$ for $\lambda \to 0$).

Warning. The above ($*$) and ($**$) do not hold true unless X and Y are quasi-periodic in a suitable sense. (Having cocompact simplicial isometry groups is more than sufficient for the needed quasi-periodicity. On the other hand, one can refine the definition of $\check{h}^k_\lambda$ which would take care of non-quasiperiodic spaces).

ℓ_p-Variation. Let us indicate another possibility of varying $h^k = h^k_{\ell_2}$. This time we suppose $h^k = 0$ and we want to define some kind of ℓ_p-Betti number for $p \to 2$. Here are several options.

(1) For each k-dimensional simplex σ let c_σ denote the unit cochain supported on σ (i.e. $c(\sigma) = 1$ and $c(\sigma') = 0$ for $\sigma' \neq \pm\sigma$) and let $\delta(\sigma)$ and $D(\sigma)$ denote the ℓ_p distances from c_σ to the subspaces $\operatorname{Ker} d_k$ and $\operatorname{Im} d_{k-1}$ respectively. Notice that $D(\sigma) \geq \delta(\sigma)$ as $\operatorname{Ker} d_k \supset \operatorname{Im} d_{k-1}$ and that these distances do not depend

on the orientation of σ. Then we set $\mathrm{tr}(\sigma) = D^2(\sigma) - \delta^2(\sigma)$ and define $h^k_{\ell_p}$ as in 8.A. by summing up over the Γ-orbits $\overline{\sigma}$ of k-simplices σ, $h^k_{\ell_p} = \sum_{\overline{\sigma}\in\overline{S}_k} (\mathrm{Ind}\,\Gamma_{\overline{\sigma}})^{-1}\,\mathrm{tr}(\overline{\sigma})$. Alternatively, we could take

$$\check{h}^k_{\ell_p} = \sup_{\sigma\in S_k} \mathrm{tr}(\sigma) \ .$$

(2) Consider all p-harmonic cochains c on X and set

$$\widetilde{h}^k_{\ell_p} = \sup_c \|c\|_{\ell_\infty} \;/\; \|c\|_{\ell_p} \ .$$

It is clear $\widetilde{h}^k_{\ell_p} \to 0$ for $p \to 2$ (as we assume $h^2_{\ell_2} = 0$) and this also seems plausible (and easy to show) for $h^k_{\ell_p}$ and $\check{h}^k_{\ell_p}$. Furthermore, the asymptotic behavior of these three numbers for $p \to 2$, probably is a Γ-homotopy invariant and even a quasi-isometry invariant under favourable circumstances.

Remarks. (a) The above "ℓ_p-Betti numbers" represent just an attempt to work out a good definition. It is unlikely that these are full fledged ℓ_p-Betti numbers for all p, but these may exist for p infinitely close to 2.

(b) There is another approach to the ℓ_p-spectral density near zero via non-linear spectra discussed in $[\mathrm{Gro}]_{17}$. For example, given a tower of finite s_i-sheeted (Galois) coverings $\widetilde{V}_i$ of a fixed compact polyhedron V one may define, for each $\lambda > 0$, the "ℓ-spectral dimension below λ", denoted $\dim_p(V_i; k, \lambda)$ in each degree k. (This can be done in a variety of ways using the function $\varphi \mapsto \|d_k\varphi\|_{\ell_p} \;/\; \|\varphi\|_{\ell_p}$ on the projectivized space $\Phi = \ell_p C^k \;/\; \mathrm{Im}\, d_{k-1}$, see $[\mathrm{Gro}]_{17}$). Then the asymptotics of $\dim_p(\widetilde{V}_i; k, \lambda)$ for $i \to \infty$ and $\lambda \to 0$ (relative to s_i) carries interesting (?) information on $\widetilde{V}_i$, $i \to \infty$, and on the infinite covering $X = \lim_{i\to\infty} \widetilde{V}_i$ of V. (In the case $p = 2$ this boils down to h^k_λ of X.) It is less clear, however, how one should define the non-linear spectral dimensions in the infinite dimensional framework where there is no preferred approximation of X by finite polyhedra.

8.B. More on vanishing and non-vanishing of $\mathbf{L_p}$-cohomology. We continue the discussion on vanishing and non-vanishing of L_p-cohomology with a closer look at maps with p-summable gradients (compare (e) at the beginning of §8). We denote, as earlier, by $\| \wedge^k_\varphi \|(x)$ for a C^1-map $\varphi : X \to Y$, the norm of k-th exterior power of the differential $d\varphi$ of φ on $T_x(X)$ (thus, $\| \wedge^1 \varphi\| = \|d\varphi\|$) and we wish to understand the overall effect on φ of the condition $\int_X \|\wedge^k\varphi\|^p (x)dx < \infty$. For example, we ask when there is such a map φ of X on $\mathbb{R}^n$, $n = \dim X$, which is proper and has non-zero degree or a map into $\mathbb{R}^n$ with summable Jacobian, i.e. $\int_X \|\wedge^n\varphi\|(x)dx < \infty$ and with *geometric degree* $\int_X \wedge^n\varphi(x)dx \neq 0$. We also want to study maps φ from X to $\mathbb{R}^N$ (where $N \neq n$ and where we are particularly interested in *embeddings* $X \to \mathbb{R}^N$ for $N \geq n$) and the size of their limit sets. More generally, we may require $\int \|\wedge^{k_i}\varphi\|^{p_i} < \infty$ for several values $0 < k_1 < k_2 < \dots k_n < n = \dim X$ and study the space of φ's satisfying these inequalities.

Examples. (a) Let the exponential map $\exp : T_{x_0}(X) \to X$ be bijective. Then the inverse map

$$\varphi = (\exp)^{-1} : X \to T_{x_0}(X) = \mathbb{R}^n$$

is a homeomorphism and if the curvature of X is pinched between $-(1+\varepsilon)^2$ and -1, then, clearly,

$$\int \|\wedge^k\varphi\|^p < \infty, \quad \text{for } p(k-1) > (h-1)(1+\varepsilon) \ .$$

(a′) Suppose $K(X) \leq 0$, let $-R^2$ denote the lower bound on the Ricci curvature and $-R^2_k$ the upper bound of the Ricci curvatures on k-dimensional subspaces $T \subset T_x(X)$ (see 7.C$_1$.). Then

$$\int \|\wedge^k\varphi\|^p < \infty \quad \text{for } pR_d > R \ .$$

For example, let X be the Cartesian product of two spaces of constant curvature, $X = H_{\mathbb{R}}^{n_1} \times H_{\mathbb{R}}^{n_2}$. Then $R_k > 0$ for $n - k < m = \min(n_1, n_2) - 1$ (where $n_1 + n_2 = n = \dim X$). In fact, constant curvature -1 makes $R_k = m - (n-k)$ for $n - k \leq m$, while $R = n_1 + n_2 - 2$.

(b) Look at the normal projection φ of X onto a unit ball in $B \subset X$ (instead of $\exp^{-1}$). Here

$$\int \|\wedge^k\|^p < \infty \text{ if } pk \geq (n-1)(1+\varepsilon) \text{ or if } pR_{k+1} > R ,$$

and now φ has summable Jacobian and non-zero geometric degree.

(b′) Let X be a symmetric space of rank k_0 or the Cartesian product of k_0 manifolds with $K \leq -1$. Then we modify the above φ to another *monotone radial* map, say $\varphi' : X \to B$ sending each geodesic ray $r \subset X$ issuing from the center $x_0 \in B$, into itself such that $\varphi|r$ is a monotone map (for $r = \mathbb{R}_+$) which is constant outside a small conical neighbourhood $X_0 \subset X$ of a *regular* ray $r_0 \subset X$ and φ' equals φ in a smaller conical neighbourhood $X_1 \subset X_0$. In fact, such a modification can be achieved by composing φ with an appropriate smooth radial map $B \to B$. This φ' still has summable Jacobian and positive geometric degree. What we gain, is p-summability of $\| \wedge^k \varphi'\|$ for all $k \geq k_0 = \operatorname{rank} X$, and sufficiently large $p = p(X,k)$. For example, if $X = H_{\mathbb{R}}^{n_1} \times H_{\mathbb{R}}^{n_2}$, then

$$\int \|\wedge^k \varphi'\|^p < \infty \quad \text{for } \frac{(k-1)p}{\sqrt{2}} > n_1 + n_2 - 2 ,$$

where we take a *diagonal* ray for r_0 (i.e. whose projections to both factors are exactly $(1/\sqrt{2})$-Lipschitz) and where the size of $X_0 \supset r_0$ may depend on p (i.e. closer p is to $(n_1 + n_2 - 2)\frac{\sqrt{2}}{d}$, smaller X_0 should be).

Remark. We have seen at the beginning of §8 that maps $\varphi : X \to \mathbb{R}^n$ may provide non-trivial reduced L_p-cohomology. Namely, if φ has summable Jacobian, non-zero geometric degree and

$$\int \|\wedge^k \varphi\|^p < \infty \quad \text{and} \int \|\wedge^{n-k}\|^q < \infty$$

for $n = \dim X$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\overline{L_pH}^k(X) \neq 0$ and also $\overline{L_qH}^{n-k}(X) \neq 0$. For example, if X is $(1+\varepsilon)^2$-pinched, then we see with (b) that $\overline{L_pH}^k \neq 0$ provided $\varepsilon < \frac{1}{n-1}$ and $\frac{k}{(n-1)(1+\varepsilon)} > \frac{1}{p} > 1 - \frac{n-k}{(n-1)(1+\varepsilon)}$. But for $\varepsilon > \frac{1}{n-1}$ the above discussion does not yield any non-vanishing information about L_p-cohomology, though it delivers pairs of closed forms, say ω of degree k and ω' of degree $n-k$, such that $\omega \in L_p$, $\omega' \in L_q$ where $\omega \wedge \omega' \in L_1$ and $\int_X \omega \wedge \omega' \neq 0$, but the equality $\frac{1}{p} + \frac{1}{q} = 1$ (or even the inequality $\frac{1}{p} + \frac{1}{q} \leq 1$) may be not valid anymore. The existence of such pairs (ω, ω') for given k, p and q, is a quasi-isometry invariant property of X which, probably, can be adequately expressed in terms of (non-vanishing of) the $L_{p,p'}$-cohomology (for $p' = 1 - \frac{1}{q}$) introduced in [Pan]$_6$. On the other hand one may believe that for every positive $k < \dim X$ there is some $p = p(X,d)$, such that $\overline{L_pH}^k(X) \neq 0$, where X is a complete simply connected manifold with pinched negative curvature. Furthermore, for certain k (most likely $k = n/2$ or $k = \frac{n \pm 1}{2}$) the number p can be, probably, taken arbitrarily close to 2. (For n even one may even conjecture the non-vanishing $\overline{L_2}H^{n/2}(X)$).

What is known in this regard is the non-vanishing of $\overline{L_pH}^1(X)$ for sufficiently large p (see [Pan]$_6$ and (d) at the beginning of §8). This implies, by the Poincaré duality, non-vanishing of $\overline{L_qH}^{n-1}(X)$ for $q = 1 + \delta$ for sufficiently small $\delta > 0$.

Remark on Poincaré duality. Exterior multiplication of forms establishes a duality between $\overline{L_pH}^k(X)$ and $\overline{L_qH}^{n-k}(X)$ for $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p$, $q < \infty$ on every smooth manifold X and this yields the Poincaré isomorphism

$$\overline{L_pH}_k(X) \rightleftarrows \overline{L_qH}^{n-k}(X) .$$

In fact, the usual proof of the Poincaré duality on (polyhedral) homology manifolds applies to non-reduced (co)homology and shows that

$$\ell_p H_k(X) \approx \ell_p H^{n-k}(X) .$$

Bounds on maps with p-summable gradients. We start with a simple bound on the $(n-1)$-volume, $n = \dim X$, of the limit set of $\varphi : X \to Y$, denoted $\inf \mathrm{Vol}_{n-1} \partial\varphi(X)$ which by definition is the infimal number V, such that for every increasing system of compact domains $X_1 \subset X_2 \subset \ldots \subset X$ exhausting X we have $\liminf_{i\to\infty} \mathrm{Vol}_{n-1} \varphi(\partial X_i) \geq V$.

Observation. *If X is a geodesically complete manifold and $\varphi : X \to Y$ has $\int \|\wedge^{n-1}\varphi\| < \infty$, then $\inf \mathrm{Vol}_{n-1} \partial\varphi(X) = 0$. Consequently, if $\int \|\wedge^k\varphi\|^p < \infty$ for some $k \leq n-1$ and $p = \frac{n-1}{k}$, then again $\inf \mathrm{Vol}_{n-1} \partial\varphi(X) = 0$. Furthermore, if X has uniformly bounded local geometry then the latter remains valid with $p \geq \frac{n-1}{k}$.*

The proof is immediate with concentric balls in X taken for X_i.

Remark. If $\dim Y = \dim X = n$ and φ has summable Jacobian and $\inf \mathrm{Vol}_{n-1} \partial\varphi(X) = 0$, then φ has zero geometric degree, i.e. for every bounded n-form ω on Y, the pull-back $\varphi^*(\omega)$ on X satisfies $\int_X \varphi^*(\omega) = 0$. In general, if $\int \|\wedge^k\varphi\|^p < \infty$ then bounded k-forms on Y go to p-summable forms on X and so one can define the subspace, in $\overline{L_pH}^k(X)$ coming from the pull-backs of closed bounded forms in Y. We denote this by $\overline{\varphi}^*_{L_p}(L_\infty H^k(Y)) \subset \overline{L_pH}^k(X)$, though the homomorphism $\varphi^* : L_\infty H^d(Y) \to L_pH^k(X)$ is not defined, unless $\int \|\wedge^{k-1}\varphi\|^p < \infty$ as well as $\int \|\wedge^k\varphi\|^p < \infty$.

Example. Let $\varphi : X \to \mathbb{R}^k$ have $\int_X (\|\wedge^{k-1}\varphi\|^p + \|\wedge^k\varphi\|^p)dx < \infty$. Then

$$\overline{\varphi}^*_{L_p}\left(L_\infty H^k\left(\mathbb{R}^k\right)\right) = 0 .$$

Proof. Let φ_i be a standard cut-off function with the support equal to the i-ball $B(i) \subset \mathbb{R}^k$ and $\omega_i = \varphi_i\omega$ for a given bounded form ω on $\mathbb{R}^k$. Then $\varphi^*(\omega_i) \underset{L_p}{\to} \varphi^*(\omega)$ while each ω_i is cohomologous to zero in $L_\infty H^k(\mathbb{R}^k)$. Q.E.D.

This example is closely related to the following

Vanishing of the cup-product. Observe that the exterior product of forms (as well as v-product of cohains) induces the product on cohomology $\overline{L_{p_1}H}^{k_1} \otimes \overline{L_{p_2}H}^{k_2} \xrightarrow{\vee} \overline{L_pH}^k$ for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and if X has uniformly bounded geometry (or is a uniformly locally finite polyhedron) then the product is defined whenever $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$.

Observation. *If the ordinary cohomology $H^1(X;\mathbb{R})$ vanishes, then the product pairing*

$$\overline{L_pH}^1 \otimes \overline{L_pH}^k \xrightarrow{\vee} \overline{L_pH}^{k+1}$$

vanishes for all k and $p < \infty$.

Proof (compare [Gro]$_{20}$). Every exact ℓ_p-form ω_1 of degree 1 can be ℓ_p-approximated by the differentials df_i where f_i are *bounded* functions on X, since the (universal) form dt on $\mathbb{R}$ admits such approximation and ω_1, being exact, is induced from dt by a function $f : X \to \mathbb{R}$. Then $\omega_1 \wedge \omega_2$ is approximated by the differentials of L_p-forms, namely by $d(f_i\omega_2)$. Q.E.D.

Remark. The above argument yields the vanishing of the product on $\overline{L_pH}^{k_1} \times \overline{L_pH}^{k_2}$, whenever every closed p-form ω_1 can be approximated by $d\lambda_i$ for *bounded* (k_1-1)-form λ_i. This is the case, for example, for hyperbolic spaces X and all k_1. Furthermore, if X is semihyperbolic with (a suitable) rank k_0 (e.g. a symmetric space of rank k_0 or the Cartesian product of k_0 hyperbolic spaces) then every closed form ω_1 of degree $\geq k_0+1$ is d(bounded).

Question. Does the v-product always vanish on L_p-cohomology for $1<p<\infty$? This is not even known for $p=2$ and X being a finitely presented group.

Von Neumann dimension for $p \neq 2$. As we have lamented earlier there is no (known) definition of ℓ_p-Betti numbers for $p \neq 2$, but the ℓ_p-structure may be used via the cup-product. For example, one can take the Von Neumann dimension of the kernel of the product pairing $L_2H^* \otimes L_2H^* \to L_4H^*$ as well as the dimensions of the images of the pairings $L_{p_1}H^* \otimes L_{p_2}H^* \to L_2H^*$ for given p_1 and p_2 satisfying $\frac{1}{p_1}+\frac{1}{p_2} \geq \frac{1}{2}$.

Question. Let an infinite covering X of a compact space V be approximated by finite coverings $\widetilde{V}_i \to V$ (compare 8.A$_3$.). Is there any connection between the above invariants and the asymptotics of the ordinary $\vee$-product in $\widetilde{V}_i$ for $i \to \infty$?

8.C. Some foundational problems concerning ℓ_p-cohomology. The ℓ_p-cohomology is designed to measure non-acyclicity of the complex of ℓ_p-cochains and we start by indicating several notions of acyclicity.

(1) *ℓ_p-acyclicity.* This means that there exists an ℓ_p-bounded operator h (chain contraction) of our ℓ_p-complex $\{\ell_pC^i, d\}$ of degree -1, i.e. $h: C^i \to C^{i-1}$, such that

$$dh + hd = \mathrm{Id} \ .$$

If this relation holds for $i=0,1\dots,k$, we speak of acyclicity in *degrees* $0,\dots,k$.

(2) *Weak ℓ_p-acyclicity.* This means, there exists a family of ℓ_p-bounded operators h_t of degree -1, such that $(h_td+dh_t)(c)$ ℓ_p-converges to c as $t\to\infty$ for every ℓ_p-cochain c.

(3) *Vanishing of the non-reduced cohomology* $\ell_pH^* = \operatorname{Ker} d/\operatorname{Im} d$.

(4) *Vanishing of the reduced cohomology* $\overline{\ell_pH}^* = \operatorname{Ker} d/\overline{\operatorname{Im} d}$ (where $\overline{\operatorname{Im} d}$ denotes the ℓ_p-closure of $\operatorname{Im} d$).

It is obvious that

$$\begin{array}{ccc} (1) & \Rightarrow & (2) \\ \Downarrow & & \Downarrow \\ (3) & \Rightarrow & (4) \end{array}$$

but it is unclear when these implications can be reversed.

In order to have (3) $\Rightarrow$ (1) we need the subspaces $\operatorname{Im} d_i \subset \ell_pC^{i+1}$ to be not only closed but also to admit complementary subspaces, $\operatorname{Im}_i^\perp \subset \ell_pC^{i+1}$. In fact, h_i can then be constructed as the composition of the projection to $\operatorname{Im} d_i$ (along $\operatorname{Im}_i^\perp$) with d_i^{-1} (from $\operatorname{Im} d_i$ to $\operatorname{Im}_{i-1}^\perp$).

Proposition. *Suppose the images $d_0(\ell_pC^0) \subset \ell_pC^0$ and $d_0(\ell_qC^0) \subset \ell_qC^0$ are closed for some p, $q<\infty$ satisfying $\frac{1}{p}+\frac{1}{q}=1$. Then $\operatorname{Im} d_0 = d_0(\ell_pC^0)$ admits a complement in ℓ_pC^1.*

Proof. Let $\partial = d^*$ be the boundary operator on chains which is the adjoint of d and let us identify chains with cochains in the obvious way. We claim that $\operatorname{Ker}\partial \subset \ell_pC^1$ is a complementary subspace to $\operatorname{Im} d_0 = d_0(\ell_pC^0)$.

Lemma. *If X is an infinite connected polyhedron then every harmonic 0-cochain in ℓ_p, $p<\infty$, vanishes, where "harmonic" refers to the equation*

$$\Delta\varphi \underset{\text{def}}{=} \partial d\varphi = 0, \ \varphi \in \ell_pC^0 \ .$$

Proof. Since $\varphi\in\ell_p$, it goes to zero at infinity and the maximum principle applies.

Now, this lemma shows that $\operatorname{Ker}\partial \cap \operatorname{Im} d_0 = 0$ in $\ell_p C^1$ and in $\ell_q C^1$ which implies (by the duality between $\ell_p C^*$ and $\ell_q C_* = \ell_q C^*$) that $\operatorname{Ker}\partial$ and $\operatorname{Im} d_0$ span $\ell_p C^1$. Q.E.D.

Corollary. *If X is an infinite connected polyhedron with* $\operatorname{Im} d_0$ *closed in ℓ_p and ℓ_q, then it is ℓ_p* and *ℓ_q-acyclic in degree zero.*

Examples and applications. The image $d_0(\ell_p C^0) \subset \ell_p C^1$ is closed if and only if every 0-cochain $\varphi \in \ell_p$ satisfies the Poincaré-Soborev inequality

$$\|\varphi\|_{\ell_p} \le \lambda \, \|d\varphi\|_{\ell_p} \qquad (*)_{\ell_p}$$

for some constant $\lambda \geq 0$ independent of φ. This is obvious. It is also easy to see that $(*)_{\ell_1}$ implies $(*)_{\ell_p}$ for every $p \geq 1$. In fact

$$\int |\varphi^p| \le \lambda_1 \int \|d(\varphi^p)\| \Rightarrow \int |\varphi^p| \le \lambda_p \int \|d\varphi\|^p$$

by the Hölder inequality. On the other hand, $(*)_{\ell_1}$ is equivalent to the *linear isoperimetric inequality* for finite subsets F, of vertices in the polyhedron X, which reads

$$\operatorname{card} F \le \lambda_{\text{is}} \operatorname{card} \partial' F \ , \qquad (*)_{\text{is}}$$

where $\partial' F \subset F$ denotes the subset of those $x \in F$ for which there exists an edge $[x, x']$ in X with x' outside F. Here, the implication $(*)_{\ell_1} \Rightarrow (*)_{\text{is}}$ follows by applying $(*)_{\ell_1}$ to the characteristic function of F (or rather of $F - \partial' F$) while $(*)_{\ell_1}$ for a function $\varphi \geq 0$ is obtained by (properly) integrating $(*)_{\text{is}}$ for the levels $F_t = \varphi^{-1}[0,t]$. (This argument was discovered by Mazia and, independently, by Cheeger many years ago).

Now we recall that $(*)_{\text{is}}$ for discrete groups amounts to *non-amenability* and so, if X admits a discrete cocompact automorphism group Γ, $d_0(\ell_p C^0) \subset \ell_p C^0$ *admits a complement for all p in the interval $1 \leq p < 1$, if and only if Γ is non-amenable.*

Another class of spaces (polyhedra) X satisfying all $(*)_{\ell_p}$, $1 \leq p < \infty$, is constituted by the hyperbolic ones which are *uniformly non-elementary.* This means, there is a constant $\alpha > 0$, such that for every vertex $x_0 \in X$ there exists an α-quasi-isometric embedding of the *infinite tripod* Tr into X sending the center of T_2 into x_0, (where Tr is the union of three copies of $[0, \infty)$ joined at zero). For example, every geodesically complete simply connected manifold X with $K(X) \leq -\varepsilon < 0$ falls into this category.

Remarks. (a) Our complement $\operatorname{Ker}\partial$ to $\operatorname{Im} d_0$ is a particularly nice one as it is *universal* in a variety of ways. For example, it is independent of p (as explained below and it is invariant under automorphisms of X. Besides it is *locally* defined as ∂ is a local operator (i.e. $\partial\varphi(\sigma)$ depends only on $\varphi(\sigma')$ for all σ' adjacent to σ). The projection operator $P : \ell_p C^1 \to \operatorname{Im} d_0$ and the homotopy operator $h : \ell_p C^1 \to \ell_p C^0$ are certain integral operators (of course, "integral" here refers to sums, but we prefer the integral language) with kernels, say $\sqcap(\sigma, \sigma')$ and $H(\sigma, \sigma')$, (where σ, and σ' run over the sets of the 1-simplices), independent of p. So it is no surprise that the boundness of these operators for certain p's interpolates to other values of p. (In our case we know, that if $\operatorname{Im} d_0$ is closed for some $p_0 \geq 1$, then it is closed for all finite $p \geq p_0$ by Hölder inequality and so the boundness of P and/or H for p_0 implies that all finite $p \geq p_0$).

(b) Using $\operatorname{Ker}\partial$ for the complement to $\operatorname{Im} d$ is common in the ℓ_2-theory where these spaces are orthogonal and together span all of $\ell_2 C^*$ (thus $\operatorname{Ker}\partial$ equals the complement of the $\ell_2$closure $\overline{\operatorname{Im} d}$). It is unclear under which circumstances $\operatorname{Ker}\partial$ complements $\operatorname{Im} d$ (or the ℓ_p-closure $\overline{\operatorname{Im} d}$) in the ℓ_p-sense in a given degree $k \geq 2$. This can be interpreted again as a boundness question for a specific integral operator which can be explicitly written down for some X, such, for example, as symmetric spaces (where we deal with the de Rham complex, rather than with a combinatorial d. See $[\text{Pan}]_{6,7,8}$ for some results in this direction.). Notice that the basic fact of the above mentioned ℓ_2-(Hodge) theory reads *if an ℓ_2-cochain φ satisfies $\partial d\varphi = 0$ then $d\varphi = 0$*, and we want to know when this remains valid for $\ell_p, p \neq 2$. But it is not even a priori clear if this property for a given p is invariant under subdivisions of X. (The existence of *some* ℓ_p-complement to $\operatorname{Im} d$ is an invariant property. In fact, it is determined by the ℓ_p-chain homotopy equivalence class of the chain complex).

About the weak acyclicity in degree zero. Let us show that *every infinite connected polyhedron X is weakly ℓ_p-acyclic in degree zero for all p in the interval* $1 \leq p < \infty$ (under the standing assumption that X

is uniformly locally finite). This is done using the usual cut-off of cochains c, denoted $c \mapsto c_t$, which is the multiplication of c by the characteristic function of the t-ball in X around a fixed point in X. It is clear that $\|c - c_t\|_{\ell_p} \to 0$ for $t \to 0$ if $1 \le p < \infty$. Since X is infinite connected, the operator d_0 is injective and therefore invertible on its image. Its inversion followed by the cut-off operator $c \mapsto c_t$ is an ℓ_p-bounded operator as it has *finite dimensional* range. This also allows us to extend the composed operator $c \mapsto (d_0^{-1}c)_t$ to the desired operator $h_t : \ell_p C^1 \to \ell_p C^0$ satisfying $\|h_t d(c) - c\|_{\ell_p} \to 0$. Q.E.D.

Remark on $\operatorname{Ker}\partial$. It would be interesting to know if the ℓ_p-closure $\overline{\operatorname{Im} d_0}$ can be ℓ_p-complemented by $\operatorname{Ker}\partial_1$ and the first thing to show is the relation $\overline{\operatorname{Im} d_0} \cap \operatorname{Ker}\partial_1 = 0$. Since every $c \in \overline{\operatorname{Im} d_0}$ is (exact) of the form $c = d\varphi$, where φ is not in general, in ℓ_p, it is sufficient to prove that

$$\Delta\varphi = 0 \quad \& \quad d\varphi \in \ell_p \Rightarrow \varphi = 0 \ .$$

It may be true (and known to analysts) in a quite *general* situation. Here we show $\overline{\operatorname{Im} d_0} \cap \operatorname{Ker}\partial_1 = 0$ for $p < \infty$, provided X *admits a cocompact automorphism group* Γ which we assume (though it is not truly necessary for our argument) discrete. If Γ has polynomial growth, then it is virtually nilpotent and so there exists an element $\gamma \in \Gamma$ of infinite order, such that $\operatorname{dist}(x, \gamma(x)) \le \text{const} < \infty$ for all $x \in X$. This inequality shows that

$$d\varphi \in \ell_p \Rightarrow \varphi - \gamma\varphi \in \ell_p \ ,$$

and as

$$\Delta\varphi = 0 \Rightarrow \Delta(\varphi - \gamma\varphi) = 0 \ ,$$

we conclude that $\varphi - \gamma\varphi = 0$ as earlier by the maximum principle. But since γ has infinite order its orbits are infinite, and so

$$\varphi = \gamma\varphi \ \text{and}\ \varphi \in \ell_p, \ p < \infty, \ \Rightarrow \varphi = 0 \ .$$

Now, let Γ grow faster than polynomially. Then by Varopoulis' isoperimetric inequality,

$$\|\varphi\|_{\ell_p} \le \lambda \|d\varphi\|_{\ell_{p'}} \quad \text{for all } p' < p < \infty \text{ and } \varphi \in \ell_p C^0 \ ,$$

(where λ depends on p and p' but not on φ). It follows that every c in the ℓ_p-closure of $\operatorname{Im} d_0$ is $d\varphi$ for $\varphi \in \ell_{p'}$ for every $p' > p$, and so the equation $\Delta\varphi = 0$ makes $\varphi = 0$ by the maximum principle.

Finally, we apply this to p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$ and conclude that $\operatorname{Ker}\partial_1$ complements $\overline{\operatorname{Im} d_0}$ in $\ell_p C^1$ for all p in the interval $1 < p < \infty$.

The case $p = 2$. It is obvious for the ℓ_2-theory that

$$\text{acyclicity} \Leftrightarrow \ell_2 H^* = 0$$

and by the spectral theory

$$\text{weak acyclicity} \Leftrightarrow \overline{\ell_2 H}^* = 0 \ .$$

But we know that a space X may have $\overline{\ell_2 H}^* = 0$ while $\ell_2 H^* \neq 0$, as it happens, for example, if $X = \mathbb{R}^n$ of $X = H_{\mathbb{R}}^{2n+1}$.

ℓ_p-Hodge. This means that $\operatorname{Im} d_{i-1}$ can be ℓ_p-complemented by some subspace $I_i^\perp \subset \ell_p C^i$ for $i = 1, 2, \ldots$. In this case the intersection $I_i^\perp \cap \operatorname{Ker} d_i$ is canonically isomorphic to $\ell_p H^i$. Then it is additionally required (with our definition of ℓ_p-Hodge), that $\ell_p H^i$ can be complemented in $I_i^\perp \supset \ell_p H^i$ which gives us projections $B_i : \ell_p C^i \to \ell_p H^i$, $i = 0, 1 \ldots$ as well as an operator of degree -1 (homotopy) $h : \ell_p C^* \hookleftarrow$, such that

$$dh + hd = 1 - B$$

for $B = \{B_0, B_1 \ldots\}$. Thus $\ell_p C^*$ is chain homotopy equivalent to the complex $\{\ell_p H^*, \ d = 0\}$.

Weakly ℓ_p-Hodge. Here we require the existence of complements. $I_i^\perp$ of $\overline{\operatorname{Im} d_{i-1}}$ in $\ell_p C^i$ (which allow us to identify the reduced cohomology $\overline{\ell_p H}^i$ with $I_i^\perp \cap \operatorname{Ker} d_i \subset \ell_p C^i$) and of complements of $\overline{\ell_p H}^i$ in $I_i^\perp$ which give us projections $B_i : \ell_p C^i \to \overline{\ell_p H}^i$. But now, instead of a single homotopy, we insist on a family h_t, such that

$$\|(dh_t + h_t d)\varphi - \varphi + B\varphi\|_{\ell_p} \underset{t\to\infty}{\to} 0$$

for all $\varphi \in \ell_p C^*$.

There is another (a priori weaker) version of "weakly Hodge" where we just require an *approximative* chain homotopy equivalence between the complexes $\{\ell_p C^*, d\}$ and $\{\overline{\ell_p H}^*, d = 0\}$. Here "approximative" refers to the following (weak) notion of homotopy between two homomorphisms a and b of a chain complex into itself, $a \underset{\text{appr}}{\sim} b \Leftrightarrow$ there exists a family of operators h_t of degree -1, such that $h_t d - dh_t$ converges for $t \to \infty$ to $a - b$ in the *strong operator topology* (i.e. $A_t \to B$ iff $\|A_t\varphi - B\varphi\| \to 0$ for every φ).

The properties "Hodge" and "weakly Hodge", whenever they hold true, greatly facilitate homological algebra in the ℓ_p-category (as it happens for ℓ_2, see [Ch-Gr]$_3$). For example, the *exact sequence* for pairs of complexes is obvious for the non-reduced ℓ_p-cohomology, but if we want it split or be valid in the reduced case, we need something of Hodge. Also it seems hard to have a satisfactory *Künneth formula* for $\ell_p H^*(X \times Y)$ without Hodge (compare 8.C$_1$. below). Probably, one could (and should) modify the definition of the ℓ_p-cohomology to have a built-in Hodge (or weakly Hodge) property.

Generic questions on reduced versus non-reduced. We do know that sometimes the ℓ_p-cohomology may be non-reduced, i.e. $\ell_p H^i(X) \neq \overline{\ell_p H}^i(X)$, and it may even happen that $\ell_p H^i(X) \neq 0$, while $\overline{\ell_p H}^i(X) = 0$. However, this appears a rather unstable phenomenon which could be distroyed by a small perturbation of p and/or the (geometry of) X. Here one should be careful when X is a group or a related space as the deformation of geometry is highly restricted. For example, if $X = \mathbf{R}^n$ (or an arbitrary group with an infinite center for this matter) then the reduced cohomology is zero for $p \neq 1, \infty$. Yet $\ell_p H^i(\mathbf{R}^n) \neq 0$ for $i = 1, \dots, n$ and $1 < p < \infty$. Here a natural deformation of $\mathbf{R}^n$ would be a nilpotent group X, but these also have $\ell_p H^i \neq \overline{\ell_p H}^i = 0$ for all p and $1 \le i \le \dim X$. This example may look rather discouraging but one should take into account that Abelian and nilpotent groups are highly non-generic by any standard.

8.C$_1$. Künneth formula and vanishing theorems. There is an obvious pairing

$$\ell_p H^*(X) \otimes \ell_p H^*(Y) \to \ell_p H^*(X \times Y)$$

which is matched by a pairing on homology

$$\ell_q H_*(X) \otimes \ell_q H_*(Y) \to \ell_q H_*(X \times Y)\,.$$

It follows, by duality for $\frac{1}{p} + \frac{1}{q} = 1$, that this pairing is non-degenerate on the *reduced* (co)homology. For example, if $\overline{\ell_p H}^i(X) \neq 0$ and $\overline{\ell_p H}^j(Y) \neq 0$ for $1 < p < \infty$, then $\overline{\ell_p H}^{i+j}(X \times Y) \neq 0$ as well. Yet, it is unclear if (or when) all of (co)homology of $X \times Y$ comes from that of X and of Y. More specifically, if $\ell_p H^i(X) = 0$ for $i = 0, \dots, i_o$ and $\ell_p H^j(Y) = 0$, $j = 0, \dots, j_0$, we want to know whether $\ell_p H^{i+j}(X \times Y) = 0$ for $i + j = 0, \dots, i_0 + j_0 + 1$ and the same question applies to the reduced cohomology $\overline{\ell_p H}^*$. This question has a positive answer on the Hodge level.

Acyclicity Proposition. *If X is ℓ_p-acyclic (weakly ℓ_p-acyclic) in degrees $0, \dots, i_0$ and Y is ℓ_p-acyclic (weakly ℓ_p-acyclic) in degrees $0, \dots, j_0$, then $X \times Y$ is ℓ_p-acyclic (respectively weakly ℓ_p-acyclic) in degrees $0, \dots, i_0 + j_0 + 1$.*

Proof. The complex $\ell_p C^*(X \times Y)$ naturally splits into double complex which is more convenient to express in the language of differential forms rather than cochains. Namely, the (i,j)-component $\ell_p C^{i,j}$ consists of (i,j)-forms that are $a(x,y)dx_1 \wedge \dots \wedge dx_i \wedge dy_1 \wedge \dots \wedge dy_i$ and the two differentials are d_x and d_y which act on $a(x,y)dx_1 \wedge \dots$, by

$$dx(a(x,y)dx_1 \wedge \dots) = \left(\sum_\mu \frac{\partial a}{\partial x_\mu} dx_\mu\right) \wedge dx_1 \wedge \dots$$

and

$$d_y\left(a(x,y)dx_1\wedge\ldots\right)=\left(\sum_\nu\frac{\partial a}{\partial y_\nu}dy_\nu\right)\wedge dx_1\wedge\ldots\ .$$

Thus $d_x d_y=-d_y d_x$ and $d_x+d_y=d$. Every endomorphism (homotopy) of $C^*(X)$ of degree -1 induces an endomorphism, denoted h_x of $C^{i,j}$ of bi-degree $(-1,0)$ (i.e. $C^{i,j}\to C^{i-1,j}$), which can be thought of as an integration with respect to x. Similarly, a homotopy of $C^*(Y)$ gives us a homotopy h_y of $C^{i,j}$ of bi-degree $(0,-1)$, which formally is an integration with respect to y. Notice that h_x *anti*-commutes with d_y and h_y *anti*-commutes with d_x. (This trivial point kept me confused for some time unless I caught a remark slipped from Ofer Gabber). Now, we define the total homotopy by

$$h=h_x+h_y(1-dh_x-h_x d)$$

and observe with the anticommutation relations that it does the job. This concludes the proof in the acyclic case and the weakly acyclic argument goes the same way. Q.E.D.

Let us combine the above with the acyclicity in degree zero established earlier and conclude to the following

Vanishing Corollaries. (a) *Let X be the Cartesian product of k-copies of uniformly non-elementary hyperbolic spaces,* e.g. *Cayley graphs of non-elementary hyperbolic groups. Then X is ℓ_p-acyclic in degrees $0,\ldots,k-1$ for $1<p<\infty$ and consequently $\ell_p H^i(X)=0$ for $i=0,\ldots,k-1$.*

(b) *Let X be the Cartesian product of k infinite connected polyhedra, each admitting a cocompact automorphism group. Then X is weakly ℓ_p-acyclic, $1<p<\infty$, in degrees $0,\ldots,k-1$ and so it has zero reduced cohomology $\overline{\ell_p H}^i(X)$, $i=0,\ldots,k-1$.*

Remarks, applications and further questions. (1) *Summable gradients.* Look again at maps $f:X\to\mathbb{R}^n$ having $\|\wedge^k f\|$ in L_p where $\wedge^k f$ denotes the k-th exterior power of the differential of f. (This may be extended from the smooth category to polyhedra X by linearly interpolating maps defined on the set of the vertices of X). We have seen in 8.B. that the existence of certain maps of this kind yields non-vanishing of $\overline{\ell_p H}^k$ and now we can turn it around and conclude to some degeneracy of f in the case $X=X_0\times\ldots\times X_k$. It would be interesting to obtain more precise information about such maps.

(2) *Symmetric spaces of non-compact type and affine buildings.* Such spaces X are similar to Cartesian products of k hyperbolic spaces for $k=\operatorname{rank}X$ and so we *expect* that

(?) If the isometry group of X is semisimple (i.e. no Euclidean factor) then $\ell_p H^i(X)=0$, $1<p<\infty$, $i=0,1,\ldots,k-1$. (If X has an Euclidean factor, then it admits an isometry with bounded displacement and so $\overline{\ell_p H}^i=0$ for *all* $i=0,1,\ldots$.) On the other hand, the group $\overline{\ell_p H}^k$, probably, is non-zero for some p. This is definitely so for buildings (see below) where one may even expect $\overline{\ell_p H}^k\neq 0$ for *all* p in the interval $1<p<\infty$. Also, we conjecture that $\ell_p H^k$ is *reduced* which means $\ell_p H^k=\overline{\ell_p H}^k$ (and is equivalent to $\operatorname{Im} d_{k-1}$ being ℓ_p-closed), provided X has no Euclidean factor.

Here is some evidence in favour of these conjectures.

(+) *Vanishing of $\ell_p H^1$.* Let us show that $\ell_p H^1(X)=0$ assuming $\operatorname{rank}X=2$. Take a singular geodesic γ in X and consider the totally geodesic subspace $Y_\gamma\subset X$ consisting of all geodesics parallel to γ. This Y_γ metrically splits, $Y_\gamma=Y'\times\mathbb{R}$ where $\mathbb{R}$ corresponds to γ and where Y' is a symmetric space (or building) of rank one. Because of $\mathbb{R}$, the reduced cohomology $\overline{\ell_p H}^1(Y_\gamma)$ vanishes while the presence of the Y'-factor insures the linear top-dimensional isoperimetric inequality for Y_γ and thus the equality $\ell_p H^1(Y_\gamma)=\overline{\ell_p H}^1(Y_\gamma)=0$. Now, we take an arbitrary closed L_p-form α on X of degree zero, write $\alpha=d\varphi$, where φ is normalized (by adding a constant, if necessary), such that φ restricted to Y_γ lies in $L_p(Y_\gamma)$ and claim φ is in L_p on all of X. (We switched the notations from ℓ_p to L_p as we use the language of differential forms but this is pure linguistics). This is proven by first showing that the restriction of this φ to $Y_{\gamma'}$, for another singular geodesic γ' is again in $L_p(Y_{\gamma'})$. Indeed, for every pair (γ,γ') there exists a chain of singular geodesics $\gamma_1=\gamma,\ \gamma_2,\ldots,\gamma_k=\gamma'$ such that Y_{γ_i} and $Y_{\gamma_{i+1}}$ intersect across a geodesic for all $i=1,\ldots,k-1$. Since every geodesic is unbounded, there is at most one (additive) normalization of φ which makes it ℓ_p (for $p < \infty$) on this geodesic and so

$$\varphi \in L_p(Y_\gamma) \Rightarrow \varphi \in L_p(Y_{\gamma_i}),\ i = 1, \dots, k .$$

What remains to show is the implication

$$\varphi \in L_p(Y_\gamma) \quad \text{for all } \gamma \Rightarrow \varphi \in L_p(X) ,$$

but this follows from the *Crofton formula* expressing integrals over X by first integrating over Y_γ and then over the space Γ of singular geodesics γ in X with some invariant measure $d\gamma$ on Γ. Namely, we start with the inequality

$$\int_{Y_\gamma} |\varphi|^p (x) dx \le \lambda \int_{Y_\gamma} \|df\|^p (x) dx \tag{$*$}$$

(which follows from $\ell_p H^1(Y_\gamma) = 0$) and conclude to the desired inequality

$$\int_X |\varphi|^p (x) dx \le \lambda \int_X \|df\|^p (x) dx$$

by integrating $(*)$ over Γ and using the Crofton formula

$$\int_X \psi(x) dx = \int_\Gamma d\gamma \int_{Y_\gamma} \psi(x) dx .$$

Remark. This argument works for rank ≥ 2 and also for buildings as well as for symmetric spaces of rank 2. Also an obvious modification (and simplification) of this proof shows that

$$\ell_p H^1(X_1 \times X_2) = 0, \quad p < \infty ,$$

whenever $\ell_p H^1(X_1) = \overline{\ell_p H}^1(X_1)$ and where X_2 is a connected space of infinite volume (thus having $\ell_p H^0(X_2) = 0$).

Question. When does a complete simply connected space X with non-positive curvture have $L_p H^1(X) = 0$ for all $p < \infty$? Probably, the relevant geometric condition is connectivity of Tits' boundary of X. (A natural generalization of this question applies to semihyperbolic spaces and, in particular, to semihyperbolic groups).

(++) *Vanishing of $\ell_2 H^i$.* If X is a symmetric space then $\overline{\ell_2 H}^i(X) = 0$ for $i = \dim X/2$ (see [...]) and I suspect it is also known that $\ell_2 H^i(X) = 0$ unless $i = \dim X/2$ or $i = (\dim X \pm 1)/2$. Furthermore, if X is a building, then $\ell_2 H^i(X) = 0$ for $i = 0, \dots, k-1$ and $\overline{\ell_2 H}^k(X) \ne 0$. This is proven in [Gar] (also see [Bor]) under certain restrictions on X, but these, I recollect, have been eventually removed. The argument given by Garland in [Gar] has a certain stability to it which seems to imply $\ell_p H^i(X) = 0$ for $i < k$ and p close to 2. On the other hand, since $\dim X = k$, the non-vanishing of $\overline{\ell_2 H}^k(X)$ amounts to the existence of a non-zero ℓ_2-cycle c of dimension k. Such a c is then contained in $\ell_p C_k$ for every $p \ge 2$ which yields $\overline{\ell_q H}^k(X) \ne 0$ for every q in the interval $1 < q \le 2$.

Symmetrization. Given an arbitrary $c \in \ell_p C_k$ which does not vanish on some k-simplex σ, one can average c over the (compact !) group G_σ of the automorphisms of X fixing σ and then the resulting averaged chain $\bar{c}$ remains in $\ell_p C_k$ and non-zero, becoming G_σ-invariant at the same time. One knows that the group G_σ is rather large: if we take an arbitrary flat $F = \mathbb{R}^k$ containing σ (and such F does exist) then the orbit $G_\sigma(F)$ equals all of X. Thus $\bar{c}$ is determined by its restriction to F. Probably, the p-adic theory of spherical functions provides sufficient information on such $\bar{c}|F$ to decide in which ℓ_p they reside. (If $k = 1$, then X is a tree and $\bar{c} \in \ell_p$ for all $p > 1$ as is seen on Fig. 20 at the beginning of §8.)

$\ell_p H^*$ for amalgamated products. The elementary homological algebra works for computation of $\ell_p H^*$ of amalgamated product of groups. For example, if $\Gamma = \Gamma_1 \underset{\Gamma_0}{*} \Gamma_2$, where $\ell_p H^i(\Gamma_1) = \ell_p H^i(\Gamma_2) = 0$ and $\ell_p H^{i-1}(\Gamma_0) = 0$, then, obviously, $\ell_p H^i(\Gamma) = 0$. In particular, if Γ_0 is infinite, then

$$\ell_p H^1(\Gamma_i) = 0,\ i = 1, 2, \Rightarrow \ell_p H^1(\Gamma) = 0\ ,$$

but the situation is not so clear for the reduced cohomology $\overline{\ell_p H}^*$ if $p \neq 2$.

8.C_2. The L_p-cohomology sheaf on the ideal boundary $\partial_\infty X$. Here we assume X is a geodesic hyperbolic space, we compactify X by adding the ideal boundary and denote this by $\overline{X} = X \cup \partial_\infty X$. If what follows we suppose X is a sufficiently regular space, for example a manifold of negative curvature or a contractible uniformly locally finite polyhedron, e.g. Rips' complex of a word hyperbolic group. Our first objectives are continuous functions φ defined on an open subset $D \subset \partial_\infty X$ which admit continuous extensions φ' to open subsets $\overline{D}' \subset \overline{X}$ containing D, such that the restrictions $\varphi'|D'$ for $D' = \overline{D}' - D$ have $\|d\varphi'\|$ in L_p (or ℓ_p if we insist on the combinatorial language). These functions form a linear subspace in the space of all continuous functions, denoted $W_p^0(D) \subset C^0(D)$, and as we vary D, these spaces form a *sheaf* $\mathcal{W}_p^0$ on $\partial_\infty X$. (Sheaves of this kind were introduced by Pansu in [Pan]$_6$ and much of our exposition here is parallel to that in [Pan]$_6$). It is useful to generalize this definition to continuous maps $\varphi : D \to Y$ where Y may be an arbitrary metric space and where $\|d\varphi'\|(x)$, for an extension φ' to $\overline{X}$, denotes the lim sup of the Lipschitz constants of φ' on a fundamental system of neighbourhoods of x in X. Thus we have the notion of L_p for $\|d\varphi'\|$ and can define $\mathcal{W}_p^0(D \to Y)$. (If $Y \subset \mathbf{R}^N$ then $\mathcal{W}_p^0(D \to Y) \subset \bigoplus_1^N \mathcal{W}_p^0(D)$ and nothing new enters the picture). Typical questions we ask in this framework are as follows.

Given an open subset D and a continuous map $\varphi_0 : D \to Y$. For which p does there exist $\varphi \in \mathcal{W}_p^0(D \to Y)$ homotopic to φ_0 or equivalent to φ_0 in a somewhat weaker (e.g. homological) sense? What is the minimal p, such that the above $\varphi \in \mathcal{W}_p^0$ exists for all (D, φ_0) or for a single "interesting" representative (which may mean that φ_0 induces a *non-trivial* homomorphism on the cohomology in a given degree) ?

There is a generalization of the above questions where $\|d\varphi'|$ is replaced by $\| \wedge^k \varphi'|$ that is the norm of the k-th exterior power of $d\varphi$. If Y is a smooth (possibly infinite dimensional) manifold or a polyhedron the definition of $\| \wedge^k \varphi^1\|$ is clear. In the general case one may define $\| \wedge^k \varphi'\|(x)$ as follows. Take a k-dimensional submanifold (or a local cycle) V with boundary containing x in the interior of V, set

$$\| \wedge^k \varphi'(V)\| = \Big(\operatorname{Fill Rad}(\varphi'(\partial V) \subset Y)\Big)^k \Big/ \operatorname{Vol}_k V$$

and then let $\| \wedge^k \varphi'\|(x)$ be $\sup\limits_V \| \wedge^k \varphi'(V)\|(x)$ over all V in X containing x.

On the continuity of extensions from D to $\overline{D}'$. The continuity condition is not indispensable. In fact, one may define another sheaf of spaces, say $\mathcal{W}_p(D)$, $D \subset \partial_\infty X$, as the inductive limit of the spaces of functions φ' on all open $\overline{D}' \supset \overline{X}$ extending D, such that $d\varphi'|D'$ is in ℓ_p. Similarly, one defines $\mathcal{W}_p(D \to Y)$ and then one may compare the connected components of the space of continuous maps $D \to Y$ to those of $\mathcal{W}_p(D \to Y)$ where paths in the latter space are defined as restrictions of suitable homotopies of maps $D' \to Y$. For example, let Y be a subset of a Banach space. Then the (continuity) condition on a homotopy of maps $h'(x,t)$, $(x,t) \in D' \times [0,1]$ can be expressed by

$$\int\limits_{D_1} \left\| d_x \big(h'(x,t_1) - h'(x,t_2)\big) \right\|^p dx \leq \varepsilon\ ,$$

where $\varepsilon \to 0$ as $|t_1 - t_2| \to 0$. This condition should be augmented by some normalization as the differential defines a map only up to an additive constant. A possible way of such a normalization is the requirement $h'(x,0) - h'(x,t) \in \ell_{p'}$ for all $t \in [0,1]$ and some (large) $p' < \infty$.

Remark. Every function φ on D' with $d\varphi \in L_p$ can be extended to $\partial_\infty D' = \overline{D}' \cap \partial_\infty X$ in a measure theoretic sense as for *almost every* (quasi)geodesic ray $r(t)$ in D' going to $\partial_\infty D'$ for $t \to \infty$ the function $\varphi(r(t))$ converges to a definite value depending only on $\lim_{t\to\infty} r(t) \in \partial_\infty X$ (but not on a choice of $r(t)$). To make this precise, one should choose a measure class on $\partial_\infty D'$ and, in fact, any reasonable measure serves our purpose. For example, one may take the measure we use to define L_p, restrict it to the R-balls around a fixed point, normalize the resulting (finite) measures to have total mass one and take a weak limit of these for $R \to \infty$. This measure is good for us as any comparable measure. The proof of that is due to Strichartz (see $[\mathrm{Str}]_1$ and $[\mathrm{Pan}]_6$, where the authors deal with a slightly different situation but their argument works in our case as well).

Cohomology of $D \subset \partial_\infty X$ representable by L_p-forms on $D' \supset D$. It is hard to speak of a "continuous extension" of a k-form (or cochain) from D to $\overline{D}'$ unless $\overline{X} = X \cup \partial_\infty X$ carries some smooth (or at least Lipschitz) structure. (Such structures may be available in some examples, e.g. if X is strictly 1/4-pinched, and continuous extension of forms may reveal something useful about such smooth structures but we do not enter this discussion.) However, a k-dimensional cohomology class α can be usually extended from D to some $\overline{D}'$ and then we look for the smallest p such that the restriction of the extended class to $D' \subset \overline{D}$ (recall that $D' = \overline{D}' - \partial_\infty X \subset X$) can be represented by a closed ℓ_p-form (or cochain) of degree k. This number $p = p(\alpha)$ gives us an interesting invariant of α (and of X as we vary α) which is similar to infδim studied in §7.

Examples justifying the definitions. We start by showing that the sheaf $\mathcal{W}_p^0$ is sufficient to distinguish points in $\partial_\infty X$ provided p is sufficiently large.

Claim. *There exists a continuous map φ of $\overline{X}$ into the Hilbert space $\mathbb{R}^\infty$ such that φ on X has $d\varphi$ in ℓ_p for some $p < \infty$ and the restriction of φ to $\partial_\infty X$ is an embedding (i.e. one-to-one).*

Idea of the proof. Change the metric in X using a positive conformal factor $\rho(x)$, so that the new distance dist_ρ between x_1 and x_2 equals the infimum of the integrals of ρ over the curves joining x_1 and x_2. If $\rho \in L_p$ then for every $x_0 \in X$ the function $\varphi(x) = \mathrm{dist}_\rho(x_0, x)$ has $\|d\varphi(x)\| \le \rho(x)$ and so it is in L_p. Moreover, the distance function from an arbitrary subset in X also has the gradient in L_p. Next, if we want such distance function to be extendable to $\partial_\infty X$, we must assume uniform summability of ρ along the geodesic rays issuing from a given point $x_0 \in X$. But the extended distance function may become constant on $\partial_\infty X$.

Conformal dimension of $\partial_\infty X$. This dimension is, by definition, the infimum of those p for which there exists a positive $\rho \in L_p(X)$ for which dist_ρ extends to a *metric* on $\overline{X} = X \cup \partial_\infty X$. This amounts to saying that $\overline{X}$ equals the metric completion of (X, dist_ρ). (This definition of the conformal dimension is essentially equivalent to that in $[\mathrm{Pan}]_5$).

Observation. *Every hyperbolic space of bounded* (exponential) *growth (e.g. a word hyperbolic group) has* $\mathrm{conf\,dim}\,\partial_\infty X < \infty$.

This is shown with $\rho(x) = \exp(-\lambda\,\mathrm{dist}(x_0, x))$, where λ is sufficiently large (compare [Flo], [Gro-Pa] and Appendix to §8.).

Remark. It follows from $[\mathrm{Pan}]_5$ that $\mathrm{conf\,dim}\,\partial_\infty X \ge \dim \partial_\infty X$.

Now, for every $p > \mathrm{conf\,dim}\,\partial_\infty X$, we have a distance function on $\partial_\infty X$ which L_p-extends to X and then it is not hard to make up our embedding to $\mathbb{R}^\infty$. (It is obvious, there exists such an embedding to L_∞ by $x \mapsto \mathrm{dist}(x, \cdot)$, but $\mathbb{R}^\infty$ is somewhat neater. In fact, it seems not hard to make such an embedding into some $\mathbb{R}^N$, $N < \infty$, as X is assumed to have uniformly bounded local geometry).

Remark. If X is an n-dimensional Riemannian manifold with pinched negative curvature, there is an obvious embedding $\varphi : \overline{X} \to \mathbb{R}^n$ with $\|d\varphi\| \in L_p$ with p depending on the pinching (compare 8.B.).

Non-vanishing of $\overline{\ell_p H}^1(X)$. *If X is uniformly non-elementary* (e.g. *a word hyperbolic group non-commensurable to $\mathbb{Z}$) then $\overline{\ell_p H}^1(X) \neq 0$ for $p >$* $\max(1, \mathrm{conf\,dim}\,\partial_\infty X)$.

Proof. Take two disjoint subsets A_0 and A_1 in $\overline{X}$, a continuous function φ on $\overline{X}$ with $\|d\varphi\| \in L_p$, such that $\varphi|A_0 = 0$ and $\varphi|A_1 = 1$ and a (quasi-isometric) infinite regular tree in X as in Fig. 22.

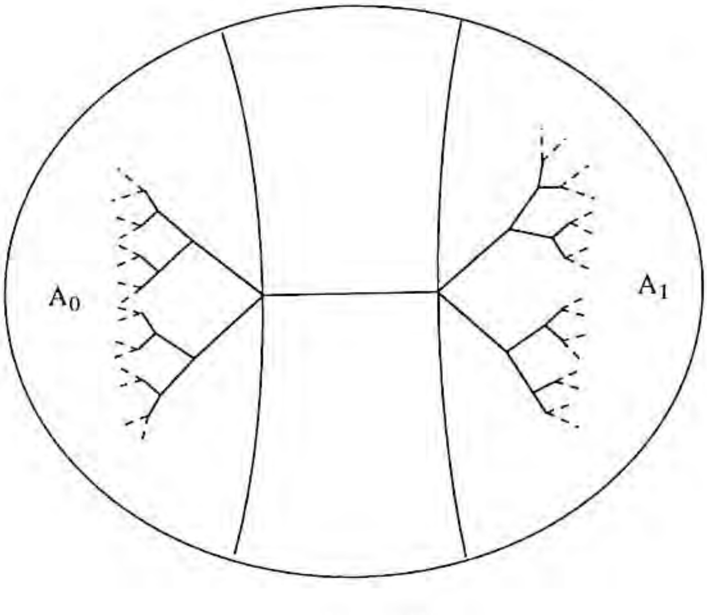

Figure 22

This tree supports an ℓ_p-cycle c (see Fig. 20 at the beginning of §8) for which $\langle d\varphi, c\rangle \neq 0$ and so $d\varphi$ gives us a non-zero L_p-cohomology class.

Remarks. (a) *Non-word hyperbolic groups.* The above discussion obviously extends to (non-quasiconvex) subsets Y in X and yields non-vanishing of $\ell_p H^1(Y)$ in-so-far as Y contains (a quasi-isometric copy of) a triadic tree. For example, *every non-elementary isometry group* Γ *of* X *has* $\ell_p H^1(\Gamma) \neq 0$ *for all sufficiently large* p. This applies, in particular, to *non-elementary subgroups* Γ *in word hyperbolic groups* as well as to the *(non-uniform) lattices in simple Lie groups of* $\operatorname{rank}_{\mathbb{R}} = 1$. In fact, this non-vanishing is shared by the groups Γ *admitting left invariant hyperbolic metrics of bounded exponential growth and having infinite* $\partial_\infty\Gamma$ *with respect to such a metric.*

(b) *Conformal boundary.* This (ideal) boundary, as defined by Floyd (see [Flo]), appears in the metric completion of $(X, \rho \operatorname{dist}_X)$ for a suitable conformal factor ρ. Let us indicate another definition which is better adjusted to the present situation (compare [Roe]$_4$). To be specific, assume X is a complete Riemannian manifold and let Φ denote the set of C^1-functions φ on X with the following two properties

(i) $\|\varphi\|_{C^1} < \infty$, i.e. φ and $\|d\varphi\|$ are L_∞.

(ii) $\|d\varphi\|$ is *uniformly summable on the quasi-geodesic rays* in X. This means, that the restriction of $\|d\varphi\|$ to every quasi-geodesic ray γ in X is in $L_1(\gamma)$ with a certain uniform bound on the L_1-norm. This bound is conveniently expressed in terms of the supremum $\delta(R)$ of $\|d\varphi(x)\|$ over the *complement* to the R-ball around a fixed point $x_0 \in X$ by requiring $\int_0^\infty \delta(R)dR < \infty$.

Now we define the conformal completion (or compactification) $CCX \supset X$ as the maximal space, such that X is dense in CCX and every function $\varphi \in \Phi$ continuously extends to CCX. More formally, CCX equals the *space of the maximal ideals* of the ring Φ. (The function space Φ *is* a ring for the usual addition and multiplication of functions).

Basic example. If X is hyperbolic then CCX is canonically homeomorphic to $\overline{X} = X \cup \partial_\infty X$ (compare [Gro-Pa]). Next, we define the *conformal boundary* $\partial_{\text{conf}} X = CCX - X$ and we want to know when $\partial_{\text{conf}} X$ is sufficiently large, e.g. contains a Cantor set. The above example shows that $\partial_{\text{conf}} X$ *is large for non-elementary hyperbolic groups* X. Another example of large $\partial_{\text{conf}} X$ is provided by *groups* X *with infinitely* *many ends* as $\partial_{\rm conf} X$ surjectively maps onto Ends(X).

Next, let L_p^1 denote the space of functions φ on X with $\|d\varphi\| \in L_p(X)$ and let us define *the conformal dimension of* X as the infimal p for which the ring $L_p^1 \cap \Phi$ has the same maximal ideals as Φ. This means, for every two points b_1 and b_2 in $\partial_{\rm conf} X$, there exists a function $\varphi \in L_p^1 \cap \Phi$ such that $\varphi(b_1) \neq \varphi(b_2)$.

It is not difficult to find a (quasi-isometric) triadic tree in X, *provided the conformal boundary of* X *is large* (i.e. contains a Cantor set), and then show as earlier that

$$p > \max(1, \operatorname{conf}\dim X) \Rightarrow \overline{L_p H}^1(X) \neq 0 .$$

Yet the following questions remain open:

(1) What are sufficient conditions for the conformal dimension to be finite ? For example, does every finitely generated group X with large $\partial_{\rm conf} X$ have $\operatorname{conf}\dim X < \infty$?

(2) When is the subset $L_p^1 \cap \Phi$ dense in L_p^1 ? When do L_p^1 and $L_p^1 \cap \Phi$ give us the same cohomology classes in $L_p H^1(X)$?

(3) Denote by $\hat{L}_p^1 \subset L_p^1$ the subset of *bounded* functions and observe that $\hat{L}_p^1$ is a ring. Let $\widehat{CC}_p X$ denote the space of the maximal ideals of $\hat{L}_p^1$ and define (compare [Hig])

$$\operatorname{corona}_p X = \widehat{CC}_p X - X .$$

Can one identify $\operatorname{corona}_p X$ with the set of the maximal ideals of some (natural) ring of (Borel) functions on $\partial_{\rm conf} X$? Observe that $L_p(X) \cap \hat{L}_p^1$ constitutes an ideal in the ring $\hat{L}_p^1$, such that the quotient ring $\widehat{H}_p^1$ equals the kernel of the natural homomorphism $L_p H^1(X) \to H^1(X; \mathbf{R})$. How does $\operatorname{corona}_p X$ relate to the space of the maximal ideals of $\widehat{H}_o^1$? (Compare $[\mathrm{Pan}]_{6,7,8}$ for the hyperbolic case.) Are there finitely presented groups X, where $\partial_{\rm conf} X$ reduces to a single point but $\overline{\ell_p H}^1(X) \neq 0$ for some p strictly between 1 and ∞ ?

Conformal cut-connectivity of $\partial_{\rm conf} X$. Let us indicate another invariant of $\partial_{\rm conf} X$ which is a priori smaller than conf dim and which measures the conformal size of cuts needed to disconnect $\partial_{\rm conf} X$. This invariant, denoted $ccc\,\partial_{\rm conf} X$, is defined as the infimum of those p for which there exists a *a* continuous function on CCX which is non-constant on $\partial_{\rm conf} X = CCX - X$ and has $\|d\varphi\|$ in $L_p(X)$ (compare $[\mathrm{Pan}]_5$). It is clear that $ccc\,\partial_{\rm conf} X \leq \operatorname{conf}\dim \partial_\infty X$ and that $\overline{L_p H}^1(X) \neq 0$ for $p > \max(1, ccc\,\partial_{\rm conf} X)$, if X is a uniformly non-elementary hyperbolic space.

Examples. (a) *Disconnectedness at infinity.* If X is disconnected at infinity, then, clearly, $ccc\,\partial_{\rm conf} X = 0$ and consequently, $\overline{L_p H}^1(X) \neq 0$ for all $p > 1$, provided X is a uniformly non-elementary hyperbolic space. In fact, this non-vanishing conclusion obviously remains valid for non-hyperbolic X with infinitely many ends in the *uniform sense* which allows a quasi-isometric embedding of an infinite regular tree into X which is injective on the set of ends (compare 8.3.L. in $[\mathrm{Gro}]_{14}$ where there is another proof of $\overline{L_p H}^1 \neq 0$).

(b) *Amalgamated products.* Let X be (or correspond to) an amalgamated product of groups $X = \Gamma = \Gamma_1 \underset{\Gamma_0}{*} \Gamma_2$. One might think that $ccc\,\partial_{\rm conf}\Gamma$ could be bounded in terms of $\operatorname{conf}\dim \partial_{\rm conf}\Gamma_0$, as it happens according to (a) for finite groups Γ_0. But this is not so for infinite Γ_0, since

$$\ell_p H^1(\Gamma_i) = 0,\ i = 1,\ 2,\ \Rightarrow \ell_p H^1(\Gamma) = 0 ,$$

as we have seen earlier. This suggests that $ccc\,\partial_{\rm conf} X = \operatorname{conf}\dim \partial_{\rm conf} X$ for most spaces X. This may be expected, in particular, for hyperbolic spaces X (where $\partial_{\rm conf} = \partial_{\rm conf}$) which are connected at infinity and admit cocompact isometry groups.

Representation of cohomology in $\partial_{\rm conf} X$ *by p-summable forms.* The above *ccc*-definition can be expressed in terms of p-summability of closed 1-forms on X representing relative cohomology classes in $H^1(CCX, D)$ for some $D \subset \partial_{\rm conf} X$. Now we want to address a similar problem (already mentioned earlier) for the (absolute) cohomology of degree $k \geq 2$. Namely, let D be a compact subset in $\partial_{\rm conf} X$ (one may think of the hyperbolic case where $\partial_{\rm conf} = \partial_\infty$, e.g. X may be a simply connected geodesically complete manifold of strictly negative curvature with $\partial_{\rm conf} X = \partial_\infty X = S^{n-1}$) and $\alpha \in H^k(D;\mathbb{R})$, where the cohomology is understood in the sense of Čech. Such an α extends (as follows from the definition of the Čech cohomology) to a cohomology class in some neighbourhood $\overline{D}' \subset CCX$ of D and we ask whether this extended class, when restricted to $D' = \overline{D}' \cap X$, can be represented by an L_p-form of degree k (or by ℓ_p-cochain) on D'. We have seen, in the hyperbolic case, that if $p > \operatorname{conf}\dim \partial_{\rm conf} X$, then CCX admits an embedding φ into the Hilbert space $\mathbb{R}^\infty$, such that $\|d\varphi\|$ is p-summable on X. (Recall that in the hyperbolic case $CCX = \overline{X}$ and $\partial_{\rm conf} = \partial_\infty$). Then every α can be represented by a bounded form in some neighbourhood $U \subset \mathbb{R}^\infty$ of D which pulls back (by φ) to a p_k-summable form on D' for $p_k = p/k$. This suggests the following

Definition of $\operatorname{conf}\dim_k$. This is the infimum of those p for which there exists an embedding $\varphi : CCX \to \mathbb{R}^\infty$, such that the k-th exterior power of the differential of φ on X is p-summable, i.e. $\| \wedge^k \varphi \| \subset L_p(X)$. Clearly, if $\operatorname{conf}\dim_k \partial_{\rm conf} X < p$, then every $\alpha \in H^k(D;\mathbb{R})$ admits a p-summable representative on some D'.

Remarks (a) The invariant $\operatorname{conf}\dim_k$ is a kind of a dual to $\inf \delta\mathrm{im}\,\mathcal{A}_k$ of 7.C$_1$.

(b) The full invariant of $\partial_{\rm conf}$ which goes along with this discussion is given by the set of strings of numbers $\{p_1, \dots, p_k, \dots\}$, such that CCX admits an embedding φ to $\mathbb{R}^\infty$ such that $\| \wedge^k \varphi \| \in L_{p_k}$ for $k = 1, 2, \dots$.

A lower bound on p. Given a cohomology class $\alpha \in H^k(D;\mathbb{R})$, we say that a subset $A \subset D$ is *α-nontrivial* if there exists a map of a k-dimensional polyhedron P into A, such that α pulls back to a non-zero class in $H^k(P;\mathbb{R})$. Denote by $\mathcal{A}_\alpha$ the set of the α-nontrivial A, and observe with the discussion in 7.C.$_1$ that in the hyperbolic case $\inf\delta\mathrm{im}\,\mathcal{A}_\alpha$ provides the following bound on p

$$\inf \delta\mathrm{im}\mathcal{A}_\alpha \geq k/p \, . \tag{$*$}$$

This means that *no extension of α from D to any open subset $\overline{D}' \subset CCX$, where the closure of $D' = \overline{D}' \cap X$ in CCX contains D, can be represented by a p'-summable k-form in D' for $p' < k^{-1} \inf \delta\mathrm{im}\mathcal{A}_\alpha$*. (We suggest the reader would generalize this to non-hyperbolic X).

On the $L_p\mathcal{H}^*$-sheaf on $\partial_{\rm conf} X$. This sheaf was introduced by Pansu (see [Pan]$_6$) for manifolds of negative curvature. This sheaf assigns to each $D \subset \partial_{\rm conf} X$ the inductive limit of the L_p-cohomologies of the open subsets $D' \subset X$ containing D in the CCX-closure. Pansu shows, in some cases, that this sheaf behaves as an ordinary sheaf of certain function spaces on $\partial_{\rm conf} X$. The sheaf $L_p\mathcal{H}^k$ for $k \geq 2$ appears especially interesting for non-hyperbolic spaces X (e.g. for semihyperbolic spaces of rank k) as it may be used to *define* an ideal boundary of X. (A naive idea would be to interpret closed k-forms at infinity as functions on k-dimensional subvarieties in the ideal boundary we want to define).

Negative remark and positive examples concerning $\mathcal{A}_\alpha$. Our definition of $\mathcal{A}_\alpha$ is rather unsatisfactory if $\partial_{\rm conf} X$ is a compact space of a general kind which receives few (if any) non-constant maps from polyhedra. On the other hand if $(CCX, \partial_{\rm conf} X)$ is homeomorphic to $(B^n,\ \partial B^n = S^{n-1})$, as is the case for manifolds of negative curvature, then our $\mathcal{A}_\alpha$ perfectly serves its purpose. For example, if $X = H^n_{\mathbb{R}}$ then

$$\inf \delta\mathrm{im}\mathcal{A}_\alpha = k/n-1 \quad \text{for every } \alpha \neq 0$$

as follows from the argument in Example (a) in 7.C$_1$. Similarly, Example (b) in 7.C$_1$. shows for $X = H^{2n}_{\mathbb{C}}$ that $\inf\delta\mathrm{im}\,\mathcal{A}_\alpha = k/2n$ for $k \leq n-1$ (which is smaller than that for $X = H^{2n}_{\mathbb{R}}$).

Translation algebra. As we mentioned earlier, the ℓ_p-cohomology goes along with uniformly proper Lipschitz maps and Lipschitz homotopies. Then, if we look at Lipschitz self-homotopies of X we see they span an algebra of operators acting on our complexes whose essential features are seen in the following

Definition. Let X_0 be a discrete metric space (e.g. the vertex space of a polyhedron X). Then the *translation algebra* $\mathcal{A}(X_0)$ consists of $(X_0 \times X_0)$-real (or complex) matrices $A = \{a(x,y)\}$, such that $a(x,y) = 0$ for $\operatorname{dist}(x,y) \geq d$ for some constant $d = d(A)$. This algebra acts on $L_p(X_0)$ for every X_0 and thus inherits a variety of topologies from the spaces of operators. The semigroup of *translations* of X (which are maps $\varphi : X_0 \to X_0$ with $\sup\limits_{x \in X_0} \operatorname{dist}(x, \varphi(x)) < \infty$) naturally embeds into $\mathcal{A}(X_0)$.

If two spaces X_0 and X_0' are bi-Lipschitz equivalent then the corresponding translation algebras are isomorphic. On the other hand ,if X_0 and X_0' are Hausdorff equivalent (i.e. $\mathrm{dist}_{\mathrm{Hau}}(X_0, X_0') < \infty$) then these algebras need not be isomorphic. For example, multiplying X_0 by a finite set F amounts to tensoring $\mathcal{A}(X_0)$ by the matrix algebra M_n for $n = \mathrm{card}\, F$. Thus the algebraic counterpart of quasi-isometry must be some kind of Morita equivalence. One can probably go quite far along these lines. For example, all of our L_p-discussion is likely to extend to the algebra theoretic framework (see [Hig], [Roe]$_3$ for some steps in this direction).

Appendix to §8 - Conformally hyperbolic groups and spaces.

Let X be a complete Riemannian manifold or a locally compact polyhedron with a complete geodesic metric (e.g. a Cayley graph of a group) and let us enumerate the properties of $\partial_{\mathrm{conf}} X$ which are immediate from the definition.

(1) For each $b \in \partial_{\mathrm{conf}} X$ there exists a ray $r \subset X$ starting from a given point $x_0 \in X$ and converging to b. (Recall that *a ray* is an isometric copy of $\mathbb{R}_+$ in X). Thus, there is a surjective continuous map $\partial_{\mathrm{geo}}(X, x_0) \to \partial_{\mathrm{conf}} X$, where the geo-boundary consists of the rays issuing from x_0 with an obvious topology.

(2) If X is connected at infinity, then $\partial_{\mathrm{conf}} X$ is connected, since $\partial_{\mathrm{conf}} X$ equals the intersections of its neighbourhoods in CCX.

(2′) There is a natural map from $\partial_{\mathrm{conf}} X$ to the space of ends $\mathrm{Ends} X$, such that the partition into the pull-backs of the points (ends) equals the partition of $\partial_{\mathrm{conf}} X$ into connected components.

(3) For every two points b_1 and $b_2 \neq b_1$ in $\partial_{\mathrm{conf}} X$ there exists a line γ (i.e. an isometric copy of $\mathbb{R}$ in X) which joins b_1 and b_2 (i.e. $\gamma(t) \to b_1$ for $t \to +\infty$ and $\gamma(t) \to b_2$ for $t \to -\infty$). Furthermore, there exists a compact subset $B \subset X$ depending on b_1 and b_2 such that every line in X between b_1 and b_2 intersects B.

(3′) Let $r_1(t)$ and $r_2(t)$ be geodesic rays in X converging to b_1 and $b_2 \neq b_1$ in $\partial_{\mathrm{conf}} X$, where t is the length parameter. Then

$$\mathrm{dist}\big(r_1(t),\ r_2(t)\big) \geq 2t - \mathrm{const}\ ,$$

for some constant depending on r_1 and r_2 (but not on t).

(I) ***Two Ends Proposition.*** *If X admits a co-compact isometry group action and $\partial_{\mathrm{conf}} X$ consists of exactly two points* then *X is quasi-isometric to $\mathbb{R}$ (and thus if X is a discrete group it is commensurable to $\mathbb{Z}$).*

Proof. If $\mathrm{card}\, \partial_{\mathrm{conf}} X = 2$ then also $\mathrm{card}\, \mathrm{Ends} X = 2$ by the properties (2) and (2′). Q.E.D.

(II) ***Convergence proposition.*** *The action of isometry group* $\mathrm{Iso}\, X$ *on $\partial_{\mathrm{conf}} X$ has the following convergence property of Gehring and Martin. Let $\mathcal{I} \subset$* $\mathrm{Iso}\, X$ *be a set of isometries such that* $\sup\limits_{g \in \mathcal{I}} \mathrm{dist}(g(x_0), x_0) = \infty$ *for some (and hence every) point $x_0 \in X$. Then there exist points b_+ and b_- in $\partial_{\mathrm{conf}} X$ (which may be equal) and a sequence $g_i \in \mathcal{I}$, $i = 1, 2, \ldots$, such that for every compact subset $B_+ \subset \partial_{\mathrm{conf}} X - \{b_-\}$ the maps $g_i | B_+$ uniformly converge to the constant map $B_+ \to b_+$ and for every compact subset $B_- \subset \partial_{\mathrm{conf}} X - \{b_+\}$ the maps $g^{-1} | B_-$ uniformly converge to $B_- \to b_-$.*

Proof. Since CCX is compact, there exists g_i such that $g_i(x_0)$ and $g_i^{-1}(x_0)$ converge for $i \to \infty$ and we take $b_+ = \lim\limits_{i \to \infty} g_i(x_0)$ and $b_- = \lim\limits_{i \to \infty} g_i^{-1}(x_0)$. Take a geodesic ray r_+ going from x_0 to b_+ and let r be a ray from x_0 to some point $b \in \partial_{\mathrm{conf}} X$. Then there are two possibilities.

(i) There are infinitely many values of i, such that $g_i(r)$ intersects a fixed compact subset in X;

(ii) $g_i(r)$ goes infinitely far away from x_0 for $i \to \infty$. In this case, clearly, $g_i(b) \to b_+$ for $i \to \infty$.

On the other hand, in case (i) one easily sees by looking at $g_i^{-1}(r_+)$ and r that r infinitely often approaches $g_i^{-1}(x_0)$ and thus converges to b_-. Thus every point $b \neq b_-$ does converge to b_+ under the action of g_i for $i \to \infty$ and this is clearly uniform on the compact subsets in $\partial_{\mathrm{conf}} X \{b_-\}$.

Remark. The convergence property was isolated by Gehring and Martin in [Ge-Ma] for actions on S^n as they were interested in (classical) quasi-conformal geometry. Yet all their general results do not use anything but compactness of S^n and thus can be applied in our case.

Let Γ be a discrete isometry group of X and $\partial^X_{\text{conf}}\Gamma \subset \partial_{\text{conf}} X$ denote the limit set of Γ, i.e. the set of the limit points of an orbit $\Gamma(x_0) \subset X$. This, clearly, does not depend on x_0 and if X/Γ is compact, then $\partial^X_{\text{conf}}\Gamma = \partial_{\text{conf}}\Gamma$ is an invariant of Γ alone.

Now the convergence property implies, according to Gehring-Martin (who have axiomatized the classical arguments from the theory of Kleinian groups) the following conclusions.

(III) *The limit set may be of the following three types.*

(i) $\text{card}\, \partial^X_{\text{conf}}\Gamma = 1$,

(ii) $\text{card}\, \partial^X_{\text{conf}}\Gamma = 2$,

(iii) $\text{card}\, \partial^X_{\text{conf}}\Gamma = \infty$,

where in case (iii) $\partial^X_{\text{conf}}\Gamma$ *is a perfect set and the action of* Γ *on* $\partial^X_{\text{conf}}\Gamma$ *is minimal (i.e. every orbit is dense). Furthermore, the action of* Γ *on* $CCX - \partial^X_{\text{conf}}\Gamma$ *is discrete.*

A transformation $\gamma \in \Gamma$ acting on CCX is called *hyperbolic* (or loxodromic) if there exist two fixed points γ_+ and $\gamma_- \neq \gamma_+$ in $\partial^X_{\text{conf}}\Gamma$, such that every compact subset in B in $CCX - \{\gamma_\pm\}$ is contracted by $\gamma^{\pm i}$ to $\gamma_\pm$ for $i \to \infty$, i.e. $\gamma^i|B$ converges, for $i \to \infty$, to the constant map $B \to \gamma_+$ and similarly, γ^{-i} converges outside γ_+ to the constant map to γ_-.

(IV) *If* $\text{card}\, \partial^X_{\text{conf}}\Gamma \geq 3$ *then the set of pairs* (γ_+, γ_-) *for all hyperbolic elements* $\gamma \in \Gamma$ *form a dense subset in the Cartesian square of* $\partial^X_{\text{conf}}\Gamma$.

This follows from Corollary 6.15 in [Ge-Ma] and the following pretty little lemma pointed out to me by D. Sullivan many years ago.

Let α *and* β *be hyperbolic transformations with* $\alpha_+ \neq \beta_-$ *and set* $\gamma(i) = \alpha^i\beta^i\alpha^{-i}\beta^{-i}$. *Then* $\gamma_+(i) \to \alpha_+$ *and* $\gamma_-(i) = \beta_-$ *as* $i \to \infty$.

Corollaries. (a) *If* $\text{card}\, \partial^X_{\text{conf}}\Gamma \geq 3$ *then the action of* Γ *on* $\partial^X_{\text{conf}}\Gamma$ *has no invariant measure and so* Γ *is non-amenable.*

(b) Γ *contains a free subgroup on two generators (assuming* $\text{card}\, \partial^X_{\text{conf}}\Gamma \geq 3$ *as earlier). In fact, if* α *and* β *are hyperbolic elements with* $\alpha_+ \neq \beta_+$, *then sufficiently high powers* α^i *and* β^j *freely generate* F_2.

Remark. One can also show with a little extra effort that α^i for large i freely generate a *free normal* subgroup in Γ (as I have not written the proof, I would feel safer with Γ having no torsion). It is also likely to be true for large *generic* elements (which are not proper powers) but I would not claim that at my present state of understanding the picture.

Definition of conformal hyperbolicity. A finitely generated group Γ is called conformally hyperbolic if $\text{card}\, \partial_{\text{conf}}\Gamma \geq 3$, where the conformal boundary refers to the action of Γ on its Cayley graph.

Basic examples. Every non-elementary word hyperbolic group is conformally hyperbolic. Every group with infinitely many ends is conformally hyperbolic.

Remarks. We could make up a more general definition by requiring the existence of some X with an isometric Γ-action, such that the limit set $\partial^X_{\text{conf}}\Gamma \subset \partial_{\text{conf}} X$ for *this* action had cardinality ≥ 3. In fact, it would be more logical to *start* with some "sufficiently hyperbolic" action of Γ on some compact (or just bounded) space B, where the hyperbolicity could be expressed either by the divergence property or by something more general as indicated in 8.2.K.-S. of $[\text{Gro}]_{14}$. Also notice that our definition of conformal hyperbolicity excludes *elementary* word hyperbolic groups, but this is just a matter of terminology.

Further examples of conformally hyperbolic groups. Floyd shows in [Flo] that every non-elementary geometrically finite discrete subgroup acting on $H^n_{\mathbb{R}}$, $n \geq 2$, is conformally hyperbolic. The interesting case here is where Γ has cusps and the basic geometric point made by Floyd (in a computational disguise, see proposition on p. 215 in [Flo] and the subsequent discussion) is as follows. Take a large sphere $S(R)$ of radius R in $H^n_{\mathbb{R}}$ and a horosphere S' passing through the center of S as in Fig. 23.

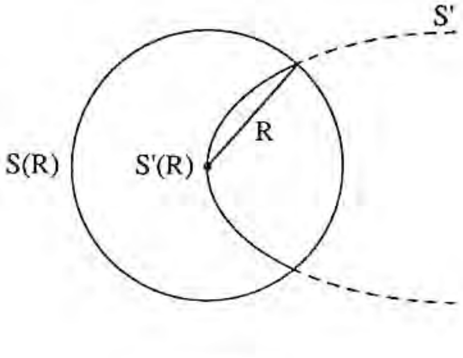

Figure 23

Then the intrinsic diameter D' of the part of the horosphere inside $S(R)$, denoted by $S'(R)$, is about D^α for the intrinsic diameter $D = D(R)$ of $S(R)$, where $\alpha = \frac{1}{2} < 1$. Using this Floyd shows that there is a (natural) continuous Γ-equivariant map of $\partial_{\text{conf}}\Gamma$ *onto* the ordinary limit set of Γ. In particular, if Γ is a (possibly non-cocompact) lattice, then there is a continuous surjective map $\partial_{\text{conf}}\Gamma \to S^{n-1}$ and for $n \geq 3$ this map is a homeomorphism.

Idea of Floyd's proof. The distortion of a geometrically finite Γ (embedded into $H^n_{\mathbb{R}}$ as an orbit) does not exceed the distortion of the horospheres. The conformal completion of H^n_R can be achieved with the conformal factor $\rho_\varepsilon(x) = D\big(R(x)\big)^{-1-\varepsilon}$, where $R(x) = \text{dist}(x_0, x)$ and $D(R)$ is the intrinsic diameter of the R-sphere. The function $\rho_\varepsilon(x)$ lifts to a function on Γ which, for a *small* $\varepsilon > 0$, is uniformly summable on "geodesic rays" in Γ as it decays with the rate $\big(\text{dist}(x_0, x)\big)^{-\frac{1}{\alpha}+\varepsilon}$. Thus ρ_ε restricted to Γ serves as a conformal factor for (the construction of the boundary of) Γ. (See [Flo] for the details.)

Generalizations. The above argument immediately extends to $H^{2n}_{\mathbb{C}}$, $H^{4n}_{\mathbb{H}}$ and $H^{16}_{\mathbf{Ca}}$ as $D'(R)$ there is also of the order of magnitude $\big(D(R)\big)^{\frac{1}{2}}$. It follows, in particular, that the conformal boundary of a lattice Γ acting on a symmetric space X of rank one is homeomorphic to the sphere $\partial_\infty X$, unless $X = H^2_{\mathbb{R}}$ (where $\partial_{\text{conf}}\Gamma$ for $X = H^2$ is a Cantor set for non-cocompact Γ which is virtually free in this case).

The estimate $\alpha < 1$ remains valid (by an easy argument) for geodesically complete manifolds X with $-4 + \varepsilon \leq K(X) \leq -1$, $\varepsilon > 0$. Thus every Γ acting on X with $\text{Vol}\, X/\Gamma < \infty$ is conformally hyperbolic and has ∂_{conf} homeomorphic to the sphere $\partial_\infty X$, unless $\dim X = 2$. (We suggest the reader would furnish the details.)

Now let X be a complete simply connected Riemannian manifold (or singular space) of non-positive curvature. If the Tits boundary of X is connected, then, clearly, the conformal boundary reduces to a single point. Then it seems likely that, in general, the conformal boundary must be equal to the set of the connected components of the Tits boundary (with possible mild extra assumptions on X, such as the existence of a cocompact Γ-action). A more specific *conjecture* reads:

Let X be the universal covering of a compact connected manifold V with $K(V) \leq 0$ having all sectional curvatures strictly negative at some point $v_0 \in V$. Then X (and hence $\Gamma = \pi_1(V)$) is conformally hyperbolic.

1. ***Encouraging example.*** *Suppose all flatness of V is confined to a locally convex subset $V' \subset V$, i.e. $K(v) \leq -\varepsilon < 0$ at every point $v \in V - V'$, where $\pi_1(V')$ has infinite index in $\pi_1(V)$. Then the universal covering X of V (and hence $\Gamma = \pi_1(V)$) is conformally hyperbolic.*

Idea of the proof. Take the product $V \times [0, \infty)$ with the Riemannian metric

$$(\exp -\lambda t) g_V + dt^2 \quad \text{for some } \lambda > 0$$

and let $V^+ \subset V \times [0, \infty)$ be the union of $V \times [0, 1]$ and $V' \times [0, \infty)$ as in Fig. 24 below.

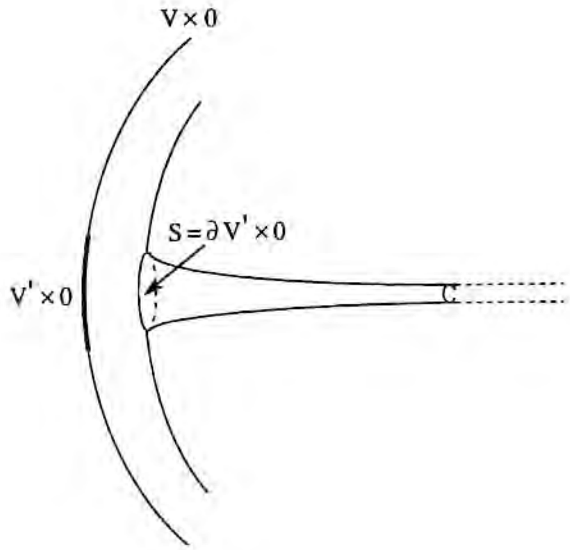

Figure 24

The universal covering X^+ of V^+ is, clearly, non-elementary hyperbolic and we claim there exists a (natural) surjective continuous equivariant map $\partial_{\mathrm{conf}} X \to \partial_{\mathrm{conf}} X^+ = \partial_\infty X^+$. This is achieved as in the previous examples with a suitable metric on X^+ with negative curvature. First we observe that the metric $(\exp -\lambda t) g_V + dt^2$ does have negative curvature, but X^+ does not because of the boundary. (Recall, the metric in X^+ is defined with the shortest curves in X^+ which may touch the boundary and the curvature is understood in the CAT-sense, see [Gro]$_{14}$). In fact, X^+ has $K < 0$ everywhere except on the subset $\widetilde{\Sigma} = \partial X' \times 1$, covering $\Sigma = \partial V' \times 1$ as an elementary consideration shows. Furthermore, if $\lambda = 0$, then V^+ (and X^+) has $K \leq 0$ being a union of *two* locally convex subsets in a space with $K \leq 0$. On the other hand, the curvature of the metric $\exp -\lambda t g_V dt^2$ for $\lambda > 0$ is strictly negative on $\Sigma = \partial V' \times 1$. It follows, by an easy argument, that there is a slight perturbation of the boundary of V^+ near Σ which makes the curvature strictly negative (in the CAT-sense) on all of the perturbed X^+. Moreover, one can arrange the matter with a sufficiently small $\lambda > 0$, such that the curvature of the perturbed X^+ becomes everywhere less than the curvature of (the hyperbolic plane with) the metric $(\exp -\lambda' t) ds^2 + dt^2$ for a given $\lambda' < \lambda$. It follows that the horospheres in $X' \times [2, \infty)$ corresponding to the rays $x' \times [0, \infty)$ have $D' \approx D^\alpha$ for $\alpha < 1$ as in the discussion around Fig. 23, and then Floyd's argument applies.

Negative Remark. The above argument is unduly artificial. There must be a more direct way to see the map $\partial_{\mathrm{conf}} X \to \partial_{\mathrm{conf}} X^+$ without any appeal to a specific metric. For example, one should prove the above whenever X^+ is hyperbolic with no assumption on the curvature of V.

Questions. (a) Let Γ act on a hyperbolic space X with an infinite limit set Λ in $\partial_\infty X$. Is then Γ conformally hyperbolic ? Does there exist a surjective Γ-equivariant map from $\partial_{\mathrm{conf}}\Gamma$ to Λ ? For example, is every finitely generated non-elementary subgroup of a word hyperbolic group conformally hyperbolic ?

(b) Let $\overline{\Gamma}$ be the quotient group of a conformally hyperbolic group Γ obtained by attaching a generic relation to Γ. Is then $\overline{\Gamma}$ conformally hyperbolic ? For example, does every conformally hyperbolic group Γ admit an infinite (free) normal subgroup N with Γ/N being conformally hyperbolic ? Summing up, does every theorem about word hyperbolic groups have a conformally hyperbolic counterpart ?

(c) Let $\Gamma = \{a, b \mid w_1, w_2, \ldots,\}$ for an infinite sequence of relations which is sufficiently sparse and generic. What is the chance of Γ to be conformally hyperbolic and/or to have $\ell_p H^1(\Gamma) \neq 0$ for large p ?

§9. Density of random groups and other speculations.

Let us start by recalling elementary (and well-known, I believe) combinatorics of random finite (sub)sets.

9.A. Density and dimension. Given two finite sets A and B we write

$$\operatorname{dens} A = \alpha \operatorname{dens} B$$

if $\operatorname{card} A = (\operatorname{card} B)^\alpha$. More generally, we write

$$\sum_i \alpha_i \operatorname{dens} A_i = 0$$

for $\prod_i (\operatorname{card} A_i)^{\alpha_i} = 1$ and

$$\sum_i \alpha_i \operatorname{dens} A_i \geq 0$$

for $\prod_i (\operatorname{card} A_i)^{\alpha_i} \geq 1$.

If we deal with subsets of a fixed set C we may normalize by setting $\operatorname{dens} C = 1$ and $\operatorname{dens} A = \exp\big((\log \operatorname{card} A)/(\log \operatorname{card} C)\big)$. Then we define *codensity* by

$$\operatorname{codens} A = 1 - \operatorname{dens} A\ .$$

Intuitively one should think of $\operatorname{dens} A$ as $\dim A/\dim C$. For example, if C is a vector space over a finite field then every affine subspace does have $\operatorname{dens} A = \dim A/\dim C$.

One knows, the *generic* affine subspaces A_1 and A_2 satisfy the *intersection formula*

$$\operatorname{codim} A_1 \cap A_2 = \operatorname{codim} A_1 + \operatorname{codim} A_2$$

with the convention

$$\operatorname{codim} A > \dim C \Leftrightarrow A = \emptyset\ .$$

Now, we want to generalize this to (arbitrary) random subsets of a finite set C, where $\operatorname{card} C \to \infty$.

Intersection formula. *Random subsets A_1 and A_2 in C satisfy*

$$\operatorname{codens}(A_1 \cap A_2) = \operatorname{codens} A_1 + \operatorname{codens} A_2$$

with the convention

$$\operatorname{codens} A > 1 \Leftrightarrow A = \emptyset\ .$$

Explanation. Let us spell out the meaning of this "random" talk. Take numbers α_1 and α_2 in the interval $[0,1]$ and three (small) positive numbers ε_1, ε_2 and δ. Let c_1 denote the cardinality of the set of subsets $A \subset C$ with $|\operatorname{codens} A - \alpha_1| \leq \varepsilon_1$ and c_2 be the same number associated to α_2 and ε_2. Then we define c as the cardinality of the set of pairs (A_1, A_2) satisfying

$$\left.\begin{array}{rcl} |\operatorname{codens} A_1 - \alpha_1| & \leq & \varepsilon_1 \\ |\operatorname{codens} A_2 - \alpha_1| & \leq & \varepsilon_2 \end{array}\right\} \tag{$*$}$$

$$|\operatorname{codens}(A_1 \cap A_2) - \alpha_1 - \alpha_2| \leq \delta\ . \tag{$**$}$$

Now, the precise statement behind the intersection formula is as follows.

If $\operatorname{card} C \to \infty$ *for* α_1, α_2, ε_1, ε_2 *and* δ *being kept fixed, then*

$$c/c_1c_2 \to 1\ . \tag{$+$}$$

Notice that the cardinality of the set of pairs (A_1, A_2) satisfying $(*)$ equals c_1c_2 and so $(+)$ tells us indeed that the relation

$$\operatorname{codens} A_1 \cap A_2 = \operatorname{codens} A_1 + \operatorname{codim} A_2$$

is satisfied with probability $p \to 1$ for $\operatorname{card} C \to \infty$.

Notice that $(+)$ does not tell us much for $\alpha_1 + \alpha_2 > 1$, but the intersection formula claims, according to our convention, that if $\alpha_1 + \alpha_2 > 1$, then the intersection $A_1 \cap A_2$ is empty with probability $p \to 1$. Namely, if we choose ε_1 and ε_2 such that $\alpha_1 + \alpha_2 - \varepsilon_1 - \varepsilon_2 > 1$, and let d denote the cardinality of the set of pairs (A_1, A_2) satisfying $(*)$ and for which $A_1 \cap A_2 = \emptyset$, then

$$d/c_1c_2 \to 1 \quad \text{for } \operatorname{card} C \to \infty .$$

Before engaging into the proof (which amounts to a trivial computation based on the fact that $1 < e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n < \infty$), we state several properties of random sets similar to the intersection formula where we assume the reader understands "random talk".

Selfintersection formula. *The selfintersection Σ of a random map $\varphi : A \to B$ satisfies*

$$\operatorname{dens} \Sigma = 2 \operatorname{dens} A - \operatorname{dens} B$$

where $\operatorname{card} B \to \infty$ *and where the relation* $\operatorname{dens} \Sigma < 0$ *signifies* $\Sigma = \emptyset$. *Thus the relation*

$$\operatorname{dens} A < \frac{1}{2} \operatorname{dens} B$$

implies that a generic map is injective. (Recall that Σ is defined as the subset in $A \times A$ consisting of the pairs (a_1, a_2) with $\varphi(a_1) = \varphi(a_2)$). *In particular, if* $\operatorname{dens} A < \operatorname{dens} B$ *then φ is injective away from a subset in A of positive codensity.*

Surjection property. *If* $\operatorname{dens} A > \operatorname{dens} B$ *then a random map $A \to B$ is onto.*

Intersection with a fixed subset. *Let $B \subset C$ be fixed and $\varphi : A \to C$ be a random map. Then if* $\operatorname{dens} A + \operatorname{dens} B < 1 (\underset{\text{def}}{=} \operatorname{dens} C)$ *then $\varphi(A) \cap B = \emptyset$. On the contrary, if* $\operatorname{dens} A + \operatorname{dens} B > 1$, *then $\varphi(A) \cap B \neq \emptyset$. Moreover,*

$$\operatorname{dens} A - \operatorname{dens} \varphi^{-1}(B) = \operatorname{codens} B .$$

It is time to make a little computation. We start with the intersection of a random map with fixed subsets. Denote by a, b and c the cardinalities of A, B and C respectively and observe that the probability P of a map $A \to C$ to miss B equals $\left(1 - \frac{b}{c}\right)^a$. Thus $p \to 1$ for $ab/c \to 0$ and $p \to 0$ for $ab/c \to \infty$. But $ab/c \to 0$ for $\operatorname{dens} A + \operatorname{dens} B < \operatorname{dens} C (= 1)$ and $ab/c \to \infty$ for $\operatorname{dens} A + \operatorname{dens} B > \operatorname{dens} C$, and so the hit-or-miss alternative is governed by the relations

$$\operatorname{dens} A + \operatorname{dens} B \lessgtr 1 = \operatorname{dens} C$$

Notice that little changes in the above computation if we apply it to random *injective* maps φ, provided $\operatorname{dens} A < 1$. Again, $\varphi(A)$ hits or misses B according to the inequalities $\operatorname{dens} A + \operatorname{dens} B \lessgtr 1 = 1 = \operatorname{dens} C$

(and again no information is available for $\operatorname{dens} A + \operatorname{dens} B = 1$). Then the same conclusion applies to random subsets $A \subset C$ which amounts to random injective maps $A \to C$ modulo permutations of A. Now, we can evaluate the density of $\varphi^{-1}(B) \subset A$ by intersecting it with a random subset $A_0 \subset A$, which gives us the relation

$$\operatorname{dens} A - \operatorname{dens} \varphi^{-1}(B) = \operatorname{dens} C - \operatorname{dens} B ,$$

and a similar argument proves the rest of the above "random" claims.

Density of multiple intersections. *Let $A_i \subset C$, $i = 1, \ldots, k$ be random subsets. Then*

$$\operatorname{codens} \bigcap_{i=1}^{k} A_i = \sum_{i=1}^{k} \operatorname{codens} A_i .$$

Let $\varphi_i{:}A_i \to C$ *be random maps. Then, the "intersection"*

$$I = \{a_1, \ldots, a_k \mid \varphi_1(a_1) = \varphi_2(a_2) = \ldots = \varphi_k(a_k)\} \subset A_1 \times \ldots \times A_k$$

has

$$\operatorname{dens} I = \sum_{i=1}^{k} \operatorname{dens} A_i - (k-1) \operatorname{dens} C \ .$$

Let $\varphi : A \to C$ *be a random map. Then the set of the k-multiple points*

$$I_k = \{a_1, \ldots, a_k \mid \varphi(a_1) = \varphi(a_2) = \ldots = \varphi(a_n)\}$$

has

$$\operatorname{dens} I_k = k \operatorname{dens} A - (k-1) \operatorname{dens} C \ .$$

The proof of these properties can be obtained by an obvious generalization of the above computation, but there is another more conceptual approach based on a (better) definition of a *densable random subset.* This is a measure on the set of all subsets $\mathrm{Sbs}(C)$ defined for every finite set C, such that the distribution of the density function $\operatorname{dens} : \mathrm{Sbs}(C) \to [0,1]$ for $A \mapsto \operatorname{dens} A$, $A \in \mathrm{Sbs}\, C$, weakly converges to a point mass located at some point $x \in [0,1]$ which is what we call *the density* of the random subset in question. The set theoretic operations over subsets obviously extend to random subsets that are measures on $\mathrm{Sbs}\, C$. For example, one may intersect random subsets A and B in C but for densable A and B the intersection is not, in general, densable and so $\operatorname{dens} A \cap B$ is not automatically defined. There are various classes of densable subsets which are stable under set theoretic operations. The basic example (which is all we need for the present paper) is constituted by the random subsets defined by measures *invariant under the permutations of* C. These are invariant under intersection and the intersection of densable A and B in this class is again densable with $\operatorname{codens} A \cap B = \operatorname{codens} A + \operatorname{codens} B$, which is just a reinterpretation of what we have said earlier. But now we can proceed by induction and conclude to a similar result for intersections of several (≥ 2) subsets.

Remark on other classes of densable random subsets. We have already mentioned that random affine subspaces in an affine space C over a finite field F satisfy the intersection formula where the randomness refers to measures which are invariant under the *affine* transformations of C and where there are several choices of possible asymptotics. Namely, one may have F fixed and let $\dim_F C \to 0$. Alternatively one may let $\operatorname{card} F \to \infty$ (or $\deg F \to \infty$ over the ground field). One may treat similarly random projective subspaces in projective spaces P over F and, more generally, one can do the same with the sets of F-points of random (non-linear) algebraic varieties. In fact, the algebraic geometry suggests an alternative (non-probabilistic) approach to the intersection theory of random sets.

Concentration of measure. The localization of density to a single value $x \in [0,1]$ is a general phenomenon in statistics which was geometrized by Paul Levy and then deeply developed by V. Milman (see [Mil-Sch], [Pis]). In the case of the intersections we have, under quite general circumstances (i.e. for measures invariant under some *transitive* group G acting on C), the averaged formula for $\operatorname{dens}(A \cap B)$ as

$$Av_A \operatorname{card} A \cap B = (\operatorname{card} A)(\operatorname{card} B) / \operatorname{card} C \ ,$$

(where B is fixed and A is moved by the group G). But it is not always true that the density function $\operatorname{dens}(A \cap B)$ stays all of the time close to the average value; yet the Levy-Milman philosophy suggests it must be so in many interesting cases.

Symmetries. These seem indispensable for a good intersection theory, though they may be more general than just group-theoretical. This is seen in the case of random algebraic varieties (either defined over a finite ground field F_0, where $\deg F/F_0 \to \infty$, or defined over $\mathbb{Z}$ with $F = \mathbb{Z}_p$ for $p \to \infty$). Another combinatorially interesting example is where C is of the form $C = \underbrace{B \times B \times \ldots \times B}_{i}$ for $i \to \infty$ and where one allows symmetries equating "fibers" $B \times b_0 \times \ldots \times b_0 \subset B$ with diagonals $(\{b_1 = b_2\} \subset B \times B) \times \underbrace{b_0 \times \ldots \times b_0}_{i-2}$.

9.B. Density of random presentations of groups. Consider groups given by $p \geq 2$ generators and some relations of a fixed length ℓ (or, may be, of length close to a given value ℓ). The set of all words C_ℓ of length ℓ has cardinality $N = (2k-1)^\ell$ and we want to study random subsets (of relations) $A \subset C_\ell$ of a given density d as $\ell \to \infty$.

Phase transition proposition. *If $d < \frac{1}{2}$ then the (random) group Γ defined by A is a hyperbolic group of dimension two and if $d > \frac{1}{2}$ then Γ is trivial.*

Proof. Let us first decide when two (cyclic) words in A have a common subword of length $\ell' < \ell$. The set $C_{\ell'}$ of the words of length ℓ' clearly has

$$\operatorname{dens} C_{\ell'} = (\ell'/\ell) \operatorname{dens} C_\ell \; .$$

The restriction of a word $a \in A$ to a subword of length ℓ' defines a map $A \to C_{\ell'}$ which (for a *random* A of density d) has no double point for

$$2d < \operatorname{dens} C_{\ell'} = \ell'/\ell \; .$$

Since the number of cyclic permutation of a word equals ℓ which is incomparably smaller than the cardinality of C_ℓ, we conclude that,

If $2d < \ell'/\ell$, then no two words, or cyclic permutations of these, in a random subset $A \subset C_C$ of density $< d$ have a common subword of length ℓ' and so Γ satisfies the ℓ'/ℓ-cancellation condition. In particular, if $d < \frac{1}{12}$ (i.e. $\operatorname{card} A$ *is significantly smaller than $(2k-1)^{\ell/12}$) then Γ is a hyperbolic small cancellation group and if $d > \frac{1}{12}$ then (the random presentation of) Γ is not 1/6.*

Now let us compute the density of the set of Dehn diagrams of a given combinatorial type. We assume our diagram is non-degenerate, i.e. is a topological disk built out of m 2-cells corresponding to words in A which meet in pairs over certain subwords (edges) of lengths $\ell_1, \ldots \ell_k$ as in Fig. 25

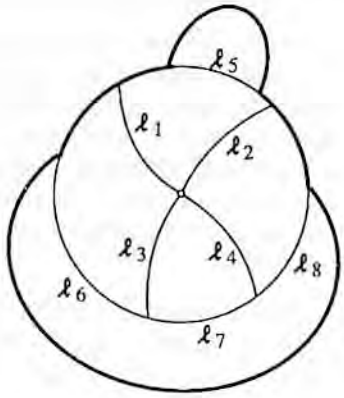

Figure 25

The density of the set of m-tuples of words in A has density md and every common edge of length ℓ_i "imposes (ℓ_i/ℓ)-conditions", i.e. reduces the density by ℓ_i/ℓ. Thus the density of the set of the diagrams equals

$$md - \sum_{i=1}^{k} \ell_i/\ell = md - m/2 + \text{perimeter}/\ell \; ,$$

where "perimeter" is the sum of the lengths of the free edges in the diagram. Thus, a diagram has negative density (i.e. does not appear at all) if

$$\text{perimeter}/\ell < m\left(\frac{1}{2} - d\right) \; . \qquad (*)$$

Now, we assume $d < \frac{1}{2}$ and observe that the reciprocal of (*) amounts to the linear isoperimetric inequality

$$\text{length (boundary)} \leq \left(\frac{1}{2} - d\right) \text{Area (Diagram)} ,$$

where we normalize the geometry by making $\ell = 1$ and the area of each 2-cell also equal one. This isoperimetric inequality applies, a priori, to the diagrams consisting of at most $m = m(\ell)$ cells where $m \to \infty$ as $\ell \to \infty$, and by elementary hyperbolic geometry (see $[\text{Gro}]_{14}$) this implies, in turn, a linear isoperimetric inequality for all Dehn diagrams. Thus Γ is indeed hyperbolic of dimension 2. What remains to show is the triviality of Γ when the density d of the generating set is $< \frac{1}{2}$ but this we leave as an easy exercise to the reader.

Remarks and open questions. (a) The sketch of the argument we gave needs filling in the details (e.g. concerning irreducibility of cyclic permutations of words) but this is not difficult.

(b) What happens at $d = \frac{1}{2}$? For example, is there some asymptotic expansion at $d = \frac{1}{2}$ which reveals the presence of non-trivial groups which are not world hyperbolic of dimension two ?

(c) The number $d = \frac{1}{2}$ appears in a closely related context in [Kes], [Gri] and [Cham]. Namely, let $N \subset \mathsf{F}_p$ be an infinite normal subgroup and $N_\ell = N \cap C'_\ell$, where C'_ℓ is the ℓ-ball in the free group F_p. Then there exists a limit

$$d(N) = \lim_{\ell \to \infty} \text{dens}\, N_{2\ell}$$

which satisfies $\frac{1}{2} \leq d(N) \leq 1$. One knows (see [Kes]) that F_p/N is amenable if and only if $d(N) = 1$. On the other hand, groups with sufficiently small cancellation have $d(N) \to \frac{1}{2}$ as the lengths of the relations in question go to ∞ (see [Cham]). It seems to imply that normal subgroups $N = N(A) \subset \mathsf{F}_p$ generated by random subsets $A \subset C_\ell$ with $\text{dens}\, A \leq d < \frac{1}{2}$ have $d(N) \to \frac{1}{2}$ for $\ell \to \infty$.

(d) Let $A \subset \mathsf{F}_p$ be a random *infinite* subset such that the intersections $A \cap C_\ell$ have densities $\leq d < \frac{1}{2}$ for $\ell \geq 1$ (where C_ℓ denotes the ℓ-sphere in F_p). Does it follow that the groups $\mathsf{F}_p/N(A \cap C'_\ell)$ are hyperbolic for all ℓ and the group $\mathsf{F}_p/N(A)$ is infinite ?

The difficulty in proving the hyperbolicity of $\mathsf{F}_p/N\,(A \cap C_\ell)$ appears as the words in $A \cap C'_\ell$ may have highly diverse lengths in the interval $[1, \ell]$ and then the present techniques only allow very sparse subsets A (see $[\text{Gro}]_{14}$, [Cham]).

(e) One can imagine the following procedure to produce subsets $A \subset C_\ell$ of given density d, which becomes especially transparent if we work in a lattice Γ_0 acting on the hyperbolic space $H^n_{\mathbb{R}}$ rather than in the free group F_p. Namely, take a subset $A_\infty \subset S^{n-1} = \partial_\infty H^n_{\mathbb{R}}$, then take a cone $\overline{A}$ over A_∞ from some point x_0 in $H^n_{\mathbb{R}}$, intersect $\overline{A}$ with the sphere of radius ℓ in $H^n_{\mathbb{R}}$ around x_0, take the ρ-neighbourhood of this intersection for a fixed large number ρ and finally intersect this neighbourhood with Γ_0 embedded to $H^n_{\mathbb{R}}$ by the orbit map $\gamma \mapsto \gamma(x_0) \in H^n_{\mathbb{R}}$. The resulting subset, say $A_\ell \subset \Gamma_0$, usually has $\text{dens}\, A_\ell = (\dim_{\text{Hau}} A_\ell)/n - 1$. However, such subsets A_ℓ are far from being random for those $A_\infty \subset S^{n-1}$ which normally appear in geometry. For example, if A_∞ is a closed set, then the subsets A_ℓ become extremely special and our counting argument does not apply. (Notice that the "random discussion" e.g. "phase transition" extends from F_p to an arbitrary non-elementary hyperbolic group Γ_0). Yet one may think that for certain (random) subsets $A_\infty \subset S^{n-1}$ the (generic) structure of the groups $\Gamma_0/N(A_\ell)$ can be understood.

(f) There are other probability settings which may lead to interesting groups. For example, let us take a projective space P (or a more general algebraic variety over a field F) for the set of generators and let the set of relations be the set of F-points in $\mathsf{P}^\ell = \underbrace{P \times \ldots \times P}_{\ell}$ of a (random) algebraic subvariety A in P^ℓ. This situation is the nearest to what we have studied earlier if F is a finite field and so, besides ℓ, the cardinality of F and the degree of F over the ground field appear as asymptotic parameters. But the groups defined in this fashion look quite pretty also for infinite (e.g. algebraically closed) fields F (and also for rings such as $\mathbb{Z}$).

(g) The probabilistic approach to the notion of a "generic" group is rather artificial as there is no apparent natural measure (or class of measures) in the space of (infinite presentations of) groups. But a natural topology is present. (Compare compactness theorems in model theory, see [Barw], Part I, §§ 2.4

and 4.2.) Generic (properties of) groups in this topology has been recently investigated by Champetier (see [Cham]). It would be interesting to understand the continuity (and measurability) property of our asymptotic invariants such as conf dim $\partial_\infty\Gamma$ and the ℓ_2-Betti numbers. For example, does the conformal dimension of a random group depend only on density or is conf dim able to distinguish groups with a given density ? This is related to the ergodicity problem of the isomorphism relation on the space of presentations. In this regard E. Ghys asks if there is any measure (or measure class) on the space of presentations of groups invariant under the isomorphism relation.

(h) What should be a hyperbolic (e.g. small cancellation) theory for finitely generated (and finitely presented) algebras (associative or Lie, for example) over a fixed field ? What about random algebras ? How are the (non)-separation properties (in the Hausdorff sense) of the space of presentations modulo the isomorphism relation connected to the (logical) classification theory ? (See [She]).

(i) We conclude with more concrete "random" questions. Does a generic group with p generators and $q \gg p$ relations admit a non-trivial linear representation ? (It is clear, such representations are numerous for $p \gg q$ but no general theory is available yet. Also notice that the most powerful geometric tool for the representation theory is provided by a higher rank generalization of Thurston's narrow simplices). Does a generic finitely presented group contain a non-free infinite subgroup of infinite index ? Our discussion on groups presented by $A \subset C_\ell$ with $\operatorname{dens} A = d < \frac{1}{2}$ indicate that these contain no surface groups of genus $\leq g$ for $g \to \infty$ with $\ell \to \infty$, but we do not know if some surface group of high genus is always present in a random (or every non-virtually free) hyperbolic group. Do generic groups with $q \gg p$ satisfy T-property? Do they, on the contrary, have $\ell_2 H^1 \neq 0$, in the case of small density of the set of relations ? Is there a relation between this density and the value of the minimal p for which $\ell_p H^1 \neq 0$?

Notice that our "generic" and/or "random" questions have a good chance to have yes-no answers since the isomorphism relation on the space of presentations is quite tangled. Namely, if the genericity refers to the topology of this space, one knows many orbits of these relations are not closed and the quotient space is highly non-Hausdorff (i.e. it strongly violates separation axioms), which makes it hard for a *non-constant continuous* function on the space of presentations to be invariant under the isomorphism relation on this space (compare [Cham]. Apparently no non-trivial function of this kind is known as was pointed out to me by E. Ghys.) Similarly, in the "random" framework, one expects certain ergodicity of the isomorphism relation which would force measurable invariants of groups to be constant (in the agreement with the concentration of the measure phenomenon of Levy-Milman which is not always directly linked to ergodicity). But even if we encounter a non (almost everywhere) constant invariant (function on the space of presentation) we may extract from it numerical information (e.g. the average over the space of presentations) which carry non-ambiguous statistical data whilst the values of this invariant at individual groups may be well beyond reach.

Let us indicate another possible use of random groups.

Conjecture. For every $p \geq 2$ and $k \geq 6$ there exists a (small cancellation) $\frac{1}{k}$-group Γ on p generators without torsion which is not isomorphic to any $\frac{1}{k+1}$-group and moreover, does not even embed into any $\frac{1}{k+1}$-group Γ'.

The generic reformulation of this conjecture claims that a *random* $\frac{1}{k}$-group is not $\frac{1}{k+1}$. In fact, one expects that a *random* group given by a (relation) set $A \subset C_\ell$ of density $\geq \frac{1}{2k}$ is not isomorphic (for large ℓ) to any $\frac{1}{k+1}$-group and may not even appear as a subgroup in a $\frac{1}{k+1}$-group. This agrees with the intuitive idea that there are significantly more isomorphism classes of groups (presented by sets $A \subset C_\ell$) with density $d < \frac{1}{2}$ than of those with density $d' < d$. Another "generic" idea behind the conjecture is that every subgroup Γ' in a random group of density $d < 1/2$ has density $d' \leq d + \varepsilon$, where $\varepsilon \to 0$ when the implied number ℓ (i.e. the length of the relations) goes to infinity.

Here are two related problems:

(1) Evaluate the number of isomorphism classes of groups on p generators presented by subsets $A \subset C_\ell$ of density $\leq d$ for a given $d < \frac{1}{2}$. Notice that the number of such *presentations* is about 2^{N^d} for $N = (2p-1)^\ell$ and the (isomorphism) equivalence relation can hardly reduce this number by much.

(2) Let us introduce a *density of an (individual) finitely generated group* Γ. This is defined as the minimal number d such that $\Gamma = \mathsf{F}_p/N$ for some p and a normal subgroup $N \subset \mathsf{F}_p$, where the following is satisfied. There exists an integer $\ell \geq 1$, such that the subset N_ℓ of the words of length $\leq \ell$ normally generate N and such that for all $k = 1, 2, \ldots, \ell$ the density of the intersection $N \cap C_k$ in C_k (where C_k is the set of all irreducible words of length k) does not exceed d. It is not hard to show that Γ presented by a random subset $A \subset C_\ell$ of density $d < \frac{1}{2}$ has $\operatorname{dens}\Gamma \leq d + \varepsilon$ where $\varepsilon \to 0$ with probability one for $\ell \to \infty$. The problem we raise reads,

Does $\operatorname{dens}\Gamma$ *converge to d with probability one for* $\ell \to \infty$?

The positive answer here amounts to saying that a random presentation is optimal in the sense that it minimizes the density.

A possible approach to the above conjectures may be based on the existence of a certain a priori bound on Lipschitz constants of isomorphisms (and monomorphisms) between hyperbolic groups (discovered by Thurston with his narrow simplices and then generalized by other people with the final result due to I. Rips). This becomes especially clear if we start with a sufficiently rigid hyperbolic group Γ_0 instead of F_p and look at what happens as we add a set A of random relations of length ℓ. A good example of a rigid Γ_0 would be a cocompact lattice acting on a hyperbolic space of dimension ≥ 3, where the outer automorphism group of Γ_0 is known to be finite by the Mostow rigidity theorem. Now, it seems plausible that for two random subsets $A \subset C_\ell(\Gamma_0)$ and $A' \subset C_{\ell'}(\Gamma_0)$ with (low) densities $\leq d_0 = d_0(\Gamma) > 0$ the isomorphism relation between the quotient groups $\Gamma = \Gamma_0/N(A)$ and $\Gamma' = \Gamma_0/N(A')$ is equivalent to the existence of an automorphism of Γ_0 mapping $N(A)$ onto $N(A')$, which makes $N(A) = N(A')$ modulo a *finite* group (i.e. Out Γ_0). (The situation here is rather opposite to the ergodicity of the isomorphism relation on the set of presentations we have emphasized earlier.) Then it becomes evident that $\operatorname{dens}\Gamma$ is essentially equal to $\operatorname{dens}A$ and that with probability $\to 1$, for $\ell \to \infty$, pairs of presentations (of low density) give mutually non-isomorphic groups, so that the number of mutually non-isomorphic groups $\Gamma = \Gamma_0/N(A)$ is, roughly, double exponential in ℓ.

Let us conclude by a Mostow type rigidity problem where the positive solution would settle the above conjecture. Let Γ_0 and Γ_0' be the fundamental groups of complete locally symmetric spaces V_0 and V_0' with negative curvatures and finite volumes. Let W_0 and W_0' be compact totally geodesic submanifolds in V_0 and V_0' respectively and let Γ and Γ' be factorgroups of Γ_0 and Γ_0' by the normal subgroups normally generated by the fundamental groups of the connected components of $W_0 \subset V_0$ in $\Gamma_0 = \pi_1(V_0)$ and of $W_0' \subset V_0'$ in $\Gamma_0' = \pi_1(V_0')$. We ask when an isomorphism between Γ and Γ' is induced by an isometry $V_0 \leftrightarrow V_0'$ which sends $W_0 \leftrightarrow W_0'$ and we conjecture this is so if $\dim V_0 \geq 3$ and the submanifolds $W_0 \subset V_0$ and/or $W_0' \subset V_0'$ are sufficiently rare, i.e. satisfy some strong small cancellation condition, (as happens, for example, for random systems of closed geodesics of low density).

Notice that there is a parallel (and easier) rigidity problem in the case of the codimension two submanifolds W_0 and W_0' where instead of the factor groups $\Gamma_0/N(\pi_1(W_0))$ and $\Gamma_0'/N(\pi_1(W_0'))$ we take the fundamental groups of the complements $V_0 - W_0$ and $V_0' - W_0'$ for Γ and Γ'. More generally, we may take the Galois groups of the universal ramified coverings of V_0 and V_0' with prescribed orders of ramifications at W_0 and W_0' (where order $= \infty$ corresponds to $\pi_1(V_0 - W_0)$).

There are three avenues to the study of these rigidity problems parallel to the approaches which proved successful in the locally symmetric case.

1. *Simplicial road.* Here one wants to understand what happens to the simplicial norm of the fundamental class of $\Gamma_0 = \pi_1(V_0)$ when one passes to $\Gamma = \Gamma_0/N(\pi_1(W_0))$ (or $\Gamma = \pi_1(V_0 - W_0)$). Observe that the case of a 3-manifold V_0 minus several closed geodesics fits into Thurston's theory and our rigidity problem for $V_0 - W_0$ (and for ramified covers) should follow from Thurston's rigidity theorem. Also notice that for many applications a compactness result would serve as well as rigidity which makes Thurston's compactness techniques (using narrow simplices) look very promising.

2. *Conformal road.* One must understand the geometry of the conformal boundary $\partial_{\text{conf}}\Gamma$. The first question here is the conformal dimension of $\partial_{\text{conf}}\Gamma$.

3. *Harmonic road.* The group Γ (and Γ') is represented by the (singular) space V obtained from V_0 by attaching cones to the connected components of W_0. In some cases this space has negative curvature

and one may look at harmonic maps of a locally symmetric space (e.g. V_0') into V. This may work if the symmetric model (for V_0') is $H^{2n}_{\mathbf{C}}$, $H^{4n}_{\mathbf{H}}$ or $H^{16}_{\mathbf{Ca}}$.

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