An introduction to combinatorial and geometric group theory

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This course is meant to be a fully self-contained introduction to combinatorial and geometric group theory, building only off the axioms of group theory.

Although the reader is expected to know what a string of symbols or a graph is and to use some arguments use intuitive topology, no in-depth knowledge is assumed and this text was written to be accessible to an incoming graduate student, or an advanced undergraduate with some exposure to algebra, analysis, and proofs. The goal of this course is to prepare and motivate a student to study word (or Gromov) hyperbolic groups.

The first part of the course gives a rigourous construction of the free group, which in turn leads to group presentations: a way to identify any group as a string of symbols from some alphabet subject equipped with rewriting rules. Although this gives a unified framework in which to study groups, it turns out that it is impossible in general to decide if two words represetning group elements in some group given by a presentation are equal, or whether two group presentations actually define isomorphic groups. In general there is no canonical presentation for a given group.

The second part presents a basic concept fo geometric group theory: every choice of generating set for a group, makes the group into a metric space, realized by a Cayley graph. Although in this way a group will admit different metrics, they will all be quasi-isometric. In this way, although a group has no canonical presentation or generating set, we can canonically associate to it a quasi isometry class. We give an example of how to show that two groups are not quasi-isometric and therefore non-isomorphic. We also explore the limitations of quasi-isometry, in particular by showing that quasi-isometry cannot distinguish virtually isomorphic groups. We end the section by showing that a group is two two ended if and only if it is virtually isomorphic to $\mathbb{Z}$, thus illustrating that it is possible to recover algebraic information about a group from purely metric information.

Third and final part returns to groups given by presentations. We introduce van Kampen's Lemma, which tells us how 2-complexes (2-dimensional generalizations of graphs) can be used to describe non trivial words representing the identity. Using intuitive planar topological arguments we deduce structural results about HNN extensions. We then present the Combinatorial Gauss-Bonnet Theorem and turn our study to group presentations satisfying the $C'(1/6)$ small cancellation condition. We show that group presentations that satisfy this property admit a fast solution to the word problem. Thus, although in general there is no method to solve the word problem given an arbitrary group presentations, if one writes out a "random" presentation, then with high probability this presentation will be $C'(1/6)$ and will admit a very simple algorithm which properly solves the word problem.

Unfortunately altough a group may admit a $C'(1/6)$ group presentation, it will also admit many, many other presentations that do not satisfy this property. Furthermore there is no method to decide whether a group has such a presentation. Presentations are not canonical. In the epilogue, we introduce δ-hyperbolicity which is a property that, similarly to small cancellation, enables us to solve the word problem, but which is robust under quasi-isometries.

From here the student will be able to consult understand many of the excellent introductions to hyperbolic groups.

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Chapter 1

Free groups and presentations

1.1 Generalized associativity

As the reader should already know, a **group** $(G, \cdot)$ is a pair consisting of a set G and a binary operation $\star : G \times G \to G$, usually called **multiplication**, which satisfies the following axioms:

- The multiplication operation is **associative**, i.e. for all $g, h, k \in G$ we have
$$(g \star h) \star k = g \star (h \star k).$$
- There exists an element $1_G \in G$ called the **identity element** such that for every $g \in, 1_G \star g = g$.
- For every $g \in G$ there exists an inverse $g^{-1} \in G$ such that $g \star g^{-1} = 1_G$.

We will always denote a group by simply G, we will almost always omit the subscript for the identity element, and we will usually denote multiplication simply by concatenation, i.e. we'll write stuff like
$$gg^{-1} = 1.$$

Throughout this course we will want to consider a generating set $A \subset G$ of a group. The cleanest definition of a generating set is to say that if $H \leq G$ is a subgroup (i.e. it is closed under multiplication and inverses) and $A \subset H$ then $H = G$.

The way we want to think of a generating set is as follows: A is a **generating set of** G if every $g \in G$ can be expressed as a product
$$g = a_1^{\epsilon_1} \cdots a_n^{\epsilon_n},$$
where $a_i \in A; i = 1, \ldots, n$ and $\epsilon_i = \pm 1$. The issue is that multiplication is a *binary operation* which means such a product must have a bunch of brackets e.g.
$$x * (y * (z * w)), (x * y) * (z * w), (x * (y * z)) * w.$$
The number of bracketings grows exponentially interms of the number of factors. This is terrible!

The associativity axiom amounts to saying that three factor long products are well-defined, i.e.
$$x * y * z = x * (y * z) = (x * y) * z.$$

Our goal is now to show that long products with arbitrarily many factors are well-defined. On the one hand, this lets us ignore those pesky brackets, on the other hand this implies that a product of long factors is well-defined, i.e. the concatenation

$$(a_1 * \cdots * a_n) * (b_1 * \cdots * b_m) = a_1 * \cdots * a_n * b_1 * \cdots * b_m$$

is well-defined.

A few words about what we mean by well-defined are in order. Consider the binary operation $\div$ of division. This operation is not associative so brackeding matters. For example

$$12 = 12 \div (12 \div 12) \neq (12 \div 12) \div 12 = \frac{1}{12},$$

so the bracket-free "long product" $12 \div 12 \div 12$ is not well defined. If the reader thinks division is fishy, then they can investigate the cross product from linear algebra which is an actual non-associative product.

For now we will keep our brackets. A product is **left-nice** if it consists of a single factor, if it consists of two factors $x * y$, or if it is of the form $x * (z)$ where z contains at least two factors and itelf left-nice.We leave as an exercise the following:

Theorem 1.1.1 Generalized associativity. *Any messy product can be turned into a nice product by a sequence of applications of the associativity axiom. Furthermore the order the factors is preserved.*

From which the following immediately follows:

Corollary 1.1.2 *In any group (or any algebraic structure with an associative binary operation) long products*

$$g_1 \cdots g_n = \prod_{i=1}^{n} g_i ; g_i \in G$$

are well-defined.

1.1.1 Exercises

1. Prove Theorem 1.1.1.
2. Let A be a subset of a group G, prove that the set of products

$$\langle A \rangle = \{ g \in G \mid g = a_1^{\epsilon_1} \cdots a_m^{\epsilon_m}, \text{ for some } a_i \in A, \epsilon_i = \pm 1 \}$$

is a subgroup.

Hint: For inverses try writing a product backwards and flipping signs.

1.2 The free group.

Let X be a set. We will call X an **alphabet** and we will call its elements **symbols**. For each symbol $x \in X$ take a formal inverse x^{-1} and we denote:

$$X^{\pm 1} = X \cup \{x^{-1} : x \in X\}.$$

We further adopt the convention that $\left(x^{-1}\right)^{-1} = x$. A **word** in $X^{\pm 1}$ is a string of symbols

$$w(X) = x_1 x_2 \cdots x_n$$

where each symbol $x_i \in X^{\pm 1}$. In the situation where alphabet is clear we will simply write w instead of $w(X)$.

So, for example, if $X = \{a, b, c\}$ then $X^{\pm 1} = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$ and the string

$$abbac^{-1}b^{-1}ba^{-1}$$

is a word in $X^{\pm 1}$.

Given two words w_1, w_2 in some alphabet X, we denote their **concatenation** by $w_1 * w_2$. For example if

$$w_1 = ab \text{ and } w_2 = ba,$$

then $w_1 * w_2 = abba$.

We can view concatenation as an associative product, and can view words as long products of single letter words. Given a word w we will say that u is a **subword** of w if there are words w', w'', which may be empty, such that

$$w = w' * u * w''.$$

We will now define an equivalence relation on the set of words in $X^{\pm 1}$ generated by rewriting rules.

Definition 1.2.1 An **elementary cancellation** in a word w is the operation of deleting a subword of the form xx^{-1} where $x \in X^{\pm 1}$, i.e.

$$w = w' * xx^{-1} * w'' \stackrel{c}{\to} w' * w''.$$

If u is obtained from w by an elementary cancellation, i.e. $w \stackrel{c}{\to} u$, then we say that w is obtained from u by an **elementary insertion**.

We the identity operation $w \to w$ to simultaneously be a **trivial cancellation** and a **trivial insertion.** $\Diamond$

So, for example, we have:

$$bca^{-1}ac \stackrel{c}{\to} bcc.$$

Is an elementary cancellation.

We now define an equivalence relation $\stackrel{c}{\sim}$ on the set of words in $X^{\pm 1}$ as follows:

1. For each word we declare $w \stackrel{c}{\sim} w$. (Reflexivity)

2. We declare $w \stackrel{c}{\sim} u$ if w can be brought to u by a sequence of elementary cancellations and elementary insertions.

We can now define the **free group $F(X)$ on the alphabet X.**

1. The set underlying $F(X)$ is the set of all words strings in the symbols $X^{\pm 1}$, modulo the equicalence relation $\stackrel{c}{\sim}$.

2. The multiplication is concatenation of words.

3. The identity element $1 = 1_{F(X)}$ is the empty word.

Although we have a definition of $F(X)$ an outstanding problem remains: although concatenation is well-defined for words, it is not immediately well-defined for equivalence classes. In particular we must exclude the following possibility

$$w \stackrel{c}{\sim} w', u \stackrel{c}{\sim} u', \text{ but } w * u \stackrel{c}{\not\sim} w' * u',$$

otherwise we will not have proved that the structure $F(X)$ we defined is actually a group.

Furthermore, working with a set modulo an equivalence relation is problematic: given two words, it's not immediately clear whether they're equal or not. Consider an analogy with the set $\mathbb{Q}$ of fractions. We can consider fractions to be formal ratios of integers, but two different formal ratios, such as $\frac{2}{4}$ and $\frac{4}{8}$, can be equal. $\mathbb{Q}$ is therefore best thought of the set of ratios of formal ratios of integers modulo some equivalence relation. Furthermore every equivalence class of ratios has a **reduced element.**. In $\mathbb{Q}$ a ratio is reduced if the numerator and denominator are relatively prime.

To this end we have the following. A word $w(X)$ in $X^{\pm 1}$ is **reduced** if it has no subwords of the form $x^{\pm 1}x^{\mp 1}$ for some $x \in X$. Now given a non-reduced word, it is possible to remove a subword of the form $x^{\pm 1}x^{\mp 1}$ via an elementary cancellation. Because words have finite length and elementary cancellations reduce length, every word can be brought to a reduced form after finitely many elementary cancellations. The outstanding issue here is that perhaps different sequences of elementary cancellations can give rise to different reduced words. Thankfully we have the following result.

Theorem 1.2.2 *Every word w in an alphabet X has a unique reduced form $\overline{w}$, i.e. if $w' \overset{c}{\sim} w \overset{c}{\sim} w''$ and both w, w'' are reduced, then $w' = w''$.*

A restatement of this theorem is that elementary reductions form a confluent rewriting system.

Denote by $\overline{w}$ the **unique reduced form of** w. We can now prove the following.

Corollary 1.2.3 *Let $F(X)$ be the free group as defined above. Then*

1. *Given two words, $w \overset{c}{\sim} w'$, so that $w =_{F(X)} w'$ if and only if $\overline{w} = \overline{w'}$ as words*

2. *Multiplication by concatenation is well-defined.*

Proof. The first item is immediate from Theorem 1.2.2. For the second item, let $w \overset{c}{\sim} w', u \overset{c}{\sim} u'$. Then $\overline{w'} = \overline{w}, \overline{u'} = \overline{u}$. Now Theorem 1.2.2 implies that

$$\overline{\overline{w} * \overline{u}} = \overline{w * u},$$

since the left hand side just means "first perform cancellations within the subwords w and u" and the Theorem implies that the order of cancellations doesn't matter. We therefore have

$$\overline{w * u} = \overline{\overline{w} * \overline{u}} = \overline{\overline{w'} * \overline{u'}} = \overline{w' * u'}.$$

So $w * u =_{F(X)} w' * u'$. ∎

1.2.1 Exercises

The purpose of these exercises, besides proving a crucial fact about free groups, is to get the reader to write arguments using words and using the technique of induction on word length.

1. Prove the diamond lemma:

Lemma 1.2.4 The diamond lemma. *Suppose we have two elementary cancellations of a word w*

$$w_1 \overset{c}{\leftarrow} w \overset{c}{\rightarrow} w_2,$$

then there exists some w' such that there are (possibly trivial) cancellations

$$w_1 \overset{c}{\rightarrow} w' \overset{c}{\leftarrow} w_2.$$

See Figure 1.2.5.

$$\begin{tikzcd}
 & w \arrow[dl, "c"'] \arrow[dr, "c"] & \\
w_1 \arrow[dr, "c"] & & w_2 \arrow[dl, "c"'] \\
 & w' &
\end{tikzcd}$$

Figure 1.2.5 The diamond lemma

Hint: There are two cases to consider: whether or not the cancelled subwords in w overlap. Be thoughtful with your notation and use of subscripts.

2. Use the diamond lemma to prove Theorem 1.2.2

Hint: Use a peak reduction argument. Suppose that w and w' were distinct reduced words with $w \overset{c}{\sim} w'$. Then there is a sequence

$$w = w_0 \overset{x}{\leftrightarrow} w_1 \overset{c}{\leftrightarrow} \cdots \overset{c}{\leftrightarrow} w_n = w'$$

where each $\overset{c}{\leftrightarrow}$ is either an elementary insertion or deletion. In particular there exists a sequence of *minimal total word length* i.e.

$$\sum_{i=0}^{n} |w_i|,$$

where $|w_i|$ is the length of a word, is minimal among all such sequences. Consider an intermediate word of maximal length and the diamond lemma.

3. Finish the showing that $F(X)$ is a group by showing that every element has an inverse.

1.3 The universal property of the free group

Recall $F(X)$, the **free group on an alphabet** X consists of the set of reduced words in X and multiplication is defined by concatenation. The free group is constructed by taking all long products of elements of X. Because we need to have inverses we also add the elements X^{-1}, and this is enough to construct a group.

The free group plays a crucial role because of the following **universal property.**

Theorem 1.3.1 The universal property of free groups. *Let G be any group and let $F(X)$ be the free group on the alphabet X. Then for any function*

$$\varphi : X \to G$$

there exists a unique group homomorphism

$$\varphi^* : F(X) \to G$$

such that $\varphi\mid_X = \varphi$, i.e. φ^* is an extension of φ.

$$\begin{tikzcd} F(X) \arrow[r, dashed, "\exists!\varphi^*"] & G \\ X \arrow[u, phantom, "\cup"] \arrow[ur, "\forall\varphi"'] & \end{tikzcd}$$

Figure 1.3.2 *The universal property of free groups.*

In particular, by the First Isomorphism Theorem, because X can be any set, this property implies that *every group is the quotient of a free group.* If, up to now, our construction of the free group (involving generalized associativity and reduced words, seemed a bit painful, this work will pay itself off by making the universal property straightforward to prove.

Let's sketch the proof and leave the details as an exercise. Suppose you are given some $\varphi : X \to G$ which maps each letter $x \in X$ of the alphabet an element $\varphi(x) \in G$. Then we can extend this to a function $\varphi^* : F(X) \to G$ as follows. Let $w = a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$, where each $a_i \in X$ and each $\epsilon_i \in \{1, -1\}$, be some arbitrary element of $F(X)$, which we can also consider a long product of elements in $X^{\pm 1} \subset F(X)$. Then we set

$$\varphi^*(w) = \varphi^*(a_1^{\epsilon_1} \cdots a_n^{\epsilon_n} = \varphi(a_1)^{\epsilon_1} \cdots \varphi(a_1)^{\epsilon_1}, \tag{1.3.1}$$

which is a product of elements of G. Generalized associativity and the fact that if $w =_{F(X)} w'$ then $\varphi^*(w) = \varphi^*(w')$ ensure that this mapping is *well-defined.* Details of showing that this is a homomorphism is left as an exercise.

We end this lecture with the following result which algebraically characterizes free groups.

Corollary 1.3.3 *Let G be some group with a subset $Y \subset G$ such that for any group H and any map $f : Y \to H$, there is an extension of f to a homomorphism $f^* : G \to H$. Then G is isomorphic to the free group $F(Y)$.*

Recall that although we would call the set Y an alphabet and its elements letters, the alphabet can be any set, even elements of another group. For clarity, however, we make a copies of sets in the proof below.

Proof. For clarity let us take X to be a copy of Y, i.e. we have a bijection $\psi : X \to Y$ and we denote its inverse $\phi : Y \to X$. We consider $Y \subset G$ and $X \subset F(X)$.

By Theorem 1.3.1 there is a homomorphism $\psi^* : F(X) \to G$ which extends ψ. By hypothesis there is also a homomorphism $\phi^* : G \to F(X)$ which extends ϕ.

Let $w = \prod_{i=1}^n x_i$, where $x_i \in X^{\pm 1}$ be an arbitrary element of $F(X)$. Then by the definition of a homomorphism and generalized associativity we have:

$$\phi^* \circ \psi^* \left(\prod_{i=1}^n x_i \right) = \prod_{i=1}^n \phi^* \circ \psi^*(x_i) = \prod_{i=1}^n x_i,$$

since on X, $\phi^* \circ \psi^* = \phi \circ \psi = Id$.

The homomorphisms ψ^*, ϕ^* are therefore inverses of each other; and thus are isomorphisms. ■

Any subset Y of a group G with this universal mapping property is called a **basis of** G, and since we're *free* to send elements of Y wherever we want, we say that G is a free group with basis Y. As is the case in linear algebra, bases are never unique, but that is for another lecture (not the next one)!

1.3.1 Exercises

1. Prove Theorem 1.3.1. Hint: This really amounts to verifying that the mapping (1.3.1) is well-defined (which is ensured by generalized associativity) and the fact that $w =_{F(X)} w'$ then $\varphi^*(w) = \varphi^*(w')$. Prove this fact by starting with the special case where w is obtained from w' by an elementary insertion and deletion and proceeding by induction. Verifying that the mapping is a homomorphism is routine.

1.4 Generating and presenting groups

Long products are group elements are important to us. The following lemma gives us a definition of **a subgroup generated by a collection of elements**

Lemma 1.4.1 *Let $H \leq G$ be a subgroup and let $S = \{h_1, \dots\} \subset G$ be a set of elements. The following are equivalent:*

- *H is the smallest subgroup of G containing S, i.e.*

$$H = \bigcap_{S \subset K \leq G} K,$$

i.e. the intersection of all subgroups of G containing S.

- *H consists of all products of elements $S^{\pm 1}$, i.e.*

$$H = \left\{ h_{i_1}^{\epsilon_1} \cdots h_{i_n}^{\epsilon_n} \in G \mid h_{i_j} \in S, \epsilon_j \in \{\pm 1\}, n \in \mathbb{Z}_{\geq 0} \right\}$$

We will denote the subgroup generated by $S = \{h_1, \dots\}$ as $\langle S \rangle$ or $\langle h_1, \dots \rangle$.

Proof. Let us denote by H the smallest subgroup containing S and by $\langle S \rangle$ set of products of elements in $S^{\pm 1}$. $\langle S \rangle$ is closed under multiplication and taking inverses so it's a subgroup of G. By definition, it's therefore immediate that $H \subset \langle S \rangle$.

To see that $H \supset \langle S \rangle$, we proceed by induction on the length of a product. By definition any product of length 1, being an element of $S^{\pm 1}$, is in H. If every product of length at most n is in H, then, because a product of length $n+1$ is a product of a product of length n and a product of length 1 and because H, being a subgroup, is closed under multiplication, every product of length $n+1$ is also in H. Therefore, by induction, $H \supset \langle S \rangle$ and equality follows. ■

Any set $S \subset G$ such that $G = \langle S \rangle$ is called a **generating set of G**. If G has a finite generating subset $\{g_1, \cdots, g_k\} \subset G$ then we way G is **finitely generated.** The **rank** of a group G is the minimal cardinality achieved by its generating sets, though this terminology is less standard and typically has other meanings in different group theoretical contexts.

Since $G = \langle G \rangle$, every finite group is finitely generated and it is a standard fact that every symmetric group S_n has rank 2. It's already more impressive when an infinite group has finite rank. Here are three examples:

- If $X = \{a, b, c\}$ then the free group $F(X)$ has rank 3. In general if $|X| = n$ then we will say that $F(X)$ is a **free group of rank n.**

- $\mathrm{GL}(2, \mathbb{Z})$, the group of 2×2 invertible matrices with $\mathbb{Z}$ coefficients also has rank 2, indeed it is generated by elementary matrices:

$$\mathrm{GL}(2, \mathbb{Z}) = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

- $\pi_1(M)$, where M is a compact finite dimensional manifold is also finitely generated.

If $\langle S \rangle = G$, then by Theorem 1.3.1, there is an epimorphism, or a surjective homomorphism, $\varphi : F(S) \twoheadrightarrow G$. So it is immediate that if G has rank n then it is the homomorphic image of a free group of rank n.

In any case if $G = \langle S \rangle$ and X is any alphabet such that $|X| = |S|$, by the First Isomorphism Theorem for groups we have

$$F(X)/\ker(\varphi) \approx G.$$

Let us fix some more terminology before continuing. If $g, h \in G$, then the **conjugate of g by h** is

$$g^h = h^{-1}gh.$$

Some people write $g^h = hgh^{-1}$, but they are *wrong* because we should have $(g^h)^k = g^{hk}$. Although it looks weird, it makes more sense to write

$${}^h g = hgh^{-1} = g^{h^{-1}}.$$

The correct notation is consistent with the concepts of left and right actions.

Recall that a normal subgroup $N \triangleleft G$ is a subgroup that is closed under conjugation by arbitrary elements in G, and the class of normal subgroups coincides with the kernels of homomorphisms from G. Consider now the concept of **normal generation**:

Proposition 1.4.2 *Let $R \subset G$ be some subset. Then the smallest normal subgroup of G containing R coincides with*

$$\langle\langle R \rangle\rangle = \left\{ \prod_{i=1}^{n} c_i^{-1} r_i^{\epsilon_i} c_i \middle| r_i \in R, \epsilon_i \in \{\pm 1\}, c_i \in G, n \in \mathbb{Z}_{\geq 0} \right\},$$

the set of products of conjugates of elements in $R^{\pm 1}$.

We leave the proof to Exercise 1.4.1.1.

Consider a subset $R \subset F(X)$ is a subset then we will call a pair $\langle X | R \rangle$ a **group presentation.** The set X of letters is called the **generators** and words in R are called **relations**.

Again we will consider $\langle X | R \rangle$ to be a set of words in $X^{\pm 1}$ subject to the following equality relation: $w =_{\langle X|R \rangle} w'$ if there is a sequence of words:

$$w = w_0 \to w_1 \to \cdots \to w_m = w'$$

such that each $w_i \to w_{i+1}$ is either

1. An elementary insertion or cancellation given in Definition 1.2.1,
2. the deletion of a subword that is a relation, i.e.
$$w_i = w'rw'' \to w'w'' = w_{i+1},$$
for some $r \in R$, or
3. the insertion of some relation into a word, i.e.
$$w_i = w'w'' \to w'rw'' = w_{i+1},$$
for some $r \in R$.

So again, elements of $\langle X|R\rangle$ are equivalence classes of words. Once again we want multiplication to be given by concatenation. Recall that for free groups we showed that concatenation gave a well defined product on the set of equivalence classes of words related by sequences of elementary insertions or deletions. Our argument relied on reduced words. In this case, we don't have reduced words so instead we will condinue to extend the relation $=_{\langle X|Y\rangle}$ as follows: if $w =_{\langle X|R\rangle} w'$ and $u =_{\langle X|R\rangle} u'$ then we impose $w * u =_{\langle X|R\rangle} w' * u'$. We can therefore continue to extend $=_{\langle X|R\rangle}$ in this way until we make $\langle X|R\rangle$ into a group. Now maybe this group collapsed down to the trivial group, but it's a group.

So on the one hand we have that every group is a quotient of a free group, on the other hand we have constructed the group $\langle X|R\rangle$ and we have.

Theorem 1.4.3

$$\langle X|R\rangle \approx F(X)/\langle\langle R\rangle\rangle$$

If $G \approx \langle X|R\rangle$ with $|X| \leq \infty$, then we say G is **finitely generated.** If both $|X|, |R|$ are finite, the we say G is **finitely presented.**

1.4.1 Exercises

1. Prove Proposition 1.4.2.

Hint: This is similar to the proof of Lemma 1.4.1 the only issue is that if you take a conjugate of a product of conjugates, i.e.

$$g^{-1}\left(\prod_{i=1}^{n}\left(c_i^{-1} r_i c_i\right)\right) g,$$

then it's not clear that the result is again a product of conjugates. See what happens when you conjugate each factor of the product by g.

2. Let $w \in F(X)$ be written as a word, let R be a subset of $F(X)$ and consider the new word w' obtained by inserting some element $r \in R$, i.e.

$$w = w_1 * w_2 \to w_1 * r * w_2.$$

Show that there are elements $h, k \in \langle\langle R\rangle\rangle$ such that:

$$hw = w' = wk.$$

Hint: conjugate r by a prefix or suffix of of w.

3. Prove Theorem 1.4.3.

Hint: Take $\varphi : F(X) \twoheadrightarrow \langle X|R\rangle$. Use the universal property of free groups as well as the previous exercise and argue that $\ker\phi = \langle\langle R\rangle\rangle$.

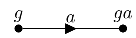
4. Use Proposition 1.4.2 and Theorem 1.4.3 to argue that equality in $\langle X|R\rangle$ is fully defined by elementary insertion and deletions and insertions and removals of relations as subwords. Forcing products to be well defined is superfluous.

Hint: By Theorem 1.4.3, $w =_{\langle X|R\rangle} w'$ if and only if there is some $n \in \langle\langle R\rangle\rangle$ such that $wn = w'$.

1.5 Cayley graphs

So far we have been very combinatorial. Let's get geometric! First recall that a directed graph Z consists of a set V of **vertices** and a set $E \subset V \times V$ of directed **edges**. If we want we can also **label** the directed edges using some symbol from some alphabet.

Given a generating set $S \subset G$ of a group G we can form a **Cayley graph**, denoted $\mathbf{Cay}_S(G)$, by taking the vertex set be the set G itself and for each $a \in S$ and each $g \in G$ we draw the directed edge labelled a from g to ga:



where the group element is drawn above the vertex. Thus $g = a_1 \cdots a_n$.

Let's do an example. Consider $D_{2\times 3}$, the dihedral group of order 6, or the symmetry group of the triangle. It is generated by ρ, the clockwise rotation by 120°, and r the reflection about the vertical axis. We let $D_{2\times 3}$ act on the left.

It is known that $D_{2\times 3}$ has 6 elements, we draw its Cayley graph $\mathbf{Cay}_{\{\rho,r\}}(D_{2\times 3})$ here where next to each vertex $g \in G$ we have the result of applying g to the triangle.

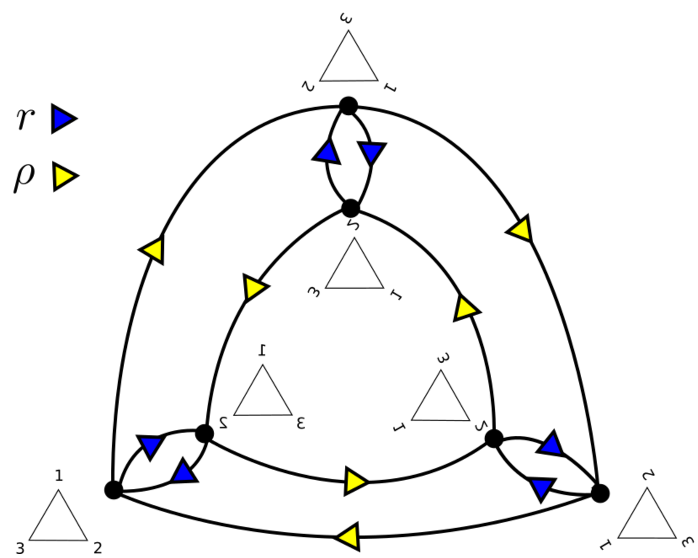


Figure 1.5.1 The Cayley graph $\mathbf{Cay}_{\{\rho,r\}}(D_{2\times 3})$

Note that our left action convention means that, say, the element $r\rho$, mean "first rotate, then reflect." The result is that the transformed triangles in Figure 1.5.1 don't immediately look in the right place.

Because they are highly symmetric, Cayley graphs are aesthetically pleasing, and they essentially play the role of a multiplication table in a group. Unfortunately (see Exercise 1.5.1.1) a group has multiple Cayley graphs, depending on the choice of generating set.

Infinite groups also have Cayley graphs. Consider for example $\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$ which is the set

$$\mathbb{Z}^2 = \{(n,m) \mid n,m \in \mathbb{Z}\}$$

equipped with component-wise addition. If we take generators $a = (1,0)$ and $b = (0,1)$ then $\mathbf{Cay}_{\{a,b\}}(\mathbb{Z}^2)$ looks like:

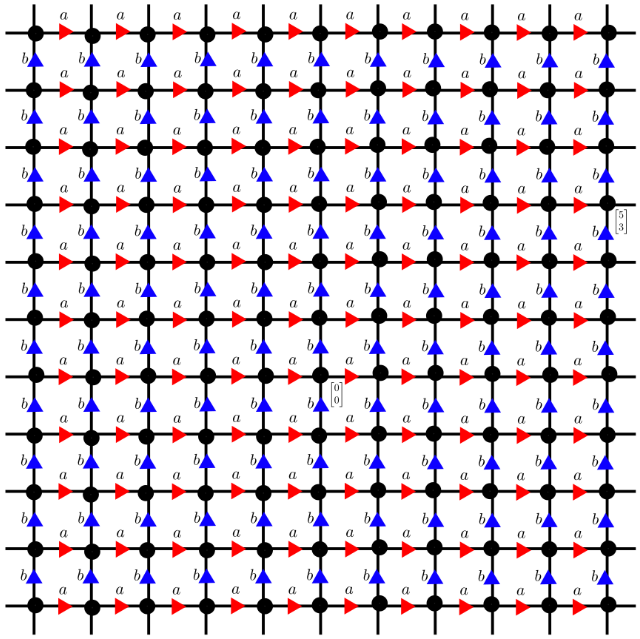


Figure 1.5.2 A Cayley graph for $\mathbb{Z}^2$

Every edge e in a directed labelled graph has an **initial** vertex $\iota(e)$, a **terminal** vertex $\tau(e)$ and a label **label**(e). For example if $e = (u, v)$, where u, v are vertices then $\iota(e) = u, \tau(e) = v$. We may define the **formal inverse** e^{-1} of an edge e so that $\tau(e) = \iota(e^{-1})$ and $\tau(e^{-1}) = \iota(e)$ and the label **label**$(e^{-1}) =$ **label**$(e)^{-1}.y$

A **path** in a directed graph Z with edge set E is a sequences of edges and formal inverses:

$$\alpha : e_1 \cdots e_n; e_i \in E^{\pm 1}$$

that "connect together", i.e. $\tau(e_i) = \iota(e_{i+1})$ for $1 \leq i < n$. The **initial point** of a path is the vertex $\iota(e_1)$ and the **terminal vertex** is the vertex $\tau(e_n)$. We say that the path α **goes from its initial vertex to its terminal vertex**. Here's an example:

A path is said to be **reduced** if it has no subpaths of the form ee^{-1} for some $e \in E^{\pm 1}$.

This is similar to our treatment of words in an alphabet. The only distinction is that you can only concatenate paths if one starts where the other ends. Otherwise there is associativity and even a reduction procedure where we successively delete subpaths of the form ee^-1 which we can call **tightening**. Everything works out like in Section 1.2, we can even define inverses. Such a structure, which is almost like a group except that multiplication is not always defined is called a **groupoid**.

If the reader has any experience with fundamental groups in topology, then this should also look familiar, as all we are doing here is giving a combinatorial treatment of a path:

$$f : [0, 1] \to Z$$

where the graph Z is thought of as a topological space. Specifically, a CW complex with 0 and 1 dimensional cells.

Given a path $\alpha = e_1 \cdots e_n$, we will say that its **length** is n, the number of edges is contains and its **label** is the word

$$\mathbf{label}(\alpha) = \mathbf{label}(e_1) \cdots \mathbf{label}(e_n).$$

The whole point of Section 1.2 was that in a free group on a specified alphabet distinct reduced words correspond to distinct elements. Once relations are added this is no longer the case. Indeed if $G = \langle X|R \rangle$ and given two distinct reduced words w and w' in the alphabet X it may be that $w \neq_{F(X)} w'$ (i.e. they are distinct reduced words) but that viewed as products of generators of G we have $w =_G w'$, i.e. they have the same image via the standard map $F(X) \to \langle X|R \rangle$.

The Cayley graph illustrates this nicely:

Lemma 1.5.3 *Let $S \subset G$ be generating set. Let β be a path in $\mathbf{Cay}_S(G)$that goes from the identity 1 to some element $g \in G$ (recall that the vertices of $\mathbf{Cay}_S(G)$ are the elements of G) then, viewed as a product of element in $S^{\pm 1}$ we have:*

$$\mathbf{label}(\beta) =_G g$$

Proof. Let $\beta : e_1 \cdots e_n$ and let $\mathbf{label}(e_i) = a_i \in S^{\pm 1}$. Then by the definition of a Cayley graph we have

■

1.5.1 Exercises

1. It is well-known that $D_{2\times 3} \approx S_3$, the group of permutations of the set $\{1, 2, 3\}$. S_3, like any symmetric group, is generated by permutations. Using cycle notation we have

$$\langle (1, 2), (2, 3) \rangle = S_3.$$

Draw the Cayley graph $\mathbf{Cay}_{\{(1,2),(2,3)\}}(S_3)$ and compare it with the Cayley graph for $D_{2\times 3}$ shown above.

2. We always have $\langle G \rangle = G$. What does $\mathbf{Cay}_G(G)$ look like?

3. Let $S \subset G$ be a generating set. Let $d : G \times G \to \mathbb{R}$ be the function defined as $d(g, h) = m$, where m is the length of the shortest path in $\mathbf{Cay}_S(G)$ from g to h.

Show that d is a metric on G. This metric is called the **word metric** on G.

4. The group $(\mathbb{Z}, +)$ is generated by 1 (here the identity is 0, so that $1_{\mathbb{Z}} = 0$). Consider the generating set $\{1, 10\} = \{a, b\}$ of $\mathbb{Z}$.

Sketch $\mathbf{Cay}_{\{a,b\}}(\mathbb{Z})$. What can you say about the ball of radius 4 about the identity in $\mathbf{Cay}_{\{a,b\}}(\mathbb{Z})$ compared to the ball of radius 4 about the identity in $\mathbf{Cay}_{\{a,b\}}(\mathbb{Z}^2)$

5. Let X be a finite alphabet, let $F(X)$ be the free group on X and consider X as a generating set of $F(X)$. Prove that $\mathbf{Cay}_X(F(X))$ is a tree.

Hint: A tree is a connected graph without any cycles. The label of a cycle in a group's Cayley graph must be reduced product that equals the identity.

1.6 Homomorphisms and Tietze transformations

Let's return to group presentations. In the previous lectures we drew Cayley graphs for $D_{6\times 3}$ and $\mathbb{Z}^2$, we will now try to compute presentations. Before continuing *a word of caution:* in general, working with presentations is very tricky. Specifically (and we will discuss this properly in a couple lectures) there is not general procedure which determines if a finite presentation gives a nontrivial group.

Let's start with a presentation for $D_{2\times 3}$. We saw in Section 1.5 that $D_{2\times 3}$ was generated by elements ρ, r. Therefore by the universal property of free groups there is a surjective homomorphism:

$$\begin{aligned} \varphi : F(r,\rho) &\twoheadrightarrow D_{2\times 3} \\ \Rightarrow D_{2\times 3} &\approx F(r,\rho)/\ker(\varphi) \end{aligned}$$

and we are left to find a normal generating set for $\ker(\varphi)$. We must be careful because adding relations has consequences, so we must not add too many. We first note that the following elements are trivial in $D_{2\times 3}$:

$$\rho\rho\rho =_{D_{2\times 3}} rr =_{D_{2\times 3}} r\rho r\rho =_{D_{2\times 3}} 1.$$

It therefore follows that $\langle\langle \rho\rho\rho, rr, r\rho r\rho\rangle\rangle \le \ker\varphi$ which means that there is a surjective homomorphism

$$\langle r,\rho | \rho\rho\rho, rr, r\rho r\rho\rangle \twoheadrightarrow D_{2\times 3}. \tag{1.6.1}$$

In other words, we have not added too many relations, we must now verify that we have added sufficiently many relations. In order to show this we will use a **normal forms argument**. We will show that for the group $H = \langle r,\rho | \rho\rho\rho, rr, r\rho r\rho\rangle$, we will be able to use the relations to rewrite any element as

$$w(r,\rho) =_H \rho^i r^j ; 0 \le i \le 2, 0 \le j \le 1 \tag{1.6.2}$$

so that H has only 6 elements and there fore the mapping (1.6.1) is not only surjective, but also injective and therefore and isomorphism.

The relations $\rho\rho\rho = 1$ and $rr = 1$ imply that any word $w(\rho, r)$ in H can be written as an alternation of powers of ρ of exponent between 0 and 2, and of the letter r, for example

$$\rho^2 r \rho^2 r \rho r \rho.$$

We now want to show that any word can be brought to the desired form.

Suppose that the final factor was of the form ρ^m so that

$$w = \cdots r\rho^m$$

then we can splice-in the inverse of a relation so that:

$$w = \cdots r\rho\rho^{m-1} = \cdots r\rho(\rho^{-1}r^{-1}\rho^{-1}r^{-1})\rho^{m-1} = \cdots \rho^{-1}r^{-1}\rho^{m-1}$$

and since

$$r^{-1} = r^{-1}(rr) = r$$

we have

$$w = \cdots \rho^{-1} r \rho^{m-1}$$

repeating this process removes the last ρ power syllable. This is progress, but the argument is tiresome!

Although we could do everything by splicing in and deleting relations, applying elementary reductions and their inverses, we can also just use group theory to replace subwords by equal elements.

In H we have

$$r\rho r\rho = 1 \Rightarrow r\rho = \rho^{-1}r^{-1} = \rho^2 r \tag{1.6.3}$$

which means we can can simply replace any subword $\cdots r\rho \cdots$ by $\cdots \rho^2 r \cdots$. In this manner, we can "commute" all the ρ symbols to the left of the word, giving $w = \rho^k r^j$, and since $\rho^3 = r^2 = 1$ the exponents k, j can be taken to be division remainders so we can rewrite any word in H as in (1.6.2). Therefore the 3 relations $\rho\rho\rho, rr, r\rho r\rho$ ensure the group has 6 elements.

Let us now consider homomorphisms given from groups with presentations.

Lemma 1.6.1 *Let G be some group and let $H = \langle X|R\rangle$ be a group given by a presentation. A mapping $\varphi^* : X \to G$ extends to homomorphism $\varphi : H \to G$ if and only if for each $r(X) = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} \in R$, where $x_i \in X, \epsilon_i \in \{\pm 1\}$ upon substitution in G we get:*

$$\varphi^*(x_1)^{\epsilon_1} \cdots \varphi^*(x_n)^{\epsilon_n} =_G 1.$$

One consequence of this equations. Let G be a group, let $x_1, \ldots, x_n$ be a collection of unknowns and consider a system of equations:

$$\mathcal{E} : \begin{cases} r_1(X) = x_{i_{11}}^{\epsilon_{11}} \cdots x_{i_{1c_1}}^{\epsilon_{1c_1}} =_G 1 \\ \cdots \\ r_m(X) = x_{i_{m1}}^{\epsilon_{m1}} \cdots x_{i_{mc_m}}^{\epsilon_{mc_m}} =_G 1 \end{cases},$$

where $x_{i_{jk}} \in X$ and $\epsilon_{jk} \in \{\pm 1\}$. Then the solutions of $\mathcal{E}$ are precisely the given by the homomorphisms

$$\langle x_1, \ldots, x_n | r_1, \cdots, r_m\rangle \to G.$$

As we saw in the previous lecture, different choices of generating sets gave different Cayley graphs. We will now describe all possible presentations of a group.

Let $\mathcal{P} = \langle X|R\rangle$ be a presentation (not a group, but a formal presentation) and consider the following three **Tietze** transformations.

- **T1: The dictionary transformation.**

  $$\langle X|R\rangle \to \big\langle X \cup \{y\} \big| R \cup \{y^{-1}w(X)\}\big\rangle,$$

  where $y \notin X$ and $w(X)$ is a word in X. Informally, add a new symbol for an element y and say that

  $$y = w(X) \Leftrightarrow y^{-1}w(X) = 1.$$

  So that they new symbol y is just "shorthand" for the element $w(X)$.

- **T2: Add a redundant relation.**

  $$\langle X|R\rangle \to \langle X|R \cup \{r'\}\rangle,$$

  where $r' \in \langle\langle R\rangle\rangle$. Informally, add a relation r' which is already a consequence of the relations in R.

- **T3: Rename a symbol.** Take some symbol $z \notin X$, take $X' = (X \setminus \{x\}) \cup \{z\}$ and do

  $$\langle X|R\rangle \to \langle X'|R'\rangle,$$

  where R' is obtained by replacing every instance of x by z in every word in R.

Exercise 1.6.1.4 is to show that these transformations yield presentations defining isomorphic groups. Note that the inverse of T1 involves deleting a generator and a very specific type of relation and the inverse of T2 involves removing a relation that is a consequence of the remaining relations.

As promised we have the following:

Theorem 1.6.2 *Two finite presentations $\langle X|R\rangle$ and $\langle Y|S\rangle$ define isomorphic groups if and only if $\langle Y|S\rangle$ can be obtained from $\langle X|R\rangle$ by a sequence of Tietze transformations and their inverses.*

The proof is deferred to Exercise 1.6.1.5.

1.6.1 Exercises

1. Prove that
$$\mathbb{Z}^2 = \big\langle a, b \big| a^{-1}b^{-1}ab\big\rangle.$$
Hint: Use the same approach as for $D_{2\times 3}$, use the normal forms $a^n b^m, n, m \in \mathbb{Z}$.

2. Prove Lemma 1.6.1.

3. Deduce the result of (1.6.3), namely that we can substitute $r\rho$ by $\rho^2 r$, only by using insertions and deletions of relations, elementary cancellations or their inverses. *Or at least try and appreciate how useful the axioms of group theory and generalized associativity are.*

4. Prove that all three Tietze transformations give isomorphisms.

5. Prove Theorem 1.6.2.

Hint: One direction is obvious. To show the converse one approach is to construct an intermediate presentation
$$\langle X \cup Y | R \cup S \cup D\rangle$$
which is obtainable from $\langle X|R\rangle$ and from $\langle Y|S\rangle$ by a sequence of Tietze transformations. D will be a bunch of dictionary relations.

Some dictionary relations will be needed. To find them consider an isomorphism $\varphi : \langle X|R\rangle \to \langle Y|S\rangle$. This will map each $x \in X$ to some word in Y. You will also need φ^{-1} and add relations.

1.7 Algorithmic problems in group theory and elements of recursion theory

Our combinatorial approach to group theory naturally turns groups into computational objects: sets of strings with rewriting rules. In the 1920s Max Dehn proposed the following three algorithmic problems:

- **The word problem:** Given a group presentation $G = \langle X|R\rangle$ and word $w(X)$ in the alphabet X determine if $w(X) =_G 1$.
- **The conjugacy problem:** Given a group presentation $G = \langle X|R\rangle$ and words $w(X), u(X)$ in the alphabet X determine if *there exists* some $g \in G$ such that $g^{-1}w(X)g =_G u(X)$.
- **The isomorphism problem:** Given two group presentations $\langle X|R\rangle$ and $\langle Y|S\rangle$ determine if they define isomorphic groups.

First the bad news:

Theorem 1.7.1 The Novikov-Boone theorem. *There exists a finitely presented group $G = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ for which the word problem is undecidable.*

Let's explain this result in more detail. There exists a group presentation which you can actually write out so that for any algorithm you come up with, for example a computer program written in Python, to determine *correctly* whether words are equal to the trivial element, there will be a word (in fact many, many words) for which your algorithm will never terminate.

We say a set S is **recursively enumerable** or **r.e.** if there is some algorithm $\mathfrak{A}$ which enumerates the elements of S.

Proposition 1.7.2 Trivial elements are recursively enumerable. *Let $G = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ be a finitely presented group. Then the set*

$$\mathcal{E} = \{w(X) \mid w(X) \text{ is reduced and } w(X) =_G 1\}$$

of reduced words equal to 1 *is recursively enumerable.*

Proof. By Proposition 1.4.2 the set $\mathcal{E}$ consists of taking all products

$$\prod_{i=1}^{n} c_i^{-1} r_{j_i}^{\epsilon_i} c_i$$

where $r_{j_i} \in \{r_1, \cdots, r_m\}, \epsilon_i \in \{\pm 1\}$ and then performing free reduction. Consider the sets $\mathcal{E}_N$ consisting of all products consisting of at most N factors $c_i^{-1} r_{j_i} c_i$, and where each conjugator also has length at most N. Then $\mathcal{E}_N$ is recursively enumerable for each N therefore so must the countable union

$$\mathcal{E} = \bigcup_{N=1}^{\infty} \mathcal{E}_N.$$

■

Although enumeration seems somewhat contrived, it is fact equivalent the more mathematically natural concept of set membership.

Proposition 1.7.3 *Let S be a set of strings of symbols in some alphabet X. The following are equivalent:*

- *There exists a* ***membership algorithm*** *$\mathfrak{A}$ which takes as inputs strings in X with the following property: $s \in S$ if and only if $\mathfrak{A}$ terminates on the input s.*

- *S is recursively enumerable.*

Proof. Suppose first you are given the algorithm $\mathfrak{A}$ as in the statement of the proposition. Let $w_1, w_2, \ldots$ be an enumeration of the strings in X and let $\mathfrak{M}_N$ be a program which performs the N first steps of $\mathfrak{A}$ on the inputs $w_1, \ldots, w_N$. If one of the parallel processes terminates then $\mathfrak{N}$ outputs the string w_j of the corresponding terminating process. Then our enumeration algorithm is an algorithm which sequentially runs $\mathfrak{N}_1, \mathfrak{N}_2, \ldots$.

Conversely suppose S is *r.e.* and let $\mathfrak{B}$ be the enumeration algorithm. Then we can make our membership algorithm $\mathfrak{A}$ a follows: gien an input s, run $\mathfrak{B}$ and wait until s appears, if it ever does, stop. ■

Now, in general, when asking

$$x \overset{?}{\in} S$$

we expect a definitive yes or no answer. So consider the finitely presented group $G = \langle X|R\rangle$, then the word problem is decidable if and only if $\mathcal{E}$, the set of reduced words representing 1 in G is r.e. and the set $F(X) \setminus \mathcal{E}$ of reduced words not representing the identity is also recursively enumerable. Indeed, in this case, there is a procedure which is guaranteed to terminate on any reduced word and correctly say whether or not the word is trivial.

So how does this work with Theorem 1.7.1? Well Proposition 1.7.2 says that all trivial elements can be enumerated, so what is difficult is to show that some reduced word represents a non-trivial element. This is equivalent to saying that some word *cannot* be written as a product of conjugates of the relations. Thus, asserting that a word is non-trivial, is a *non-existence* statement: there does not exist a product of conjugates of relators that is freely equal to some word.

It is a common theme in mathematics that non-existence proofs are difficult. One of the triumphs of geometric group theory is that we can solve all three of Dehn's problems for many interesting classes of groups. Next lecture we will do this for free groups.

1.7.1 Exercises

1. Show that the word problem is solvable for $F(X)$, the free group on the alphabet X.

Hint: This problem is not difficult.

2. The **equality problem** for $G = \langle X|R\rangle$ is do decide, given two words, whether:
$$u \stackrel{?}{=} w.$$
Show that solvability of the equality problem is equivalent to solvability of the word problem.

3. The **conjugacy search problem** is a variation of the conjugacy problem which, given elements h, k, not only outputs YES or NO to the question of whether they are conjugate, but, if YES, will also produce an element such that
$$g^{-1}hg = k$$
Show that solvability of the conjugacy search problem is equivalent to solvability of the conjugacy problem.

4. Let $G = \langle X|R\rangle$ be a finite presentation.

Show that (up to relabeling the generators) the set of all finite presentations which define groups isomorphic to G is recursively enumerable.

5. We say that $G = \langle X|R\rangle$ is **recursively presented** if there are algorithms $\mathfrak{X}$ and $\mathfrak{R}$ which enumerate the sets $X = \{x_1, x_2, \ldots\}$ and $R = \{r_1, r_2, \ldots\}$ (respectively.)

Show that if $G = \langle X|R\rangle$ is a recursive presentation, then the set of finite words is recursively enumerable.

1.8 Solving Dehn's algorithmic problems in the class of finitely generated free groups.

We already saw that the isomorphism class of $F(X)$ was determined by the cardinality $n = |X|$, so we'll fix the following notation
$$F_n = \langle a_1, \ldots, a_n | \rangle.$$

Solving the word problem in F_n is straightforward: bring a word to reduced form and make the obvious conlcusion.

Although the isomorphism problem is undecidable if you're working over arbitrary finite presentations. In practice we usually consider the isomorphism problem resricted to a class of groups. For finitely generated free groups we are asking whether it is possible that $m \neq n$ but that $F_n \approx F_m$. We will see that the answer is "no", but before doing so a couple observations are in order.

First of all, we will see later on that for any m, n (in fact we could even take the countably infinite cardinal $m = \aleph_0$) there is an injective homomorphisms

$$F_m \hookrightarrow F_n.$$

In fact in the early 2000s it was proved that all finite rank free groups have the same elementary theory. All this to say that directly showing that free groups of different rank are non-isomorphic is difficult.

Given any group G we can consider the **commutator subgroup**

$$[G, G] = \langle\langle \{g^{-1}h^{-1}gh \mid g, h \in G\}\rangle\rangle.$$

The quotient:

$$G \twoheadrightarrow G/[G, G]$$

is called the **abelianization** of G and it is an abelian group since

$$g^{-1}h^{-1}gh = 1 \Rightarrow gh = hg.$$

Fans of category theory will rejoice to observe that the abelianization map is a functor from the category fo groups to the category of abelian groups.

For free groups, the abelianization is

$$F_n \twoheadrightarrow F_n/[F_n, F_n] \approx \mathbb{Z}^n.$$

Now if this were linear algebra we would have $\dim(\mathbb{Z}^n) = n$ and vector spaces are isomorphic if and only if they have the same dimension. Thus

$$F_n \approx F_m \Leftrightarrow F_n/[F_n, F_n] \approx F_m/[F_m, F_m] \Leftrightarrow \mathbb{Z}^n \approx \mathbb{Z}^m \Leftrightarrow m = n.$$

Unfortunately $\mathbb{Z}$ isn't a field. However, there is a canonical algebraic constructions which embeds $\mathbb{Z}^n$ into $\mathbb{Q}^n$ (or any $\mathbb{Z}$-module into a $\mathbb{Q}$-vector space) is called **extension of scalars** and we get can use linear algebra again:

$$\dim_{\mathbb{Q}} \left((F_n/[F_n, F_n]) \otimes_{\mathbb{Z}} \mathbb{Q} \right) = n.$$

So the number n in F_n is determined by algebraic structure, and that solves the isomorphism problem.

Actually there is a name for this group invariant.

Definition 1.8.1 Let G be a finitely generated group then $b_1(G)$, the **first Betti number** of G is the torsion-free rank of the the abelianization $G/[G, G]$, viewed as a $\mathbb{Z}$-module or

$$b_1(G) = \dim_{\mathbb{Q}} \left(G/[G, G] \otimes_{\mathbb{Z}} \mathbb{Q} \right)$$

$\Diamond$

Let's now tackle the conjugacy problem. A useful tool for free groups is cancellation diagrams. Note that for the remainder of the section we will sometimes put a $\cdot$ to denote multiplication for emphasis. Let w_1 and w_2 be reduced words then we have the cancellation diagram:

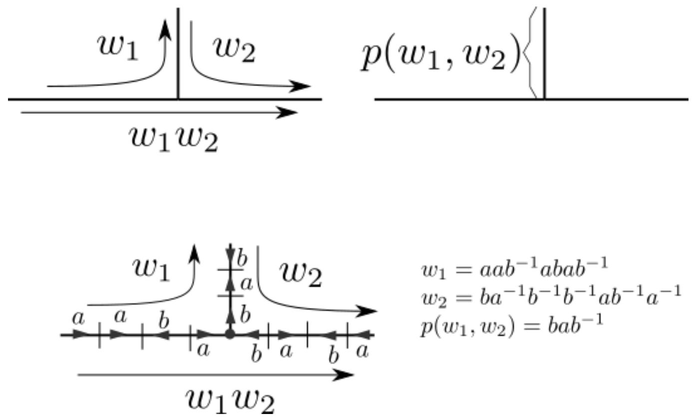


Figure 1.8.2 The cancellation bewteen two words.

And we denote by $p(w_1, w_2)$ the maximal co-terminal subword of w_1 that cancels in the product $w_1 \cdot w_2$. We will call this the **pinch in the product** $w_1 \cdot w_2$**.**

When conjugating $u^{-1} \cdot w \cdot u$ we can obtain different combinatorial types of cancellation diagrams.

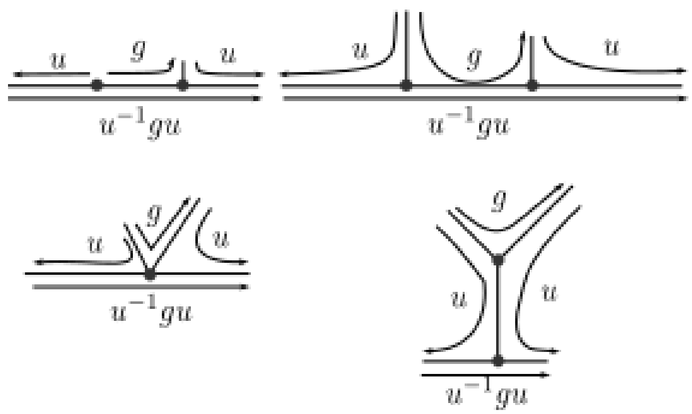


Figure 1.8.3 The cancellations in a conjugation.

Let's now solve the conjugacy problem. The conjugacy class of w consists of the set $\{g^{-1}wg \mid g \in G\}$.

Definition 1.8.4 A reduced word $w = x_1 \cdots x_m \in F_n$, $x_i \in \{a_1, \ldots, a_n\}^{\pm 1}$ is **cyclically reduced** if $x_1 \neq x_m^{-1}$. $\diamondsuit$

Lemma 1.8.5 *Given a reduced word $w \in F_n$, there is an algorithm to find a word u such that uwu^{-1} is cyclically reduced.*

Lemma 1.8.6 *In F_n, a cyclically reduced word has minimal length within its conjugacy class.*

Lemma 1.8.7 *Given any cyclically reduced word $w = x_1 \cdots x_m \in F_n$,*

- *any cyclic permutation $w' = x_j \cdots x_m x_1 \cdots x_{j-1} \in F_n$ of w can be obtained by conjugation, and*

- *all cyclically reduced conjugates of w are cyclic permutations.*

Corollary 1.8.8 *The conjugacy problem in F_n is solvable.*

Proof. Given two words $u, w \in F_n$, by Lemma 1.8.5 we can algorithmically conjugate them to cyclically reduced forms u', w' respectively. By Lemma 1.8.6 and Lemma 1.8.7u and w are conjugate if and only if u' is equal to one of the finitely many cyclic permutations of w'. ■

1.8.1 Exercises

1. Let $G = \langle a, b, c \big| aa, abc, bab^{-1}a^{-1} \rangle$. Compute $b_1(G)$.

Hint: Work in abelianizations. Let a, b, c be the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ respectively. The relations also correspond to vectors, e.g. $abc = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

Now $G/[G, G] \otimes_{\mathbb{Z}} \mathbb{Q}$ is the vector space

$$V = \mathbb{Q}^3/U$$

where U is the subspace spanned by the relation vectors, and $\dim(V) = 3 - \dim(U)$.

2. Consider this more elementary approach to showing that $b_1(G)$ is a group invariant. If $G = \langle X|R \rangle$ is a finite presentation, assign to each $x_i \in X$ the standard basis vector $e_i \in \mathbb{Q}^{|X|}$ and for each $r \in R$ take the vector $v_r \in \mathbb{Q}^{|X|}$ to be the vector whose ith entry is the sum of the exponents of the letter x_i. Let $U = \mathrm{span}\{v_r \mid r \in R\}$.

Show that for every elementary Tietze transformation, although you change the presentation the number

$$|X| - \dim(U)$$

remains unchanged.

Explain why this impies that $b_1(G)$, as calculated in the previous problem, depends on the group G and not on some particular choice of presentation.

3. Prove Lemma 1.8.5

4. Prove Lemma 1.8.6

5. Prove Lemma 1.8.7

Chapter 2

First elements of geometric group theory

2.1 Quasi isometries

Back in Section 1.5 we saw that a choice of generating set $G = \langle A \rangle$ gave rise to a Cayley graph $\mathbf{Cay}_A(G)$ which in turn endows G with a metric $d_{\mathbf{Cay}_A(G)}$. However, the subject matter in the previous sections should have reinforced the fact that there should be no expectation that such a metric be *canonical* for the group. A surprisingly successful strategy has been to simultaneously consider all possible metrics on G that arise from choices of finite generating sets. This motivates the following definition.

Spoiler: Quasi isometry is an unreasonably effective tool in group theory.

Definition 2.1.1 Quasi isometric embedding. Let (X, d_X) and (Y, d_Y) be metric spaces. We say a function $f : X \to Y$ is a (K, C)**-quasi isometric embedding** where K, C are strictly positive real numbers and we have

$$\frac{1}{K} d_y(f(x_1), f(x_2)) - C \leq d_X(x_1, x_2) \leq K d_Y(f(x_1, f(x_2)) + C, \tag{2.1.1}$$

for every $x_1, x_2 \in X$. ◊

We can always make the numbers bigger.

Lemma 2.1.2 *If $f : X \to Y$ is a (K, C)-quasi isometric embedding and $K' > K, C' > C$ then f is also a (K', C')-quasi isometric embedding.*

Proof. Exercise 2.1.1.1. ■

Often we will omit the bounds (K, C) and simply talk of a quasi isometric embedding, i.e. we'll say that $f : X \to Y$ is a **quasi isometric embedding** if there exist numbers (K, C) that satisfy the requirement of Definition 2.1.1.

Definition 2.1.3 Coarsely dense. Let $(X, d_x), (Y, d_y)$ be metric spaces and let $f : X \to Y$ be a quasi-isometric embedding. We'll say its image is D**-coarsely dense** if there is some $D > 0$ such that for every $y \in Y$ there exists some $x_y \in X$ such that:

$$d_Y(y, f(x_y)) \leq D.$$

◊

As before, we will say that the image of a quasi isometry is **coarsely dense**, if it is D-coarsely dense for some $D > 0$. In this business, the actual parameters don't always matter.

Definition 2.1.4 Quasi isometry. Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be a quasi isometric embedding. We say that f is a quasi-isometry if there exists a quasi isometric embedding $g : Y \to X$ called a **quasi-inverse** such that there is some number D such that:

- For all $x \in X$, $d_X(x, g(f(x)) \leq D$, and
- For all $y \in Y$, $d_Y(y, f(g(y)) \leq D$.

$\diamondsuit$

Another way to say this is that the compositions $g \circ f$ and $f \circ g$ are **D-close to the identity**.

We will finally state two results and leave proofs and examples as exercises.

Proposition 2.1.5 *Let A, B be two different finite generating sets of a group G, then the Cayley graphs $\mathbf{Cay}_A(G)$ and $\mathbf{Cay}_B(G)$ are quasi isometric.*

Proof. Exercise 2.1.1.5 ∎

Proposition 2.1.6 *Let $(X, d), (Y, d)$ be metric spaces. Prove that if $f : X \to Y$ is quasi isometric embedding with a coarsely dense image, then f is a quasi-isometry.*

Proof. Exercise 2.1.1.6. ∎

2.1.1 Exercises

1. Prove Lemma 2.1.2.

2. Show that quasi isometry gives an equialence relation between metric spaces.

3. The following question is stated in the context of function $f : X \to Y$ between metric space.

By a **linear growing function** we mean a linear function $\ell(x) = mx + b$ where $m, b \in \mathbb{R}$ and $m > 0$ (like in Calculus 1.) Show the following:

(a) If there is a linear growing function $\ell_r(x)$ such that

$$d_Y(f(x), f(y)) \leq \ell_r(d_X(x, y)),$$

then there is a linear growing function $\ell_l(x)$ such that

$$\ell_l(d_Y(f(x), f(y))) \leq d_X(x, y)$$

(b) Show that if there are linear growing functions ℓ_1, ℓ_2 such that for all $x, y \in X$ we have

$$d_X(x, y) \leq \ell_1(d_Y(f(x), f(y)))$$

and

$$d_Y(f(x), f(y)) \leq \ell_2(d_X(x, y)),$$

then $f : X \to Y$ is a quasi isometric embedding.

4. Let Γ be a graph and let $V(\Gamma)$ be it's vertex set. Let $V(\Gamma)$ be equipped with the metric induced by Γ. Suppose every edge has length 1 and consider a retraction

$$r : \Gamma \to V(\Gamma)$$

where each edge interior is mapped to some adjacent vertex. Show that r is a quasi-isometry.

5. Prove Proposition 2.1.5.

Hint: Look back at Exercise 1.6.1.5

Other hint: In this case you are considering the a group G with two metrics d_A and d_B (from two finite generating sets of G) and function you want to use is the identity. Show that there are linear functions ℓ_1, ℓ_2 such that

$$d_A(x, y) \leq \ell_1(d_B(x, y)) \text{ and } d_B(x, y) \leq \ell_2(d_A(x, y)),$$

for all $x, y \in G$.

6. Prove Proposition 2.1.6.

Hint: Although the statement of Proposition 2.1.6 doesn't have any parameters, you should start by fixing parameters.

Other hint: for a quasi-inverse define $g(y)$ to be some $x \in X$ such that $d_Y(f(x), y)$ is minimal, or at least unifomly bounded for all y (this is where D-coarsely dense becomes important.)

7. Prove that with the standard Euclidean metric that the mapping

$$\begin{aligned} \mathbb{R} &\to \mathbb{R}^2 \\ x &\mapsto (x, 0) \end{aligned}$$

Is a quasi-isometric embeding, but not a quasi-isometry.

8. Consider the graph $\Gamma = \mathbf{Cay}_{\{1\}}(\mathbb{Z})$, i.e. the real line with vertices at integer points. Let's conisider what happens when we change the lengths of edges.

(a) Fix $0 < a < b \in \mathbb{R}$. Show that if for each edge e of Γ we randomly change its length to some number $a < \ell_e < b$ then the resulting metric space Γ' is quasi-isometric to Γ.

(b) Show that if we let the length of edges be any $0 < l_e \leq 1$, then the resulting graph may not be quasi-isometric to Γ

Hint: if edges schrink too much maybe the graph will have finite diameter.

2.2 Can anything be expressed by quasi isometry?

Two metric spaces are the same if they are isometric. In the previous section we saw although there is no canonical **word metric** on a group, if we consider metric spaces only up to quasi isometry, then we do obtain something canonical: namely the **quasi isometry class of word metrics coming from finite generating sets.** The issue then becomes that maybe quasi isometry is too vague and that all groups are quasi isometric. *For quasi isometry to be useful, it must be able to tell groups apart.*

All finite groups are quasi isometric to the metric space consisting of a single point, so this viewpoint is not useful for the study of finite groups. We also see that quasi isometry is able to distinguish finite groups from infinite groups, but that isn't particularly impressive.

In this lecture we will show that the groups $\mathbb{Z}$ and $\mathbb{Z}^2$ are not quasi isometric and we will prove Theorem 2.2.5, the Svarc-Milnor Lemma, which has far reaching consequences such as the fact that if $K \leq H$ is a finite index subgroup then H and K are quasi isometric. This gives us a limitation of how much can be determined by quasi isometry.

Let x be a point in a metric space (X, d), let $r > 0$ be some number, then we define the **closed ball of radius** r **about** x to be the set

$$B(x, r) = \{y \in X \mid d(x, y) \leq r\}.$$

When it is clear from the context in which metric space a given ball lies, we will not bother explicitly giving the metric.

It is worth recalling from Exercise 1.5.1.4 that for any ball about the identity in the standard Cayley graph of $\mathbb{Z}^2$, we can find a generating set of $\mathbb{Z}$ which gives the exact same ball about the identity. In other words, at small and medium scales word metrics will not distinguish $\mathbb{Z}$ and $\mathbb{Z}^2$

The proof of the following is known as an **asymptotic** argument.

Proposition 2.2.1 *The groups $\mathbb{Z}$ and $\mathbb{Z}^2$ are not quasi isometric.*

Proof. We take the standard cayley graphs $\mathbf{Cay}_{\{1\}}(\mathbb{Z})$, the line, and $\mathbf{Cay}_{\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}}(\mathbb{Z}^2)$, the infinite grid, and equip $\mathbb{Z}$ and $\mathbb{Z}^2$ with the induced word metric.

Suppose towards a contradiction that $\mathbb{Z}$ and $\mathbb{Z}^2$ were quasi-isometric. Then there would exist a (K, C)-quasi isometric embeding $f : \mathbb{Z}^2 \to \mathbb{Z}$. Consider the balls $B(1, r) \subset \mathbb{Z}^2$, on the one hand they look like diamonds and contain $|B(1, r)| = O(r^2)$ points. On the other hand, because we have a quasi-isometric embedding we have a containment.

$$f(B(1, r)) \subset B(f(1), Kr + C) \subset \mathbb{Z}.$$

Since $\mathbb{Z}$ is a line we have

$$|B(f(1), Kr + c)| = O(r).$$

Because the cardinality of $B(1, r) \subset \mathbb{Z}^2$ grows faster than the cardinality of $B(f(1)), Kr + C) \subset \mathbb{Z}$ the restrictions $f|_{B(1,r)}$ will be far from injetive, in partiular as $r \to \infty$ then maximal number of preimages of a point in $B(f(1), Kr + C)$ via the restrictions $f|_{B(1,r)}$ must also go to infinity.

Pick M so large that any subset of $\mathbb{Z}^2$ with M or more elements must have diameter at least $K \cdot (1 + C)$ and pick r so large that some element $z_0 \in B(f(1), Kr + C)$ has at least M $f|_{B(1,r)}$-pre images.

On the one hand there must be elements $y, y' \in \mathbb{Z}^2$ such that $f(y) = f(y') = z_0$ but $d(y, y') \geq K \cdot (1 + C)$. But by definition of a quasi-isometric embedding we must have

$$\frac{1}{K} d(y, y') - C \leq d(f(y), f(y'))) = d(z_0, z_0) = 0, \tag{2.2.1}$$

but

$$d(y, y') \geq K \cdot (1 + C) \Rightarrow \frac{1}{K}(K(1 + C)) - C = 1 \leq 0$$

which is a contradiction. It follows that there can be no quasi-isometric embedding from $\mathbb{Z}^2$ to $\mathbb{Z}$. ∎

So at least now we have established that quasi isometry is maybe not a completely useless notion as it can tell at least two groups apart. Quasi isometry, however, does have limitations and we will now explore these.

From now on we will asssume our spaces X are path connected metric spaces, the readier at this point is free to assume that X is a connected graph, all of whose edges have a given length (usually they have unit length). A **path** is a continuous mapping

$$p : \underbrace{[0, L]}_{\subset \mathbb{R}} \to X.$$

and we can define its speed at s as

$$|\dot{p}(s)| = \lim_{h \to 0^+} \frac{d(p(s+h), p(s))}{h}.$$

If X is a graph, viewing edges as isometric copies of intervals of $\mathbb{R}$ we can make sense of this for almost all values of $s \in [0, L]$.

Integrating speed gives **arc length** and we say that p is **arc length parameterized** if $|\dot{p}(s)| = 1$ for all $s \in [0, L]$. We say that a path is a **geodesic** if it is a shortest path between its endpoints $p(0)$ and $p(L)$ and we can define the **path metric** to be given by the minimal length of all paths joining two points. A metric space is called a geodesic metric space if its metric coincides with the path metric.

We assume the reader knows what a left group action

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

on a space is. We will use the $\cdot$ symbol to denote the action of a group element on a point. We say that G **acts by isometries** on (X, d) if for every $g \in G$ and every $x, y \in X$ we have:

$$d(x, y) = d(g \cdot x, g \cdot y).$$

For our current purposes it will be sufficient to only consider our metric spaces to be path connected graphs, but the results we state will be be more general. For our purposes a **compact set** is a finite subgraph. An **action by isometries** is an action by graph automorphisms. The action of G on a graph X is **proper** if the stablizer of any vertex is finite. The **quotient** $G \backslash X$ of an action has vertices that conist of equivalence classes

$$[v] = \{u \in V(X) \mid \text{for some } g \in G, g \cdot u = v\}$$

and edges

$$[e] = \{f \in E(X) \mid \text{for some } g \in G, g \cdot f = e\}$$

and is itself a graph. The action of G on X is **cocompact** if $G \backslash X$ is finite and a **coarse fundamental domain** is compact subset $K \subset X$ whose G translates cover X, i.e.

$$X = G \cdot K.$$

With all this terminology out of the way, we can start proving results.

We will first state a useful result without proof. It follows from a careful consideration of the definitions.

Lemma 2.2.2 *If a group G acts properly, cocompactly and by isometries on a graph X, then X is locally finite, i.e. all vertices are adjacent to only finitely many edges. In particular balls of finite radius only contain finitely many* *elements.*

Lemma 2.2.3 *Let G act properly and cocompactly on a path connected graph X with a distinguished vertex p_0 by isometries. Then the following are true:*

1. *There is some $r_0 > 0$ such that $X = G \cdot B(p_0, r_0)$, i.e. G-translates of $B(p_0, r_0)$ cover X.*

2. *The subset*

$$R_s = \{g \in G \mid g \cdot B(p_0, s) \cap B(p_0, s) \neq \emptyset\} \subset G$$

is finite *for all radii $s > 0$.*

3. *Let r_0 be as in Item 1. Then set*

$$S = R_{r_0} = \{g \in G \mid g \cdot B(p_0, r_0) \cap B(p_0, r_0) \neq \emptyset\}$$

is a generating set for G, i.e. $G = \langle S \rangle$.

Proof. Let's first prove Item 1. Because the action is cocompact the quotient $G \backslash X$ is a finite graph. Let $[v_1], \ldots, [v_k]$ and $[e_1], \ldots, [e_l]$ be the vertices and edges of $G \backslash X$ respectively. For vertex each $[v_i]$ of the quotient pick some vertex $[v_i] \ni \hat{v}_i \in V(X)$ and for each edge $[e_j]$ of the quotient pick some edge $[v_j] \ni \hat{e}_j \in E(X)$.

Since cells of $G \backslash X$ corespond to G-orbts, for every $v \in V(X)$ there is some $g \in G$ such that $g \cdot v = v_i$ for some i and the analogous statement also holds for edges. Because the set of cells

$$C = \{\hat{v}_1, \ldots, \hat{v}_k, \hat{e}_1, \ldots \hat{e}_l\} \subset X$$

is finite and X is path connected there must be some sufficiently large r_0 so that $C \subset B(p_0, r_0)$.

Let's now prove Item 2. Suppose towards a contradiction that for some set $s > 0$ the subset $R_s \subset G$ is infinite. Since $B(p_0, s)$ is finite there must be some vertex $v_0 \in B(p_0, s)$ such and an infinite subset $\{g_1, g_2, \ldots\}$ such that

$$v_0 \in \bigcap_{i=1} g_i \cdot B(p_0, s)$$

Let $w_i = g_i^{-1} \cdot v_0$. On the one hand $w_i \in B(p_0, s)$, on the other hand since $B(p_0, s)$ is finite there must be infinitely many w_i's that are equal, say

$$w_{i_1} = w_{i_2} = \cdots$$

but then it follows that the elements

$$g_{i_j} g_{i_1}^{-1}; j = 1, 2, 3, 4, \ldots$$

are all distinct, but all fix the the point v_0. This contradicts the hypothesis that the action of G on X is proper.

Finally we can prove Item 3. Consider first the set

$$V = G \cdot p_0 = \{v \in V(X) \mid v = g \cdot p_0, \text{ for some } g \in G\}.$$

And suppose towards a contradiction that $\langle S \rangle \subsetneq G$.

Claim: $\langle S \rangle \cdot p_0 = V$. Suppose this was not the case, let $w_0 = g \cdot p_0$ be an element of minimal distance from $\langle S \rangle \cdot p_0$. Let $u_0 \in \langle S \rangle \cdot p_0$ be a point closest to w_0 and let α be a path from u_0 to w_0 that realizes their distance.

On the one hand there is some $s \in \langle S \rangle$ such that $s \cdot p_0 = u_0$, on the other hand there is some vertex u_1 a distance 1 from u_0 along the path α which is strictly closer to w_0 than u_0. The element $s^{-1} \cdot u_1$ is distance 1 from p_0. There is some $h \in G$ such that $h \cdot p_0 = (s^{-1} \cdot u_1)$ and since $s^{-1} \cdot u_1 \in h \cdot B(p_0, 3 \cdot r_0) \cap B(p_0, r_0) \neq \emptyset$ we have that $h \in S$. It now follows from equations above that

$$sh \cdot p_0 = s \cdot (s^{-1} \cdot u_1) = u_1.$$

Thus, $u_1 \in \langle S \rangle \cdot p_0$, but this contradicts that $u_0 \in \langle S \rangle \cdot p_0$ is as close as possible to w_0. *This proves the claim.*

So suppose finally towards a contradiction that $\langle S \rangle \subsetneq G$. Then there is some $G \ni g \notin \langle S \rangle$. By the claim above there is some $s \in \langle S \rangle$ such that $s^{-1} \cdot (g \cdot p_0) = p_0$, so that $s^{-1} g$ fixes p_0, but then $s^{-1} g \in S$ which implies that $g \in \langle S \rangle$ which is a contradiction. ■

The proof above illustrates the potent combination of a proper, cocompact action by isometries. Such an action is typically called **a geometric action**. The lemma above gives an unexpected consequence:

Corollary 2.2.4 *If a group G acts geometrically on a path connected graph X then G is finitely generated.*

Recall that the quasi isometry class of a group G is the smallest quasi isometry class which contains all word metrics on G. In particular the statement below makes complete sense without having to specify some metric on G.

Theorem 2.2.5 The Svarc-Milnor Lemma. *If G acts geometrically on a geodesic metric space (e.g. a graph) X, then G is quasi-isometric to X*

sketch. First note that by hypthesis $G \cdot p_0$ is coarsely dense in X and in fact quasi isometric to X. So we need need only show that G is quasi isometric to $G \cdot p_0 \subset X$. We will show that the following map

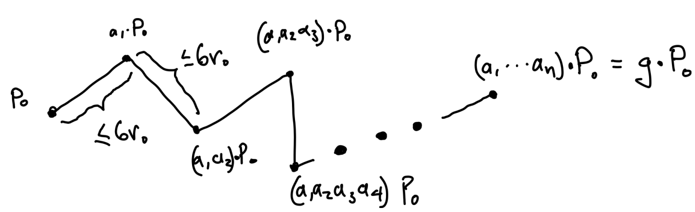
$$\begin{aligned} f : G &\to p_0 \\ g &\mapsto g \cdot p_0 \end{aligned}$$

is a quasi-isometric embedding. Since it is surjective the result will follow.

By Lemma 2.2.3, Item 3 $S = R_{r_0}$ is a finite generating set of G, therefore(by Lemma 2.2.3, Item 3) so must be $A = R_{3r_0}$. We equip G with the finite generating set A. Note that for each $a \in A$ we have

$$d_X(p_0, a \cdot p_0) \leq 6 \cdot r_0$$

So if $g = a_1 \cdots a_n$ is a product of minimal length representing $g \in G$, i.e. $d_A(1, g) = n$ then considering the broken path:



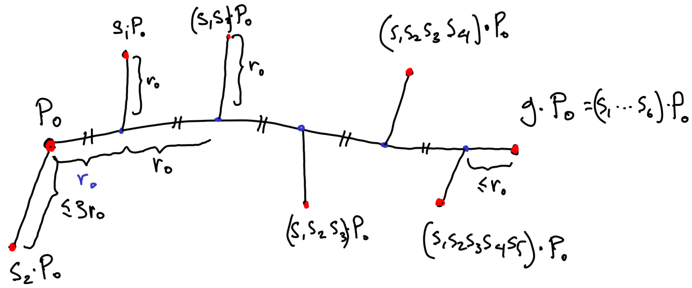
we have by the triangle inequality that

$$d_X(p_0, g \cdot p_0) \leq 6 \cdot r_0 d_A(1, g),$$

which is one inequality we need to show.

We now need to bound $d_A(1, g)$ above by some linear function of $d_X(p_0, g \cdot p_0)$. This follows by noting that every point of X is a distance at most r_0

from some point in $G \cdot p_0$ (Lemma 2.2.3, Item 1) and that every point in $v \in G \cdot p_0 \cap B(p_1, 3r_0)$ there is some a such that $a \cdot p_0 = v$. The picture below which is obtained by dividing the shortest path from P_0 to $g \cdot P_0$ into a minimal number of segments of length at most r_0 implies that $d_A(1, g) \leq \frac{1}{r_0} d_X(p_0, g \cdot p_0) + 1$.



At this point showing that f is a quasi isometry is straightforward. ■

From this we get the following, which gives us the ultimate limitiation of quasi-isometry

Corollary 2.2.6 *Any finitely generated group G is quasi isometric to all its finite index subgroups.*

The result above seems trivial for the following reason. If (X, d) is a metric space and have a subset $S \subset X$ and we equip S with the subspace metric $d|_S$, then S is isometrically embedded into X with respect to this metric. In particualr if S is coarsely dense in X, as is the case with a finite index subgroup, then $(S, d|_S)$ will be quasi isometric to (X, d_X). Let us call this the **extrinsic** metric on S from the embedding $S \subset X$. *Above corollary is stronger than that.* In particular any finite index subgroup of H is finitely generated so a finite generating set $\langle A \rangle = H$ endows H with an **intrinsic** quasi isometry class.

The above corollary states that if $H \leq G$ is a finite index subgroup of of a finitely generated group. Then H and G are quasi isometric with respect to their respective *intrinsic* quasi isometry classes.

Now it may happen that we have an injective homomorphism $H \hookrightarrow G$ where both H and G are finitely generated but given aword metric d on G the induced extrinsic metric $d|_H$ induced on H is not quasi isometric to any (intrinsic) word metric on H. In this case we say H is a **distorted subgroup.** We illustrate this with an example

Example 2.2.7 $BS(1, 2)$ **the simplest weird group.** $BS(1, 2)$, the Baumslag-Solitar group is a weird group. It has the following presentation:

$$BS(1, 2) = \left\langle a, t \middle| tat^{-1}a^{-2} \right\rangle.$$

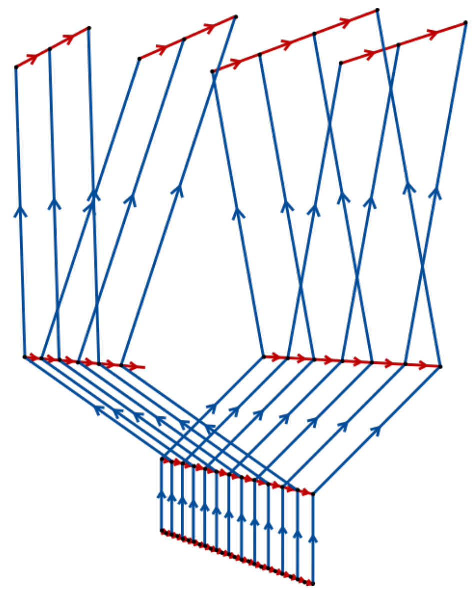
Notice that the single relation gives the following commutation relation:

$$ta = a^2 t$$

which means that every element can be expressed as a word of the form:

$$a^n t^m; n, m \in \mathbb{Z}.$$

So far that doesn't seem weird, but its Caley graph looks like this:



.

Consider the homomorphism $\phi : \mathbb{Z} \hookrightarrow BS(1,2)$ given by $\psi(n) = a^n$. In the next chapter we will see that ϕ is injective. For now we can see this as $\langle a \rangle \leq BS(1,2)$ as being a copy of $\mathbb{Z}$ sitting in $BS(1,2)$.

We can now compare the standard metric $d_{\mathbb{Z}}$ on $\mathbb{Z}$ to the extrinsic metric $d|_{\langle a \rangle}$ induced by its inclusion into $BS(1,2)$.

On the one hand we have $d_{\mathbb{Z}}(1, a^n) = |n|$, which is straightforward. Now the relation lets us rewrite, for example,

$$a^2 = tat^{-1} \Rightarrow a^{2^3} = aaaaaaaa = tttat^{-1}t^{-1}t^{-1}.$$

More generally we have $a^{2^n} = t^n a t^{-n}$, so on the other hand we have two different metrics:

$$d_{\mathbb{Z}}(1, a^{2^n}) = 2^n, d|_{\langle a \rangle}(1, a^{2^n}) = 2n + 1.$$

In other words this embedding $\phi : \mathbb{Z} \hookrightarrow BS(1,2)$ exponentially distorts the intrinsic metric on $\mathbb{Z}$. In particular this embedding is not a quasi isometric embedding, as the latter only allows linear distortion. $\square$

We will say that groups G and H are **abstractly commensurable** if they have finite index subgroups $K \leq G$ and $K' \leq H$ which are isomorphic, i.e $K \approx K'$. Corollary 2.2.6 tells us that *commensurable groups are quasi-isometric* therefore we have the limitation that *quasi isometry* cannot distinguish between groups in a commensurability class.

2.2.1 Exercises

1. In the proof of Proposition 2.2.1 in equation (2.2.1), we didn't use the same formulation of quasi isometric embedding as in Definition 2.1.1. Explain why this is still okay.

Comment: If anything this is to emphasize that even the definition of a quasi isometric embedding is best left vague.

2. A group acts on itself on the left and the right. Consider a generating set $\langle A \rangle = G$, and let d be the word metric on G.

- Show that the left action of G on itself is an action by isometries with respect to this metric.

- Show that the right action of G on itself is not generally by isometries with respect to this metric.

Hint: For part 2, G needs to be non abelian. Take a free group and just check out a few examples.

3. Look at the proof of Proposition 2.2.1. Informally explain whether or not the argument could be generalized to distinguish $\mathbb{Z}^n$ and $\mathbb{Z}^m$ for general distinct $m, n \in \mathbb{Z}_{\geq 0}$

4. Consider $\mathbf{Cay}_{\{1\}}(\mathbb{Z})$, the standard Cayley graph for $\mathbb{Z}$. Consider the subgroup $3\mathbb{Z} \leq \mathbb{Z}$.

(a) Sketch $3\mathbb{Z} \leq \mathbb{Z}$ inside $\mathbf{Cay}_{\{1\}}(\mathbb{Z})$.

(b) Draw the quotient $3\mathbb{Z}\backslash\mathbf{Cay}_{\{1\}}(\mathbb{Z})$

5. Let G be a finitely generated group and let $K \leq G$ be a finite index subgroup. Let A be a finite generating set of G.

Prove that the induced action of K on $\mathbf{Cay}_A(G)$ is co-compact.

Hint: The vertices of the quotient $K\backslash\mathbf{Cay}_A(G)$ correspond to cosets.

6. Prove Corollary 2.2.6.

Hint: Just combine all the previous results.

7. Prove the commensurablity of finitely generated groups is an equivalence relation.

Hint: If $K_1, K_2 \leq G$ are finite index subgroups of G, then the intersection $K_1 \cap K_2$ is a finite index subgroup of K_1, K_2, and G.

8. Let $f : G \to H$ be a surjective group homorphism such that $\ker(f)$ is finite. Prove that G is quasi isometric to H

Hint: G acts on the Cayley graph of H via the action $g \cdot x = f(g) \cdot_H x$ where $\cdot_H$ is the standard action of H on its Cayley graph.

2.3 Many ended graphs

Our motivation: quasi isometric rigidity. We say that two groups G_1, G_2 are **virually isomorphic** if they contain finite index subgroups $G_i' \overset{\text{f.i.}}{\leq} G_i$ and such that there are finite normal subgroups $K_i \triangleleft G_i'$ $(i = 1, 2)$ such that:

$$G_1'/K_1 \approx G_2'/K_2.$$

In the previous lecture, we saw that quasi isometry could not distinguish between virtually isomorphic groups. The question still remains: *Can quasi isometry determine a group up to virtual isomorphism?* The short answer is: no, not always.

The longer answer is *sometimes, and even negative results in this matter tend to be non-trivial.* The phenomenon where quasi isometry (purely geometric/metric) determines a group property up to virtual isomorphism (purely algebraic) is known as **quasi isometric rigidity**. Our goal for the next few lectures is will be to prove the simplest possible such result.

Theorem 2.3.1 *$\mathbb{Z}$ is quasi isometrically rigid. That is to say if H is some group that is quasi isometric to $\mathbb{Z}$ then H contains a finite index subgroup $H' \overset{\text{f.i.}}{\leq} H$ such that $H' \approx \mathbb{Z}$.*

We first note that all non trivial subgroups of $\mathbb{Z}$ are isomorphic, so there is no need to pass to finite index subgroups of $\mathbb{Z}$. The example below shows that there are infnite groups (called periodic groups) which do not contain subgroups isomorphic to $\mathbb{Z}$.

Example 2.3.2 The Burnside Problem is related to this issue. Consider the group

$$\langle a, b, c | w^n ; w \in F(a, b, c)\rangle$$

obtained by declaring every n^{th} power to be trivial. For $n = 2, 3, 4, 6$ this group is known to be finite. For $n = 10^{10}$ it is known to be infinite, for $n = 5$, it is unknown. In particular, infinite Burnside groups (which exist) do not contain subgroups isomorphic to $\mathbb{Z}$. $\square$

In particular we will need to show that such a group H actually contains an infinite order element. The way we will achieve this is via group actions, and a quasi isometry invariant known as ends.

2.3.1 Essential disconnecting sets and many endedness

In graph theory, a **cutset** is a minimal (w.r.t inclusion) set of edges whose removal disconnects a graph. Such sets of edges abound, for example the set of edges incident to a vertex contains a cutset. For infinite graphs we have something more interesting: a subset of a graph is an **essential disconnecting set** if its removal disconnets the graph into two *infinite* components. Of most interest to us will be **finite essential disconnecting sets.**

Example 2.3.3

- In an infinite tree, any finite subset is an essential disconnecting set.
- The grid $\mathbf{Cay}_{\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}}\left(\mathbb{Z}^2\right)$ doesn't admit any finite essential disconnecting sets.

$\square$

Before defining what an **end** is we have the following.

Definition 2.3.4 We say a graph X is

- zero ended if it is finite,
- many ended if it admits a finite essential disconnecting set
- one ended if it none of the above are satisfies, i.e. it is infinte but withuot an essential disconnecting set.

We say a finitely generated group G is zero, one, or many ended if it has a Cayley graph (w.r.t. a finite generating set) which is zero, one, or many ended, respectively. $\Diamond$

We should pause to ponder whether this definition is even well-defined: *Why couldn't a group have two Cayley graphs, one of which is one ended and the other one isn't?*

2.3.2 A digression about paths

At this point we want to provide a perspective on compact essential disconnecting sets that is compatible with quasi-isometry.

If X is a path connected space (e.g. a connected graph) then we already encountered the notion of a geodesic

$$p : [0, L] \to X.$$

We can relax this notion to that of a **quasi geodesic**

$$q : [0, L] \to X$$

where q is a (K, C)-quasi isometric embedding (where we are equipping the interval $[0, L]$ with the standard metric of length on $\mathbb{R}$.) Now q is not necessarily continuous and can do all kinds of strange things. However we can do the following:

Lemma 2.3.5 *Let $q : [0, L] \to X$ be a (K, C)-quasi geodesic, where $L \in \mathbb{Z}_{>0}$, and consider a path $q_p : [0, L] \to X$ satisfying the following:*

- *For $i \in [0, L] \cap \mathbb{Z}$ $q_p(i) = q(i)$*
- *On the open intervals $(i, i + 1)$, $i = 0, \ldots, L - 1$ the restriction $q_p|_{(i,i+1)}$ is a geodesic joining $q_p(i)$ and $q_p(i + 1)$,*

*i.e. q_p is a **piecewise geodesic**.*

Then for every $z \in [0, L]$ we have

$$d_X(q(x), q_p(x)) \leq 2(K + C).$$

The concept of a geodesic path is a global notion, and is a bit too restrictive. The right notion we want is **rectifiability**, but that takes too much work to properly define and justify. If we restrict ourselves to the case where X is a graph, then we can make the following assumption: *every path in X is a piecewise geodesic path, specifically it travels through every edge at constant unit speed.* This way we can ignore analysis and focus on combinatorics.

2.3.3 Many endedness as a quasi-isometry invariant

Now let us give a more metric characterization of a finite essential cut set.

Proposition 2.3.6 *A locally finite graph X admits a finite essential cutset if and only if there is:*

- *some point $x_0 \in V(X)$ and a fixed number $D > 0$,*
- *two sequences of vertices $(x_n), (y_n)$ with*

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_0, x_n) = \lim_{n\to\infty} d(x_0, y_n) = \infty, \textit{and}$$

- *the following property: every path p_n from x_n to y_n intersects the ball $B(x_0, D)$.*

In particular this result says that a finite essential disconnecting set is a "bottleneck in the graph".

Corollary 2.3.7 *If X and Y are quasi isometric graphs, then X is many ended if and only if Y is many ended.*

In particuar, being 0, 1, or many ended is a quasi isometric invariant.

2.3.4 Exercises

1. Work out the two examples in Example 2.3.3. It's best to use pictures and to be informal.

2. Prove Lemma 2.3.5

Hint: Use the triangle inequality, don't try to be optimal.

3. Prove Proposition 2.3.6

4. Use Proposition 2.3.6 to give a rigourous argument that $\mathbb{Z}^2$ is one ended.

Hint: Fix some x_0, D take sequences $(x_n), (y_n)$ satisfying the requrements of and show how to construct $B(x_0, D)$-avoiding paths from x_n to y_n, provided n is sufficiently large.

5. Prove Corollary 2.3.7

Hint: Let $q : X \to Y$ be a (K, C) quasi-isometry. The piecewise geodesic paths $p : [0, L] \to X$ are sent via $q \circ p$ to piecewise quasigeodesic paths, which can then be turned by Lemma 2.3.5 into piecewise geodesics which stay close to $q \circ p$. Assume that X is many-ended and Y is one ended and use Proposition 2.3.6 to derive a contradiction.

2.4 What is an end, anyway?

In the previous lecture we presented the notion of a **many ended graph**, but didn't quite define what an end was. We will do this in this lecture and apply the concept to quasi-isometric rigidity.

2.4.1 Ends are like essential components at infinity.

Previously we defined a locally finite connected graph X to be many ended if there was some point x_0 such that the deletion of some ball $B(x_0, r)$ in X produced multiple infinite components.

Consider some sequence $0 < r_1 < r_2 < \ldots$ of radii growing to infinity and denote by

$$\mathcal{K}^{\rangle} = \{K^i_j \subset X; 1 \leq j \leq N_i\}$$

the infinite connected components of $X \setminus B(x_0, r_i)$. Common sense about sets implies that since we have $B(x_0, r_1) \subset B(x_0, r_2) \subset \cdots$ then if $n > m$ then each element of $\mathcal{K}^n$ is contained in a unique element of $\mathcal{K}^m$ which gives a well defined surjective function $f_{nm} : \mathcal{K}^n \to \mathcal{K}^m$.

If you want you can make a rooted tree where level i is a component in $\mathcal{K}^i$ and the ancestor of each component is the element of $\mathcal{K}^{i-1}$ which contains it.

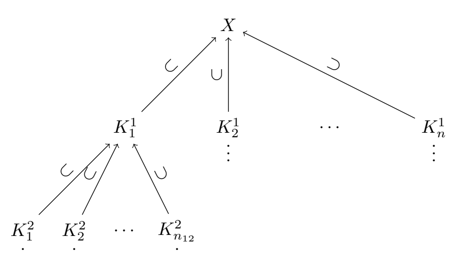


Figure 2.4.1 Essential components

The **ends of the graph** X correspond to the infinite sequences $K^1_1 \supset K^2_{n_2} \supset K^3_{n_3} \supset \cdots$ of nested components. Formally the set of ends is the defined as the set of inverse limits of components

$$\mathrm{Ends}(X) = \lim_{\leftarrow} \mathcal{K}^n$$

with respect to the system of functions $f_{nm}, n > m$. The elements of the inverse limit are formally sequences of infinite connected components, and so can be seen as "components" at infinity and they are in bijective correspondence with the infinite branches of the tree shown in Figure 2.4.1.

We suppress the concern that ends seem to be highly dependent on the choice of growing balls (we will confront this momentarily) and will consider two examples.

Example 2.4.2 Let L be the infinite line (i.e. the standard Cayley graph for $\mathbb{Z}$) and let T_4 be the infinite regular tree of valence 4 (i.e. $T_4 = \mathbf{Cay}_{\{a,b\}}(F(a,b))$.) In both metric spaces consider the balls $B(x_0, 1) \subset B(x_0, 2) \subset B(x_0, 3) \subset \cdots$.

- For L we always have that $L \setminus B(x_0, n)$ has exactly two components, so as the balls grow we get the following containements of components

$$\cdots \subset K_1^2 \subset K_1^1 \subset L \supset K_2^1 \supset K_2^2 \supset \cdots$$

  so the diagram given in figure Figure 2.4.1 has precisely two infinite branches and, so L has two ends, which we should think of as being $\pm\infty$.

- In T_4, the complement $T_4 \setminus B(x_0, n)$ has precisely $4 \cdot 3^n$ components. In particular each $K_j^n \in \mathcal{K}^n = T_4 \setminus B(x_0, n)$ contains exactly 3 components in $\mathcal{K}^{n+1}$. It follows that the ends of T_4 are in bijective correspondence with infinite strings of the form:

$$ab_1b_2b_3\cdots, a \in \{1,2,3,4\}, b \in \{1,2,3\}$$

  and therefore form an infinite set of continuum size.

□

2.4.2 Ends are robust

A **compact (or finite)** exhaustion of a set X is a sequence of compact (or finite) subsets $K_1 \subset K_2 \subset \cdots$ such that:

$$X = \bigcup_{i=1}^{\infty} K_i.$$

A sequence of balls with a common center whose radii grow to infinity is an example of a compact exhaustion.

The main issue about our definition of ends is that it seems to depend on our choice of compact exhaustion. We will state our main robustness result, but only sketch a proof. The first item is a straightforward, yet non trivial exercise in inverse limits, which are beyond the scope of the prerequisites, and the second item is obvious given the material that was previously covered, but also ultimately depends on inverse limits.

Theorem 2.4.3 *Let X be connected locally finite graph. Then*

- *the set* $\mathrm{Ends}(X)$ *does not depend on the choice of compact exhaustion, and*

- *if $f : X \to Y$ is a quasi-isometry, then there is a natural induced bijection*

$$\hat{f} : \mathrm{Ends}(X) \to \mathrm{Ends}(Y).$$

sketch. We first sketch the proof of the first point. Let $A_1 \subset A_2 \subset \cdots$ and $B_1 \subset B_2 \subset \cdots$ be two different compact exhaustions of X and denote by $\mathcal{A}^i$ and $\mathcal{B}^j$ the sets of infinite components of $X \setminus A_i$ and $X \setminus B_i$ we want to show an identification between the inverse limits

$$\lim_{\leftarrow} \mathcal{A}^n, \lim_{\leftarrow} \mathcal{B}^n$$

but the difficulty is that we may not have a well defined map $\mathcal{A}^n \to \mathcal{B}^m$ where we want to send $Z \in \mathcal{A}^n$ to the unique component $W \in \mathcal{B}^m$ such that $Z \subset W$ even if $n > m$.

The issue is resolved by observing that for some sufficiently large $N(m)$ such maps $\mathcal{A}^k \to \mathcal{B}^m$ will exist provided $k > N(m)$ and then applying the abstract nonsense of inverse limits which enables us to consider the inverse limits of the union $\{\mathcal{A}^n\} \cup \{\mathcal{B}^n\}$ equipped with a compatible system of surjective functions.

We now sketch a proof of the second point. If $K \subset X$ is a finite essential disconnecting set, then it's quasi isometric image $f(K) \subset Y$ can be enlarged (by a uniformly bounded amount) to a finite essential disconnecting set. Furthermore the quasi isometric image of a compact $A_1 \subset A_2 \subset \cdots$ exhaustion of X gives rise (perhaps after bounded enlargement) to a compact exhaustion $\widehat{f(A_1)} \subset \widehat{f(A_2)} \subset \cdots$of Y. Again, the image $f(K_j^i) \subset Y$ of an infinite component of $K_j^i \subset X \setminus A_i$ may not quite lie in a component of the counterpart $Y \setminus \widehat{f(A_i)}$, but if $n >> m$ then we will have $f(K_j^n) \subset Y \setminus \widehat{f(A_m)}$. The bijection then follows from the universal property of inverse limits. ! ∎

Here's an even more informal explanation:

- given different compact exhaustions of a space, we can always find a way to eventually match things up so that we have the same ends, and
- Quasi isometries sent compact exhaustions to compact exhaustions and finite essential disconnecting sets to finite essential disconnecting sets, and everything works out great.

The main point of all this is that we get a quasi-isometric invariant of groups.

Corollary 2.4.4 *Any finitely generated group G has a well defined (up to bijection) set of ends* $\mathrm{Ends}(G)$.

2.4.3 2 endedness and the quasi isometric rigidity of $\mathbb{Z}$

We have now developed just enough machinery to show that $\mathbb{Z}$ is quasi isometrically rigid, what we will prove is precisely the following:

Theorem 2.4.5 *If G is two ended (i.e. $|\mathrm{Ends}(G)| = 2$) then it contains a subgroup of finite index isomorphic to $\mathbb{Z}$.*

Here's the first step in the proof. Note that in this proof, quasi isometry is not used, but rather action by isometries.

Lemma 2.4.6 *Let G be two ended and let $X = \mathbf{Cay}_A(G)$ be some Cayley graph of G. Then G permutes the set $\mathrm{Ends}(G) = \{\pm\infty\}$ and $H \leq G$, the kernel of this permutation representation is a subgroup of index at most 2.*

Furthermore, if $C \subset X$ is a finite essential disconnecting such that $E^+, E^- \subset X \setminus C$ are the two infinite components then there is no $h \in H$ such that $h \cdot E^+ \subset E^-$.

Proof. Since G acts on X by isometries, then in particular it acts on X by quasi isometries, and every compact exhaustion is sent to another compact exhaustion. By Theorem 2.4.3, this implies that G acts on $\mathrm{Ends}(G)$. Now if

$G \twoheadrightarrow F$ is a homomorphism to a finite group then the kernel has index $|F|$, an the first point follows from the action of G on the two element set $\{\pm\infty\}$.

Suppose now that we passed to H (note that we allow $H = G$) and let

$$\cdots \subset K_1^2 \subset K_1^1 \subset L \supset K_2^1 \supset K_2^2 \supset \cdots$$

be some nested sequence of components of complements of sets in a compact exhaustion starting with C. Without loss of generality (reversing E^+ and E^- if necessary) we may assume that $K_1^n \subset E^-$ and $K_2^n \subset E^+$, for all n. Then in particular the point ∞ corresponds to the chain of inclusions $L \supset K_2^1 \supset K_2^2 \supset \cdots$. Now if for some $h \in H$ we have $h \cdot E^+ \subset E^-$ then this means that the translated chain $L \supset h \cdot K_2^1 \supset h \cdot K_2^2 \supset \cdots$ is contained in E^- and its terms converge to $-\infty$ so we get $h \cdot \infty = -\infty$ contradicting our assumption that $H \leq G$ didn't permute the ends of G. ∎

We also have the following useful fact.

Lemma 2.4.7 *Suppose that X admits an essential finite disconnecting set K, then X also admits an essential finite disconecting set $K' \supset K$ such that all the components of $X \setminus K'$ are infinite.*

sketch. Let $F_1, \ldots, F_m$ be the the finite components of $X \setminus K$ and let $I_1, \ldots, I_n$ be the infinite components of $X \setminus K$. Then every component F_i touches K (actually, to be accurate the closure of each component intersects K in a non-empty set.) Take the set

$$K' = K \cup F_1 \cup \cdots \sup F_m.$$

It is an exercise in point-set topology to see that K' is also compact. Graph theoretically, finiteness of K' s obvious. Either the complement $X \setminus K'$ consists of a finite union of infintie components. ∎

2.4.4 Exercises

1. Draw the complements of the first three balls in T_4 as described in Example 2.4.2.

2. Prove Theorem 2.4.5

Hint: Follow these steps.

(a) Start with G and take $X = \mathbf{Cay}_A(G)$ for some finite generating set.

(b) Take $H \leq G$ as in Lemma 2.4.6.

(c) Let $C \subset X$ be a finite essential cutset and prove that there exists some $h \in H$ such that $h \cdot C \cap C = \emptyset$.

(d) By Lemma Lemma 2.4.7 we can assume that $X \setminus C$ consists of precisely two *infinite* components.

(e) If K_1, K_2 are the infinite components of $X \setminus C$, argue that $g \cdot C$ must lie in one of these components an by the assumption that g doesn't permute ends that (w.l.og.)$g \cdot K_1 \subset K_1$.

(f) Iterate this nesting, i.e $K_1 \supset g \cdot K_1 \supset g^2 \cdot K_1 \supset \cdots$, to argue that g has infinite order so that $\langle g \rangle \approx \mathbb{Z}$.

(g) Argue that $\langle h \rangle$ is finite index in G by showing that the action of $\langle h \rangle$ on X is co-compact. This can be done by showing that every component of

$$X \setminus \left(\bigcup_{n \in \mathbb{Z}} h^n \cdot C \right) = Y$$

is finite, so that every $g \in G$ can be joined to an element of $\langle h \rangle$ by a finite path. Show that if this is not the case, then it is possible to find an essential finite disconnecting set whose complement has at least 3 infinite components (one of these components must lie in a connected component of Y), contradicting 2-endedness.

2.5 Appendix: Some final facts about quasi-isometry

Although we will now move on to other topics, there are a few facts the author cannot continue to ignore.

2.5.1 Counting ends, easy as $0, 1, 2, 2^{\aleph_0}$

Although it's easy to come up with graphs that any number of ends, take for example an infinite tripod, for groups the situation is more restrictive. Suppose that G has more than two ends, then for some Cayley graph $X = \mathbf{Cay}_A(G)$ there is some finite ball $B = B(1, r_0)$ about the identity such that $X \setminus B$ has at least three infinite components:

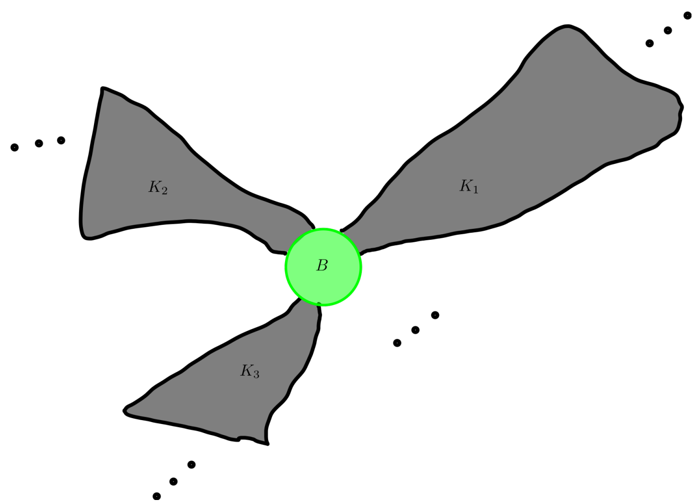


Figure 2.5.1 A finite essential disconnecting ball in a Cayley graph with more than two infinite complementary components

Because a finitely generated group always acts freely (and therefore properly) on its own Cayley graphs we find that we can always find elements that will "push" B off itself and we have the following tree-like structure:

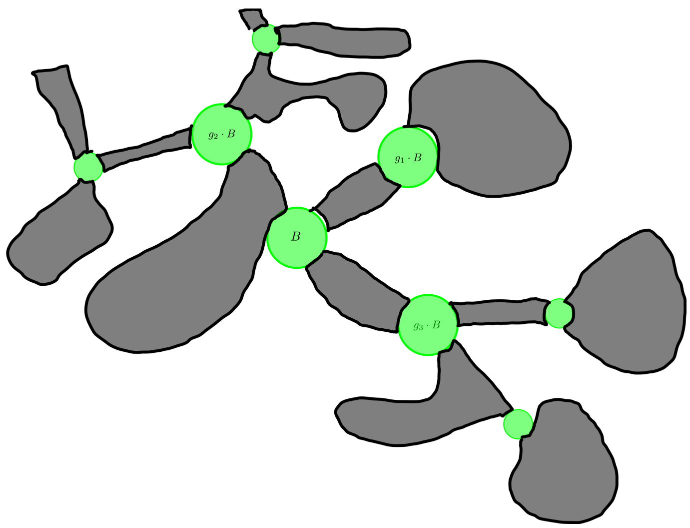


Figure 2.5.2 Disjoint translates of an essential disconnecting ball.

Given this picture, it is immediate that as we take compact exhaustions of X the number of infinite components of complements goes to infinity, therefore if $|\text{Ends}(G)| > 2$, then $|\text{Ends}(G)| = \infty$, and it is easy to see that this infinite cardinality must be continuum or $2^{\aleph_0}$.

2.5.2 The topology of $\text{Ends}(G)$

As we saw, ends correspond to an infinite chains $K_1 \supset K_2 \supset \cdots$ of infinite components. Since these are complements of *exhaustions* we must have for a given end

$$\bigcap_{i=1}^{\infty} K_i = \emptyset.$$

The inverse limit construction, is what lets us get "something from nothing". Nonetheless, given some compact essential disconnecting set $C \subset X$ it makes sense to think of an end as lying in some component of $X \setminus C$, formally we can think of some decreasing sequence of infinte components eventually lying in some component of $X \setminus C$. This lets us put a topology on $\text{Ends}(X)$, for some graph. Let $C \subset X$ be a compact set and let $K \subset X \setminus C$ be an infinite component. Then we have an open set $U_C \subset \text{Ends}(X)$ consisting of all ends that "lie" in C. The topology can be informally, yet accurately, described as follows: two ends are "close" only if they are separated by a "large" ball about the identity.

Here are interesting facts:

- A quasi isometry $f : X \to Y$ induces not only a bijection, but in fact, homeomorphism $\hat{f} : \text{Ends}(X) \to \text{Ends}(Y)$
- In the case where $X = \mathbf{Cay}_A(G)$, then the topology on $\text{Ends}(X)$ is that of a Cantor set.
- When X is an infinite tree $X \cup \text{Ends}(X)$ can be topologized as a compactification of X, which is natural in the context of metric spaces. An example we say is compactifying $\mathbb{R}$ with the points $\pm\infty$.
- Certain groups admit refinements of ends which are also quasi isometric invariants, known as boundaries, which enable many stronger quasi isometric rigidity results.

2.5.3 Gromov's polynomial growth theorem

Let us denote the commutator $[x, y] = x^{-1}y^{-1}x, y$, and the n-fold commutator

$$[x_1, x_2, \cdots, x_m] = [x_1, [x_2, \cdots, x_n]].$$

A group N is said to be nilpotent of class c if every $(c+1)$-fold commutator is trivial. For example abelian groups are precisely nilpotent of class 1.

The reader may have seen the class of solvable groups before and it would be natural to think that solvable and nilpotent are the same thing, but they're not. Nilpotent implies solvable, but not the other way around.

Previously we saw that for the abelian groups $\mathbb{Z}^n$, the amount group elements contained in a ball $B(1, r)$ with respect to some word metric grew like a polynomial. Since quasi isometries distort distances linearly and that the composition $f \circ l$ of a degree d polynomial f and a linear function ℓ is still a degree d polynomial, it can be shown that the *degree* of polynomial growth of balls is a quasi isometric invariant of groups. This phenomenon is called **polynomial growth.**

Not all groups have polynomial growth, for example, the number of elements a ball of radius r in $F(a, b)$ grow *exponentially.*

So far we have focused on quasi isometry as applied to a single group, and asked to which extent does a quasi isometry class determine the algebraic properties of the group. There are many other results about classes of groups such as the following.

Theorem 2.5.3 *If G has polynomial growth, then G contains a finite index subgroup $H \leq G$ which is nilpotent of some class $c \in \mathbb{Z}_{\geq 0}$.*

Equivalently, if G has polynomial growth, the G is virtually nilpotent.

We first note that due to the fundamental limitations of quasi isometry, it is impossible to make such a statement without the "virtually" qualification. Secondly, we note that this is a very strong algebraic conclusion based solely geometry. Finally, the way this is proved uses very cool machinery that is unlike anything that will be covered in this course.

Chapter 3

Diagram methods

3.1 van Kampen diagrams and the geometry of the word problem

Let us now turn our attention back to finitely generated groups given by relations:

$$G = \langle X|R \rangle.$$

In Section 1.4 we say that any word representing the identity in G was equal in $F(X)$ to a product of conjugates of elements of $R^{\pm 1}$. In Section 1.8 in we saw that every element in $F(X)$ was conjugate to a cyclically reduced element and that all cyclic permutations of an element are conjugate. With this in mind we have the following two-dimensional notion: an **X-directed 2-cell**, or just briefly **a 2-cell**, is a cyclic X-digraph enclosing a topological disc.

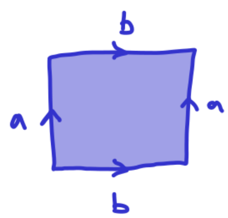


Figure 3.1.1 A 2-cell

As defined, we note that a 2-cell has no preferred orientation or vertex. Thus, for a given boundary word r, the same 2-cell represents every cyclic conjugate of $r^{\pm 1}$. A **diagram** is 2-complex, i.e. a graph to which is attached 2-cells. A **planar embedding** of a diagram is a specific way of embedding/ drawing a diagram in the plane. We will often abuse notation say **planar diagram** instead of "diagram equipped with a planar embedding". We will say that a planar diagram is **simply connected** if, informally, it has no holes, or if every cycle in the underlying graph is filled with discs. Noting that a diagram is a CW 2-complex, "simply connected" is perfectly well-defined.

We point out that most of graphs shown in the previous section are definitely not simply connected.

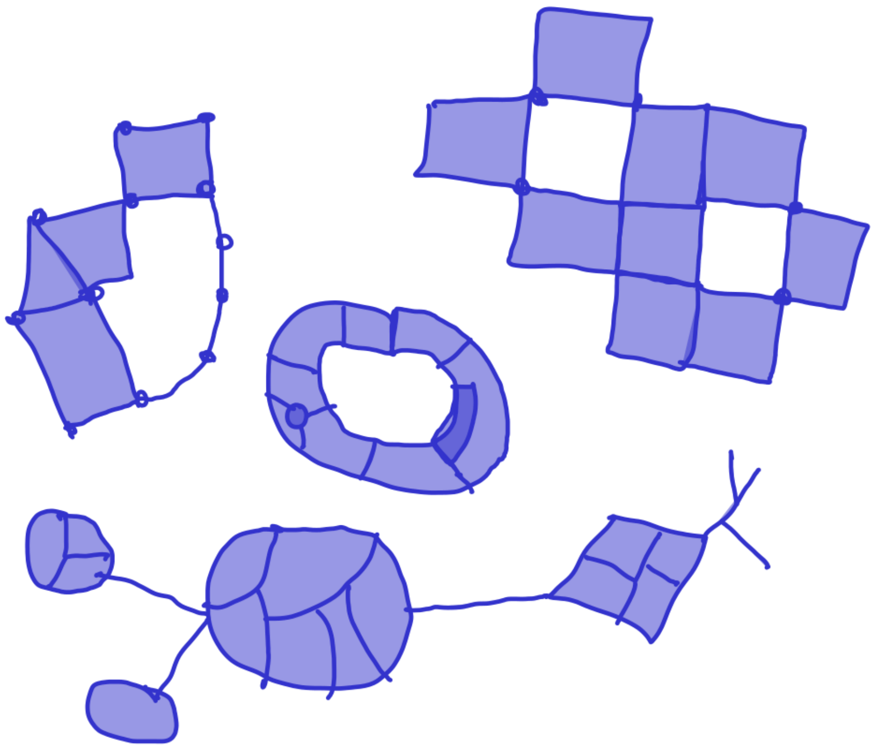


Figure 3.1.2 Some planar diagrams. Can you pick out the simply connected one? Note that we will sometimes omit drawing directions and labels of edges.

3.1.1 van Kampen diagrams

A **van Kampen diagram** is a simply connected planar diagram. The planar embedding gives rise to a **boundary word**, which is the label of the path that can be read *clockwise* around the boundary. We note that the boundary word is well up to cyclic permutation.

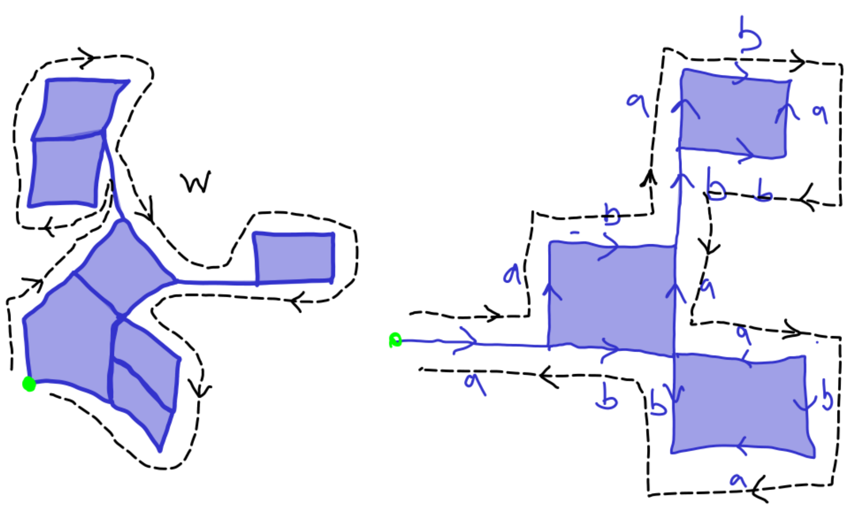


Figure 3.1.3 The boundary word is the label of the path read clockwise around the boundary.

Van Kampen diagrams are the most important tool in studying the word problem in groups. First note that if some word w represents the identity in $\langle X|R\rangle$, then

$$w =_{F(X)} \prod_{i=1}^{k} w_i r_i^{\epsilon_i} w_i^{-1}$$

for some conjugates of $r_i^{\pm 1} \in R$. We can therefore form a **balloon diagram** as in Figure 3.1.4 whose boundary word is precisely the (unreduced product) $\prod_{i=1}^{k} w_i r_i^{\epsilon_i} w_i^{-1}$.

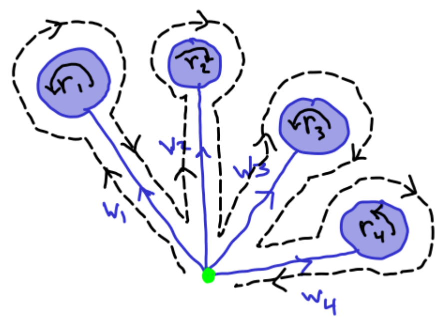


Figure 3.1.4 A balloon diagram whose boundary word $w_1 r_1^{-1} w_1^{-1} w_2 r_2 w_2^{-1} w_3 r_3^{-1} w_3^{-1} w_4 r_4^{-1} w_4^{-1}$.

Let $\mathcal{D}$ be a van Kampen diagram whose 2 cells are bounded by words $r \in R$. Then the boundary word w of $\mathcal{D}$ represents the trivial element in $\langle X|R\rangle$. See Figure 3.1.5, for example.

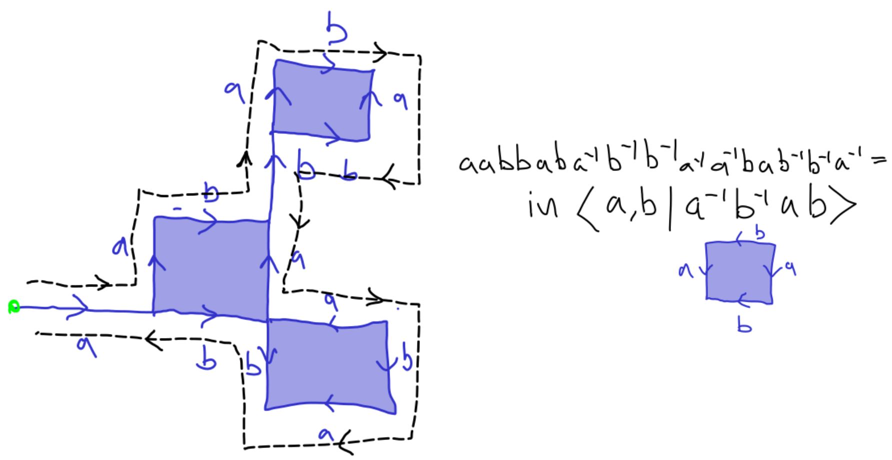


Figure 3.1.5 A balloon diagram whose boundary word $w_1 r_1^{-1} w_1^{-1} w_2 r_2 w_2^{-1} w_3 r_3^{-1} w_3^{-1} w_4 r_4^{-1} w_4^{-1}$.

The reason is that any such diagram can be "unfolded" to a balloon diagram (see Figure 3.1.6). This witnesses that a boudary word w is equal to a product

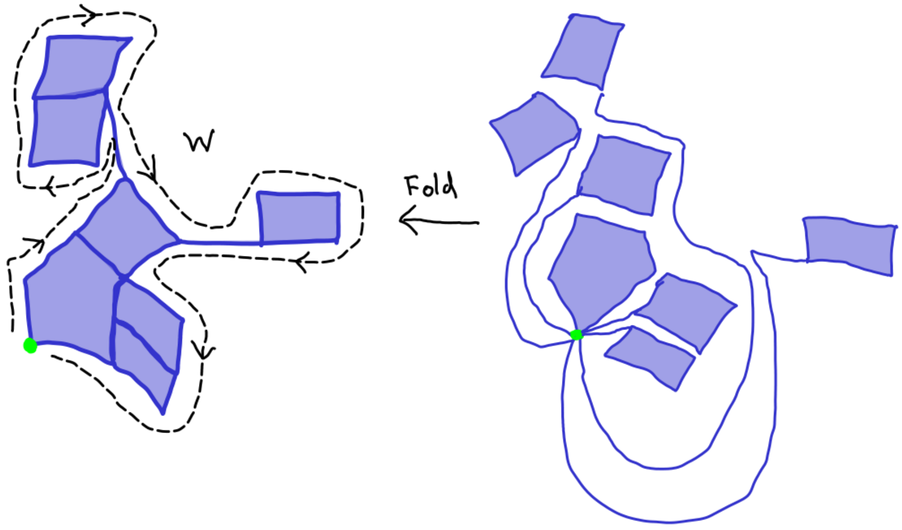


Figure 3.1.6 Unfolding a van Kampen diagram to a balloon diagram.

This unfolding fact and its converse give van Kampen's Lemma. The main difficulty resides in the fact that when folding a balloon diagram, some 2-cells can get folded "all the away around" creating spheres (think of a soccer ball.) The argument is to show that if we start with a balloon diagram with a minimal number of 2-cells, by performing the foldings corresponding to elementary reductions in boundary words, we remain planar at every step. Otherwise we could have removed 2-cells.

Theorem 3.1.7 van Kampen's Lemma. *Let w be a reduced word in alphabet X. Then $w = 1$ in $\langle X|R\rangle$ if and only if w is the boundary of a van Kampen diagram whose 2-cells are bounded by words in R.*

3.1.2 Exercises

1. Let X be some alphabet and let T be an X-digraph which is a tree. Prove that the boundary label is a word that is equal in $F(X)$ to the identity.

Hint: A **spur** is a vertex with degree 1. Perform elementary cancellations one at a time. Show at each step that the word obtained from an elementary cancellation can be obtained by deleting a spur and incident edge in a tree and taking the new boundary word.

2. "Unfold the van Kampen diagram in Figure 3.1.5 an explicitly express that word as a product of 3 conjugates of $(a^{-1}b^{-1}ab)^{\pm 1}$

3. Show that if two word w, w' are reduced words that are equal in $\langle X|R\rangle$ then there is a "possibly pinched bigon" along whose top we can read w and along whose bottom we can read w' that can be filled with 2-cells bounded by words in R.

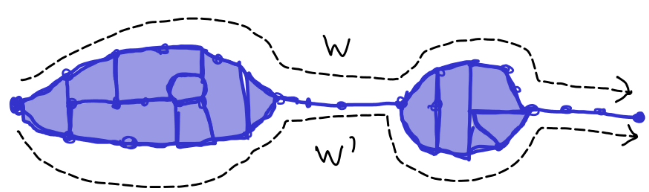


Figure 3.1.8 A possibly pinched bigon

4. In Section 1.6 you were asked to show that in $\langle r, \rho | \rho\rho\rho, rr, r\rho r\rho \rangle$ we have $r\rho = \rho^2 r$. Construct the bigon witnessing this fact.

3.2 HNN extensions and dual tracks

In this lecture we will develop a method to understand the famous Baumslag-Solitar group

$$\mathrm{BS}(1,2) = \langle a, t \big| tat^{-1}a^2 \rangle.$$

It worth stressing again that there is no general procedure which can take an arbitrary finite presentation $\langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ and will decide if this is a presenation of the trivial group.

For this reason, techniques to study groups from presentations are necessarily ad-hoc, and will only work in certain circumstances. We start with a construction in group theory

3.2.1 HNN extensions

HNN extensions are named after mathematicians Higman, Neumann, and Neumann (the Neumanns were married.) They are formed by putting a group G inside a larger group $G*_t$ with an extra generator called the **stable letter**, typically we use t, which conjugates elements in G. More precisely, suppose that G contains two subgroups A, A' such that $A \approx A'$, i.e. A, A' are isomorphic and let $\varphi : A \xrightarrow{\sim} A'$ be a specific isomorphism. See Figure 3.2.1.

$$\begin{tikzcd} & G & \\ A \arrow[ur, hook] \arrow[rr, "\sim", "\varphi"'] & & A' \arrow[ul, hook'] \end{tikzcd}.$$

Figure 3.2.1 The setup for an HNN extension

If $G = \langle X | R \rangle$, then the **HNN extension of** G **(with associates subgroup** $A, A' \leq G$ **and attachment isomorphism** φ**)** is given by the following (possibly) presentation

$$G*_t = \langle X, t \big| R, t^{-1}at = \varphi(a); a \in A \rangle, \tag{3.2.1}$$

i.e. we add a new generator t (we assume $t \notin X$), the stable letter, and for each $a \in A$ we add a relation which says that conjugation of an element of a by t produces the same result as making taking the image $\varphi(a)$. Incidentally we slightly changed the **convention for relations**: instead of writing $t^{-1}at(\varphi(a)^{-1}$ we wrote $t^{-1}at = \varphi(a)$, which is equivalent. Also note that for these relations we actually mean that we wrote words in $X^{\pm 1}$ represeting elements in $a, \varphi(a) \in G$. In particular we are not assuming $A, A' \subset X$.

Needless to say, adding extra relations to a group can have unexpected consequences. We want to show that $G \leq G*_t$, i.e. that $G*_t$ is an extension of G. Note that by Lemma Lemma 1.6.1 there is a natural homomorphism $G \hookrightarrow G*_t$. Our first task is to show that this homomorphism is *injective*.

3.2.2 Dual tracks and van Kampen diagrams

Comapring the presentation $G = \langle X | R \rangle$ and the presentation given in (3.2.1) we see that we have extra **HNN relations** which give rise to the following 2-cells:

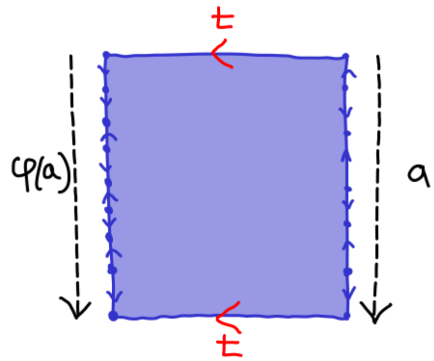


Figure 3.2.2 An HNN 2-cell.

Within each such 2-cell, we can draw a **dual t-track** which a segment in each HNN-2-cell joining the midpoints of the t-labeled edges. Within a van Kampen diagram, because the *only place* t-labelled edges occur is either as fully external edges, or inside HNN-2-cells, these segments join up to make **dual t-tracks in van Kampen diagrams.**

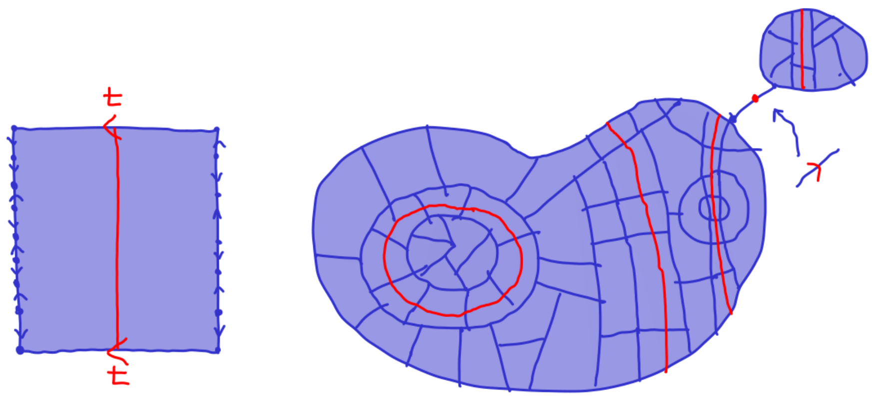


Figure 3.2.3 Dual t-tracks.

This next lemma is a topological consequence of our requirements that van Kampen diagrams are planar and simply connected, that dual tracks are curves, and the Jordan curve theorem. One could probably also come up with a simpler purely combinatorial proof... but whatever, the lemma is completely obvious.

Lemma 3.2.4 The dual track lemma. *Dual tracks within a van Kampen diagram $\mathcal{D}$ are either points, arcs with endpoints in the boundary of the diagram, or circles. Every track divides $\mathcal{D}$ into two disjoint regions. Distinct tracks are disjoint.*

We note that if a point type or an arc type track did not divide $\mathcal{D}$ into two distinct components we would violate simple connectivity. If a circular track did not divide into two components we would have an embedded Möbius strip.

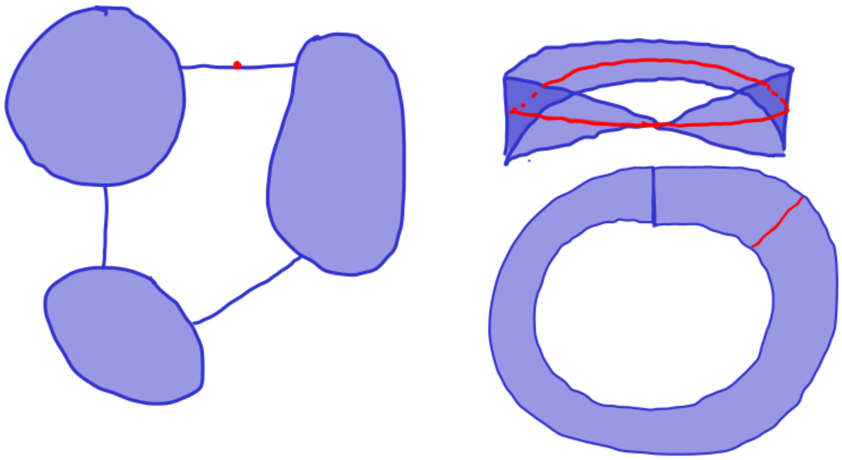


Figure 3.2.5 When tracks do not divide into 2 components.

This topologically flavoured lemma is the key to proving things about HNN extensions. We start with the following:

Theorem 3.2.6 *If $G = \langle X|R\rangle$ and we have an HNN extension*

$$G*_t = \big\langle X, t \big| R, t^{-1}at = \varphi(a); a \in A\big\rangle,$$

then G is naturally isomorphic to a subgroup of $G*_t$.

Proof. Consider the two presentation. The inclusions $X \hookrightarrow X \cup \{t\}$ and $R \hookrightarrow R \cup \{t^{-1}w_a t(w_{\varphi(a)})^{-1} \mid a \in A\}$, where $w_a, w_{\varphi(a)}$ are choices of words in $X^{\pm 1}$ representing the elments $a, \varphi(a)$ in G, induce by Lemma Lemma 1.6.1 a natural homomorphism $\psi : G \to G*_t$. We need to show that ψ is injective.

Suppose towards a contradiction that ψ was not injective. Then there is some $g \in G \setminus \{1\}$, of minimal word length, such that $\varphi(g) \neq 1$. By van Kampen's Lemma (Theorem 3.1.7) there is some reduced word w_g (realizing minimal word length) which is the boundary word of a van Kampen diagram $\mathcal{D}$ whose 2-cells have boundary words from $R \hookrightarrow R \cup \{t^{-1}w_a t(w_{\varphi(a)})^{-1} \mid a \in A\}$.

Because $g \neq_G 1$, $\mathcal{D}$ cannot consist solely of 2-cells with boundaries from R, therefore $\mathcal{D}$ must contain HNN-2-cells and therefore at least one dual t-tracks. Let $\mathcal{D}$ such a diagram with a *minimal* number of tracks. Because the boundary word of $\mathcal{D}$ is a word containing letters only in X none of the tracks can reach the boundary of $\mathcal{D}$, so they must all be cicrcles.

Let $\tau \subset \mathcal{D}$ be an innermost circular track, i.e. it doesn't enclose any other circular tracks. The union of 2-cells containg it for an annulus $\mathcal{A}$ with an inner boundary word w^{in} and an outer boundary word w^{out}. Now note on the one hand that, without loss of generality, w^{in} is a product of elements in A therefore $w^{in} \in A$ and $w^{out} = \varphi(w^{in})$. On the other hand w^{in} is the boundary word of a van Kampen (sub) diagram, and since τ is innermost there are no HNN-2-cells, so all 2-cells in the subdiagram have boundary labels in R. This means that $w^{in} =_G 1$ and since $\varphi A \to A'$ is an isomorphism we have that $w^{out} =_G 1$ as well. Therefore by van Kampen's lemma we can find a van Kampen diagram $\mathcal{D}'$with 2-cells with boundary in R whose boundary word is w^{out}.

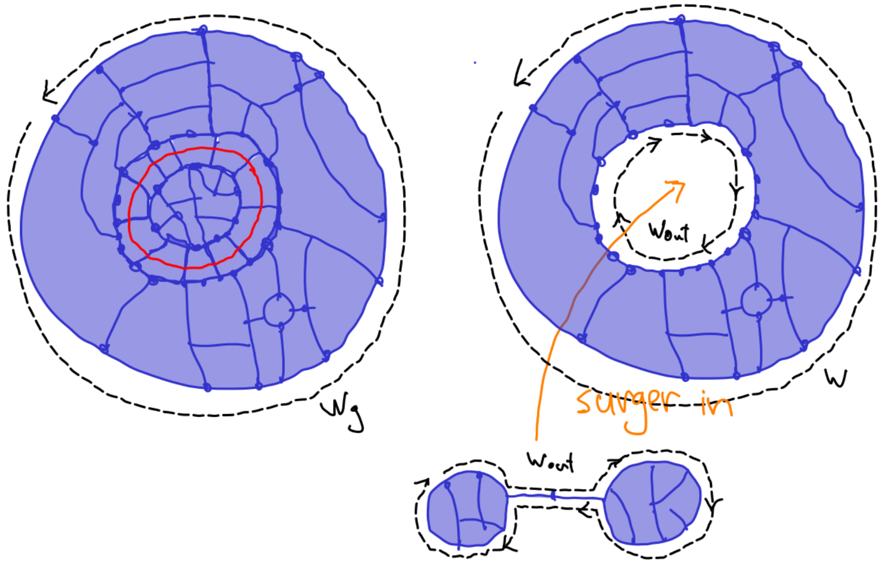


Figure 3.2.7 Surgically decreasing the number of tracks, by taking out an innermost track.

It follows that we can cut out the annulus $\mathcal{A}$ and surgically re-attach $\mathcal{D}'$. The resulting van Kampen diagram is a witness of the triviality of w_g in

$$G*_t = \langle X, t \big| R, t^{-1}at = \varphi(a); a \in A \rangle,$$

with one less track than $\mathcal{D}$, but this contradicts the fact that $\mathcal{D}$ had a minimal number of tracks. ■

In fact, the main thrust of the proof of the previous Theorem can be restated as the following lemma:

Lemma 3.2.8 *Among all van Kampen diagrams witnessing the triviality of a word in an HNN extension, a diagram $\mathcal{D}$ with a minimal number of dual t-tracks will not contain any circlar tracks.*

Next we have this other injectivity result for the mapping $\mathbb{Z} \to \langle t \rangle$:

Proposition 3.2.9 *Let $G*_t$ be an HNN extension with stable letter t, then the element t has infinite order in $G*_t$.*

Proof. Suppsose towards a contradiction that $t^n = 1$ form some $n > 0$. Then t^n is the boundary word of a van Kampen diagram $\mathcal{D}$. This means that there are tracks starting and ending on the boundary of $\mathcal{D}$, but since the 2-cells containing containing tracks are HNN-2-cells, Figure 3.2.10 below shows that any such track τ must lie in a "twisted strip".

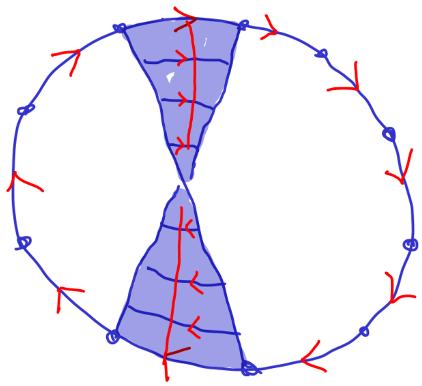


Figure 3.2.10 A twisted strip.

This contradicts the planarity of $\mathcal{D}$, or the fact that τ cuts $\mathcal{D}$ into two components. ■

3.2.3 Economical presentations and (some of) the secrets of $\mathrm{BS}(1,2)$

The presentation (3.2.1) for $G*_t$ may have infinitely many relators, one for each element in A. Until now, this infinite presentation has been convenient, but seeing as we are primarily interested in finitely presented groups, it will be nice to get a criterion for when an HNN extension is also finitely presented.

Proposition 3.2.11 *Suppose that the associated subgroup $A, A' \leq G$ are finitely generated and that $G = \langle X|R\rangle$ is finitely presented. Then the HNN extension $G*_t$ is finitely presentable as*

$$G*_t = \left\langle X, t \middle| R; t^{-1}a_i t(a_i')^{-1} : 1 \leq i \leq n\right\rangle$$

where $\{a_1, \dots a_n\}$ are words in $X^{\pm 1}$ representing a finite generating set for A and the a_i' are words satisfying $\varphi(a_i) =_G a_i'$.

So now let's turn our attention to the following little presentation:

$$\mathrm{BS}(1,2) = \left\langle a, t \middle| t^{-1}ata^{-2}\right\rangle.$$

Let's show that it is an HNN extension.

We first note that $\mathbb{Z} \approx \langle a|\rangle$. Now $\langle a|\rangle$ contains the subgroups $\langle a\rangle$, i.e. the whole thing, and $\left\langle a^2\right\rangle$, i.e. $2\mathbb{Z}$. Both of these subgroups are isomorphic to $\mathbb{Z}$ and we can take the isomorphism:

$$\begin{aligned} \varphi : \langle a\rangle &\to \left\langle a^2\right\rangle \\ a &\mapsto a^2. \end{aligned}$$

Recall that a homomorphism is fully determined by the image of a generating set.

Therefore $\mathrm{BS}(1,2)$ is an HNN extension of the cyclic group. In particular we have an injective homomorphisms

$$\mathbb{Z} \approx \langle a|\rangle \hookrightarrow \left\langle a, t \middle| t^{-1}ata^{-2}\right\rangle,$$

so a has infinite order in $\mathrm{BS}(1,2)$. This fact was what was missing in Example 2.2.7. Now we have proved the existence of a group that is "algebraically" embedded into another group, but such that the inclusion is not a quasi isometric embedding.

Actually, the simplest examples of distorted subrgroups (i.e. subgroups that are not quasi isometrically embedded) arise from HNN extensions.

Similarly, we see that the other Baumslag-Solitar groups

$$\mathrm{BS}(n,m) = \left\langle a,b \middle| b^{-1}a^n b a^{-m} \right\rangle$$

are also infinite and the element a has infinite order. Note that here although we used the letter b instead of t, we still have the HNN extensions structure.

3.2.4 Syllables and Britton's lemma.

Our treatment of HNN extensions, using van Kampen diagrams and tracks, is not historical. In this final section we will give some classical notation and the main classical result regarding HNN extensions.

Suppose that we are given $G = \langle X|R \rangle$ then every element in $G{*_t}$ can be represented as a word of the form

$$g =_{G*_t} w = a_0 t^{n_1} a_1 \cdots t^{n_m} a_m, \text{ or } a_0 t^{n_1}, \text{or} t^{n_m} a_m$$

where the subwords a_i are words in $X^{\pm 1}$, t is the stable letter, and the $n_i \in \mathbb{Z}$. The words a_i are called G-syllables.

If we are more explicit and identify $A, A' \leq G$ and $\varphi : A \to A'$ so that $\varphi(a) = t^{-1}at; a \in A$ then we say that a word w can be **pinched** if it contains a subword

$$ta_it^{-1} \text{ or } t^{-1}a_jt,$$

where $a_i \in A'$ or $a_j \in A$. In this case we can reduce the numner of occurrences of t via

$$ta_it^{-1} = \varphi^{-1}(a_i) \text{ or } t^{-1}a_jt = \varphi(a_j).$$

For this reason we will say that a word in an HNN extension is **reduced** if it doesn't contain any pinches. We now finish by stating what is the classical fundamental theorem of HNN extensions:

Theorem 3.2.12 Britton's Lemma. *If a word win an HNN extension > $G{*_t}$ represents the trivial element, then either it is a word in $X^{\pm 1}$ representing the trivial element in G or it can be pinched.*

In particular reduced words in HNN extensions are non-trivial. At this point there is enough machinery so that an interested reader could consult Chapter 9 of [6] to see how to contruct a group with undecidable word problem.

3.2.5 Exercises

1. Consider $\mathbb{Z} = \langle a \rangle$. $\langle a \rangle$ has two automorphisms, the identity automorphism and the automorphism perscribed by $a \mapsto a^{-1}$.

(a) Identify the group you get when you use the identity automorhpism. I assure you, you know this group.

(b) Show that the group G'obtained by forming a HNN extension using the non-trivial automorphism of $\mathbb{Z}$ is not isomorphic to the group you found previously.

(c) Give a critique of the notation $G{*_t}$ used to denote an HNN extension.

2. Prove Proposition 3.2.11.

Hint: Start by showing that for each of the (possibly infinitely many) old HNN-2-cells, you can construct a van Kampen diagram with the same boundary word using your finite set of relations. Use this and van Kampen's Lemma to apply Lemma 1.6.1 in order to get an isomorphism between the two groups corresponding to the finite and infinite presentation.

3. Prove Britton's Lemma (Theorem 3.2.12)

Hint: Draw a van Kampen diagram. Seriously, MAKE A DRAWING. Explain why you can assume there are no circular tracks. Find an "outermost" track and show that the piece of the boundary word travelling between the endpoints of the outermost track must lie in one of the associated subgroups.

4. Take the free group $F_3 = F(a, b, c)$ and find two isomorphic subgroups H, H' of rank at least 4.

(a) Give an isomorphism between H, H' and write out the full presentation of the corresponding HNN extension $G = (F_3 *_t)$

(b) Referring to results of the previous chapters, give a convincing explanation of how to solve the word problem in this group.

Hint: Subgroup membership and pinching.

3.3 A pause: why are doing any of this anyway?

In the previous chapter we presented HNN extensions. The motivation for doing so was to demonstrate the effectiveness of van Kampen diagram arguments. That said why should anybody care about HNN extensions to begin with? In this section we will try to place HNN extensions into a larger (and more interesting) context.

One thing HNN extensions are good for are to construct groups with interesting properties, for example we constructed $\mathrm{BS}(1, 2)$ as an HNN extension of $\mathbb{Z}$ and in so doing were able to show that some subgroups are not quasi-isometrically embedded.

Another important application is as follows. A state of a Turing machine (i.e. a theoretical computer) consists : one of finitely many an internal state $\{Q_1, \ldots, Q_r\}$, and the string of symbols written on a tape. It is possible, by a clever choice of normal forms, to encode the state of a as elements of some group H so that, given a state s, there is a corresponding a word w_s and corresponding group element. Crucially different states give different *group elements*. A transition $s_1 \to s_2$ is achieved by an HNN extension $H*_t$ such that

$$t^{-1} w_{s_1} t =_H w_{s_2}.$$

From there, using elementary but clever arguments, it is possible to construct a group with undecidable word problem and to show that the isomorphism problem is undecidable. An interested reader can consult Chapter 9 of [6] for a thorough account.

Besides creating designer groups, HNN extensions play a much deeper role.

3.3.1 The amalgamated product: the HNN extension's more popular sibling

Given groups G, H it is possible to form their Cartesian product $G \times H = \{(g, h) \mid g \in G, h \in H\}$ with multiplication defined componentwise. Another way of combining groups is via a **free product** which we define in terms of presentations as follows:

1. Let $G = \langle X|R \rangle$ and $H = \langle Y|S \rangle$, then the free product is:
2. $$G * H = \langle X, Y|R, S \rangle$$

Where we abuse notation and write, for example, X, Y instead of $X \cup Y$. As for HNN extensions, elements of $G * H$ are represented as words

$$a_1 b_1 \cdots a_n b_n$$

where the a_i, b_j are (possibly empty) words in $X^{\pm 1}, Y^{\pm 1}$, respectively. Call them G **and** H**-syllables** if you want. Here there are no relations involving letters from both X and Y, so an van Kampen diagram argument similar to, but easier than, the one used to prove Theorem 3.2.6 shows that we can assume $G, H \leq G * H$.

Note that if we present $\mathbb{Z} = \langle a| \rangle \approx \langle b| \rangle$ then we find that

$$\mathbb{Z} * \mathbb{Z} = \langle a, b| \rangle = F(a, b).$$

Even better, consider the trivial group $\{1\} = \langle c|c \rangle$ (i.e. a group with one generator that is declared to be trivial.) The trivial group admits the identity automorphism, in this case, given by $c \mapsto c$ so we can form the HNN extension of the trivial group

$$\{1\}*_t = \left\langle c, t \middle| c, t^{-1}ctc^{-1} \right\rangle \overset{\text{Tietze}}{\approx} \langle t| \rangle \approx \mathbb{Z}.$$

So in this way, starting with the trivial group, it is possible to construct all free groups just by using HNN extensions free products. In fact only HNN extensions are sufficient.

Suppose now we are given two group $G = \langle X|R \rangle, H = \langle Y|S \rangle$ and suppose we have a pair of injective homomorphisms:

$$G \overset{\psi}{\hookleftarrow} A \overset{\phi}{\hookrightarrow} H, \tag{3.3.1}$$

i.e. G and H contains isomorphic copies of A as subgroups. Then these isomorphisms prescribe a way of making a new group by "gluing" G and H along the images of A. The **amalgamated product of** G **and** H **over** A is the group

$$G *_A H = \langle X, Y|R, S, \psi(a) = \varphi(a); a \in A \rangle \tag{3.3.2}$$

Note here that although A is not a subgroup of G or H we have for all $a \in A, \psi(a) \in G, \varphi(a) \in H$ so that the set of **amalgamating relations** $\{\psi(a)\varphi(a)^{-1} \mid a \in A\}$ can in fact be written as words in $(X \cup Y)^{\pm 1}$.

Similarly to the case with HNN extensions, we can observe that ther only relations containing letters from both X and Y are the amalgamating relations, so in van Kampen diagrams we have **amalgamating 2-cells**. Similarly to the case for **HNN-e-cells** we can draw tracks in such 2-cells.

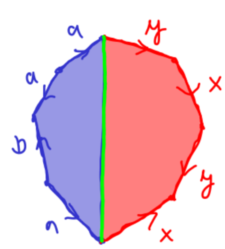


Figure 3.3.1 A track in an amalgamating 2-cell.

Using the arguments of the previous chapter we can prove analogous results for amalgamated products, like embedding theorems, i.e. $G \leq G *_A H$ and the analogue of Britton's lemma, i.e. if a word

$$a_1 b_1 \cdots a_n b_n =_{G*_A H} 1$$

Then either some syllable $G \ni a_i \in \psi(A)$ or $H \ni b_j \in \varphi(A)$. Finally if the amalgamating subgroup A is finitely generated and both G and H are finitely presented, then the amalgamated product is finitely also presented (again same argument as for HNN extensions.) *The only difference is that tracks are nicer to draw in the HNN case.*

Ultimately the reason why amalgamated products are more popular is that gluing two groups together along a subgroup is more intuitively easier to understand than whatever is done for HNN extensions. Both are equally important.

3.3.2 Stop avoiding the question! Why should I care?

It is now time to have the talk about category theory. So, um... when a mother and a father, or any couple whose members may or may not have biary genders, care alot for each other... oops wrong talk.

A **category** consists of a collection of **objects** and a collection of arrows between these objets, sometimes called **morphisms**. In practice categories consist of objects of the same **type.** Here are some examples:

1. The category **set** whose objects consists of all sets (so the collection of objects in a category need not be a set [ignore this parenthesis if you don't know about Russell's paradox]) and the morphisms are functions.

2. The category **group** whose objects are groups and whose morphisms are homomorphisms.

3. The category **AbGroup** whose objects are abelian groups and whose morphisms are homomorphisms.

4. The category **Zmod** whose objects are $\mathbb{Z}$-modules abelian groups and whose morphisms are $\mathbb{Z}$-linear homomorphisms.

5. A **based topological space** consists of a pair (X, x_0) where x_0 is some point in X called a basepoint and we can form a category out of these objects by taking **based continuous functions** denoted

$$(X, x_0) \to (Y, y_0)$$

are continuous map $X \to Y$ that map basepoints to basepoints, i.e. $x_0 \mapsto y_0$.

6. A directed graph forms a **small category** the objects are the vertices and arrows are the edges (which are literally arrows.)

We also have rings, field extensions, etc.

Certain categories have features that others don't. For example for all the listed categories, except the small category, each object admits an arrow to itself: the identity arrow, which acts as the identity function. Again, with the exception of the small category, all other morphisms come from functions so we have a composition operation

$$f \circ g$$

on morphisms, provided their domains and codomains match up.

One advantage of category theory is that we can universally define concepts. For example the categories of groups, abelian groups, $\mathbb{Z}$-modules, and rings are all algebraic we have for example the following:

1. An isomorphism is a morhpism $f : A \to B$ such that there exists another morphism g such that $g \circ f = 1_A$, the identity morphism on A.

2. Up to unique isomorphism (i.e. there may be multiple objects that have this property, but for any two there is a unique isomorphism between them), the trivial object T is the object such that for any other object A there exists a unique surjective morphisms $f : A \to T$. I.e. the trivial object is the universal receiver.

3. If we define the **cokernel** of a morphism to be it's image then we have the first isomorphism theorem

$$\mathrm{coker}(f) \approx \mathrm{dom}(f)/\ker(f)$$

Note that free groups play the opposite role as a universal sender.

3.3.3 Come one! You just gave me yet another super abstract thing not to care about!

I assure you there is a point to this!

Certain constructions can also be defined categorically for example. Take two objects A, B we want to define a third object C which satisfies the following property: There is a pair of surjective morphisms (also called **epimorphisms**) $p_A : C \to A, p_B : C \to B$ such that for any object T and any pair of morphisms $\psi : T \to A$ and $\phi : T \to B$, there to exist a unique morphism ρ which makes the following diagram commute:

$$\begin{tikzcd} & T \arrow[dl, "\forall\psi"'] \arrow[d, dashed, "\exists! f"'] \arrow[dr, "\forall\phi"] & \\ A & C \arrow[l, two heads, "p_A"'] \arrow[r, two heads, "p_B"] & B \end{tikzcd}$$

Figure 3.3.2 The specification of a product

Such an object, if it exists, is called the *product* of A and B. Note that in most familiar algebraic structures, the Cartesian product $A \times B$ satisfies all these properties. Indeed, we have the projection morphisms onto factors, e.g. $A \times B \overset{p_A}{\twoheadrightarrow} A$, and the maps ϕ, ψ, from and an arbitrary object T specify the images on each factor, so the only possibility for f is

$$\begin{aligned} f &= \psi \times \phi \\ x &\mapsto (\psi(x), \phi(y)) \end{aligned}$$

Note that if another object C' also satisfies this property, we can put in the place of T in the diagram and use the projection functions and we'll get that $C' \approx C$ and that this isomorphism is unique.

In this way we have a completely categorical definition of a **(direct) product**. In category theory a cool thing to do is to put the word co in front of stuff and reverse arrows. Compare the diagram below with Figure 3.3.2.

$$\begin{tikzcd}
 & T & \\
A \arrow[ur, "\forall\psi"] \arrow[r, hook, "i_A"] & C \arrow[u, dashed, "\exists! f"'] & B \arrow[ul, "\forall\phi"'] \arrow[l, hook', "i_B"']
\end{tikzcd}$$

Figure 3.3.3 The specification of a coproduct

where i_A, i_B (which reverse the surjective p_A, p_B) are injective. The object C satisfying this diagram is called the coproduct.

In the category of groups the coproduct of A, B is the direct product $A * B$ given at the very beginning of this section. Indeed for $A * B$ we mentioned a van Kampen diagram argument which gives the inclusion $i_A : A \hookrightarrow A * B$, furthermore the maps ψ, ϕ fully specify images of a generating set of $A * B$ into T, finally by Lemma 1.6.1 it is immediate that all the relations on $A * B$ vanish, thus giving us our unique homomorphism, compatible with the given ψ, ϕ.

Thus, even though a direct product seems just like stupidly smashing two presentations together, in the category of groups, the free product realizes the coproduct, i.e. the dual of the direct product. Amalgamated products and HNN extensions can also be similarly be defined, albeit with more complicated diagrams. For example amalgamated products are the result of dualizing fibre products. Have another look at (3.3.1).

Incidentally, in the categories of $\mathbb{Z}$ modules and rings, coproducts are direct sums, i.e. $A \oplus B$ and products are $A \times B$, which unless when taking infinitely many terms, are interchangeable. In group theory coproducts are more complicated because of the non-commutativity.

3.3.4 Okay, so maybe smashing presentations together isn't that random after all. What is it good for?

We must now discuss **functoriality**. A functor F is a map $F : C_1 \to C_2$ between categories such that for any arrow $A \xrightarrow{f} B$ in C_1, in C_2 we have $F(A) \xrightarrow{F(f)} F(B)$. Let's give some examples.

1. Every group has an underlying set, and every homomorphisms is a function. We therefore have the **forgetful functor** F : **group** $\to$ **set**. We say it's forgetful because we forgot about the algebra.

2. Every $\mathbb{Z}$ module is an abelian group via the addition operation. Conversely any ableian group A admits a $\mathbb{Z}$ multiplication via the exponentiation map $g \mapsto g^n$. Since A is abelian exponentiation is $\mathbb{Z}$-linear, i.e. $(gh)^n = g^n h^n$, for all $n \in \mathbb{Z}$. Since $\mathbb{Z}$-linear homomorphism are group homomorphism, and abelian group homomorphisms commute with exponentiation (and therefore have a linear structure). We have functors F : **AbGroup** $\to$ **Zmod** and G : **Zmod** $\to$ **AbGroup** with $F \circ G = Id$, so the categories of ablian groups and $\mathbb{Z}$-modules are isomorphic. Functionality thus gives a rigorous way of saying abelian groups and $\mathbb{Z}$ modules are the same thing.

3. In Section 1.8 we mentionned abelianization: take any group G and add relations so that all its generators commute. This gives an ableian quotient group G_{ab}. Again Lemma 1.6.1 tells us that given a morphism $f : G \to H$, following the images of generators gives another morphism $f_{ab} : G_{ab} \to H_{ab}$, so abelianizaition is a functor. Moreover since abelian

groups are isomorphic to $\mathbb{Z}$-modules. We can use ableianization and linear algebra as a tool to study groups. Recall that this was sufficient to prove that free groups of different rank are not isomorphic, and can be used to show that if a group has fewer relations then generators, then it is non-trivial and in fact infinite.

The importance of amalgamated free products and HNN extensions shows up in topology. Let (S^1, s_0) be a based circle and let (X, x_0) be a based space. Then a contiuous based map $\gamma : (S^1, s_0), (X, x_0)$ is a loop in X based at x_0. The set of based loops of (X, x_0) (modulo homotopy), with the concatenation opeation form the fundamental group of (X, x_0) denoted

$$\pi_1(X, x_0).$$

Remarkably, although we have a continuum of based loops, if we only consider homotopy classes, then in many cases we have a countable set, in fact:

Theorem 3.3.4 *A group G is finitely presented if and only if it acts properly and cocompactly by isometries on a simply-connected geodesic space*

Basically the fundamental group of any compact manifold or CW complex will be finitely presented. *It is in this context that the study of groups with generators and relations originated.*

Now if $\gamma : (S^1, s_0) \to (X, x_0)$ is a loop and $f : (X, x_0) \to (Y, y_0)$ is a continuous map, then $f \circ \gamma$ is a loop in (Y, y_0). In a topology course it is shown that the map:

$$\pi_1(X, x_0) \ni [\gamma] \mapsto [f \circ \gamma] \in \pi_1(Y, y_0),$$

where $[\gamma]$ represents a based homotopy class in fact is a *homomorphism*

$$f_\sharp : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

Thus π_1 gives a functor from the category of based topological spaces to the category of groups.

Consider not the following generalization of a coproduct. Supose we have a pair of injective morphisms $i_X : Z \hookrightarrow X, i_Y : Z \hookrightarrow Y$ then the **coproduct over** (i_X, i_Y) is an object C equipped with two inclusions $\rho_X : X \hookrightarrow C, \rho_Y : Y \hookrightarrow C$such that givem any triple of commuting maps ϕ_X, ϕ_Y from X, Y (respectively) to any object T, there exists a unique a unique map $f : C \to T$ making the following diagram commute:

$$\begin{tikzcd}
& T & \\
X \arrow[ur, "\forall\phi_X"] \arrow[r, hook, "\rho_X"] & C \arrow[u, dashed, "\exists! f"] & Y \arrow[ul, "\forall\phi_Y"'] \arrow[l, hook', "\rho_Y"'] \\
& Z \arrow[ul, hook, "i_X"] \arrow[ur, hook', "i_Y"'] &
\end{tikzcd}$$

Figure 3.3.5 The a coproduct over a pair of monomorphisms

Note that our diagram commutation requirement means that we have an equality of compositions

$$\phi_X \circ i_X : Z \to T = \phi_Y \circ i_Y : Z \to T$$

If the standard coproduct was the free product, it should not come as a surprise to the reader that the the coproduct ovar a pair of monomorphisms is nothing else than the amalgamated product.

We therefore have two definitions of the amalgamated product: a concrete one involving an explicit presentation and one using the language of category theory, also affectionately known as **abstract nonsense.**

What good is abstract nonsense? Let us now turn our attention to based spaces. Let us first define an injective continuous map $f : (Z, u_0) \hookrightarrow (X, x_0)$ to be π_1**-injective** if it's functorial image $f_\sharp \pi_1(U, u_0) \to \pi_1(X, x_0)$ is also injective. And consider the diagram Figure 3.3.5 in the context of of based spaces. Then the universal object in the category of based spaces is the topological space $X \cup_Z Y$ obtained by gluing X to Y as prescribed by the pair of functions of from Z to X, Y.

As an immediate consequence of functoriality, we get the **Seifert van Kampen** (yes the same van Kampen) theorem: the fundamental group of space obtained by gluing spaces together is obtained by gluing the fundamental group together.

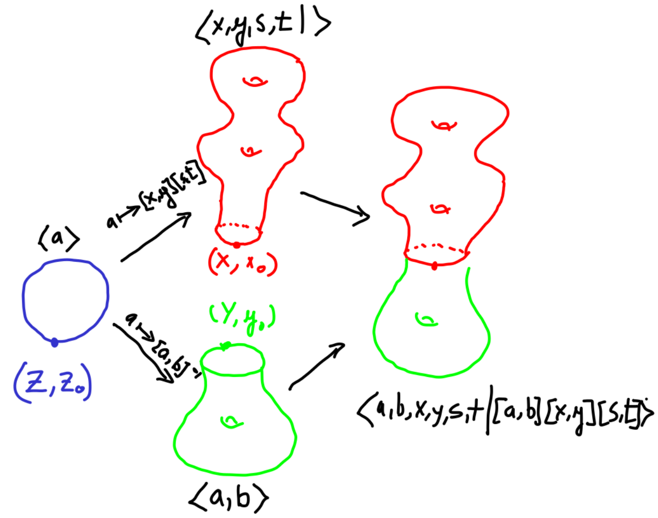


Figure 3.3.6 Gluing spaces together, glues together fundamental groups. This picture depicts spaces, fundamental groups and morphisms.

Actually, the real Seifert van Kampen theorem handles when the inclusion maps are not π_1-injective, but is proved the same way (it's just that we would have to generalize amalgamated products and all that stuff, but there is nothing new needed.)

The mysterious HNN extensions occurs when a space gets glued to itself, basepoint issues make things awkward, however.

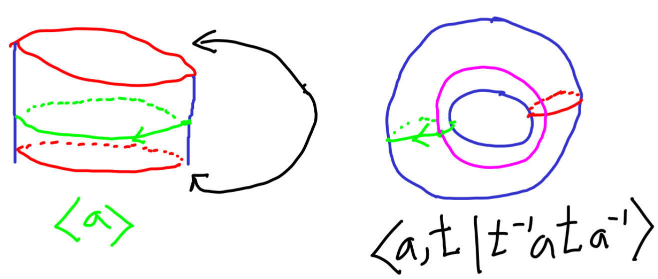


Figure 3.3.7 Glueing a space to itself. Notice how an extra loop gets created.

3.3.5 Final remark.

In the field of combinatorial and geometric group theory, we typically require that the amalgamating subgroup is mapped injectively. We could relax the definition of an amalgamated product and no longer require the maps i_X, i_Y to be injective as below:

$$\begin{tikzcd}
 & T & \\
X \arrow[ur, "\forall\phi_X"] \arrow[r, "\rho_X"] & C \arrow[u, dashed, "\exists! f"] & Y \arrow[ul, "\forall\phi_Y"'] \arrow[l, "\rho_Y"'] \\
 & Z \arrow[ul, "g_X"] \arrow[ur, "g_Y"'] &
\end{tikzcd}$$

Figure 3.3.8 Relaxing to coproducts over a pair of possibly non-injective morphisms

Again we would have a universal object, actually given by the same kind of presentation as (3.3.2), only here since ψ, φ are no longer injective, the factors G, H may no longer embed in the cofibered coproduct (or whatever it should be called). Still Lemma 1.6.1 can be used to show that this presentation will give the corresponding universal object. This generlaization of an amalgamated product is what is used in the full Seifert van Kampen Theorem, which covers the case where inclusions are no longer π_1-injective.

For example: the unit disc D has $\pi_1(D) = \{1\}$ whereas the unit circle S^1 has $\pi_1(S_1) = \mathbb{Z}$ we certainly have the inclusion $S^1 \subset D$, and this inclusion is not π_1-injective. The full Seifert van Kampen Theorem covers this case.

This Seifert van Kampen Theorem is the one of the main tool used to study the fundamental groups of topological spaces, a field in which one of the most important results of this century has been made:

Theorem 3.3.9 Poincaré conjecture. *Let M be a closed 3-manifold. Then M is homeomorphic to the the 3-sphere S^3 if an only if $\pi_1(M) = \{1\}$.*

3.4 The combinatorial geometry and topology of van Kampen diagrams.

Although we've been using them for a few lectures already, we will now give formal definitions of 2-complexes.

An **open n-cell** is a copy of the open ball in $\mathbb{R}^n$. For example an open 1-cell is an open interval, and open 2-cell is an open disc. We will now define a cell complex inductively as follows:

1. A closed 0-cell is a point. A zero-complex is a disjoint union of points.

2. A closed 1-cell consists of an open 1-cell (an interval) and a boundary consisting of two closed zero-cells, i.e. an open interval with it's two endpoints. We think of a 1-cell as being embeddable into $\mathbb{R}$. Note that if an open cell is the 1-ball, then the 0-sphere (i.e. the points of distance 1 from the origin in $\mathbb{R}$) consists of two points. A 1 complex is obtained by taking a collection of 1-cells and identifying points in their boundaries.

Note that this is precisely the definition of an undirected graph that can contain multiedges and loops. **A combinatorial mapping of 1-complexes** is a function between 1 complexes which sends 0-cells to 0-cells and 1-cells to 1-cells.

3. A closed 2-cell D consists of an open 2-cell and a boundary ∂B which is 1-complex homeomorphic to a circle (i.e. a closed polygon). Again we think of a 2-cell as being embeddable into $\mathbb{R}^2$. A 2-complex X is obtained by taking $X^{(1)}$, called a 1-skeleton, and attaching 2-cells via the identification given by a combinatorial mapping $\partial B \to X^{(1)}$.

The reader will probably see that this construction generalized to arbitrarily high dimensions. For our purposes, however, it is sufficient to stay in dimension 2, i.e. to work with 2-complexes.

Now we note that 2-complexes can be realized, and should be thought of, as CW complexes, also called **cell complexes**. That said the data that goes into describing them is completely discrete. It actually turns out that many interesting topological spaces can be studied from a combinatorial viewpoint.

3.4.1 Euler characteristic of van Kampen diagrams.

Given a finite (i.e. with finitely many cells) connected 2-complex X we have the following invariant called **Euler Characteristic**

$$\chi(X) = V_X - E_X + F_X \tag{3.4.1}$$

Were V_X, E_X, F_X denote the number of vertices (0-cells), edges (1-cells), faces (2-cells) of X (respectively).

We will now define **collapses** which are transformations of cell complexes that are useful for inductive arguments. A **free n-face** in a cell complex X is an n-cell f which is contained in the boundary $\partial\sigma$ of single open $n+1$-cell σ.

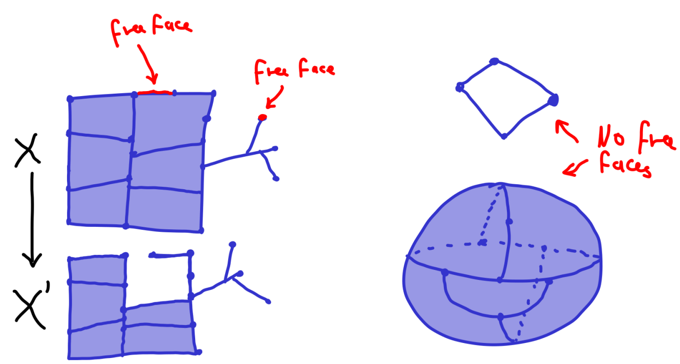


Figure 3.4.1 Some free faces and complexes without free g faces.

An **elementary collapse of cell complex X at a free face f** is the process of passing to $X' \subset X$ which is optained by deleting the free $n - face$ f and the unique open $(n+1)$-cell σ containing f. We note that although $X' \subset X$ we actually have a continuous function

$$r : X \twoheadrightarrow X'$$

such that the restriction $r|_{X'} = Id_{X'}$, so that r is a retraction. In fact r is a *deformation retraction*. The reader should not panic if they don't know what a deformation retraction is, as the following consequence of this fact (which can be proved directly from definitions) is all we will need.

Lemma 3.4.2 *If $X \to X'$ is an elementary collapse then*

$$\chi(X) = \chi(X').$$

We now turn our attention specifically to van Kampen diagrams, which we have defined as finite planar simply connected 2-complexes. We start with the following lemma.

Lemma 3.4.3 *Any van Kampen diagram, which is not a point, contains a free face and therefore admits an elementary collapse.*

Proof. We first consider the case where X contains 2-cells. First note that because X is planar every edge is contained in at most two 2-cells.

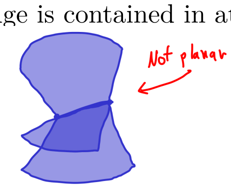


Figure 3.4.4 Edges in van Kampen diagrams can only be contained in more than two 2-cells.

So X has 2-cells, but no free faces, then every connected union of 2-cells must be a closed 2-manifold (i.e. a compact 2 manifold without boundary) contradicting that X is planar.

We now consider the situation where X doesn't have any 2-cells. Then X is a finite graph. Since X is simply connected it doesn't contain any cycles, so X is a finite tree, therefore it contains vertices of degree 1, which are precisely free 0-faces.

Having exhausted all possibilities, the result follows. ■

From this we immediately have, which we will use later.

Corollary 3.4.5 *If X is a van Kampen diagram then $\chi(X) = 1$.*

3.4.2 The combinatorial Gauss-Bonnet theorem for planar diagrams.

The classical Gauss-Bonnet theorem relates the integral of curvature over a Riemannian manifold to the manifold's Euler characteristic. We will now present a combinatorial version of this fact. We'll start by stating the theorem and then define what is meant by curvature.

Theorem 3.4.6 *Let X be an angled 2-complex, then*

$$\sum_{f \in 2-\text{cells}} \text{Curvature}(f) + \sum_{v \in 0-\text{cells}} \text{Curvature}(v) = 2\pi\chi(X).$$

An angled 2-complex is a 2-complex such that an angle has been assigned corner. We note that this is done abstractly, we don't actually care about whether it is realizable.

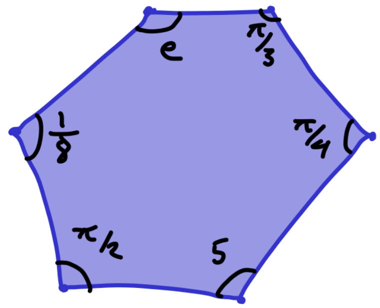


Figure 3.4.7 An angled 2-cell. Angles are in radians.

Given a 2-cell f we denote it's **perimeter**, the length of ∂f, by $|\partial f|$. We note that if f is such that two of it's boundary edges are identified as e, then we count e twice in f. We can now define the **curvature of a face**:

$$\text{Curvature}(f) = \sum_{c \in \text{corners}(f)} \angle c - (|\partial f| - 2)\pi.$$

We note that for a regular polygon P with $|\partial P|$ side in $\mathbf{E}^2$, the total angle sum is $(|\partial P| - 2)\pi$ (e.g. the sum of the internal angles of a triangle is π, for a square it's 2π, etc). The curvature of a 2-cell therefore measures the excess angle sum compared to a Euclidean polygon.

The curvature of a vertex is a bit more subtle. On the one hand it is set up so that Theorem 3.4.6 holds, but it also has geometrical significance. Given a vertex $v \in X$ in a 2-cell, we can consider a spheres (i.e. a sets of points equidistant to v) shrinking to v. These sphere will intersect incident edges in points and incident 2-faces in arcs. We call the resulting, persisting 1-complex the **link of** v, denoted link(v).

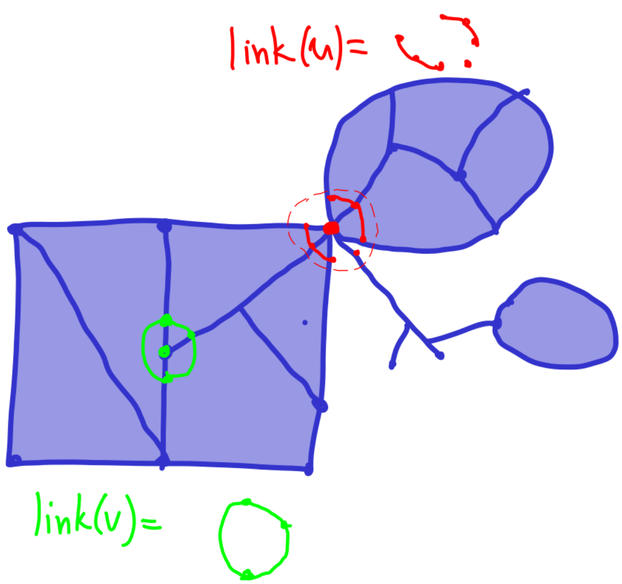


Figure 3.4.8 The link of a vertex.

We note, for example, that if e forms a loop at v, then it contributes two vertices to link(v). We also note that because X is an angled complex, each edge in link(v) corresponds to the corner of some 2-cell and therefore carries an angle. I.e. if $e \in E(\text{link}(v)$ then there is a well defined $\angle e$

We can now define the curvature of a vertex:

$$\text{Curvature}(v) = \pi(2 - \chi(\text{link}(v))) - \left(\sum_{e \in E(\text{link}(v))} \angle e\right)$$

We note that in the case where $\text{link}(v)$ is just a cycle, then if the picture is realized in the plane, the sum of the angles should add up to 2π, and in this case the cuvature measures the total angle deficiency. This definition, however, extends to arbitrary angled 2-complexes.

Ultimately Theorem 3.4.6 is simply a double counting argument: the terms in these sums of curvature cancel out to give $2\pi\chi(X)$.

3.4.3 Exercises

1. Consider the (topological)2-sphere realized as a tetrahedtron, T consisting of 4 triangles.

(a) Sketch T and compute $\chi(T)$.

(b) Angle the triangles so that they are Euclidean equilateral triangles, i.e. each corner has angle $\pi/3$. What is the curvature of the faces, what is the curvature of the vertices?

(c) Now angle the complex so that each vertex has curvature 0. This can be done by making all the angles adjacent to a vertex v add up to 2π. Verify that this indeed makes vertices have curvature 0. What is the curvature of each face?

2. Repeat the previous exercise with a cube

3. Consult the following link for a hyperbolic blanket, which is an infinite planar 2-complex whose 2-cells are 5 sided and such that every vertex is adjacent to 4 2-cells.

Assume that the pentagons are regular Euclidean i.e. each angle is $\frac{3\pi}{5}$, what is the curvature of the vertices?

Why would people say a hyperbolic blanket is a negatively curved space?

3.5 The $C'(1/6)$ small cancellation condition and Dehn's algorithm

Let us now put combinatorial curvature to work. Let $G = \langle X | R \rangle$ and suppose X, R are finite, that all words in R are cyclically reduced and furthermore that no $r \in R$ is a consequence of the remaining relations. A **piece** is a subword of some $r \in R$ which either occurs in some other $r' \in R$ or which occurs in a different location in r.

Another way of considering pieces is to consider a van Kampen diagram $\mathcal{D}$ without cancellable pairs (see Figure 3.5.1).

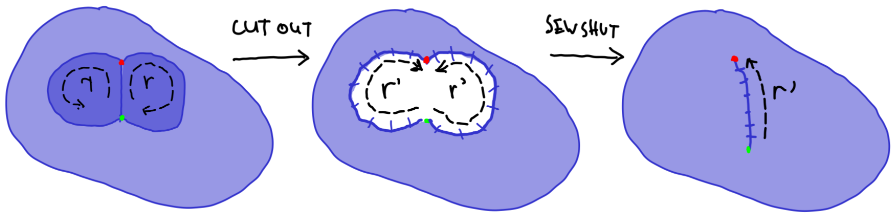


Figure 3.5.1 A cancellable pair in a van Kampen diagram and how to remove it.

A **arc** is a maximal connected subgraph whose vertices have degree at most 2, an **internal edge** is an edge contained in the boundaries of two 2-cells, and an **internal arc** is an arc consisting of internal edges. Then **pieces** are exactly the labels internal arcs of van Kampen diagrams over $\langle X|R\rangle$.

Obviously there are many pieces, any symbol that occurs twice in R is a piece. The small cancellation condition has to do with proportionately large pieces.

Definition 3.5.2 A presentation $\langle X|R\rangle$ satisfies the $C'(\lambda)$ **metric small cancelation condition** if for any $r \in R$ and any piece p which is subword of r we have

$$\frac{|p|}{|r|} < \lambda,$$

where $|p|, |r|$ denote word lengths. $\lozenge$

Now this condition interests us because whenever it is satisfied by a presentation, we will have a uniform method to solve the word problem. Before going into details let's explain the what will happen.

The word problem on $\langle X|R\rangle$ is difficult to solve when there are words representing the trivial which are relatively short but whose van Kampen diagrams contain many, many 2-cells. We can think of this as a small perimeter enclosing a large area. For example one could justifiably draw a circle on the ground around oneself and declare that this circle encloses the rest of the surface of the earth.

In a space, the relationship between areas enclosed by some perimeter is given by something called an isoperimetric function. For example in the Euclidean plane $\mathbb{E}^2$ given closed loop of length ℓ, the maximal area that can be enclosed is $A = \frac{1}{4\pi}\ell^2$. So we'll say that $\mathbb{E}^2$ has a **quadratic** isoperimetric function.

Now as we saw, on a sphere (which is positively curved) we have regions whose areas are much larger than the length of bounding curve. On the plane (which has zero curvature) area is at most quadratic in the length of a bounding curve. In the presence of negative curvature, area is a *linear* function of the length of a bounding curve.

As far as group theory is concerned, these concepts appear in van Kampen diagrams. We consider the length of the boundary word of a van Kampen diagram to be the length of an enclosing curve and we consider the area to be the number of 2-cells. In this case negative curvature manifests itself as follows: a reduced word w representing the identity in $\langle X|R\rangle$can be written as the product of at most $C \cdot |w|$ conjugates of relations. We will now present something which is a special case of a result known as Greendlinger's Lemma.

3.5.1 Finding and removing shells

For this section we'll assume that $\langle X|R\rangle$ be a $C'(1/6)$ presentation. Before considering we shall make the following modification to van Kampen diagrams: we will merge all the arcs appearing in a diagram to a single edge, the label of the resulting edge will be the word read along the arc. We will call such a diagram **arc reduced**. We note that there are no longer any vertices of degree 2 in an arc reduced diagram. For the moment we will also restrict our attention to van Kampen diagrams that are homeomorphic to discs, i.e. which cannot be disconnected by removing a vertex.

Let $\mathcal{D}$ be an arc reduced van Kampen diagram homeomerphic to a disc. We will say that a vertex v is **interior** if is in the interior of $\mathcal{D}$, otherwise we say that v is **exterior**. Given a 2-cell f we will say that f is internal if all the (open) edges in its boundar contained in the interior of $\mathcal{D}$, otherwise we say it is **external.** We note that for an internal 2-cell f, all the arcs in its boundary consist of pieces. By the $C'(1/6)$ condition, all these arcs have length strictly less than 1/6 the length of the boundary word of the 2 cell, therefore *internal 2-cells* must have at least 7 sides. We now come to a more technical definition:

Definition 3.5.3 Let $\mathcal{D}$ be an arc-reduced van Kampen diagram homeomorphic to a disc. We say that a 2-cell f is an ***i*-shell** if its boundary only contains one external arc, and if it is joined to the rest of the diagram by a path consisting of *i*arcs. $\Diamond$

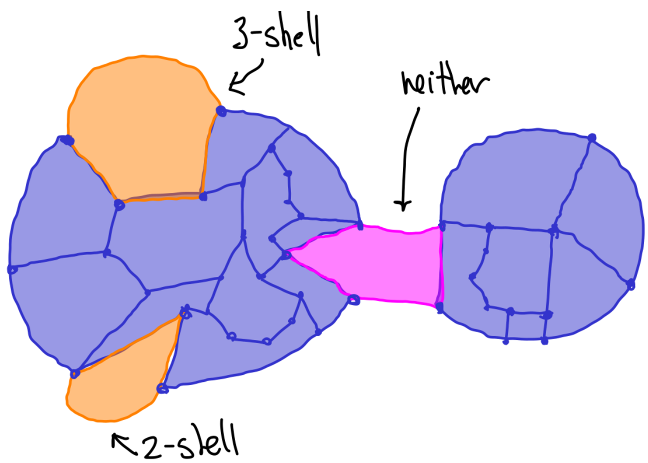


Figure 3.5.4 Some i-shells and an external 2-cell isn't an i-shell.

Now, recalling that labels of arcs pieces and that pieces are less that 1/6 the length of any relator they are part of, for an $i \leq 3$ shell we see that the length of the union of internal arcs attaching it to the rest of the diagram is less than the length of the external arc $(3 \times (< 1/6) < 1/2)$ so the effect of removing an i-shell so to produce a new van Kampen diagram with fewer 2-cells and a strictly shorter boundary word.

Our goal is now to show that the $C'(1/6)$ small cancellation condition always forces arc-reduced van Kampen diagrams homeomorphic to circles to have i-shells for some $1 \leq i \leq 3$. We will find these using curvature.

In the previous lecture we saw that $\chi(\mathcal{D}) = 1$, therefore Curvature$(\mathcal{D}) = 2\pi > 0$. For reference consider the following regular Euclidean polygons:

1. Triangle, angle sum π, average angle $\pi/3$.
2. Square, angle sum 2π, average angle $\pi/2$.
3. Pentagon, angle sum 3π, average angle $3\pi/5$.

4. Hexagon, angle sum 4π, average angle $2\pi/3$.
5. Heptagon, angle sum 5π, average angle $5\pi/7$

In our context, recall that for a 2-cell we have that Curvature(f) is the angle sum in excess of the expected Euclidean angle sum.

For vertices either v is interior, in which case $\chi(\text{link}(v)) = 0$ and the expected Euclidean angle sum is 2π, or v is exterior in which case $\chi(\text{link}(v)) = 1$ and the expected Euclidean angle sum is π. Now if the angle sum of a vertex is equal to the expected Euclidean angle sum, then it has curvature 0. Recall the the combinatorial Gauss-Bonnet theorem holds for *any angle assignment*. In particular we are free to pick angle assignments which will be suitable to our purposes.

Proposition 3.5.5 *Let $\mathcal{D}$ be an arc-reduced van Kampen diagram homeomorphic to a disc. Then it contains an i-shell for $1 \leq i \leq 3$*

Proof. We start by assigning angles so that all vertices have curvature 0.

- If v is interior, we set all adjacent angles to $\frac{2\pi}{\deg(v)}$
- If v is exterior, we set all adjacent angles to $\frac{\pi}{\deg(v)-1}$

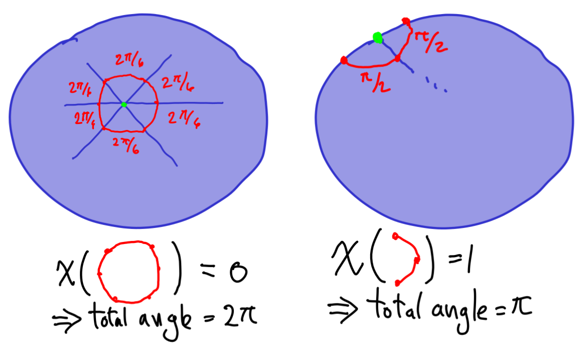


Figure 3.5.6 Assigning angles to have zero curvature interior and exterior vertices.

We note that this gives angles adjacent to interior vertices, we'll call these **interior angles**, a maximum possible value of $\frac{2\pi}{3}$, the other angles (i.e. the **exterior angles**) have a maximum possible angle of $\frac{\pi}{2}$.

From this it immediately follows that any internal 2-cell must have negative curvature as it must have at least seven sides and to non-negative curvature its angles must be on average at least $\frac{5\pi}{7} > \frac{2\pi}{3} > \frac{\pi}{2}$.

Now $\mathcal{D}$ has positive curvature overall, and this curvature comes from exterior 2-cells. Let us first consider an exterior 2-cell f that is not a shell.

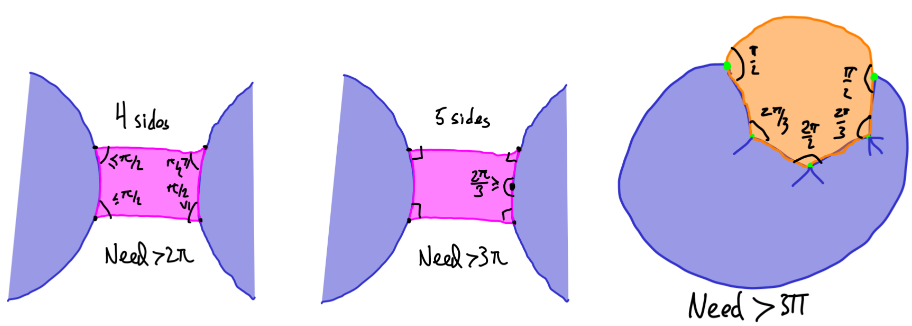


Figure 3.5.7 Exterior 2-cells which are not shells and a 4-shell

We first see that four of the corners are exterior and therefore have angle at most $\pi/2$. Looking at the figure, we see that a 4-sided 2 cell has curvature at most zero. A five sided 2-cell has an angle sum of at most

$$4 \cdot \pi/2 + 2\pi/3 = (2 + 2/3)\pi < 3\pi$$

and therefore has negative curvature. Finally, seeing as some of the angles are at most $\pi/2$ and none of the others can exceed $2\pi/3$, it is impossible to achieve even the expected Eclidean angle sum for 2-cells that have 6 or more sides, i.e. any such 2-cells will also have negative curvature.

We are therefore forced to accept that $\mathcal{D}$ contains shells. Suppose finally towards a contradition that $\mathcal{D}$ only contained i-shells for $i > 3$. On the one hand, if $i > 5$ then an $i - shell$ gives rise to a 2-cell with $i + 1$ sides, and in the previous paragraph we saw that these never have positive curvature. On the other hand a 4-shell gives rise to a Pentagon with two external vertices so the maximal angle sum is

$$2 \cdot \frac{\pi}{2} + 3 \cdot \frac{2\pi}{3} = 3\pi$$

and the resulting curvature is at most zero. Therefore the only 2-cells that can contribute positive curvature are $(1, 2, 3)$-shells and since the total curvature is positive, these must exist. ■

This result immediately gives us the following corollary.

Corollary 3.5.8 *Let $\langle X|R\rangle$ satisfy the $C'(1/6)$ small cancellation condition. Any boundary word of a van Kampen diagram $\mathcal{D}$ which is homeomorphic to a circle must contain a subword which is more than half of some cyclic permutation of some $r \in R^{\pm 1}$*

Just to be clear the *reason why we chose* $1/6$ *in the first place* is precisely so that the bit sticking out of a 3-shell is longer than the other side.

3.5.2 Dehn's algorithm

We now present an algorithm which solves the word problem in $c'(1/6)$ presentation.

Definition 3.5.9 Dehn's Algorithm. Let w be some word representing an element of $\langle X|R\rangle$ consider the following algorithm for an input word w:

1. If $w = w'r'w''$ and $r'r''$ is some cyclic permutation of some $r \in R$ with $|r''| < |r'|$ then repeat with $w'' = w'(r'')^{-1}w''$, which is shorter.

2. Otherwise, w cannot be shortened by replacing "half relations" and we stop.

◊

Now note that at each step, the algorithm produces shorter words or terminates. Also the sequence of words it produces are all equal in the group. Furthermore, this algorithm will terminate for every word and every finite presentation. We would really like it if the algorithm reduces the word to the identity if an only if the word was trivial, or equivalently, if a word can no longer be shortened in this simplistic way that is non-trivial. But if this were the case we would have an algorithm to solve the word problem which will work for all groups, which is impossible. In fact, if we take the familiar

$$\mathbb{Z}^2 = \langle a, b | a^{-1}b^{-1}ab \rangle$$

and take the word $w = aaabbba^{-1}a^{-1}a^{-1}b^{-1}b^{-1}b^{-1}$, then we will see that

Dehn's algorithm terminates on w, thus giving a "false positive" on the non-triviality of w.

That said for a $C'(1/6)$ presentation, the presence of $1, 2, 3$-shells provide exactly the subwords of words representing the identity that can be replaces by shorter halves of relations. Now we previously only considered van Kampen diagrams which were homeomorphic to discs. More generally van Kampen diagrams are made up of discs that are either joined by vertices or graphs. We will say that a vertex v is **semi-exterior** if $\text{link}(v)$ is not connected. In this case either v doesn't lie in a 2-cell, or it is where some maximal disc is attached to the rest of the diagram.

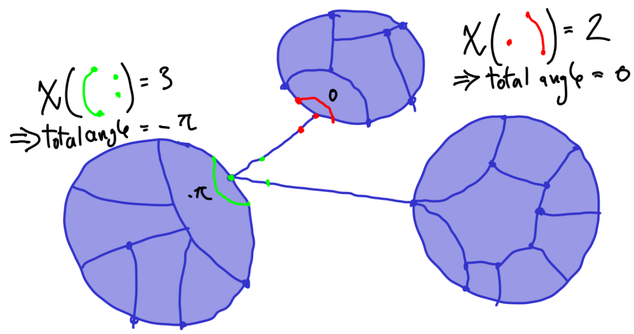


Figure 3.5.10 Semi-exterior vertices

In all such cases $\chi(\text{link}(v) \geq 2$, so in all cases the "expected" Euclidean angle sum is 0 or negative, which means that they contribute 0 or negative angles. A quick verification shows that, if anything, semi-exterior vertices make the case for $1, 2, 3$-shells even stronger and we get the general statement.

Theorem 3.5.11 Dehn's algorithm works for $C'(1/6)$ small cancellation. *If $G = \langle X|R\rangle$ satisfies the $C'(1/6)$ small cancellation, then Dehn's algorithm correctly solves the word problem, i.e. it will reduce a word w to the identity if an only if $w =_G 1$. If w is not reduced to 1, then it is non-trivial.*

This is a quintessential combinatorial and geometric group theory result: it is an algorithmic result about group presentations which relies on the geometric notion of curvature. We also note the the failure of Dehn's algorithm on $\mathbb{Z}^2$ can be explained by the fact that $\mathbb{Z}^2$ is the fundamental group of the torus, which has curvature 0 and Dehn's algorithm is a negative curvature phenomenon.

Finally although algorithmic results are great we are doing algebra after all, and it would be desirable to obtain algebraic results, we have.

Theorem 3.5.12 *If $G = \langle X|R\rangle$ satisfies the $C'(1/6)$ condition then G is infinite and contains elements of infinite order.*

sketch. The simplest proof of this fact follows from using deterministic finite automata. We say that a word w is **locally geodesic** if it contains no subwords which are long halves of cyclic permutations of generators.

A set of words which is forbidden to contain a finite list of poison subwords forms a regular language L which can be encoded by a finite automaton $\mathfrak{A}$. It is easy to show that we can get the pumping lemma to apply and we get a sequence of geodesic words

$$u * (w^m) * t; m \in \mathbb{N}.$$

All of these are non-trivial and it is easy to see that $w^m, m \in \mathbb{N}$ is in fact a sequence of distinct G-non-trivial (in fact locally geodesic) words and the result follows. ■

3.5.3 On the abundance of $C'(1/6)$ presentations.

In this final part I want to informally explain why $C'(1/6)$ presentations are, in a sense, abundant. First we need to define what we mean by a random presentation. We will define 4 parameters for a presentation $\langle X|,\rangle R$:

$$n = |X|, m = |R|, M = \max_{r\in R}(|r|), \mu = \min_{r\in R}(|r|),$$

i.e. the number of generators, the number of relations, the maximal generator length and the shortest generator length. We denote by $\mathcal{P}(m, m, M, \mu)$ the set of all presentations satisfying these parameters, and it is clear that we can fix a specific alphabet X of the appropriate size so that $\mathcal{P}(n, m, M, \mu)$ becomes a finite set.

Now if we fix $M = 1$, i.e all relations have length 1 and we let $m >> n$, then with high probability every generator will show up as a relation and we'll have a trivial group. So in this random model we have trivial groups with high probability.

The few relators model random is when we fix n, m and, for example, set $M \leq 2\mu$. In this case, as $\mu \to \infty$, we'll find that the probability that a presentation selected from $\mathcal{P}(n, m, 2\mu, \mu)$ at random will be $C'(1/6)$.

To see why this is true, consider the one relator case ($m = 1$). Failure of $C'(1/6)$ means that there is some subword of r of length at least $\frac{|r|}{6}$ occurs twice. To get a sense of how unlikely this is consider tossing a coin 1 000 000 times, what is the probability of the same $16000 \approx \frac{1000000}{6}$ streak of heads and tails occurring twice? It is small. In general the probability of failure decreases exponentially as $\mu \to \infty$.

Now the actual argument has some technicalities involving conditional probabilities that arise, for example, from considering when subwords overlap with themselves, or shifts of cyclic words, but the situations which involve tricky technicalities only constitute a negligible portion of all possibilities and, guided by the coin-toss heuristic above, it is actually quite doable to work out an exponentially decaying *upper bounds* for the probability of $C'(1/6)$ failure. In fact this will also be the case for $C'(\lambda)$ for any $\lambda > 0$. Basically, if the number of relations is fixed and their lengths go to infinity, you will most likely get a small cancellation presentation.

This reason alone mean that small cancellation presentations are somehow important.

3.5.4 Exercises

1. Let $\langle X|R\rangle$ be a $C'(1/6)$-presentation. Prove that it is impossible to tile a sphere 2-cells with boundary words in R.

Hint: The point of this problem is to get you to read the proof Proposition 3.5.5. Note that all 2-cells on a sphere must be"internal".

Comment: This in particular implies that $C'(1/6)$ presentations are **aspherical**, which is significant when considering group cohomology.

2. The fundamental group of a closed surface of genus 2 has the following presentation:

$$\left\langle a, b, c, d \middle| aba^{-1}b^{-1}cdc^{-1}d^{-1}\right\rangle.$$

How would you go about solving the word problem for this group?

3. In your own words, using drawings, examples, and citing results, explain why Theorem 3.5.11 is true.

3.6 Epilogue: word hyperbolic groups.

We started out this course with a rigorous construction of $F(X)$ the free group on X which led to group presentations: a unified way of working with finitely presented groups. Immediately, it became apparent that a group doesn't have a distinguished generating set, nor a distinguished, finite, collection of relations. That is to say, a group presentation $\langle X|R\rangle$ is not determined by the (isomorphism class) of the group G, or in other words, presentations are not canonical. What would be canonical for finitely presented group G would be the infinite set of *all finite presentation* of G, which we described using Tietze transformations.

We ended the first chapter with Dehn's algorithmic problems:

1. The word problem.

2. The conjugacy problem.

3. The isomorphisms problem.

All three of these problems are undecidable in general. This, combined with the fact that a group generally has no distinguished presentations, reinforce the conception that group presentations are completely useless. We ended the first chapter by trying to temper this impression by showing that when we restrict to a class of groups, in this case to the class of finitely generated free groups, we are able to solve all three of these algorithmic problems.

At the end of the first chapter we also presented an invariant, namely the first Betti number. We say it is an invariant because, unlike some presentation, it is determined by the isomorphism class of a group G. The second chapter was devoted to presenting the concept of quasi-isometry, which we motivated as follows: all finite generating sets of a group G are equally valid, and can give rise to different Cayley graphs. These Cayley graphs can look very different, *but they are all going to be quasi-isometric.* This leads us to consider spaces up to quasi isometry.

If we consider a finitely generated group G equipped with a word metric and forget about everything else this is nothing more than some countable set G with some function distance function $d : G \times G \to \mathbb{Z}_{\geq 0}$. Topologically, these are all just countable discrete sets, i.e. they are boring. It is only when considering such spaces up to quasi-isometry that they become interesting. For example we show that as metric spaces, $\mathbb{Z}^2$ and $\mathbb{Z}$ are not quasi-isometric.

Any group invariant will only give partial information about the group. The rest of chapter 2 is devoted to exploring the limitations of quasi-isometry and as a consequence of the Svarc-Milnor lemma it is shown that *at best* quasi-isometry can distinguish groups up to virtual isomorphism. An accurate analogy is that *quasi-isometry is like a special pair of goggles:* it lets you see entire groups, but your vision is blurry so some things you can tell apart like $\mathbb{Z}$ and $\mathbb{Z}^2$, but other groups are indistinguishable, like $\mathbb{Z}$ and $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$.

We ended Chapter 2 by showing that if we are only given a word metric on a group

$$d : G \times G \to \mathbb{Z}_{\geq 0}$$

and the assurance that this metric space is quasi-isometric to $\mathbb{Z}$, then G is itself is virtually isomorphic to $\mathbb{Z}$, so that quite a bit of algebraic structure can be inferred from this purely metric information.

Chapter 3 is about actually calulating things. It starts by soliving yet another algorithmic problem within the class of free groups: the subgroup membership problem and introduces the concept of folding. We then introduce 2-complexes and van Kampen diagrams, which give us a geometric tool to study words representing the identity in a group presentation $\langle X|R\rangle$. The concept of folding is used here because it is what enables us to construct a van Kampen diagram from a product of conjugates of relations.

Van Kampen diagrams allow us to use 2 dimensional topological arguments regarding groups. In particular, using tracks, we were able to prove fundamental results about HNN extensions and by using commbinatorial analogues of curvature we were able to show that Dehn's algorithm, a greedy approach to solving the word problem given a group presentation $\langle X|R\rangle$, will almost surely work correctly provided the relations are chosen at random with length much longer than the number of relations.

In particular the $C'(1/6)$ small cancellation condition is "generic", but it depends on a particular choice of presentation. Given a $C'(1/6)$, it is easy to ransform it using Tietze transformations so that it no longer satisfies the criteron. Also, how does this condition play with quasi-isometry?

3.6.1 Isoperimetric inequalites and word hyperbolicity

Given $G = \langle X|R\rangle$ and some word $w =_G 1$ we can define

$$\operatorname{area}(w) = \min\left(\left\{ n \in \mathbb{Z}_{\geq 0} \middle| w = \prod_{i=1}^{n} a_i r_i^{\epsilon_i} a_i^{-1} \right\}\right)$$

or, equivalently, the minimal number of R-2-cells among all van Kampen diagrams witnessing the triviality of w in $\langle X|R\rangle$. Now given, $\langle X|R\rangle$ we can define the **isoperimetric function** of the presentation as the following :

$$\begin{aligned} f_{\langle X|R\rangle} &: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \\ & f(n) = \min\left(\{\operatorname{area}(w) \in \mathbb{Z}_{\geq 0} | |w| \leq n\}\right). \end{aligned}$$

Clearly such functions are nonnegative and non-decreasing.

We say a presentation $\langle X|R\rangle$ is a **Dehn presentation** if Dehn's algorithm correctly solves the word problem. Recall that would take a $C'(1/6)$ presentation and, while preserving the generating set, change the relations so that Dehn's algorithm no longer works.

We will now return to geometry. Let x, y, z be three points in a geodesic metric space (i.e. for any two points there is a path joining them realizing the distance), a **geodesic triangle** $\Delta(x, y, z)$ is a union of three geodesics, one from x to y, one from y to z, and one from z to x. Note that in a graph, unlike in Euclidean space, there may by multiple geodesics joining two points. (See Figure 3.6.1.)

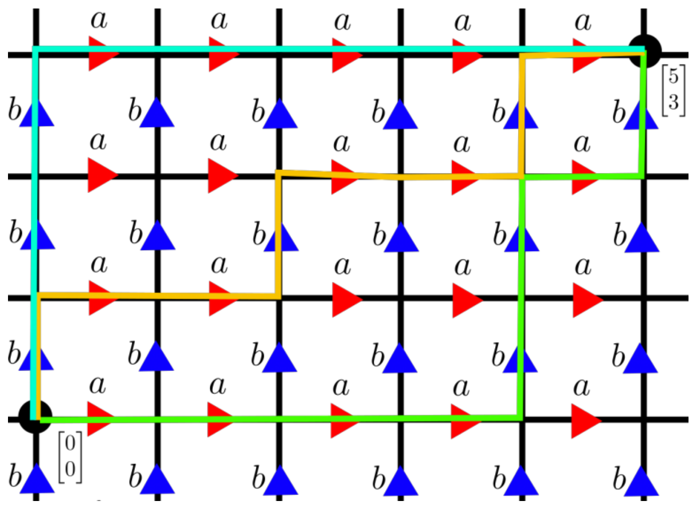


Figure 3.6.1 Three distinct geodesics joining $\begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}5\\3\end{bmatrix}$ in $\mathbf{Cay}_{\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}}(\mathbb{Z}^2)$

We say a geodesic triangle is **δ-thin** if every side is contained in the union of the δ-neighbourhood of the two other sides (see figure Figure 3.6.2.)

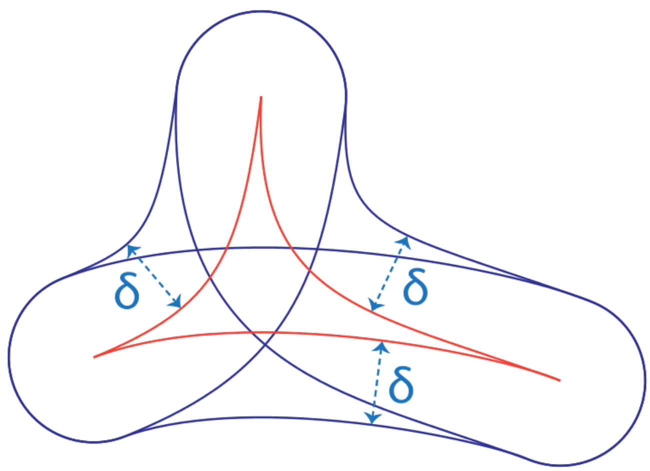


Figure 3.6.2 A δ-thin triangle. Each side stays δ-close to the union of the other two sides.

Recall that a geodesic metric space X is a space such that for any pair of points $x, y \in X$ there is a path joining these points realizing this distance. A geodesic space is said to be **Gromov hyperbolic** if there exists a uniform δ such that every geodesic triangle is δ-thin. If X is a tree for example, then every geodesic triangle is a "tripod" so every side is contained in the 0-neighbourhood of union of the other two sides.

This notion of hyperbolicity plays well with quasi-isometry, in fact it is easy to show the following: If X is Gromov hyperbolic and is quasi-isometric to Y then Y is also Gromov hyperbolic, though possibly with a different δ-parameter.

As far as group theory is concerned we have the following equivalent characteristics.

Theorem 3.6.3 *Let G be a finitely generated group. The following are equivalent.*

1. *For every finite generating set $\langle S\rangle = G$, the Cayley Graph, $\mathbf{Cay}_S(G)$ is Gromov hyperbolic. (Different generating sets will give different δ parameters.)*

2. *For every finite generating set $\langle S\rangle = G$, there exists a finite collection of relations $D_G(S)$ such that*

$$G = \langle S | D_G(S)\rangle$$

   and $\langle S|D_G(S)\rangle$ is a Dehn presentation.

3. *For any finite presentation $G = \langle S|R\rangle$, the isperimetric function is at most linearly growing, i.e. $f_{\langle S|R\rangle}$ is $O(n)$.*

We first note that this theorem says something about a group and not specifically about some presentation of the group. We will say that G is **word hyperbolic** if it satisfies any of the equivalent conditions of this theorem. A proof of this result (and much more) can be found in [2], which should now be understandable after doing this course.

We note that an argument using Tietze transformations and van Kampen diagrams immediately tell us that 2. $\Rightarrow$ 3. We will now make a few remarks about how 1. $\Rightarrow$ 2. is proved.

3.6.2 A local to global property of Gromov hyperbolic spaces.

In normal differential geometry, shortest paths tend to be "straight lines". For example if p, q are two points in $\mathbb{R}^2$ equipped with the Euclidean geometry, then any path $t \mapsto (x(t), y(t))$ joining p and q will be a shortest path if and only if the velocity $(x'(t), y'(t))$ is always parallel to the acceleration $(x''(t), y''(t))$. In other words, and this is how it is classically, geodesics are defined via local properties: if a path looks "straight" at every point, then it is globally straight.

Now figure Figure 3.6.1 shows that in some geodesic metric spaces, we can have multiple geodesics joining two points, and furthermore, looking at a path locally doesn't tell us anything about its large scale behaviour. For example a path in a graph could close up on itself, but there will be no indication of this looking at edges and vertices individually. A consequence of Gromov-hyperbolicity is that there is some way to pass from local to global. Consider this "medium scale" local to global result:

Theorem 3.6.4 Local geodesics are quasi-geodesics. *Let X be a δ-hyperbolic geodesic metric space and let γ be a k-local geodesic for $k > 8\delta$. Then γ is a (K, C)-quasi-geodesic for $K = \frac{k+4\delta}{k-4\delta}$ and $C = 2\delta$.*

To grasp the significance of this result. Note that this property is not enjoyed by the standard Cayley graph of $\mathbb{Z}^2$ as shown in Figure 3.6.5, in fact for any k it is possible to construct arbitrarily long closed loops (so definitely not quasi-geodesics with reasonable constants) in this Cayley graph that are k-local geodesics.

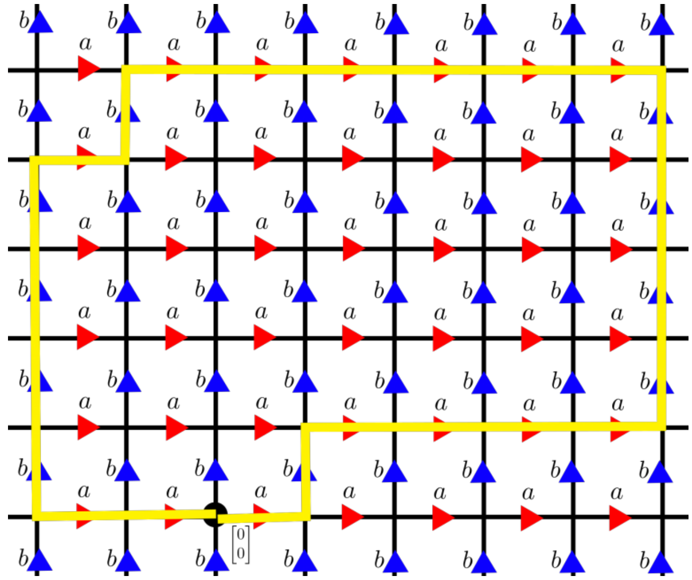


Figure 3.6.5 A 4-local geodesic in $\mathbf{Cay}_{\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}}(\mathbb{Z}^2)$ which is definitely not geodesic.

This leads to Dehn presenations as follows: an word w in some generating set $\langle S\rangle = G$ is the label of some path $\gamma_w$in the Cayley graph, which is δ-hyperbolic. Any word representing the identity is the label of a closed loop.

If all subwords of length $8\delta+1$ of w are geodesic, this forces γ_w to be quasi-geodesic and since we know the parametrs, if w is sufficiently long its endpoints will be far from one another, so w cannot be a closed loop and therfore is not trivial. It follows that words representing the identity are either short (one of the finitely many words representing the identity of length at most some bound M) or long, but containing a subword of length at most $8\delta+1$ that is not "geodesic". In the latter case, such subwords can pre replaced by a shorter words, decreasing the length of w. Repeating this gives Dehn's algorithm.

In fact this local to global property is more than just a consequence of Gromov hyperbolicity. For groups, possessing this property is in fact equivalent to hyperbolicity! This is just one reason why the class of hyperbolic groups is remarkable.

3.6.3 Dehn's algorithmic problems in hyperbolic groups

The solution to all three of Dehn's algorithmic problems in hyperbolic groups has been a major achievement of geometric group theory. The solution to the word problem follows from Dehn presentations. A solution to the conjugacy problem (given g, h decide if there is some x such that $xgx^{-1}h^{-1} = 1$) actually follows from δ-hyperbolicity and a study of geodesic *quadrilaterals*.

The isomorphism problem for hyperbolic groups involved work spanning from 1995 to 2011. There is no simple way to explain how it works as it involves asymptotic methods to create exotic topological spaces arising from limits of spaces and canonical group decomposition theories. In other words, it's super cool!

Finally, it is worth pointing out that not every hyerpbolic group G admits a $C'(1/6)$ presentation, even though they all admit Dehn presentations. The proof goes like this: there exist hyperbolic groups which satisfy something called Kazhdan's property (T), but it was also shown that $C'(1/6)$ groups never satisfy this property. Once again, the proofs of these facts involve fascinating math.

References

Much of the material in these lectures is covered in [5], but this book is difficult to read. Stillwell's book [6] has an excellent treatment of algorithmic problems and covers many crucial interactions between topology and group theory that will not be not covered in this course. Both of these books also give important historical accounts.

Although the ideas of small cancellation theory have been around for many decades the paper [7] gives a thorough and modern treatment of the topic. Most material on hyperbolic groups will be taken from [2].

The book, which as of 2020, gives the best description of the field is Drutu and Kapovich's *Geometric Group Theory* [4]. This book also has many historical references. Many important topics not covered in this book are covered in [3] which is also called... *Geometric Group Theory.*

As far as accessible contemporary introductions go, the texts [1] and [9] are at a level similar to this course, but cover different topics.(And yes, so far there are three books with the same title.) Office Hours with a Geometric Group Theorist apparently also gives a good idea of the field. And finally [10] is another good introductory text which covers substantially different topics.Oleg Bogopolski. *Introduction to Group Theory.* February 2008.J. M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short. *Notes on word hyperbolic groups. In Group theory from a geometrical viewpoint (Trieste, 1990),* pages 3–63. World Sci. Publ., River Edge, NJ, 1991.Mladen Bestvina, Michah Sageev, and Karen Vogtmann, editors. *Ge- ometric group theory.* Number volume 21 in IAS/Park City math- ematics series. American Mathematical Society ; Institute for Ad- vanced Studyb, Providence, RI : [Princeton, N.J.], 2014.Cornelia Druţu and Michael Kapovich. *Geometric Group Theory.* American Mathematical Soc., March 2018.Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory.* Classics in Mathematics. Springer-Verlag, Berlin, 2001.John Stillwell. *Classical topology and combinatorial group theory,* volume 72 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1993.Jonathan P. McCammond and Daniel T. Wise. *Fans and Ladders in Small Cancellation Theory..* Proceedings of the London Mathematical Society, 84(3):599–644, May 2002. Publisher: Cambridge University Press.Matt Clay and Dan Margalit, editors. *Office hours with a geometric group theorist.* Princeton University Press, Princeton, NJ, 2017.Clara Löh. *Geometric group theory.* Universitext. Springer, Cham, 2017.Vaughn Climenhaga and Anatole Katok. *From Groups to Geometry and Back.* American Mathematical Soc., April 2017