Geometric group theory
Lecture Notes*

M. Hull

1 Introduction

One of the main themes of geometric group theory is to study a (finitely generated) group G in terms of the geometric properties of the Cayley graph of G. These "geometric properties" come in the form of quasi-isometry invariants. Our goal this semester is to look as some specific quasi-isometry invariants such as Dehn functions, hyperbolicity, growth functions, and amenabiliity and try to understand what algorithmic, algebraic, and analytic properties of groups they are capturing.

We begin by giving necessary definitions and establishing the notation that we will use throughout.

Word metric and Cayley graphs Let G be a group generated by $S \subseteq G$; for convenience, we will always assume that our generating sets are *symmetric*, that is $S = S^{-1}$. A *word* in S is a finite concatenation of elements of S. For such a word W, let $\|W\|$ denote its length. If two words W and U are letter for letter equivalent, we write $W \equiv U$, and if W and U represent the same element of the group G, we write $W =_G U$. For an element $g \in G$, let $|g|_S$ denote the length of the shortest word in S which represents g in the group G. Given $g, h \in G$, let $d_S(g,h) = |g^{-1}h|_S$. d_S is called the *word metric* on G with respect to S.

We let $\Gamma(G,S)$ denote the *Cayley graph of* G *with respect to* S. This is the graph whose vertex set is G and there is an oriented edge e labeled by $s \in S$ between any two vertices of the form g and gs. We typically identify the edges labeled by s and s^{-1} with the same endpoints and consider these as the same edge with opposite orientations. $\mathbf{Lab}(e)$ denotes the label of the edge e; similarly, for a (combinatorial) path p, $\mathbf{Lab}(p)$ will denote the concatenation of the labels of the edges of p. Also for such a path p, we let p_- and p_+ denote the intial and the terminal vertex of p respectively, and $\ell(p)$ will denote the number of edges of p. The metric obtained on the vertices of $\Gamma(G,S)$ by the shortest path metric is clearly equivalent to the word metric d_S; identifying each edge with the unit interval $[0,1]$ in the natural way allows us to extend this metric to all of $\Gamma(G,S)$.

Metric spaces Throughout these notes, we denote a metric space by X and its metric by d (or d_X if necessary). For $x \in X$ and $n \geq 0$, let $B_n(x) = \{y \in X \mid d(x,y) \leq n\}$, that is the closed ball

*Disclaimer: Nothing in these notes is my own original work. However, most of the material is standard so I will not attempt to provide citations for every result. The interested reader is referred to the standard text [5]. For further resources, see [8, 16, 18, 24, 26, 34].

of radius n centered at x. For a subset $A \subseteq X$, we usually denote the closed n-neighborhood of A by A^{+n}, that is $A^{+n} = \{x \in X \mid d(x, A) \leq n\}$.

A *path* in X is a continuous map $p\colon [a, b] \to X$ for some $[a, b] \subseteq \mathbb{R}$. We will often abuse notation by using p to refer to both the function and its image in X. As above, we let $p_- = p(a)$ and $p_+ = p(b)$. Similarly, a *ray* is a continuous map $p\colon [a, \infty) \to X$, and a *bi-infinite path* is a continuous map $p\colon (-\infty, \infty) \to X$.

A *geodesic* is a path p which is also an isometry onto its image (see below for the definition of an isometry). Equivalently, A path p is a geodesic if $\ell(p) = d(p_-, p_+)$ where $\ell(p)$ denotes the *length* of p, and is defined as

$$\ell(p) = \sup_{a \leq t_1 \leq \ldots \leq t_n \leq b} \sum_{i=1}^{n-1} d(p(t_i), p(t_{i+1}))$$

where the supremum is taken over all $n \geq 1$ and all possible choices of $t_1, ..., t_n$[1]. Geodesic rays and bi-infinite geodesics are defined similarly. X is called a *geodesic metric space* if for all $x, y \in X$, there exists a geodesic path p such that $p_- = x$ and $p_+ = y$. Note that geodesic metric spaces are clearly path connected. For x, y in a geodesic metric space X, we let $[x, y]$ denote a geodesic from x to y.

We will usually assume throughout these notes that X is a geodesic metric space. However, most statements and proofs will also work under the weaker assumption that X is a *length space*, that is a path connected space such that for any $x, y \in X$, $d(x, y) = \inf\{\ell(p) \mid p_- = x, p_+ = y\}$.

Let X and Y be metric spaces and $f\colon X \to Y$. If f is onto and for all $x_1, x_2 \in X$, $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$, then f is called *isometry.* If f is onto and there is a constant $\lambda \geq 1$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2).$$

Then f is called a *bi-lipschitz equivalence.* In this case, we say that X and Y are bi-lipschitz equivalent and write $X \sim_{lip} Y$. Now suppose there are constant $\lambda \geq 1$, $C \geq 0$, and $\varepsilon \geq 0$ such that $f(X)$ is ε-*quasi-dense* in Y, i.e. $f(X)^{+\varepsilon} = Y$, and for all $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + C.$$

Then f is called a *quasi-isometry*, or a $(\lambda, c, \varepsilon)$-quasi-isometry if we need to keep track of the constants. In this case we say X and Y are *quasi-isometric* and write $X \sim_{qi} Y$. Note that unlike isometries and bi-lipschitz equivalences, quasi-isometries are not required to be continuous.

If the condition that f is onto (or $f(X)$ is quasi-dense) is dropped from the above definitions, then f is called an *isometric embedding*, *bi-lipschitz embedding*, or a *quasi-isometric embedding* respectively.

Exercise 1.1. Show that $\sim_{lip}$ and $\sim_{qi}$ are both equivalence relations on metric spaces.

Exercise 1.2. Let X and Y be bounded metric spaces. Prove that $X \sim_{qi} Y$.

Exercise 1.3. Suppose $S \subseteq G$ and $T \subseteq G$ are two finite generating sets of G. Show that $(G, d_S) \sim_{lip} (G, d_T)$, and hence $\Gamma(G, S) \sim_{qi} \Gamma(G, T)$.

[1] In general this may be infinite, but we will usually only consider *rectifiable* paths, that is paths p for which $\ell(p) < \infty$.

It follows from this exercise that any finitely generated group is canonically associated to a $\sim_{qi}$-equivalence class of metric spaces. We will often abuse notation by considering the group G itself as a metric space, but it should be understood that the metric on G is only well-defined up to quasi-isometry.

If P is a property of metric spaces such that whenever $X \sim_{qi} Y$, X has P if and only if Y has P, then P is called a *quasi-isometry invariant*. If G is a finitely generated group, then for any two choices of finite generating sets the corresponding Cayley graphs will have exactly the same quasi-isometry invariants. Hence these invariants are inherent properties of the group G.

Group actions Let G be a group acting on a metric space X. We will always assume that such actions are by isometries, that is for all $x, y \in X$ and $g \in G$,

$$d(x, y) = d(gx, gy).$$

There is a natural correspondence between actions of G on X and homomorphisms $\rho\colon G \to Isom(X)$, where $Isom(X)$ denotes the group of all isometries of X. We say the action is *faithful* if the corresponding homomorphism is injective. This is equivalent to saying for all $g \in G$, there exists $x \in X$ such that $gx \neq x$. The action is called *free* if for all $x \in X$, $Stab_G(x) = \{1\}$, where $Stab_G(x) = \{g \in G \mid gx = x\}$; equivalently, for all $x \in X$ and for all $g \in G$, $gx \neq x$. The action is called *proper*[2] if for any bounded subset $B \subseteq X$, $\{g \in G \mid gB \cap B \neq \emptyset\}$ is finite. The action is called *cobounded* if X/G is bounded, or equivalently there exists a bounded subset $B \subseteq X$ such that

$$X = \bigcup_{g \in G} gB$$

The following lemma is fundamental to geometric group theory. It was first proved by Efremovic.

Lemma 1.4 (Milnor-Švarč Lemma)**.** *Let G be a group acting properly and coboundedly on a geodesic metric space X. Then G has a finite generating set S and*

$$\Gamma(G, S) \sim_{qi} X.$$

Proof. Fix a point $o \in X$. Since the action of G is cobounded, there exists a constant K such that for all $x \in X$, there exists $g \in G$ such that $d(x, go) \leq K$. Let $S = \{g \in G \mid d(o, go) \leq 2K + 1\}$. By properness, the set S is finite. Note that if $s_1, s_2 \in S$, then $d(o, s_1 s_2 o) \leq d(o, s_1 o) + d(s_1 o, s_1 s_2 o) = d(o, s_1 o) + d(o, s_2 o) \leq 2(2K + 1)$. Similarly, it is easy to show by induction that for all $g \in \langle S \rangle$, $d(o, go) \leq |g|_S (2K + 1)$.

Now fix $g \in G$, and let p be a geodesic from o to go. Choose points $o = x_0, x_1, ..., x_n = go$ on p such that $d(x_i, x_{i+1}) = 1$ for $0 \leq i \leq n - 2$ and $d(x_{n-1}, x_n) \leq 1$. For each $1 \leq i \leq n - 1$, choose $h_i \in G$ such that $d(x_i, h_i o) \leq K$, and set $h_0 = 1$ and $h_n = g$. By the triangle inequality, $d(o, h_i^{-1} h_{i+1} o) = d(h_i o, h_{i+1} o) \leq 2K + 1$ for all $0 \leq i \leq n - 1$. Hence $h_i^{-1} h_{i+1} \in S$. Furthermore,

$$h_1 (h_1^{-1} h_2)(h_2^{-1} h_3)...(h_{n-1}^{-1} h_n) = h_n = g$$

[2]This is the metric version of properness. There is also a topological version, where bounded is replaced by compact.

Thus $g \in \langle S \rangle$, and since g is arbitrary we get that S generates G. Furthermore, $|g|_S \leq n$ and by our choice of x_i, $n-1 < d(o, go) \leq n$. Let $f \colon G \to X$ be the function defined by $f(g) = go$. Then we have shown that

$$|g|_S - 1 \leq d(o, go) \leq (2K+1)|g|_S.$$

Furthermore, our choice of K implies that $f(G)$ is K-quasi-dense in X. It follows easily that the map f is a quasi-isometry from G with the metric d_S to X. □

Corollary 1.5.

1. *If G is finitely generated and H is a finite index subgroup of G, then H is finitely generated and $G \sim_{qi} H$.*
2. *If $N \trianglelefteq G$ is a finite normal subgroup of G and G/N is finitely generated, then G is finitely generated and $G \sim_{qi} G/N$.*
3. *If M is a closed Riemannian manifold with universal cover $\widetilde{M}$, then $\pi_1(M)$ is finitely generated and $\pi_1(M) \sim_{qi} \widetilde{M}$.*
4. *If G is a connected Lie group with a left-invariant Riemannian metric and $\Gamma \leq G$ is a uniform lattice in G, then Γ is finitely generated and $\Gamma \sim_{qi} G$. It follows that any two uniform lattices in the same connected Lie group are quasi-isometric to each other.*

One important example that follows from Corollary 1.5(3) is if S_g is a closed, orientable surface of g with $g \geq 2$, then S_g can be equiped with a hyperbolic Riemannian metric which allows us to identify the universal cover $\widetilde{S_g}$ with the hyperbolic plane $\mathbb{H}^2$. Thus, $\pi_1(S_g) \sim_{qi} \mathbb{H}^2$.

In geometric group theory, we are often concerned with connections between the algebra and the geometry of a group G. When $G = \pi_1(M)$ for a Riemannian manifold M, then the algebra of G is determined by the topology of M while the geometry of G is determined up to quasi-isometry by the geometry of $\widetilde{M}$. The geometry/topology of $\widetilde{M}$ is determined locally, and in some ways globally, by the geometry/topology of M. These connections lead to a fruitful interplay between geometry, topology and (geometric) group theory.

Definition 1.6. Fintely generated groups G_1 and G_2 are *commensurable* if each G_i contains a finite-index subgroup H_i such that $H_1 \cong H_2$. If each G_i contains a finite-index subgroup H_i and each H_i contains a finite normal subgroup N_i such that $H_1/N_1 \cong H_2/N_2$, then G_1 and G_2 are called *weakly commensurable*.

Examples 1.7.

1. Any two finite groups are commensurable.
2. If G is any finitely generated group and K is any finite group, then G and $G \times K$ are commensurable.
3. For all $n, m \geq 2$, F_n is commensurable with F_m.
4. For all $g, g' \geq 2$, $\pi_1(S_g)$ is commensurable with $\pi_1(S_{g'})$.

It is easy to see that (weak) commensurability is an equivalence relation on finitely generated groups. From Corollary 1.5, any two (weakly) commensurable groups are quasi-isometric. The above examples allow us to see some group theoretic properties which are not quasi-isometry invariants. For example, from (1), many algebraic properties such as being abelien, nilpotent, solvable, or even simple are not quasi-isometry invariants. Similarly, from (2) it follows any group theoretic property P which passes to subgroups and for which there exists a finite group which does not have P is not a quas-isometry invariant, even for infinite groups. However, some of these properties only fail to be quasi-isometry invariants becuase of this issue with finite groups. The following definition captures this notion:

Definition 1.8. Let P be a property of groups. A group G is said to be *virtually* P if G has a finite index subgroup H such that H has the property P.

It turns out that being virtually abelian and virtually nilpotent are both quasi-isometry invariants. This is a consequence of Gromov's polynomial growth theorem which we will discuss later. On the other hand, being virtually solvable is not a quasi-isometry invariant, though some important subclasses of solvable groups are preserved under quasi-isometries and this is still an active area of research (see, for example, [11, 14]).

From examples (3) and (4) above we can also see that the rank of a groups, that is the minimal size of a finite generating set is not quasi-isometric invariant. In addition the ordinary group homology/cohomology is not a quasi-isometry invariant, though there do exist "coarse" versions of homology and cohomology which do provide quasi-isometry invariants (see [32]).

2 Dehn functions and algorithmic problems

2.1 Group presentations

Given a set S, we denote the free group on S by $F(S)$. Recall that the elements of this group are equivalence classes of words in S, where words two words are equivalent if you can obtain one from the other by adding or removing subwords of the form ss^{-1} finitely many times. Equivalently, $F(S)$ can be defined as the unique group (up to isomorphism) such that for any group G and any function $f\colon S \to G$, there is a unique homomorphism $\bar{f}\colon F(S) \to G$ extending f. If $S = \{s_1, ..., s_n\}$, we typically denote $F(S)$ by F_n.

Given a subset $R \subseteq G$, where G is a a group, the *normal closure* of R, denoted $\langle\langle R\rangle\rangle$, is defined as the intersection of all normal subgroups of G which contain R. Equivalently,

$$\langle\langle R\rangle\rangle = \{f_1^{-1}r_1f_1f_2^{-1}r_2f_2...f_k^{-1}r_kf_k \mid k \geq 0, f_i \in G, r_i \in R^{\pm 1}\}.$$

Given a set S and $R \subseteq F(S)$, we say that

$$\langle S|R\rangle \tag{1}$$

is a *presentation* of the group G if $G \cong F(S)/\langle\langle R\rangle\rangle$. In this case S is called the set of *generators* and R is called the set of *relations* of the presentation. The presentation is called *finite* if both S and R are finite sets, and G is called *finitely presentable* if G has a finite presentation. For convenience we will always assume that our set of relations is symmetric, that is $r \in R$ implies $r^{-1} \in R$. We will also abuse notation and write $G = \langle S|R\rangle$ to indicate that $\langle S|R\rangle$ is a presentation for the group G.

Exercise 2.1. Identify the groups given by the following presentations:

1. $\langle a \mid a^n = 1\rangle$.
2. $\langle a_1, ..., a_n \mid \emptyset\rangle$.
3. $\langle a, b \mid [a, b] = 1\rangle$.
4. $\langle a, b \mid a^n = b^m = 1\rangle$.

Given a group presentation $\langle S|R\rangle$ for a group G, there is an associated CW-complex Y with $\pi_1(Y) \cong G$, called the *presentation complex*. This Y contains a single vertex v, one edge (labeled by s) with both ends glued to v for each $s \in S$, and one 2-cell Π for each $r \in R$, glued to the 1-skeleton of Y such that $\partial\Pi$ is labeled by r.

The universal cover $\tilde{Y}$ is called the *Cayley complex* associated to $\langle S|R\rangle$. Note that the 1-skeleton of the Cayley complex can be naturally identified with $\Gamma(G, S)$, and the 2-skeleton of $\tilde{Y}$ is obtained by gluing, for each $g \in G$ and $r \in R$, a 2-cell with boundary a loop based at g and labeled by r.

2.2 Van Kampen Diagrams

Suppose $\langle S \mid R\rangle$ is a presenation for a group G and W is a word in S. Then $W =_G 1$ if and only if there exist $r_1, ..., r_k \in R$ and $f_1, ..., f_k \in F(S)$ such that

$$W =_{F(S)} f_1^{-1} r_1 f_1 ... f_k^{-1} r_k f_k. \tag{2}$$

We now show how this can be encoded geometrically. Let Δ be a finite, connected, simply connected, planar 2-complex in which every edge is oriented and labeled by an element of S. If e is an edge of Δ with label s and $\bar{e}$ is the same edge with the opposite orientation, then $\mathbf{Lab}(\bar{e}) = s^{-1}$. Labels of paths in Δ are defined the same as in Cayley graphs. If Π is a 2-cell of Δ, then $\mathbf{Lab}(\partial\Pi)$ is the word obtained by choosing a base point $v \in \partial\Pi$ and reading the label of the path $\partial\Pi$ starting and ending at v. Note that a different choice of basepoint results in a cyclic permutation of the word $\mathbf{Lab}(\partial\Pi)$, so we consider $\mathbf{Lab}(\partial\Pi)$ as being defined only up to cyclic permutations. $\mathbf{Lab}(\partial\Delta)$ is defined similarly. Δ is called a *van Kampen diagram* over the presentation $\langle S \mid R\rangle$ if for every 2-cell Π of Δ, (a cyclic permutation of) $\mathbf{Lab}(\partial\Pi)$ belongs to R. In this case it can be shown by a reasonably straightforward induction on the number of 2-cells of Δ that $\mathbf{Lab}(\partial\Delta) =_G 1$. It turns out the converse is also true.

Exercise 2.2. Suppose G is a group with presentation $\langle S \mid R\rangle$ and Δ is a van Kampen diagram over $\langle S \mid R\rangle$. Prove that $\mathbf{Lab}(\partial\Delta) =_G 1$.

Lemma 2.3 (van Kampen Lemma). *Suppose $\langle S \mid R\rangle$ is a presentation for a group G and W is a word in S. Then $W =_G 1$ if and only if there exists a van Kampen diagram Δ over the presentation $\langle S \mid R\rangle$ such that* $\mathbf{Lab}(\partial\Delta) \equiv W$.

Proof. If W is the boundary label of a van Kampen diagram, then $W =_G 1$ be the previous exercise. Now suppose that $W =_G 1$. Then there exist $r_1, ..., r_k \in R$ and $f_1, ..., f_k \in F(S)$ such that

$$W =_{F(S)} f_1^{-1} r_1 f_1 ... f_k^{-1} r_k f_k.$$

Each word of the form $f_i^{-1}r_if_i$ is the label of a van Kampen diagram consisting of a path labeled by f_i connected to a 2-cell with boundary label r_i. glueing the initial points of each of these paths together produces a van Kampen diagram with boundary label $f_1^{-1}r_1f_1...f_k^{-1}r_kf_k$ (sometimes called a "wedge of lollipops"). Now $f_1^{-1}r_1f_1...f_k^{-1}r_kf_k$ can be transformed into the word W by a finite sequence of moves consisting of adding or deleting subwords of the form ss^{-1}. One can check that if one of these moves is applied to a word U produces U' and U is the boundary label of a van Kampen diagram, then there is a natural move on the diagram which produces a new van Kampen diagram with boundary label U'. It follows that the "wedge of lollipops" diagram can be modified by a finite sequence of moves to produce a van Kampen diagram with boundary label W.

□

Exercise 2.4. Suppose Δ is a van Kampen diagram over a presentation $\langle S|R\rangle$ for a group G, and p is a closed (combinatorial) path in Δ. Prove that $\mathbf{Lab}(p) =_G 1$.

From this exercise, it follows that if you fix a vertex $v \in \Delta$, there is a well-defined, label preserving map from the 1-skeleton of Δ to $\Gamma(G, S)$ which sends v to 1.

A van Kampen diagram can be interpreted topologically as follows: Any word W defines a path in the Cayley graph $\Gamma(G, S)$, and $W =_G 1$ if and only if this path is a loop. In this case, then there is a homotopy which contracts this loop to a point in the Cayley complex corresponding to the presentation $\langle S|R\rangle$. A van Kampen diagram is a combinatorial descriptions of such a homotopy.

2.3 Dehn functions

Given a van Kampen diagram Δ, let $Area(\Delta)$ be the number of 2-cells of Δ. For a fixed group presentation $\langle S \mid R\rangle$ and a word W in S such that $W =_G 1$, let

$$Area(W) = \min\{Area(\Delta) \mid \Delta \text{ is a van Kampen diagram over } \langle S \mid R\rangle \text{ and } \mathbf{Lab}(\partial\Delta) \equiv W\}.$$

equivalently, $Area(W)$ is equal to the minimal k such that is equal to a product of k conjugates of elements of R (see (2)).

Exercise 2.5. Let W be a word in S such that $W \equiv W_1UW_2$ and let (some cylic shift of) $UV^{-1} \in R$ with V possibly the empty word. We say that W' is obtained from W by an *R–move* if $W =_{F(S)} W_1VW_2$. Given $G = \langle S|R\rangle$ and a word S such that $W =_G 1$, prove that $Area(W)$ is equal to the minimal number of R moves needed to transform W into the empty word.

The *Dehn function* of a finitely presented group G, denoted δ_G, is the function $\delta_G \colon \mathbb{N} \to \mathbb{N}$ defined by

$$\delta_G(n) = \max_{\|W\| \leq n} Area(W)$$

Of course, this depends not only on G, but also on the chosen presentation of G. In order to make the Dehn function of G independent of the presentation (as is suggested by the notation δ_G), we consider this function as defined only up to the following equivalence relation: functions $f, g \colon \mathbb{N} \to \mathbb{N}$ are equivalent if there exist constant A_1, B_1, C_1 and A_2, B_2, C_2 such that for all $n \in \mathbb{N}$,

$$f(n) \leq A_1g(B_1n) + C_1n \text{ and } g(n) \leq A_2f(B_2n) + C_2n.$$

Note that the linear term in the above equivalence is indeed necessary, since even the trivial group has the presentation $\langle s \mid s = 1\rangle$ and $Area(s^n) = n$.

Exercise 2.6. (a) Show that this is indeed an equivalence relation.

(b) Show that $f_1(n) = 1$, $f_2(n) = \log n$, and $f_3(n) = n$ are all equivalent.

(c) Show that two polynomials p and q are equivalent if and only if they have the same degree.

(d) Show that 2^n and 3^n are equivalent.

Exercise 2.7. Prove that a finite group has at most linear Dehn function.

Exercise 2.8. Prove that a finitely generated abelian group has at most quadratic Dehn function, and that the Dehn function of $\mathbb{Z}^2$ is equivalent to n^2.

Examples 2.9. 1. If G is nilpotent of class c, then $\delta_G(c) \leq n^{c+1}$

2. If G is the fundamental group of a compact, orientable surface of genus $g \geq 2$, then Dehn's algorithm shows that δ_G is linear.

3. The Dehn function of $BS(1, 2) = \langle a, t \mid t^{-1}at = a^2\rangle$ is equivalent to 2^n

4. For $G = \langle a, b, c \mid a^b = c, a^c = a^2\rangle$, δ_G is equivalent to $2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}$, where this tower has length $\log_2(n)$.

5. $SL(2, \mathbb{Z})$ has linear Dehn function. $SL(3, \mathbb{Z})$ has exponential Dehn function. For $m \geq 5$, the Dehn function of $SL(m, \mathbb{Z})$ is quadratic [35]. The Dehn function of $SL(4, \mathbb{Z})$ is unknown, but is conjectured to be quadratic.

If M is a Riemannian manifold and $G = \pi_1(M)$, then the Dehn function δ_G is equivalent to the isoperimetric function on the universal cover $\widetilde{M}$, that is the function which measures the maximal area of a disc whose boundary is a curve of length at most n [6].

Definition 2.10. Give an presentation $\langle S|R\rangle$ for a group G, A *Tietze transformation* on $\langle S|R\rangle$ one of the following four types of operations:

1. (Add a generator) $\langle S|R\rangle \to \langle S \cup \{t\}|R \cup \{t^{-1}W\}\rangle$, where $t \notin S$ and W is any word in S.

2. (Remove a generator) $\langle S|R\rangle \to \langle S \setminus \{s\}|R'\rangle$, where $s \in S$ and there exist a word W in $S \setminus \{s\}$ such that $s =_G W$ and R' is obtained from R by replacing each occurence of $s^{\pm 1}$ with $W^{\pm 1}$.

3. (Add a relation) $\langle S|R\rangle \to \langle S|R \cup \{W\}\rangle$ where $W \in \langle\langle R\rangle\rangle$ in the free group $F(S)$.

4. (Remove a relation) $\langle S|R\rangle \to \langle S|R \setminus \{U\}\rangle$ where $U \in R$ such that $U \in \langle\langle R \setminus U\rangle\rangle$ in the free group $F(S)$.

Exercise 2.11. Check that applying one Tietze transformation to a group presentation produces a presentation with equivalent Dehn function.

Theorem 2.12. *[24, Proposition 2.1] Suppose $G_1 = \langle S_1|R_1\rangle$ and $G_2 = \langle S_2|R_2\rangle$ with both presentation finite. Then G_1 is isomorphic to G_2 if and only if there is a finite sequence of Tietze transformations which turn $\langle S_1|R_1\rangle$ into $\langle S_2|R_2\rangle$.*

It follows from the previous theorem and exercise that the Dehn function of a finitely presented group G is independent of the choice of finite presentation up to the equivalence relation given above. We next prove that the Dehn function is also a quasi-isometry invariant.

Theorem 2.13. *Suppose G is finitely presented and H is finitely generated. If $G \sim_{qi} H$, then H is finitely presented and δ_G is equivalent to δ_H.*

Proof. Let $\langle S|R\rangle$ be a finite presentation for G and let T a finite generating set for H. Let $M = \max\{\|r\| \mid r \in R\}$. Let $f\colon \Gamma(H,T) \to \Gamma(G,S)$ be a $(\lambda, c, \varepsilon)$ quasi-isometry. Let p be a closed (combinatorial) path in $\Gamma(H,T)$, and let $v_1, v_2, ..., v_n, v_{n+1} = v_1$ denote the vertices of p. Let q be the closed path in $\Gamma(G,S)$ formed by connecting each $f(v_i)$ to $f(v_{i+1})$ by a geodesic. Since $d_T(v_i, v_{i+1}) = 1$, $d_S(f(v_i), f(v_{i+1})) \leq \lambda + c$, and hence $\ell(q) \leq (\lambda + c)n$. Since q is a closed path, $\mathbf{Lab}(q) =_G 1$, so there exists a van Kampen diagram Δ with $\mathbf{Lab}(\partial\Delta) \equiv \mathbf{Lab}(q)$. We also choose Δ such that $Area(\Delta) \leq \delta_G((\lambda+c)n)$. We identify the 1-skeleton of Δ with its image in $\Gamma(G,S)$ under the natural map $\Delta^{(1)} \to \Gamma(G,S)$ which sends $\partial\Delta$ to q. Now we build a map $g\colon \Delta^{(1)} \to \Gamma(H,T)$ for each interior vertex $v \in \Delta$, choose a vertex $u \in \Gamma(H,T)$ such that $d_S(v, f(u)) \leq \varepsilon$, and set $g(v) = u$. Each exterior vertex $v \in \Delta$ lies on some geodesic $[f(v_i), f(v_{i+1})]$; if v is closer to $f(v_i)$ we set $g(v) = v_i$, otherwise we set $g(v) = v_{i+1}$. Now if two vertices v and u are adjacent, then we join $g(v)$ and $g(u)$ by geodesics in $\Gamma(H,T)$. Note that for such u and v, $d_S(f(g(u)), f(g(v))) \leq 2\varepsilon + 1$, and hence $d_T(g(u), g(v)) \leq (2\varepsilon + 1 + c)\lambda$. It follows that if Π is a 2-cell of Δ, there is a closed loop in $g(\Delta^{(1)})$ corresponding to the image of $\partial\Pi$ of length at most $(2\varepsilon+1+c)\lambda\ell(\partial\Pi) \leq (2\varepsilon+1+c)\lambda M$.

Let $R' = \{r \in F(T) \mid \|r\| \leq (2\varepsilon + 1 + c)\lambda M \text{ and } r =_H 1\}$. From above, we have that there is a van Kampen diagram Δ' whose 1-skeleton is $g(\Delta^{(1)})$ and each two cell is labeled by an element of R'. Hence Δ' is a van Kampen diagram over $\langle T \mid R'\rangle$ and $W \equiv \mathbf{Lab}(\partial\Delta')$. Thus $\langle T \mid R'\rangle$ is a presentation for H, in particular H is finitely presented. Furthermore,

$$Area(W) \leq Area(\Delta') = Area(\Delta) \leq \delta_G((\lambda + c)n)$$

since W is an arbitrary word of length n, we get that $\delta_H(n) \leq \delta_G((\lambda + c)n)$. Reversing the roles of G and H in the above proof will result in the reverse inquality (with possibly different constants), hence δ_G is equivalent to δ_H.

□

Corollary 2.14. *Finite presentability is a quasi-isometry invariant.*

2.4 Algorithmic problems

The following algorithmic problems were introduced by Max Dehn in 1911.

Word Problem: Given a presentation $\langle S \mid R\rangle$ of a group G, find an algorithm such that for any word W in S, the algorithm determines whether or not $W =_G 1$.

Conjugacy Problem: Given a presentation $\langle S \mid R\rangle$ of a group G, find an algorithm such that for any two words W and U in S, the algorithm determines whether or not W and U represent conjugate elements of the group G.

Isomorphism Problem: Find an algorithm which accepts as input two group presentations and determines whether or not they represent isomorphic groups.

Exercise 2.15. Describe an algorithm which solves the word problem for the standard presentation of $\mathbb{Z}^n$, that is $\langle a_1, ..., a_n \mid [a_i, a_j], 1 \leq i < j \leq n \rangle$.

Note that the word problem is equivalent to deciding whether a given element of $F(S)$ belongs to the normal subgroup $\langle\langle R \rangle\rangle$. It is not hard to see that there is an algorithm for listing all elements of the normal subgroup $\langle\langle R \rangle\rangle$, but by itself this algorithm will not be able to certify that a given element $g \notin \langle\langle R \rangle\rangle$.

The following is a classical result in computability and group theory, first proved by Novikov in 1955; another proof was given by Boone in 1958.

Theorem 2.16 (Novikov-Boone)**.** *There exists a finite group presentation for which the word problem is undecidable.*

Note that the word problem can be viewed as a special case of the conjugacy problem, since $W =_1 G$ if and only if W is conjugate to 1 in G. It follows that any group with undecidable word problem will also have undecidable conjugacy problem. There do, however, exist group presentations with decidable word problem but undecidable conjugacy problems.

Similarly, the isomorphism problem is undecidable in general, though as with the other algorithmic problems it can solved in certain special cases, that is if one only considers presentations which represent groups belonging to a specific class of groups.

Proving the existence of a group with undecidable word problem is quite difficult, but once it is known that such a group exists many other algorithmic questions about groups can be reduced to the word problem and hence proved to be undecidable in general. For more details, see the Adian-Rabin theorem.

Given a van Kampen diagram Δ, we define the *type* of Δ by the ordered pair of natural numbers $(Area(\Delta), \ell(\partial\Delta))$.

Exercise 2.17. Show that for a finite presentation $\langle S|R \rangle$ and a fixed type (k, n), there are only finitely many van Kampen diagrams over $\langle S|R \rangle$ of type (k, n).

A function $f \colon \mathbb{N} \to \mathbb{N}$ is called *recursive* is there exists an algorithm which computes $f(n)$ for all $n \in \mathbb{N}$.

Theorem 2.18. *Let G be a finitely presented group. The following are equivalent.*

1. *δ_G is recursive.*
2. *There exists a recursive function $f \colon \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\delta_G(n) \leq f(n)$.*
3. *The word problem in G is solvable.*

Proof. Fix a finite presentation $\langle S|R \rangle$ for the group G.

(1) $\implies$ (2)

Trivial.

(2) $\implies$ (3)

Let W be a word in S with $W =_G 1$ and $\|W\| = n$. By assumption, there exists a van Kampen diagram Δ with $\mathbf{Lab}(\partial\Delta) \equiv W$ and $Area(\Delta) \leq \delta_G(n) \leq f(n)$. However, by the previous exercise there are only finitely many van Kampen diagrams of type (k, n) with $1 \leq k \leq f(n)$. Hence one can list all of these diagrams; if a some diagram in this list has boundary label W, then $W =_G 1$, otherwise $W \neq_G 1$.

(3) $\implies$ (2).

Fix $n \in \mathbb{N}$, and let R_n be the set of words W in S such that $\|W\| \leq n$ and $W =_G 1$. This set can be explicitly computed by applying the algorithm which solves the word problem in G to each word of length at most n. Now for each $W \in R_n$, we can compute $Area(W)$ by listing all van Kampen diagrams of type $(1, \|W\|)$, then type $(2, \|W\|)$ etc. Since we know $W =_1 G$, there must be some k such that this list produces a van Kampen diagram with boundary label W and area k; if k is the smallest natural number for which such a diagram occurs, then $Area(W) = k$. Hence we can compute the area of each of the the finitely many words in R_n, and by definition $\delta_G(n)$ is the maximum of these areas.

□

Corollary 2.19. *Solvability of the word problem is a quasi-isometry invariant.*

The Dehn function can be interpreted as a measure of the geometric complexity of the word problem in a group G. In particular, the Dehn will give an upper bound on the time complexity of the word problem in G. In general, however, this upper bound is far from being sharp as can be seen in the following example:

Example 2.20. Recall that for $G = BS(1, 2) = \langle a, t | t^{-1}at = a^2\rangle$, $\delta_G \sim 2^n$. However, the time complexity of the word problem in G is at most polynomial in n. Indeed, G is isomorphic to the subgroup of $GL(2, \mathbb{Q})$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$, so the time complexity of the word problem in G is at most the time complexity of multiplying 2×2 matrices.

3 Hyperbolic groups

3.1 Hyperbolic metric spaces

Before we define hyperbolic groups, we need to defined hyperbolic metric spaces and study some basic properties of their geometry. In particular, we need to show that hyperbolicity is invariant under quasi-isometry in order for hyperbolicity to be well-defined in the world of groups.

The following is the mostly commonly cited definition of hyperbolicity and is attributed to Rips.

Definition 3.1 (Slim Triangles or the Rips Condition)**.** Let $\delta \geq 0$. We say that a geodesic metric space X is *δ–hyperbolic* if for any geodesic triangle T in X with sides p, q, r and any point $a \in p$, there exists $b \in q \cup r$ such that $d(a, b) \leq \delta$.We say X is *hyperbolic* if it is δ–hyperbolic for some $\delta \geq 0$.

A triangle T which satisfies the conditions in this definition is called *δ-slim.*

Exercise 3.2. Let X be a δ-hyperbolic geodesic metric space and $P = p_1p_2...p_n$ a geodesic n-gon in X for $n \geq 3$. Let a be a point on p_i for some $1 \leq i \leq n$. Prove that there exists $j \neq i$ and $b \in p_j$ such that $d(a, b) \leq (n-2)\delta$. (In fact, $n-2$ can be replaced by $\log_2(n)$).

Examples 3.3.

1. If X is a bounded metric space, then X is δ-hyperbolic for $\delta = diam(X)$.
2. $\mathbb{R}$ with the standard metric is 0-hyperbolic.
3. Is X is a simplicial tree, that is a connected graph with no cycles equipped with the combinatorial metric, then X is 0-hyperbolic (equivalently, every triangle is a tripod).
4. Generalizing the previous two examples, a 0-hyperbolic geodesic metric space is called a $\mathbb{R}$-*tree*. Some more examples of $\mathbb{R}$-trees:
   (a) $X = \{(x,y) \mid x \in [0,1], y = 0\} \cup \{(x,y) \mid x \in \mathbb{Q}, y \in [0,1]\}$ with the metric $d((x_1,y_1),(x_2,y_2)) = |y_1|+|x_2-x_1|+|y_2|$ when $x_1 \neq x_2$ and $d((x_1,y_1),(x_2,y_2)) = |y_2-y_1|$ otherwise.
   (b) $X = \mathbb{R}^2$ with the following metric: If the line containing (x_1,y_1) and (x_2,y_2) passes through the origin, then $d((x_1,y_1),(x_2,y_2))$ is the usual Euclidean distance. Otherwise, $d((x_1,y_1),(x_2,y_2)) = \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$.
5. $\mathbb{R}^n$ with the Euclidean metric is not hyperbolic for any $n \geq 2$.
6. The classical hyperbolic space $\mathbb{H}^2$ is δ-hyperbolic. Recall that a triangle T in $\mathbb{H}^2$ with angles α, β, and γ has area $= \pi - \alpha - \beta - \gamma$. For a point x on T, consider the largest semi-circle contained in T and centered at x. This semi-circle has area at most the area of T which is at most π; this provides a bound on the radius of the semi-circle, which can be explicity computed to show that $\mathbb{H}^2$ is δ–hyperbolic for $\delta = 4\log\varphi$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.
7. From the previous example, it follows that $\mathbb{H}^n$ is δ-hyperbolic for all $n \geq 2$.
8. If (X,d) is any metric space, then we can define a new metric $\hat{d}$ on X by $\hat{d}(x,y) = \log(1 + d(x,y))$. Then $(X,\hat{d})$ is $2\log 2$ hyperbolic.
9. $O(n,1)$, $U(n,1)$, $SP(n,1)$ are all hyperbolic when given left-invariant Riemannian metrics.

Definition 3.4. Let X be a metric space and A, B closed subsets of X. The *Hausdorff distance* between A and B is the infimum of all ε such that $A \subseteq B^{+\varepsilon}$ and $B \subseteq A^{+\varepsilon}$. We denote this distance by $d_{Hau}(A,B)$.

Exercise 3.5. Verify that d_{Hau} is a metric on the set of closed subsets of X.

Definition 3.6. Suppose X is a metric space and $p\colon [a,b] \to X$ is a (λ, C)-quasi-isometry onto its image in X. Then p is called a (λ, C) *quasi-geodesic.*

If p is continuous (and hence a path), then we say that p is a (λ, C) quasi-geodesic path if for any subpath q of p,

$$\ell(q) \leq \lambda d(q_-, q_+) + C$$

Note that, in general, quasi-geodesic are not required to be continuous.

Given these definitions, being a quasi-geodesic path depends only on the image of p while being a quasi-geodesic depends on the chosen parameterization. However, any quasi-geodesic path is in fact a quasi-geodesic when it is parameterized by arc length, that is when $p\colon [a,b] \to X$ is such that for all $a \leq s < t \leq b$, $\ell(p|_{[s,t]}) = |t-s|$. Furthermore, the next lemma shows that every quasi-geodesic is close to a quasi-geodesic path.

Lemma 3.7. *Let $p\colon [a,b] \to X$ be a (λ, c)–quasi-geodesic. Then there exists λ', c', and D depending only on λ and c and there exits a (λ', c') quasi-geodesic path p' with the same endpoints as p such $d_{Hau}(p, p') < D$*

sketch. Define $p'(t) = p(t)$ for all $t \in \mathbb{Z} \cap [a,b]$. Now "connect the dots" by geodesics. Verifying that p' satisfies the above conditions is straightforward. □

Exercise 3.8. Let $p\colon [a,b] \to X$ be a geodesic and let $f\colon X \to Y$ be a (λ, c) quasi-isometric embedding. Prove that $f \circ p$ is a (λ, c) quasi-geodesic.

Definition 3.9. Suppose X is a metric space and p is a path in X. p is called a *k-local geodesic* if every subpath of p of length $\leq k$ is a geodesic.

Remark 3.10. *Quasi-geodesic rays*, *bi-infinite quasi-geodesics*, *local geodesic rays*, and *bi-infinite local geodesics* are all similarly defined in the obvious ways.

We will assume for the rest of this section that X is a geodesic and δ-hyperbolic metric space.

Lemma 3.11. *Let p be a (rectifiable) path in X from x to y. Then for any geodesic $[x,y]$ and any point $a \in [x,y]$, there exists $b \in p$ such that*

$$d(a,b) \leq \delta|\log_2(\ell(p))| + 1$$

Proof. We assume that p is paramterized such that $p\colon [0,1] \to X$ and for all $0 \leq i < j \leq 1$, $\ell(p|_{[i,j]}) = \frac{1}{j-i}\ell(p)$. Choose N such that $2^N \leq \ell(p) \leq 2^{N+1}$. Let $z_1 = p(\frac{1}{2})$. Let T_1 be a triangle with sides $[x,y]$, $[x,z_1]$, and $[z_1,y]$. Since T_1 is δ-slim, there exists a point $b_1 \in [x,z_1] \cup [z_1,y]$ with $d(a,b_1) \leq \delta$. If $b_1 \in [x,z_1]$, let $z_2 = p(\frac{1}{4})$ and $T_2 = [x,z_1][x,z_2][z_1,z_2]$; if $b_1 \in [z_1,y]$, let $z_2 = p(\frac{3}{4})$ and $T_2 = [z_1,y][z_1,z_2][z_2,y]$. We apply slimness to T_2 and b_1 to find a point b_2 on one of the other two sides of T_2 that is δ close to b_1. We then define z_3 as the midpoint of the subpath of p that is "above" the side of T_2 containing b_2 and T_3 as the triangle which contains the side of T_2 that contains b_2 and geodesics connecting the endpoints of this side to z_3. Continue this process inductively until we obtain b_N.

Note that by construction, for each $1 \leq i \leq N-1$, $d(b_i, b_{i+1}) \leq \delta$, and hence $d(a, b_N) \leq N\delta \leq \delta|\log_2(\ell(p))|$. Furthermore, b_N belongs to a geodesic q with endpoints on p such that $\ell(q) \leq \frac{\ell(p)}{2^N}$. Let $b \in p$ be the closest endpoint of q to b_N, hence $d(b, b_N) \leq \frac{1}{2}\ell(q) \leq 1$. Thefore,

$$d(a,b) \leq d(a,b_N) + d(b_N, b) \leq \delta|\log_2(\ell(p))| + 1.$$

□

Theorem 3.12 (Morse Lemma)**.** *Let X be a δ-hyperbolic metric space. Then for any $\lambda \geq 1$, $C \geq 0$, there exists $K = K(\delta, \lambda, C)$ such that for any geodesic p and any (λ, C)-quasi-geodesic q with $p_- = q_-$ and $p_+ = q_+$, $d_{Hau}(p, q) \leq K$.*

Proof. Note that, by Lemma 3.7, we can assume that q is a quasi-geodesic path. Let $D = \sup_{x \in p}\{d(x, q)\}$; our first goal will be to bound D in term of δ, λ and C. Since p and q are compact, there is point $x_0 \in p$ which realizes this supremum. In particular, the the interior of $B_D(x_0)$ does not intersect q. Now choose $y \in [p_-, x_0]$ such that $d(x_0, y) = 2D$, or if no such y exists then set $y = p_-$. Choose $z \in [x_0, p_+]$ similarly. By definition of D, there exists some $y', z' \in q$ such that $d(y, y') \leq D$ and $d(z, z') \leq D$. By the triangle inequality,

$$d(y', z') \leq d(y', y) + d(y, z) + d(z, z') \leq 6D$$

If q' is the subpath of q joining y' to z', then since q is a (λ, C)-quasi-geodesic, $\ell(q') \leq 6\lambda D + C$. Let $c = [y, y']q'[z', z]$, and note that $l(c) \leq 6\lambda D + C + 2D$ and $d(x_0, c) = D$. By Lemma 3.11, $d(x_0, c) \leq \delta|\log_2(\ell(c))| + 1$, and combinging this with the previous estimates gives

$$D \leq \delta|\log_2(6\lambda D + 2D + C)| + 1.$$

This equation implies that D must be bounded in terms of δ, λ and C.

It remains to show that q is contained in a bounded neighborhood of p. Suppose $q = q_1 q_2 q_3$ such that q_2 is a maximal subpath of q which lies outside p^{+D}. Now every point of p is within D of some point on either q_1 or q_3; by connectedness of p, there must exist some $x \in p$ and $y \in q_1$, $z \in q_3$ such that $d(x, y) \leq D$ and $d(x, z) \leq D$. In particular, this means that $\ell(q_2) \leq \lambda(2D) + C$. It follows that q is contained in the $2\lambda D + D + C$ neighborhood of p. □

Corollary 3.13. *Let X be a δ-hyperbolic metric space. Then for any $\lambda \geq 1$, $C \geq 0$, there exists $\kappa = \kappa(\delta, \lambda, C)$ such that for any (λ, C)-quasi-geodesics p and q with $p_- = q_-$ and $p_+ = q_+$, $d_{Hau}(p, q) \leq \kappa$.*

Exercise 3.14. Prove that there exist $\lambda \geq 1$ and $C \geq 0$ such that for any $K \geq 0$, there exists a (λ, C)-quasi-geodesic q in $\mathbb{R}^2$ such that $d_{Hau}(q, [q_-, q_+]) \geq K$.

Exercise 3.15. Let X be a geodesic metric space. Prove X is hyperbolic if and only if for all $\lambda \geq 1$, $C \geq 0$ that there exists δ' such that for any triangle T in X whose sides are (λ, C)-quasi-geodesics is δ'-slim.

Proposition 3.16. *Let X be a hyperbolic metric space. Suppose Y is a geodesic metric space and $f\colon Y \to X$ is a quasi-isometric embedding. Then Y is hyperbolic.*

Proof. Let $f\colon Y \to X$ be a (λ, c) quasi-isometric embedding and let $p_i\colon [a_i, b_i] \to Y$ be geodesics for $1 \leq i \leq 3$ such that $T = p_1 p_2 p_3$ is a geodesic triangle in Y. By Exercise 3.8 $f(T) = f(p_1)f(p_2)f(p_3)$ is a (λ, c) quasi-geodesic triangle in X, and hence $f(T)$ is δ'–slim for some δ' depending only on δ, λ, and c by Exercise 3.15.

Now let x be a point on p_1 and let $x' = f(x) \in f(p_1)$. Then there exists some $y' \in f(p_2) \cup f(p_3)$ such that $d_X(x', y') \leq \delta'$. Without loss of generality, suppose $y' \in f(p_2)$ and $y' = f(p_2(s))$ for some $a_2 \leq s \leq b_2$. Let $y = p_2(s)$. Then $d_Y(x, y) \leq \lambda\delta' + \lambda c$, hence T is $(\lambda\delta' + \lambda c)$–slim.

□

Corollary 3.17. *Suppose X and Y are geodesic metric spaces and $X \sim_{qi} Y$. Then X is hyperbolic if and only if Y is hyperbolic.*

Definition 3.18. A finitely generated group G is *hyperbolic* if for some (equivalently, any) finite generating set S, $\Gamma(G,S)$ is a hyperbolic metric space.

Remark 3.19. By the Milnor-Svarc Lemma, a group G is hyperbolic if and only if G admits a proper, cobounded action on a geodesic hyperbolic metric space.

Examples 3.20.

1. Finite groups are hyperbolic.
2. $\mathbb{Z}$ is hyperbolic. More generally, any group which is virtually $\mathbb{Z}$, such as $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.
3. F_n is hyperbolic for any $n \geq 1$.
4. $SL(2,\mathbb{Z})$ is virtually free and hence hyperbolic.
5. If M is a closed hyperbolic manifold, then $\pi_1(M)$ is hyperbolic. In particular, if S is an orientable surface of genus g, then $\pi_1(S)$ is hyperbolic if and only if $g \geq 2$.
6. $\mathbb{Z}^n$ is hyperbolic if and only if $n = 1$.

Definition 3.21. A group is called *elementary* if it contains a cyclic subgroup of finite index

There are three types of elementary groups: every elementary group is either finite, finite-by-$\mathbb{Z}$, or finite-by-D_∞, where D_∞ is the infinite dihedral group. All elementary groups are hyperbolic, but there will be a number of results which only hold for the non-elementary hyperbolic groups.

Lemma 3.22. *Suppose p is a k-local geodesic in X from x to y for $k > 8\delta$. Then*

1. $p \subseteq [x,y]^{+2\delta}$.
2. $[x,y] \subseteq p^{+3\delta}$.
3. *p is a (λ, c)-quasi-geodsic path for $\lambda = \frac{k+4\delta}{k-4\delta}$ and $c = 2\delta$.*

Proof. (1) Choose a point $a \in p$ which maximizes the distance to $[x,y]$. Choose $b, c \in p$ such that a is the midpoint of the subpath of p from b to c and $8\delta < d(b,c) \leq k$. (if such points do not exist, we use the endpoints of p instead and an obvious modification of the following arguement will work). Choose b', c' as the points on $[x,y]$ closest to b and c respectively, and consider the quadrilateral (b', b, c, c'). a must be 2δ from one of the other sides of this quadrilateral by hyperbolicity. If a is within 2δ of a point on $[b', b]$ or $[c, c']$, it would contridict our choice of a as the point which maximizes the distance to $[x,y]$. Hence a is within 2δ of a point on $[b', c'] \subseteq [x,y]$.

(2) Now let $a \in [x,y]$. Since p is connected, there exists some $b \in p$ such that $d(b, [x,a]) \leq 2\delta$ and $d(b, [a,y]) \leq 2\delta$. Applying hyperbolicity to the triangle spanned by b and the two points which realize these inequalities produces the desired result.

(3) We subdivide p into subpaths $p = p_1 p_2 ... p_{n+1}$ such that $\ell(p_i) = k' = \frac{k}{2} + 2\delta$ for $1 \leq i \leq n$ and $0 \leq \ell(p_{n+1}) = \eta < k'$. Note that

$$\ell(p) = nk' + \eta$$

Now let $a_i = (p_i)_-$, and let a'_i be a point on $[x, y]$ with $d(a_i, a'_i) \leq 2\delta$. We first need to show that each a'_i is "between" a'_{i-1} and a'_{i+1} on $[x, y]$, which will imply that $x = a'_1, a'_2 ... a'_{n+1}, y$ forms a monotone sequence along $[x, y]$.

Let $x_0 \in p_{i-1}$ with $d(a_{i-1}, x_0) = 2\delta$ and $y_0 \in p_i$ with $d(a_{i+1}, y) = 2\delta$. Note that $d(x_0, y_0) = 2k' - 4\delta = k$, hence a geodesic $[x_0, y_0]$ can be chosen as a subpath of p. Consider the triangle T with endpoints a_{i-1}, a'_{i-1}, and x_0. By hyperbolicity, $T \subseteq B_{3\delta}(a_{i-1})$. Since $d(a_{i-1}, a_i) = k' > 6\delta$, T does not intersect $B_{3\delta}(a_i)$. Similarly, a triangle with endpoints a_{i+1}, a'_{i+1}, and y_0 will not intersect $B_{3\delta}(a_i)$. Now we apply hyperbolicity to the quadrilateral with vertices $a'_{i-1}, x_0, y_0, a'_{i+1}$ and the point a_i, we get a point $a''_i \in [a'_{i-1}, a'_{i+1}]$ with $d(a_i, a''_i) \leq 2\delta$. By hyperbolicity of the triangle (a_i, a'_i, a''_i), $d(a_i, z) \leq 3\delta$ for any point z which is between a'_i and a''_i. In particular, neither a'_{i-1} nor a'_{i+1} are between a'_i and a''_i, and since $a''_i \in [a'_{i-1}, a'_{i+1}]$, we must also have $a'_i \in [a'_{i-1}, a'_{i+1}]$.

Since $x = a'_1, a'_2 ... a'_{n+1}, y$ forms a monotone sequence along $[x, y]$, we get that

$$d(x, y) = \sum_{i=1}^{n} d(a'_i, a'_{i+1}) + d(a'_{n+1}, y)$$

Now for each $1 \leq i \leq n$, $d(a'_i, a'_{i+1}) \geq k' - 4\delta$, and $d(a'_{n+1}, y) \geq \eta - 2\delta$. Hence,

$$d(x, y) \geq nk' - 4\delta n + \eta - 2\delta = \ell(p) - 4\delta n - 2\delta$$

Finally, since $n \leq \frac{\ell(p)}{k'}$,

$$d(x, y) \geq (\frac{k' - 4\delta}{k'})\ell(p) - 2\delta.$$

Finally, it only remains to note that every subpath of p is again a k-local geodesic to which the above proof applies. □

Corollary 3.23. *Suppose p is a k-local geodesic in X for $k > 8\delta$. Then either p is constant or $p_- \neq p_+$.*

3.2 Algorithmic and isoperimetric characterizations of hyperbolic groups

Given a group presentation $\langle S|R \rangle$ and a word W in S, *Dehn's algorithm* is the following procedure: First freely reduce W; if this produces the empty word, the algorithm stops. Now if W is freely reduced and non-empty, search W for subwords U such that U is also a subword of relation (or a cyclic shift of a relation) $r \in R$ and $\|U\| > \frac{1}{2}\|r\|$. If no such subword exists, the algorithm stops. If such a U exists, then there is a (possibly empty) word V (the complement of U in r) such that $UV^{-1} =_G 1$ and $\|V\| < \|U\|$. In this case, the algorithm replaces U with V and repeats.

If the presentation $\langle S|R \rangle$ is finite, then Dehn's algorithm terminates after finitely many steps for any word W.

Definition 3.24. Let $\langle S|R \rangle$ be a finite presentation for a group G. Dehn's algorithm solves the word problem for $\langle S|R \rangle$ if for any non-empty word W for which Dehn's algorithm stops, $W \neq_G 1$.

Exercise 3.25. Find a group presentation for which Dehn's algorithm does not solve the word problem.

Exercise 3.26. Suppose $\langle S|R\rangle$ is a finite presentation for a group G for which Dehn's algorithm solves the word problem. Prove that G has linear Dehn function.

Theorem 3.27. *For any finitely generated group G, the following are equivalent.*

1. *G is hyperbolic.*
2. *G has a finite presentation $\langle S|R\rangle$ for which Dehn's algorithm solves the word problem.*
3. *G is finitely presented and has linear Dehn function.*
4. *G is finitely presented and has subquadratic Dehn function.*

Exercise 3.28. Suppose G and H are hyperbolic. Prove that $G * H$ is hyperbolic.

Proof. (1) $\implies$ (2)

Let S be a finite, symmetric generating set of G, and let δ be the hyperbolicity constant of $\Gamma(G,S)$. Let $R = \{U \mid U \text{ is a word in } S\ , \|U\| \leq 16\delta, U =_G 1\}$. We will show that $\langle S|R\rangle$ is a presentation for G for which Dehn's algorithm solves the word problem.

Let W be a non-empty word in S such that $W =_G 1$. Let p be the path in $\Gamma(G,S)$ with $p_- = 1$ and $\mathbf{Lab}(p) \equiv W$. By assumption, p is not constant and $p_- = p_+$, so by Corollary 3.23, p is not a k-local geodesic for any $k > 8\delta$. This means that p contains a subpath q with $\ell(q) \leq 8\delta$ such that q is not a geodsic. Let r be a geodesic from q_- to q_+. Then $\ell(r) < \ell(q) \leq 8\delta$, so qr^{-1} is a closed loop with $\ell(qr^{-1}) \leq 16\delta$. This means that $\mathbf{Lab}(qr^{-1}) \in R$, and since $\ell(r) < \ell(q)$, Dehn's algorithm will not stop on W.

Thus we have shown that every word in S which is equal to 1 in G by be reduced to the empty word via Dehn's algorithm using only relations from R. Therefore, $\langle S|R\rangle$ is a finite presentation for G and Dehn's algorithm solves the word problem for $\langle S|R\rangle$.

(2) $\implies$ (3) by Exercise 3.26.

(3) $\implies$ (4) is trivial.

□

We will need a few auxillary results before proving the final implication. First, however, I would like to highlight the following consquence of the above theorem, which is a purely algebraic consequence of the geometric assumption of hyperbolicity.

Corollary 3.29. *If G is a hyperbolic group, then G is finitely presented.*

In fact, this corollary is a special case of a more general finiteness phenomenon for hyperbolic groups which we will see later when we introduce the *Rips complex.*

Now, we return to the proof of Theorem 3.27.

Given a polygon $P = p_1p_2...p_n$ in X, we say P is *t-slim* if for any point $a \in p_i$, there exists $j \neq i$ and a point $b \in p_j$ such that $d(a,b) \leq t$. We define the *thickness* of P, denoted $t(P)$, as the minimal constant t such that P is t-slim. Clearly, if X is non-hyperbolic it will have triangles of arbitrarily large thickness. We will show that in this case there are polygons or arbitrarily large thickness t whose perimeter length is linear in t. Next we will show that the area of a polygon P is bounded below by a quadratic function of the thickness of P. These results together will finish the proof of Theorem 3.27.

For the next two lemmas I am following the proofs from [28].

Lemma 3.30 (Thick polygons with linear perimeter)**.** *Suppose a geodesic metric space X is not hyperbolic. Then for all $t_0 \geq 0$, there exists $t \geq t_0$ such that X contains a polygon of thickness t whose perimeter length is at most $46t$.*

Proof. Let $T = pqr$ be a geodesic triangle in X with $x = p_- = q_-$, $y = q_+ = r_-$, and $z = p_+ = r_+$. Let $a \in p$ a point such that $d(a, q) = d(a, r) = t \geq t_0$, and $b \in q, c \in r$ such that $d(a, b) = t = d(a, c)$. Let $e \in [b, y]$ such that $d(d, e) = 7t$ (or $e = y$ if no such point exists). Let f be the point of $[y, c]$ which is closest to e.

Case 1: $d(e, f) \geq 4t$. In this case we analyze the triangle with vertices b, c, and y. Choose a point $o \in [b, y]$ which maximizes $d(o, [y, c])$. Note that by our assumption, $d(o, [y, c]) \geq 4t \geq \frac{1}{2}d(b, c)$ (hence this is an example of a *wide* triangle). Let $D = d(o, [y, c])$, and let $g \in [o, y]$ with $d(o, g) = \frac{3d}{2}$ and $i \in [o, b]$ with $d(o, i) = \frac{3D}{2}$ (as usual, choose g and i to be the endpoints if needed). By definition of o, there exist $h, j \in [y, c]$ with $d(g, h) \leq D$ and $d(i, j) \leq D$. In case $i = b$, we set $j = c$, and if $g = y$, then $h = g = y$.

Now we show the quadrilateral Q with vertices $[g, h, i, j]$ is $\frac{D}{2}$-thick. Indeed, $d(o, [h, j]) \leq d(o, [y, c]) = 2D$. Since $d(o, g) = \frac{3D}{2}$ and $d(g, h) \leq D$, $d(o, [g, h]) \geq \frac{D}{2}$. Similarly, if $i \neq b$, $d(o, [i, j]) \geq \frac{D}{2}$. For the case $i = b$, observe that $d(b, c) \leq \frac{D}{2}$, and since $d(o, c) \geq D$ we must have $d(o, [b, c]) \geq \frac{D}{2}$. Hence, $\frac{t}{2} \leq \frac{D}{2} \leq t(Q)$.

Finally, $d(g, h)$ and $d(i, j)$ are both bounded by D, $d(g, i) \leq 3D$, and hence the triangle inequality gives $d(h, j) \leq 5D$. Therefore the length of the perimeter of Q is at most $10D \leq 20t(Q)$.

Case 2: $d(e, f) \leq 4t$. First, we are going to show that for any $k \in [e, f]$, $d(a, k) \geq t$. First note that $d(f, c) \geq d(b, e) - d(b, c) - d(e, f) \geq 7t - 2t - 4t = t$.

Now, the following two inequalities can be extracted via applying the triangle inequality to the relevent sequences of points, which can be easily traced out if the right picture is drawn.

$$d(x, z) \leq d(x, b) + d(b, c) + d(c, z) \leq d(x, b) + d(c, z) + 2t. \tag{3}$$

$$d(x, e) + d(z, f) \leq d(x, a) + d(a, k) + d(k, e) + d(z, a) + d(a, k) + d(k, f)$$

substituting $d(b, e) + d(c, f) \geq 7t + t$, $d(e, f) \leq 4t$, and $d(x, z) = d(x, a) + d(a, z)$ into the above equation gives

$$d(x, b) + d(z, c) + 8t \leq d(x, z) + 2d(a, k) + 4t$$

Summing this with 3 produces $d(a, k) \geq t$, as desired.

We now continue constructing the desired polygon. Let $g \in [x, a]$ and $i \in [a, z]$ with $d(g, a) = d(a, i) = 3t$ (as usual, we may need to choose the end points, and the proof is easily modified to work in this case). Furthermore, we can assume that there are points $h \in [x, b]$ and $j \in [z, c]$ such that $d(g, h) \leq 2t$ and $d(i, j) \leq 2t$. If these points do not exists, then we will get a wide triangle, and we can then procede as in Case 1.

Now the inequalites $d(a, g) = 3t$ and $d(g, h) \leq 2t$ implie that $d(a, [g, h]) \geq t$. Similarly, $d(a, [i, j]) \geq t$. It follows that the hexigon H with vertices $[g, h, e, f, j, i]$ is at least t-thick, since the distance from a to any other side of H is at least t.

It only remains to esitmate the perimeter of H. I will leave this as an exercise, but using known lengths and estimating the rest with the triangle inequality will produce a bound on the perimeter of $46t$. □

Exercise 3.31. Show the hexegon H constructed in the above proof as perimeter $\leq 46t$.

The following lemma can be proved in the context of general geodesic metric spaces. However, we will restrict our attention to the case of Cayley graphs of finitely presented groups. This restriction is purely for convenience of notation, there are no essential differences in the following proofs for general geodesic metric space once a suitable notion of area is defined.

Given a polygon $P = p_1...p_n$ in a Cayley graph $\Gamma(G, S)$, let $\mathbf{Lab}(P) \equiv \mathbf{Lab}(p_1)...\mathbf{Lab}(p_n)$. Also, we slightly modify our notion of thickness for such polygons by only measuring distance between points which are vertices of the Cayley graph. This change decreases the thickness of a polygon by at most 1, so it clearly does not affect our previous result.

Lemma 3.32 (Thick polygons have quadratic area)**.** *Let G be a group given by a finite presentation $\langle S|R\rangle$, and let $M = \max_{r\in R}\{\|r\|\}$. Let P be a polygon in $\Gamma(G, S)$ of thickness t with $W \equiv \mathbf{Lab}(P)$. Then Area$(W) \geq \frac{4}{M^3}t^2$.*

Proof. By definition of thickness, there exists some side p of P and a vertex $a \in p$ such that $d(a, P \setminus p) \geq t$. Let q be the remaining sides of P, so $P = pq$.

We now fill the closed loop pq with a van Kampen diagram Δ. We now set $x_0 = y_0 = a$ and inductively define a sequence of simple closed paths, $z_i = x_i y_i$ for $0 \leq i \leq \frac{2t}{M} + 1$ which satisfy the following properties:

1. x_i is a subpath of p containing x_{i-1}.
2. For every vertex $b \in y_i$, $d(a, b) \leq \frac{Mi}{2}$.
3. The subdiagram Δ_i bound by $z_i = x_i y_i$ contains the maximal area over all simple closed paths which satisfy the first two properties.

Increasing x_i is necessary, we can assume that each y_i has no edges in common with p. Furthermore, if $i \leq \frac{2t}{M}$ then y_i does not intersect q, since if $b \in y_i$, $d(a, b) \leq \frac{Mi}{2} < t$. Suppose b is a vertex of both y_{i-1} and y_i. Then b is a vertex of $\partial\Delta_i$, and since b does not belong to the boundary of Δ, there must exist some 2-cell Π such that $b \in \partial\Pi$ but Π does not belong to Δ_i. But since $b \in y_{i-1}$, $d(a, b) \leq \frac{M(i-1)}{2}$, and the definition of M gives that for any vertex $c \in \partial\Pi$, $d(b, c) \leq \frac{M}{2}$. Hence Δ_i could be enlarged by adding Π without violating the first two conditions, which contradicts the third condition of the definiton of Δ_i. Thus the vertices of y_i and y_{i-1} are disjoint.

It follows that every edge of y_i belongs to the boundary of a 2-cell which is contained in Δ_i but not in Δ_{i-1}. Let m_i be the number of such faces , and note that m_i is at least $\frac{\ell(y_i)}{M}$. Since y_i and y_{i-1} are disjoint, x_i must contain at least 2 more vertices then x_{i-1}, one on each end. Thus,

$\ell(x_i) \geq 2i$, and since each x_i is a subpath of a geodesic, $d((x_i)_-, (x_i)_+) \geq 2i$. y_i has the same endpoints as x_i, so $\ell(y_i) \geq 2i$, which implies that $m_i \geq \frac{2i}{M}$. Finally, we get

$$Area(\Delta) \geq \sum_{i=1}^{\frac{2t}{M}+1} m_i \geq \sum_{i=1}^{\frac{2t}{M}+1} \frac{2i}{M} \geq \frac{4t^2}{M^3}.$$

$\square$

Combining the previous two lemmas gives the following corollary, which finishes the proof of Theorem 3.27 (in particular, it shows that (4) $\implies$ (1) part the proof of Theorem 3.27).

Corollary 3.33. *Let G be a finitely presented group which is not hyperbolic. Then the Dehn function of G is at least quadratic.*

Exercise 3.34. Suppose G has a presentation $\langle S|R\rangle$ with sublinear Dehn function. Prove that $R = \emptyset$, so in fact G is the free group on S.

3.3 More properties of hyperbolic groups

In addition to having solvable word problem, the other two classical algorithmic questions of Dehn are solvable for hyperbolic groups.

Theorem 3.35. *If G is a hyperbolic group, then the conjugacy problem is solvable in G.*

Theorem 3.36 (Sela,...)**.** *The isomorphism problem is solvable for presentations of hyperbolic groups.*

Hyperbolicity also implies a number of algebraic properties of the group. In particular, there are strong restrictions on subgroups of hyperbolic groups:

Theorem 3.37 (Strong Tits alternative)**.** *Let G be a hyperbolic group and let $H \leq G$. Then either H is virtually cyclic or H contains a subgroup isomorphic to F_2.*

It is also important to note that subgroups of hyperbolic groups, even finitely presented subgroups, are not necessarily hyperbolic.

The above dichotomy implies, for example, that G contains no subgroups isomorphic to $\mathbb{Z}^2$. It also implies that G cannot be decomposed as a direct product of two infinite subgroups.

Theorem 3.38. *If G is hyperbolic and $G = A \times B$, then either A is finite or B is finite.*

Morever, all infinite normal subgroups of hyperbolic groups must fall into the second case in this dichotomoy.

Theorem 3.39. *If G is hyperbolic and non-elementary then every infinite normal subgroup of G contains a subgroup isomorphic to F_2. In particular, $|Z(G)| < \infty$.*

Furthermore non-elementary hyperbolic groups have lots of these normal subgroups and hence are very far from being simple. A group G is called *SQ-universal* if every countable group can be embedded in some subgroup of G. It is a classical theorem of HNN and F_2 is SQ-universal. Olshanskii extended this to all non-elementary hyperbolic groups.

Theorem 3.40. *[29] Every non-elementary hyperbolic group is SQ-universal.*

Since there are uncountably many finitely generated groups and every finitely generated group has only countably many finitely generated subgroups, this theorem implies the following.

Corollary 3.41. *Every non-elementary hyperbolic group has uncountably many normal subgroups.*

Theorem 3.42. *If G is a hyperbolic group, then G contains only finitely many conjugacy classes of elements of finite order. More generally, G contains only finitely many conjugacy classes of finitie subgroups.*

Finally, hyperbolicity also has implications for the homology of a group. This is based on the following result of Rips.

Proposition 3.43. *Let G be generated by a finite set S such that $\Gamma(G,S)$ is δ-hyperbolic. Then for all $d \geq 4\delta + 1$, the Rips complex $P_d(X)$ is contractible. In particular, if G is torsion-free, then the quotient $P_d(X)/G$ is a finite $K(G,1)$.*

Corollary 3.44. *If G is a torsion-free hyperbolic group, then there exists $N \in \mathbb{N}$ such that $H_n(G) = \{0\}$ for all $n > N$.*

If G is a hyperbolic group with torsion, then the action on the Rips complex is still sufficient to prove the following, see [15, Remark 7.3.2].

Corollary 3.45. *If G is any hyperbolic group, then $H_n(G)$ is finitely generated for all $n \geq 1$.*

4 Growth in groups

4.1 Basic properties and examples

Definition 4.1. Let G be a group generated by a finite set S. Then the *growth function of G*, $\gamma_G \colon \mathbb{N} \to \mathbb{N}$, is defined by

$$\gamma_G(n) = |\{g \in G | |g|_S \leq n\}$$

Exercise 4.2. Let G be a group generated by a finite set S. Let $x, y \in G$. Show that for all $n \geq 0$, $|B_n(x)| = |B_n(y)|$ where these balls are taken in the metric space (G, d_S).

We consider these functions up to the following equivalence relation: Given $f, g \colon \mathbb{N} \to \mathbb{N}$, we say that$f \preceq g$ if there exist constants A and B both ≥ 1, such that for all $n \in \mathbb{N}$,

$$f(n) \leq Ag(Bn).$$

We say that $f \sim g$ if $f \preceq g$ and $g \preceq f$.

Exercise 4.3.

1. Check that this is an equivalence relation.
2. Check that the equivalence class of a polynomial only depends on its degree.
3. Check that for any $a, b > 1$, $a^n \sim b^n$.

Up to this equivalence it is straightforward to check that γ_G is independent of the choice of finite generating set of G. Morevoer,

Lemma 4.4. *If $G \sim_{qi} H$, then $\gamma_G \sim \gamma_H$.*

Proof. Let S and T be finite generating sets for G and H respectively and let $f\colon (G, d_S) \to (H, d_T)$ be a (λ, c)–quasi-isometric embedding. Fix $x \in G$ and $y = f(x) \in H$.

First observe that for any $p, q \in G$, if $f(p) = f(q)$ then $d(p, q) \leq \lambda c$. Hence f is at most m–to-one, where $m = |\{g \in G||g|_S \leq \lambda c\}$.

Now for any point $p \in B_n(x)$, $d(f(p), y) \leq \lambda n + c$. Hence $f(B_n(x)) \subseteq B_{\lambda n+c}(y)$. Since f is at most m–to-one, we get

$$|B_n(x)| \leq m|B_{\lambda n+c}(y)|$$

And hence $\gamma_G(n) \leq m\gamma_H(\lambda n+c) \leq m\gamma_H((\lambda+c)n)$. By symmetry of the quasi-isometry relation the result follows.

□

Corollary 4.5. *If G and H are finitely generated groups and G is quasi-isometrically embedded in H, then $\gamma_G \preceq \gamma_H$.*

Note that the embedding in this corollary is geometric and not necessarily algebraic; that is, (G, d_S) may be quasi-isometrically embedded in H as a metric space but G may not be isomorphic to a subgroup of H.

Remark 4.6. If M is a Riemannian manifold, then $\gamma_{\pi_1(M)}(n) \sim Vol_{\widetilde{M}}(n)$, where $Vol_{\widetilde{M}}(n)$ is the volume of a ball of raduis n in $\widetilde{M}$.

If $\gamma_G(n) \sim 2^n$, we say that G has *exponential growth*; otherwise, G has *sub-exponenital growth* (see Lemma 4.12). If $\gamma_G(n) \sim n^k$ for some $k \in \mathbb{N}$, we say G has *polynomial growth.* If $n^k \preceq \gamma_G(n)$ for all $k \in \mathbb{N}$, we say that G has *super-polynomial growth*

Exercise 4.7. Compute the growth functions for F_n and $\mathbb{Z}^n$.

Corollary 4.8. *$\mathbb{Z}^n \sim_{qi} \mathbb{Z}^m$ if and only if $n = m$.*

Recall that in the case of (non-abelian) free groups, $F_n \sim_{qi} F_m$ for all $n, m \geq 2$.

Exercise 4.9. The (integral) *Heisenberg group* is the group of matricies of the form

$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\},$$

or equivalently the group given by the presentation $\langle a, b, c \mid ac = ca, bc = cb, c = [a, b]\rangle$. Show that the Heisenberg group has growth equivalent to n^4.

Exercise 4.10. Show that if H is finitely generated and $H \leq G$, then $\gamma_H \preceq \gamma_G$.

Exercise 4.11. Show that if H is a quotient of G, then $\gamma_H \preceq \gamma_G$.

The above exercises imply the following.

Lemma 4.12. *For any finitely generated group G, $\gamma_G(n) \preceq 2^n$.*

Example 4.13. Suppose G is a hyperbolic group. If G is elementary, then $G \sim_{qi} \mathbb{Z}$ and hence $\gamma_G(n) \sim \gamma_{\mathbb{Z}}(n) \sim 2n+1$. If G is non-elementary, then by the Tit's alternative for hyperbolic groups G contains a subgroup isomorphic to F_2, so by the above exercises $\gamma_G(n) \sim 2^n$.

4.2 Nilpotent groups

[3]

We begin by recalling the definition and some basic properties of nilpotent groups. Given a group G and $g, h \in G$, the *commutator* of g and h is $[g, h] := g^1 h^{-1} gh$. Given subgroups H_1 and H_2 of G, let $[H_1, H_2]$ be the subgroup generated by $\{[g, h] \mid g \in H_1, h \in H_2\}$.

Definition 4.14. Given a group G, let $\gamma_1(G) = G$ and let

$$\gamma_i(G) = [\gamma_{i-1}(G), G]$$

This produces a sequence of normal subgroups $G \trianglerighteq \gamma_1(G) \trianglerighteq \ldots$ called the *lower central series.*

Definition 4.15. A group G is *nilpotent* if $\gamma_{k+1}(G) = \{1\}$ for some $k \geq 1$. The nilpotenecy class of G is the minimal such k.

Note that every nilpotent group is solvable and a group is abelian if and only if it is nilpotent of class 1.

It is straigtforward to check that the identities $[xy, z] = [x, [y, z]][y, z][x, z]$ and $[x, yz] = [x, y][y, [x, z]][xz]$[4] hold in any group (or perhaps some slight variation of these).

Suppose G is generated by a finite set S. Applying this identity inductively gives that for any $g, h \in G$, $[g, h]$ is equal to a product of commutators of elements of S and an element of $\gamma_2(G)$. A similar argument shows that any element of $\gamma_2(G)$ is equal to a product of commutators of the form $[[s_1, s_2], s_3]$ with each $s_i \in S$ and an elemement of $\gamma_3(G)$. Continuing inductively, we get a similar statement for each $\gamma_i(G)$. Now if G is nilpotent of class k, then $\gamma_{k+1} = \{1\}$ and hence the above argument shows that $\gamma_k(G)$ will be generated by a finite set of k-fold commutators of elements of S. Hence $\gamma_{k-1}(G)$ will also be finitely generated, continuing in this way we get the following.

Lemma 4.16. *If G is nilpotent and finitley generated, then each $\gamma_i(G)$ is finitely generated.*

It is relatively straightforward to prove that a nilpotent group has at most polynomial growth. Later we will compute the precise degree of growth of a nilpotent group, but this is a little more complicated, and we will only give a sketch of the proof.

[3]The primary reference I am using for this section is [9]

[4]these are wrong, but something similar works

Proposition 4.17. *Let G be a finitely generated nilpotent group. Then for some $d \in \mathbb{N}$, $\gamma_G(n) \preceq n^d$.*

Proof. We proceed by induction on the nilpotency degree of G. Suppose G is nilpotent of class k and the proposition holds for all groups of nilpotency class at most $k-1$. Let G be generated by a finite set $S = \{s_1, ..., s_m\}$. Let T be a finite generating set for $[G, G]$ which contains all i–fold commutators of elements of S for all $1 \leq i \leq k$. Let $g \in G$ such that $|g|_S \leq n$, and let W be a word in S such that $W =_G g$. Our goal will be show that

$$g = s_1^{k_1} ... s_m^{k_m} b$$

Where each k_i is equal to the number of times s_i occurs in the word W minus the number of occurences of s_i^{-1} and $b \in [G, G]$ with $|b|_T \leq n^k$. Since $[G, G]$ is nilpotent of class $k-1$ we get a polynomial bound on the number of such b by induction and hence a polynomial bound on the number of g with $|g|_s \leq n$.

We start by commuting each occurence of s_1 in the word W to the front. When we commutate s_1 by an element s_i, we add a commutator $s_i s_1 = s_1 s_i [s_i, s_1]$. When we commute a occurence of s_1 with one of these commutators we add an element of $\gamma_2(G)$, and so on. Eventually, we get a word of the form $s_1^{k_1} U$ where U is a word in $S \setminus \{s_1\}$ and elements belonging to the terms of the lower central series. Then we repreat this for all other letters in the original word until we get the desired normal form.

Note that we have to commute at most n^2 generators with generators. But then we commute at most n^3 generators with commutators. In the end we get that b is a product of at most $n^2 + ... + n^k$ elements, and each of these elements is at worst a k-fold commutator of elements of S. Hence $|b|_T \leq n^k$. If $[G, G]$ has growth bounded above by n^l, then $\gamma_G(n) \preceq n^{m+kl}$ □

Note that from the definition of $\gamma_{i+1}(G)$, $[\gamma_i(G), \gamma_i(G)] \leq \gamma_{i+1}(G)$, hence $\gamma_i(G)/\gamma_{i+1}(G)$ is an abelian group. If G is finitely gnerated and nilpotent of class k, then each of these quotients is a finitely generated abelian group and hence isomorphic to $\mathbb{Z}^{m_i} \times A_i$ where each $m_i \geq 0$ and each A_i is a finite abelian group. Define the *homegeneous dimension* of G by

$$d(G) = \sum_{i=1}^{k} i m_i$$

Remark 4.18. For nilpotent groups, the numbers m_i above are quasi-isometry invariants.

Theorem 4.19. *If G is nilpotent, then $\gamma_G(n) \sim n^d$ where $d = d(G)$.*

The proof again proceeds by induction on the nilpotency degree of G. If G is nilpotent of class k, then let $K = \gamma_k(G)$. The G/K is nilpotent of class $k-1$ and $d(K) = d(G) - km_k$, where m_k is defined as above. Hence the growth of G/K is $\sim n^{d-km_k}$ by induction. Let S be a finite generating set for G and consider the quotient G/K with the image of S as its finite generating set. Then the quotient map $\varphi\colon G \to G/K$ surjects $B_n^G(1)$ onto $B_n^{G/K}(1)$. If $B_n^{G/K}(1) = \{q_1, ..., q_N\}$, for each q_i let $g_i \in \varphi^{-1}(q_i)$. Note that by assumption $N \sim n^{d-km_k}$.

For the lower bound, note that every element g of $B_n^G(1)$ is equal to an element of the form $g_i k$ for some $1 \leq i \leq N$ and some $k \in K$. Since $k = g_i^{-1} g$, $|k|_S \leq 2n$. Hence $|B_n^G(1)| \geq N \cdot |B_{2n}^G(1) \cap K|$

For the upper bound, $B^G_{2n}(g)$ contains all of the elements of the for g_ik where $1 \le i \le k$ and $|k|_S \le n$, hence $|B^G_{2n}(1)| \le N \cdot |B^G_n(1) \cap K|$.

For both cases, we need to be able to count the number of elements in $B^G_n(1) \cap K$. Now K is abelian and hence has growth n^{m_k}, however in order to count $B^G_n(1) \cap K$ we need to understand the *distortion* of K in G.

Definition 4.20. Let G be generated by a finite set S and let H be a subgroup of G generated by a fintie set T. Then the distortion of H in G is

$$\Delta^H_G(n) = \max\{|h|_T \mid |h|_S \le n\}$$

It is straightforward to show that up to the equivalence relation defined on growth functions that the distortion of H in G is independent of the choices of finite generating sets. Hence one can assume that $T \subseteq S$, and it follows easily that $n \preceq \Delta^H_G(n)$, for any finitely generated groups G and $H \le G$. In case $\Delta^H_G(n) \sim n$, we say that H is *undistorted* in G. This is equivalent to the inclusion map $(H, d_T) \to (G, d_S)$ being a quasi-isometric embedding.

Examples 4.21.

1. If $G = BS(1,2) = \langle a, t \mid t^{-1}at = a^2\rangle$ and $H = \langle a \rangle$, then $\Delta^H_G(n) \sim 2^n$.
2. If G is the Heisenberg group and $H = \langle c \rangle$, then $\Delta^H_G(n) \sim n^2$.
3. A fundamental property of hyperbolic groups is that every cyclic subgroup is undistorted.

We refer to [9] for a proof of the following proposition.

Proposition 4.22. *Let G be a nilpotent group of class k let $K = \gamma_k(G)$. Then $\Delta^K_G \sim n^k$.*

This proposition together with the above argument complete the proof of Theorem 4.19.

4.3 Solvable and Linear groups

Definition 4.23. A group G is called *solvable* if there exists a sequence of normal subgroup of G

$$G = G_0 \trianglerighteq G_1 \trianglerighteq ... \trianglerighteq G_k = \{1\}$$

Such that each quotient G_i/G_{i+1} is abelian. If such a sequence exists with each quotient G_i/G_{i+1} cyclic, then G is called *polycyclic.*

If G is nipotent, then since each term in the lower central series is finitely generated this sequence can be refined to a sequence with cyclic quotients, hence all nilpotent groups are polycyclic. Clearly all polycyclic groups are solvable.

Theorem 4.24 (Milnor-Wolf)**.** *If G is solvable, then either G has exponential growth or G is virtually nilpotent.*

Definition 4.25. A group G is called *linear* if for some field k and some $n \ge 1$, G is isomorphic to a subgroup of $GL(n, k)$.

Theorem 4.26 (Tit's altenerative)**.** *If G is finitely generated and linear, then either G is virtually solvable or G contains a subgroups isomorphic to F_2.*

Theorem 4.27. *If G is either solvable or linear, then either G has exponential growth or G is virtually nilpotent and hence has polynomial growth.*

In light of the above results, Milnor asked whether or not there exists groups whose growth is both super-polynomial and sub-exponential. These groups are called *groups of intermediate growth*, there existence was proven by Grigorchuk.

Theorem 4.28. *Groups of intermediate growth exist.*

4.4 Gromov's polynomial growth theorem-overview

Theorem 4.29. *Suppose G is a finitely generated group with $\gamma_G \preceq n^d$ for some $d \in \mathbb{N}$. Then G is virtually nilpotent.*

Since we will be frequently passing to subgroups of finite index throughout the proof, we first observe the following basic properties of this process.

Exercise 4.30. Let G be a finitely generated group.

1. If H_1 and H_2 are finite index subgroups of G, then $H_1 \cap H_2$ is a finite index subgroup of G.
2. If H_1 is a finite index subgroup of G and H_2 is a finite index subgroup of H_1, then H_2 is a finite index subgroup of G.
3. For any $i \geq 1$, G has only finitely many subgroup of index i.
4. If $\varphi\colon G \to H$ is a homomorphism and H_1 is a finite index subgroup of H, then $\varphi^{-1}(H)$ is a finite index subgroup of G.

The main step in proving this theorem is the following:

Theorem 4.31. *Suppose G is a finitely generated group with $\gamma_G \preceq n^d$ for some $d \in \mathbb{N}$. Then there exists a non-trivial homomorphism $\varphi\colon G_1 \to \mathbb{Z}$. Where G_1 is a fintie index subgroup of G.*

Note that passing to a subgroup of finite index is necessary in this theorem. The infinite dihedral group D_∞ is generated by two elements of order two, hence there is no non-trival map $D_\infty \to \mathbb{Z}$. However D_∞ is itself virtually $\mathbb{Z}$ and hence has linear growth. Note also that D_∞ is solvable of class 2.

Exercise 4.32. Prove that for a finitely generated group G, there exists a non-trivial homomorphism $\varphi\colon G \to \mathbb{Z}$ if and only if the abelianization $G/[G,G]$ is infinite.

Exercise 4.33. Let G be virtually solvable. Prove that there exists a non-trivial homomorphism $\varphi\colon G_1 \to \mathbb{Z}$ where G_1 is a fintie index subgroup of G

Gromov's theorem follows easily from Theorem 4.31 together with the following lemma.

Lemma 4.34. *Let G be finitely generated and let $\varphi\colon G \to \mathbb{Z}$ be a non-trivial homomorphism with kernel K. Then*

1. *If $\gamma_G(n) \preceq n^d$ then K is finitely generated and $\gamma_K(n) \preceq n^{d-1}$.*

2. *If K is virtually solvable then G is virtually solvable.*

Proof. (1). Let $s \in G$ such that $\varphi(s) = 1$, and let $t_1, ..., t_m \in K$ such that $G = \langle s, t_1, ..., t_m \rangle$. Let $s_{j,i} = s^{-j} t_i s^j$. Since K is normal each $s_{j,i} \in K$. Morever, if W is a word in $S = \{s, t_1, ..., t_m\}$, then W represents an element of K if and only if the number of occurences of s minus the number of occurences of s^{-1} is zero. It is straightforward to show that any such word is equal to a product of the elements $s_{j,i}$ for $j \in \mathbb{Z}$ and $1 \leq i \leq m$. Hence this set of elemets generates K.

Now for a fixed $1 \leq i \leq m$ consider $A = \{s_{0,i}^{\varepsilon_0} s_{1,i}^{\varepsilon_1} ... s_{n,i}^{\varepsilon_n} \mid \varepsilon_j \in \{0,1\} \text{ for } 1 \leq j \leq n\}$. Since each $|s_{j,i}|_S \leq 2j+1$, each $g \in A$ satisfies $|g|_S \leq \sum_{j=0}^n (2j+1) \leq n^2$. There are $\preceq n^{2d}$ elements of G inside $B_{n^2}(1)$ but there are 2^{n+1} sequences $(\varepsilon_0, ..., \varepsilon_n)$ with each $\varepsilon_j \in \{0,1\}$. Hence for some sufficiently large n, where are two distinct such sequences $(\varepsilon_0, ..., \varepsilon_n)$ and $(\delta_0, ..., \delta_n)$ with $\varepsilon_n \neq \delta_n$ such that

$$s_{0,i}^{\varepsilon_0} s_{1,i}^{\varepsilon_1} ... s_{n,i}^{\varepsilon_n} = s_{0,i}^{\delta_0} s_{1,i}^{\delta_1} ... s_{n,i}^{\delta_n}.$$

Assuming $\varepsilon_n = 1$ and hence $\delta_n = 0$, we get

$$s_{n,i} = s_{n-1,i}^{-\varepsilon_{n-1}} s_{1,i}^{-\varepsilon_1} s_{0,i}^{-\varepsilon_0} s_{0,i}^{\delta_0} s_{1,i}^{\delta_1} ... s_{n-1,i}^{\delta_{n-1}}.$$

Hence $s_{n,i} \in \langle s_{0,i}, ..., s_{n-1,i} \rangle$. Conjugating the above equality by s gives that $s_{n+1,i} \in \langle s_{0,i}, ..., s_{n,i} \rangle = \langle s_{0,i}, ..., s_{n-1,i} \rangle$, so by induction $s_{j,i} \in \langle s_{0,i}, ..., s_{n-1,i} \rangle$ for all $j \geq n$. Repeating this argument for each $1 \leq i \leq m$ gives a finite generating set for K.

Hence we can assume that $T = \{t_1, ..., t_m\}$ is a finite generating set for K. Then $\{gs^i \mid g \in K, |g|_T \leq \frac{n}{2}, 1 \leq i \leq \frac{n}{2}\} \subseteq B_n^G(1)$, and this set contains $|B_{\frac{n}{2}}^K(1)| \cdot \frac{n}{2}$ elements. Hence $\gamma_K(\frac{n}{2}) \leq \frac{2}{n} \gamma_G(n) \preceq n^{d-1}$.

(2). Let K_1 be a finite index solvable subgroup of K. Let K_2 be the intersection of all conjugates of K_1 in K; since there are only finitley many of these K_2 will again be a finite index subgroup, but now K_2 is also normal in K. Let $G_1 = \langle K_2, s \rangle$. Since K_2 is normal in K, $K \cap G_1 = K_2$, that is K_2 is equal to the kernel of $\varphi|_{G_1}$. Since K_2 is solvable and $G_1/K_2 \cong \mathbb{Z}$, G_1 is solvable.

If $k_1, ..., k_n$ are a set of coset representatives for K_2 in K, then $K = \cup_{i=1}^n k_i K_2$. But $G = KG_1$ and $K_2 \subset G_1$, so $G = \cup_{i=1}^n k_i G_1$, hence G_1 has finite index in G.

□

Proof of Gromov's Theorem given Theorem 4.31. We induct on d. If $d = 0$ then G is finite and the theorem is trivial. Assume now that the theorem holds for all finitely generated groups of growth $\preceq n^{d-1}$ and G satisfies $\gamma_G(n) \preceq n^d$. Hence G has a non-trivial homomorphism to $\mathbb{Z}$ by Theorem 4.31, and by part 1 of the previous lemma the kernel K of this map has growth $\preceq d-1$. Hence the inductive hypothesis implies that K is virtually nilpotent, so by part 2 of the lemma G is virtually solvable. Hence the Milnor-Wolf Theorem implies that G is virtually nilpotent. □

In order to prove Theorem 4.31, Gromov constructs a space Y on which G acts by isometries. This space Y is called an *asymptotic cone*, we will study the construction in the next subsection.

Theorem 4.35. *If G is finitely generated and then there exists a metric space Y together with an action of G on Y by isometries such that Y is complete, geodesic, and homogeneous. If, in addition, $\gamma_G(n)$ is bounded by a polynomial, then Y is locally compact and finite-dimensional.*

Remark 4.36. Since Y is a complete, locally compact, geodesic metric space it is also *proper*, that is all closed balls in Y are compact, by the Hopf-Rinow Theorem.

Given such a space we can apply deep results of Gleason-Mongomery-Zippin which gave a solution to Hilberts 5'th problem.

Theorem 4.37. *Give Y as in Theorem 4.35, $Isom(Y)$ is a lie group with finitely many connected compenents.*

Gromov's proof of Theorem 4.31 is based on studying the action on G on Y and using properties of lie groups together with the Tit's alternative.

4.5 Asymptotic cones

Gromov's construction is the space Y is based on creating a notion of limits for metric spaces; That is Gromov described what it means for (certain types of) sequences of metric spaces (X_i, d_i) to converge to a limiting metric space Y. This convergence is now called *Gromov-Hausdorff convergence*. Van den Dries and Wilkie showed how some tools from non-standard analysis could be used to give another, similar type of limiting procedure on metric spaces which simplified Gromov's construction. This is the approach we will take.

Definition 4.38. An *ultrafilter* (on $\mathbb{N}$) is a finitely additive probability measure $\mu\colon 2^{\mathbb{N}} \to \{0,1\}$. That is, $\mu(\mathbb{N}) = 1$ and for any disjoint sets A and B, $\mu(A \cup B) = \mu(A) + \mu(B)$.

An ultrafilter is called *principal* if for some finite $A \subseteq N$, $\mu(A) = 1$. Equivalently, μ is principle if for some $x \in \mathbb{N}$, for any $A \subseteq \mathbb{N}$ $\mu(A) = 1$ if and only if $x \in A$.

A *non-principal ultrafilter* is an ultrafilter such that $\mu(A) = 0$ for all finite $A \subseteq \mathbb{N}$.

Throughout the rest of this section, we will fix a non-principal ultrafilter ω. Note that the existence of such an ultrafilter is equivalent to the axiom of choice. Given a property P which depends on a natural number i, we say P holds μ–almost surely if $\mu(\{i \mid P \text{ holds for } i\}) = 1$.

Definition 4.39. Let a_i be a sequence of real numbers. Then $\lim^{\mu} a_i = a$ if for all $\varepsilon > 0$, $\mu(\{i \mid |a_i - a|)\}) = 1$. In this case, a is called the μ–ultra-limit of the sequence (a_i). If for all $N \geq 0$, $\mu(\{i \mid a_i \geq N\}) = 1$ then we say $\lim^{\mu} a_i = \infty$. $\lim^{\mu} a_i = -\infty$ is defined similarly.

If the ordinary limit exists then it is equal to the μ–ultra-limit (since μ is non-principal). The advantage of using ultra-limits is that they always exist; that is, for any sequence (a_i) of real numbers $\lim^{\mu} a_i$ is a well-defined element of $\mathbb{R} \cup \{\pm\infty\}$. This follows easily from the same argument as the standard proof that every sequence on a closed interval has a convergent subsequence. Similarly, the standard argument that limits are unique also shows that ultra-limits are unique. In addition, It is easy to see that if $\lim^{\mu} a_i = a$ then some subsequence of a_i converges to a. From this is is easy to derive that all of the standard limit laws from calculus still apply to μ–ultra-limits (the limit of a sum is the sum of the limits, etc).

Given a sequence of sets X_i, the *ultra-product* of X_i with respect to μ is defined as the set of sequences (x_i) with $x_i \in X_i$ modulo the equivalence relation $(x_i) \sim (y_i)$ if $\mu(\{i \mid x_i = y_i\}) = 1$. We denote this ultra-product by $\Pi^\mu X_i$. When each X_i has some fixed (first-order) structure i.e. if each one is a group, ring, poset, graph, ect, then the ultra-product inherits that same structure. When each X_i is equal to a fixed X, this is called an *ultra-power* of X and denoted by X^μ.

Suppose now that (X_i, d_i) is a sequence of metric spaces. Let us first consider the ultra-product $\Pi^\mu X_i$ with the function $d\colon \Pi^\mu X_i \to \mathbb{R} \cup \{\infty\}$ which is $d((x_i),(y_i)) = \lim^\mu d_i(x_i, y_i)$. This function d does satisfy the triangle inequality, but it fails to be a metric for a number of reasons. First of all, there can be distinct sequences (x_i) and (y_i) such that $d((x_i),(y_i)) = \infty$. For each space, let o_i be some fixed basepoint in X_i. and consider only the $\{(x_i) \in \Pi^\mu X_i \mid \lim^\mu d_i(o_i, x_i) < \infty\}$. On this subset the triangle inequality implies that all distances will be finite. However, there may still be distinct sequences (x_i) and (y_i)) such that $d((x_i),(y_i)) = 0$[5]. This condition defines an equilvalence relation $\sim$ on the set of sequenes; that is $(x_i) \sim (y_i)$ if and only if $d((x_i),(y_i)) = 0$. If idenitify any two sequences which are equivalent in this way then we will get that the induced function d on is an honest, well-defined metric on the quotient. That is,

$$\lim^\mu (X_i, o_i) = \{(x_i) \in \Pi X_i \mid \ \lim^\mu d_i(o_i, x_i) < \infty\}/\sim$$

With the metric d defined by $d((x_i),(y_i)) = \lim^\mu d_i(x_i, y_i)$. This space is called the μ–ultra-limit of the metric spaces X_i. Points in the space $\lim^\mu(X_i, o_i)$ are equivalence classes of sequences, but we will often abuse notation by thinking of the sequences themselves as the points. However, it is important to remember that not every sequence in ΠX_i defines a point in the ultra-limit, only sequences which are a finite distance from the sequence of basepoints.

The asympotic cone of a space X is a special case of the above construction. Let (X, d) be a metric space, let $o_i \in X$, and let e_i be a sequence of real numbers with $e_i \to \infty$. Let X_i be the metric space $(X, d/e_i)$, that is X_i is the same as X with the metric re-scaled by $\frac{1}{e_i}$. The the asymptotic cone of X with respect to o_i and e_i is defined by $Cone^\mu(X, e_i, o_i) = \lim^\mu(X_i, o_i)$.

If $X' \subseteq X$, and each $o_i \in X'$, then $Cone^\omega(X', e_i, o_i) \subseteq Cone^\omega(X, e_i, o_i)$.

Lemma 4.40. *If X' is an ε-quasi-dense subset of X, then there exists $o_i' \in X'$ such that $Cone^\omega(X', e_i, o_i') = Cone^\omega(X, e_i, o_i)$.*

Proof. Choose $o_i' \in X'$ such that $d(o_i, o_i') \leq \varepsilon$. Then $\lim^\mu \frac{d(o_i,o_i')}{e_i} \leq \lim^\mu \frac{\varepsilon}{e_i} = 0$. Hence (o_i) and (o_i') are the same point in $Cone^\omega(X, e_i, o_i)$, and it follows that $Cone^\omega(X, e_i, o_i) = Cone^\omega(X, e_i, o_i')$. Now $Cone^\omega(X', e_i, o_i')$ is a subset of $Cone^\omega(X, e_i, o_i')$ by definition. However, for any point $(x_i) \in Cone^\omega(X, e_i, o_i')$, we can choose $x_i' \in X'$ with $d(x_i, x_i') \leq \varepsilon$. By the same arguement as above (x_i) and (x_i') define the same point in $Cone^\omega(X, e_i, o_i')$, hence $Cone^\omega(X', e_i, o_i') = Cone^\omega(X, e_i, o_i)$. □

Example 4.41. $Cone(\mathbb{Z}, e_i, (0))$ is isometric to $\mathbb{R}$. First, by the previous Lemma, $Cone(\mathbb{Z}, e_i, (0)) = Cone(\mathbb{R}, e_i, (0))$. Now consider the function $f\colon \mathbb{R} \to Cone(\mathbb{R}, e_i, (0))$ defined by $f(x) = (xe_i)$.This is a well-defined point of $Cone(\mathbb{R}, e_i, (0))$ since $\lim^\mu \frac{|xe_i|}{e_i} = |x| < \infty$. Furthermore, for any $x, y \in \mathbb{R}$, $d(f(x), f(y)) = \lim^\mu \frac{|xe_i - ye_i|}{e_i} = |x - y|$. Hence f is an isometric embedding.

[5]At this point d is a *pseudo-metric*. What follows is the standard way to build a metric space out of a pseudo metric space.

It only remains to show that f is surjective. Let $(x_i) \in Cone(\mathbb{R}, e_i, (0))$ with $d((0), (x_i)) = x$. This means that $\lim^\mu \frac{|x_i|}{e_i} = x$. Suppose that $x_i \geq 0$ μ.–almost surely. Then $f(x) = (x_i)$, otherwise $f(-x) = (x_i)$.

Generalizing the above example, for $\mathbb{Z}^n$ the the word metric coming from the standard generating set and any scaling sequence (e_i), $Cone(\mathbb{Z}^n, e_i)$ is isometric to $\mathbb{R}^n$ with the ℓ^1–metric.

In general, the construction of the asymptotic cone depends on all of the choices made in the construction.

We start by listing some basic properties of the asymptotic cone.

Definition 4.42. A metric space X is called *homogeneous* if for any $x, y \in X$ there exists an isometry $f\colon X \to X$ such that $f(x) = y$.

Lemma 4.43. *Let X be any metric space, o_i a sequence of points in X and e_i a sequence of real numbers $e_i \to \infty$. Then $Y = Cone(X, e_i, o_i)$ is complete, and*

1. *If X is homogeneous then Y is homogeneous and independent of the sequence of basepoints (up to isometry).*
2. *If X is geodesic then Y is geodesic.*

Proof. The proof that Y is complete is a standard diagonal argument, only a slight variation of the proof that $\mathbb{R}$ constructed as Cauchy sequences of rationals is complete. For a detailed proof, see [22].

Suppose X is homogeneous and $(x_i), (y_i) \in Y$. Let f_i be an isometry of x_i such that $f(x_i) = y_i$. Let $f\colon Y \to Y$ be defined by $f((a_i)) = (f_i(a_i))$. Since each f_i is an isometry it is easy to see that f is an isometry and clearly $f((x_i)) = (y_i)$. Similarly, if o_i and o_i' are two sequences of basepoints and $f_i(o_i) = o_i'$, then the corresponding f is an isometry from $Cone(X, e_i, o_i)$ to $Cone(X, e_i, o_i')$

Suppose X is a geodesic metric space and $x = (x_i)$, $y = (y_i)$ are points in Y. Let $l_i = \frac{d(x_i, y_i)}{e_i}$, and let $\gamma_i\colon [0, l_i e_i] \to X$ be a geodesic from x to y. Note that $l = \lim^\mu l_i$ is equal to $d(x, y)$ Let $\gamma\colon [0, l] \to Y$ be defined by $\gamma(t) = (\gamma_i(te_i))$, where if $t > l_i$ we let $\gamma_i(te_i) = y_i$. It is a straightforward computation to show that γ is a geodesic from x to y.

□

Note that the final part of the proof shows that any two points in Y can be connected by a geodesic which is a limit of geodesics in X. However, Y may also contain geodesics which are *not* limits of geodesics from X.

We will assume from now on that all spaces under consideration are homogeneous, hence the previous Lemma implies that each asymptotic cone is independent of the choice of basepoints. Hence from now on we will supress this from the notation, that is we will denote the asymptotic cone asociated to a scaling sequence (e_i) by $Cone^\mu(X, (e_i))$.

Lemma 4.44. *If $X_1 \sim_{qi} X_2$, then for any scaling sequence (e_i), $Cone^\mu(X_1, (e_i))$ and $Cone^\mu(X_2, (e_i))$ are bi-lipscitz equivalent. In particular, they are homeomorphic.*

Proof. Exercise. □

Remark 4.45. This Lemma implies that topological properties of the asymptotic cone are quasi-isometry invariants of the space X.

We return now to the case where G is group generated by a finite set S. By Lemma 4.40, we will get the same space whether we consider an asymptotic cone of (G, d_S) or of $\Gamma(G, S)$. Since (G, d_S) is homogeneous we get that any asymptotic cone is homogeneous and independent of the choice of basepoints; for convenience we will always choose the basepoints to be the constant seqence (1). Since $\Gamma(G, S)$ is geodesic we get that any asymptotic cone is geodesic. Note that geodesic spaces are always (path) connected and locally (path) connected. By Lemma 4.44 the asymptotic cones of G are independent of the choice of finite generating set up to bi-lipschitz equivalence. In general, however, the asymptotic cone does depend on the choice of scaling sequence (e_i). Hence we will denote the asymptotic cone of G with respect to the scaling sequence (e_i) by $Cone^{\mu}(G,(e_i))$, and note that this is well-defined up to bi-lipshitz equivalence.

Note that the action of G on itself by left multiplication induces an action of $G^{\mu}_{(e_i)} = \{(g_i) \in G^{\mu} \mid \lim^{\mu} \frac{|g_i|_S}{e_i} < \infty\}$ on $Cone(G,(e_i))$; that is for any $(g_i) \in G^{\mu}$ and any $(x_i) \in Cone(G,(e_i))$, $(g_i) \cdot (x_i) = (g_i x_i)$. Since G naturally embeds into G^{μ} as the (equivalence classes of) constant sequences, we get an induced action of G on $Cone(G,(e_i))$ by isometries.

Thus we have shown the first part of Theorem 4.35. We now show the second part.

Definition 4.46. A subset S of a metric space X is called η-separated if for all $x, y \in S$, $d(x, y) > \eta$.

Lemma 4.47. *If G is finitely generated and $\gamma_G(n) \leq n^d$, then for some (e_i) $Cone(G,(e_i))$ is locally compact and finite dimensional.*

Proof. Fix $\varepsilon > 0$. We first show the following:

Claim: For some sequence (e_i) satisfying $\log i \leq e_i \leq i$, for any $j \geq 4$ any $\frac{e_i}{j}$–separated subset in $B_{\frac{e_i}{4}}(1)$ has at most $j^{d+\varepsilon}$ elements ω–almost surely.

The proof of this claim is quite technical, we refer to [9] for a full proof. The basic idea is to choose some j_i for which the claim does not hold and then cover each $B_{\frac{i}{4}}(1)$ by disjoint balls of radius $\frac{i}{2j_1}$. Then we can find j_2 such that the claim does not hold for the sequence $\frac{i}{j_1}$ and cover the corresponding balls by disjoint balls of radius $\frac{i}{2j_1 j_2}$. Repeating this process using non-standard induction will eventually yield some large ball which contains too many points for the given polynomial bound on the growth function of the group.

Let e_i be the sequence for which the claim holds. Let $\delta > 0$ and choose $j \geq 4$ such that $\frac{1}{j} < \delta$. The claim gives a uniform bound of $j^{d+\varepsilon}$ on the size of a $\frac{1}{j}$–separated subset of $B_{\frac{1}{4}}(1)$ in the metric space $(G, \frac{d_S}{e_i})$. Hence every $\frac{1}{j}$ separated subset of $B_{\frac{1}{4}}(1)$ in $Cone(G,(e_i))$ also has at most $j^{d+\varepsilon}$ elements. Choosing a maximal $\frac{1}{j}$–separated subset we get that this ball is covered by finitely many balls of size δ for any $\delta > 0$, i.e. $B_{\frac{1}{4}}(1)$ is *totally bounded*. Since $Cone(G,(e_i))$ is a complete metric space, the Heine-Borel theorem implies that the closure of $B_{\frac{1}{4}}(1)$ is compact, and since $Cone(G,(e_i))$ is homogeneous it follows that it is locally compact.

Next we show that the asymptotic cone is finite dimensional. Since we are working in a metric space the usual topological covering dimension is bounded above by Hausdorff dimension. Hence is suffices to show that the asymptotic cone has finite Hausdorff dimension. Now, suppose $\alpha > d+\varepsilon$. As above, for all $j \geq 4$ $B_{\frac{1}{4}}(1)$ can be covered by t balls of radius $\frac{1}{j}$ with $t \leq j^{d+\varepsilon}$. This implies that the Hausdorff measure

$$\mu_\alpha(B_{\frac{1}{4}}(1)) \leq \sum_{i=1}^{t}(\frac{1}{j})^\alpha \leq j^{d+\varepsilon-\alpha} \to 0 \text{ as } j \to \infty$$

Hence the Hausdorff dimension of $B_{\frac{1}{4}}(1))$ is at most α. Now $Cone(G,(e_i)$ is a complete, locally compact, geodesic metric space, and it follows that all closed balls in $Cone(G,(e_i))$ are compact. Hence $Cone(G,(e_i)$ can be covered by countably many balls of radius $\frac{1}{4}$, since μ_α is countably additive we get that $\mu_\alpha(Cone(G,(e_i))=0$. Hence $Dim_{Haus}(Cone(G,(e_i)) \leq \alpha$.

$\square$

Remark 4.48. Since α and ε are abitrary in the above proof, we in fact get that $Dim_{Haus}(Cone^\mu(G,(e_i)) \leq d$.

Given (e_i) as in the previous Lemma, we let Y denote $Cone^\mu(G,(e_i)$ from now on.

Finally, in order to complete the proof of Gromov's polynomial growth theorem we need the following lemma about the action of G on the asymptotic cone. Let $\eta\colon G^\mu_{(e_i)} \to Isom(Y)$ be the homomorphism corresponding to this action, and let K denote the kernel of η; that is, $K = \{g \in G \mid gx = x \;\forall\; x \in Y\}$. Note that $g \in K$ if and only if for all $x = (x_i) \in Cone^\mu(G,(e_i)$, $\lim^\mu \frac{d(x_i,gx_i)}{e_i} = 0$.

$Isom(Y)$ is a lie group with a topology generated by basic open sets around id of the form $U_{N,\varepsilon} = \{f \mid d(x,f(x)) \leq \varepsilon \;\forall\; x \in B_N(1)\}$.

Lemma 4.49. *Suppose K is a finite index subgroup of G, and suppose K is not virtually abelian. Let T be a finite generating set for K. Then for all neighborhoods U of id in $Isom(Y)$ there exists a sequence $(g_i) \in G^\mu$ such that*

1. *For all $k \in K$, $(g_i^{-1}kg_i) \in G^\mu_{(e_i)}$.*
2. *For some $t \in T$, $\eta(t) \in U$.*

In particular, there is a homomorphism $\eta_U\colon K \to Isom(Y)$, namely $\eta_U(k) = (g_i^{-1}kg_i)$, such that the image of η_U has a non-trivial intersection with U.

Proof. For a fixed $t \in T$, if $\{|g^{-1}tg|_S \mid g \in G\}$ is bounded then t has finitely many conjugates in K, hence the centralizer of t is a finite index subgroup of K. Since the center K is equal to the intersection of the centralizers of the elements of T and K is not virtually abelian, at least one $t \in T$ has infinite index centralizer and hence $\{|g^{-1}tg|_S \mid g \in G\}$ is unbounded.

Next we note that for $k \in K$, $g, x \in G$,

$$d_S(x, g^{-1}kgx) \leq d_S(x,kx) + 2|g|_S \tag{4}$$

The proof of this inequality is straitforward and left as an exercise for the reader.

Now let $U_{\varepsilon,N}$ be a basic open set around id in $Isom(Y)$. From above there exists a sequence (h_i) such that for some $t \in T$, $|h_i^{-1}th_i|_S \geq \varepsilon e_i$. Let $h_i = t_{1,i}...t_{l_i,i}$ with each $t_{j,i} \in T$. Let $g_{j,i} = t_{1,i}...t_{j,i}$ $(g_{0,i} = 1)$. Let

$$M_i^j = \max_{t \in T, x \in B_{Ne_i}(1)} d(x, g_{j,i}^{-1}tg_{j,i}x)$$

By assumption, $M_i^{l_i} > \varepsilon e_i$ and since $t \in K$, $M_i^0 < \varepsilon e_i$ μ–almost surely. By 4, $|M_i^j - M_i^{j+1}| \leq 2|t_{i,j}|_S \leq 2C$, where $C = \max_{t \in T} |t|_S$. Hence there exists some $1 \leq j_i \leq l_i$ such that

$$|M_i^{j_i} - \varepsilon e_i| \leq 2C. \tag{5}$$

Define $g_i := g_{j_i,i}$. 5 implies that for any $t \in T$, $(g_i^{-1}tg_i) \in G^{\mu}_{(e_i)}$. Since any $k \in K$ is a finite product of elements of T, the same holds for all $k \in K$ by the triangle inequality. Hence $k \to (g_i^{-1}kg_i)$ is a homomorphism $K \to G^{\mu}_{(e_i)}$, and hence we get an induced action of K on Y. We denote the corresponding homomorphism $\eta_U \colon K \to Isom(Y)$. Applying 5 again we get that for all $t \in T$ and for all $x \in B_N(1)$, $d(x, tx) \leq \varepsilon$, hence $\eta_U(t) \in U$. Furthermore, if t and x_i realize the maximum in the definition of $M_i^{j_i}$ μ–almost surely, then $d(x_i, tx_i) = \varepsilon$, thus $\eta_U(t) \neq id$.

□

Finally, the last ingredient in the proof of Gromov's theorem is the following theorem of Jordan about finite subgroups of Lie groups.

Theorem 4.50 (Jordan's Theorem)**.** *Let L be a lie group with finitely many connected components. Then there exists $q = q(L)$ such that every fintie subgroup of L has an abelian subgroup of index $\leq q$.*

Proof of Theorem 4.31. Given a finitely generated group G, we have constructed a space Y as above and a homomorphism $\eta \colon G \to Isom(Y)$, where $Isom(Y)$ is a lie group with finitely many connected components. Let L be the connected component of the identity in $Isom(Y)$; replacing G by a finite index subgroup we can assume that $\eta \colon G \to L$. Now there are two cases.

Case 1: The image of G in L is infinite. In this case we consider the *adjoint representation*, $ad \colon L \to GL(n, \mathbb{R})$, whose kernel is the center of L. If $ad(\eta(G))$ is finite, then $\eta(G)$ has a finite index subgroup which is contained in the center of L. Hence a fintie index subgroup of maps to an abelian group (namely $Z(L)$) with infinite image, so we are done by Exercise 4.32. If $ad(\eta(G))$ is infinite, then since it is a quotient of G its growth is polynomially bounded, and hence the Tit's alternative implies that $ad(\eta((G))$ is virtually solvable. Thus a finite index subgroup of $ad(\eta(G))$ admits a non-trivial map to $\mathbb{Z}$ by Exercise 4.33, and so a finite index subgroup of G itself admits such a map.

Case 2: The image of G in L is finite. In this case $K = Ker(\eta)$ is a finite index subgroup of G. If K is virtually abelian, then a finite index subgroup of K is an infinite abelian group which clearly admits a non-trivial map to $\mathbb{Z}$. Hence we can assume K is not virtually abelian and apply Lemma 4.49. Let q be given by Jordan's theorem and let H be the intersection of all subgroups of K of index $\leq q$.

Since L is a lie group, it has the *no small subgroups property*, that is for every n there exists an open neighborhood U around id which contains no non-trivial elements of order n (this can be easily derived from the fact that for any open set U around the origin in $\mathbb{R}^n$ there exists an open set V such that for all $v \in V$, $\{v, 2v, ..., nv\} \subset U$ and the fact that the exponential map is a diffeomorphism from a small neighborhood of the origin in $\mathbb{R}^n$ to a small neighborhood around id in L). Hence for any $n \geq 1$ there a homomorphism $\eta_U : K \to L$ and an element $t \in K$ such that $\eta_U(t) \in U \setminus \{id\}$.Hence $|\eta_U(K)| \geq |\langle \eta_U(t) \rangle| \geq n$. If some $\eta_U(K)$ is infinite then we proceed as in Case 1, so we can assume from now on that each $\eta_U(K)$ is finite. By Jordan's Theorem, $\eta_U(K)$ has a subgroup A of index at most q which is abelian. Hence $\eta_U^{-1}(A)$ has index at most q in K, which means that $H \leq \eta_U^{-1}(A)$. Thus $\eta_U(H)$ is an abelian group with $|\eta_U(H)| = \frac{|\eta_U(K)|}{[\eta_U(H):\eta_U(K)]} \geq \frac{n}{[H:K]}$. In particular, H (which is a finite index subgroup of G) has arbitrarily large finite abelian quotients, hence $H/[H : H]$ must be infinite. Thus H has a non-trivial map to $\mathbb{Z}$ by Exercise 4.32.

□

Corollary 4.51. *The following are equivalent for a finitely generated group G:*

1. *γ_G is equivalent to a polynomial.*
2. *γ_G is bounded by a polynomial.*
3. *G is virtually nilpotent.*

Corollary 4.52. *If G is quasi-isometric to a nilpotent group then G is virtually nilpotent.*

Theorem 4.53. *If G is quasi-isometric to an abelian group then G is virtually abelian.*

Before leaving the world of asymptotic cones, we mention that they can be used to characterize hyperbolic groups.

Theorem 4.54. *G is hyperbolic if and only if every asymptotic cone is an $\mathbb{R}$–tree.*

5 Amenable Groups

[6]

5.1 Invariant measures and Folner sequences

Definition 5.1. A (discrete) group G is amenable if G has a finitely additive left-invariant probability measure. That is there exists a function $\mu : 2^G \to [0, 1]$ such that

1. For all disjoint sets $A, B \subseteq G$, $\mu(A \cup B) = \mu(A) + \mu(B)$
2. For all $A \subseteq G$ and $g \in G$, $\mu(gA) = \mu(A)$.
3. $\mu(G) = 1$.

[6] The main references for this section are [9] and [20].

Example 5.2. Let G be a finite group. For any $A \subseteq G$, definte $\mu(A) = \frac{|A|}{|G|}$. Clearly μ is a finitely additive left-invariant probability measure, hence G is amenable.

Exercise 5.3. Show that G has a finitely additive left-invariant probability measure if and only if G has a finitely additive right-invariant probability measure if and only if G has a finitely additive bi-invariant probability measure.

Exercise 5.4. Let G amenable, prove that every subgroup of G is amenable and every quotient of G is amenable.

This definition is short and beautiful, but for infinite groups it is difficult to verify. The following more geometric definition is often useful. For conveience we give the definition for finitely generated group but it can be easily adapted to all countable groups.

Let G be generated by a finite set S. Given $A \subseteq G$, define $\partial A = \{g \in G \mid d_S(g, A) = 1\} = A \cdot S \triangle A$.

Definition 5.5. Let G be generated by a finite set S. A Folner sequence for G is a sequence of finite subsets of G, F_n, such that

$$\frac{|\partial F_n|}{|F_n|} \to 0 \text{ as } n \to \infty$$

Exercise 5.6. Prove that F_n is a Folner sequence for G if and only if for all $g \in G$,

$$\frac{|gF_n \triangle F_n|}{|F_n|} \to 0 \text{ as } n \to \infty$$

As a consequence of this exercise, Folner sequences are independent of the choice of generating set of G.

Remark 5.7. The condition in this exercise is used to define a Folner sequence for a countable group.

Exercise 5.8. Consider $\mathbb{Z}$ with the standard generating set. Let $F_n = [-n, n]$. Then for all n, $|\partial F_n| = 2$ but $|F_n| = 2n + 1$. Hence F_n is a Folner sequence for $\mathbb{Z}$.

Theorem 5.9. *G is amenable if and only if it contains a Folner sequence.*

We will prove one direction of this theorem and refer to [20] for the other.

Proof. Suppose G contains a Folner sequence F_n. Let ω be a non-principal ultra-filter, and for each $A \subseteq G$ define

$$\mu(A) = \lim_{n\to\infty}{}^{\omega} \frac{|A \cap F_n|}{|F_n|}$$

Clearly $\mu(G) = 1$. If A and B are disjoint, then for all F_n $|(A \cup B) \cap F_n| = |A \cap F_n| + |B \cap F_n|$, hence $\mu(A \cup B) = \mu(A) + \mu(B)$.

Now, given $g \in G$,

$\mu(A) - \mu(gA) = \lim^{\omega} \frac{|A \cap F_n| - |gA \cap F_n|}{|F_n|} = \lim^{\omega} \frac{|A \cap F_n| - |A \cap g^{-1}F_n|}{|F_n|} \leq \lim^{\omega} \frac{|(A \cap F_n) \triangle (A \cap g^{-1}F_n)|}{|F_n|} \leq \lim^{\omega} \frac{|F_n \cap g^{-1}F_n|}{|F_n|} = 0.$

Thus, μ is a finitely additive left-invariant probability measure on G.

$\square$

Given a left-invariant probability measure μ on G, there exists a left-invariant linear functional $\ell^\infty(G) \to \mathbb{R}$ given by integration with respect to μ, that is $f \to \int f d\mu$. This functional is left-invariant with respect to the action of G on $\ell^\infty(G)$, that for all $f \in \ell^\infty(G)$ and for all $g \in G$, $\int g \cdot f d\mu = \int f d\mu$. Here $g \cdot f(x) = f(g^{-1}x)$.

We now prove two more closure properties of amenable groups.

Lemma 5.10. *Suppose H and K are amenable and there is a short exact sequence*

$$1 \to K \to G \to H \to 1.$$

Then G is amenable

Proof. Let $\varphi\colon G \to H$ with $\ker(\varphi) = K$. Let μ_K and μ_H be finitely-additive, left-invariant probability measures on H and K respectively. Note that μ_K can be naturally extended to any coset gK by defining $\mu_K(A) = \mu_K(g^{-1}A)$ for any $A \subseteq gK$. Now, given a subset $A \subseteq G$, we define a function $f_A\colon H \to \mathbb{R}$ by $f_A(h) = \mu_K(A \cap \varphi^{-1}(h))$. Clearly $f_A \in \ell^\infty(H)$, so we can define $\mu(A)$ by the formula

$$\mu(A) = \frac{1}{2} \int f_A d\mu_H$$

Clearly $\mu(G) = 1$. If A and B are disjoint, then $f_{A \cup B} = f_A + f_B$, and hence $\mu(A \cup B) = \mu(A) + \mu(B)$. Now, given $g \in G$ and $A \subseteq G$, for each $h \in H$ $gA \cap \varphi^{-1}(h) = A \cap \varphi^{-1}(\varphi(g^{-1})h)$, that is $f_{gA} = \varphi(g) \cdot f_A$. Hence the left-invariance of integration on H gives

$$\int f_g A d\mu_H = \int \varphi(g) \cdot f_A d\mu_H = \int f_A d\mu_H$$

Hence $\mu(gA) = \mu(A)$. □

Lemma 5.11. *If $G_1 \leq G_2 \leq \ldots$ and $G = \bigcup_{i=1}^{\infty} G_i$ where each G_i is amenable, then G is amenable.*

Proof. If μ_i is a finitely additive left-invariant probability measure on G_i, then for each $A \subseteq G$ we can define $\mu(A) = \lim^\omega(\mu_i(A \cap G_i)$ where ω is a non-principle ultrafilter. □

Combining together some previous theorems and exercises we get the following.

Theorem 5.12. *The class of amenable group is closed under taking subgroups, quotients, extensions, and direct limits.*

Corollary 5.13. *Every solvable group is amenable.*

Definition 5.14. The smallest class of groups which contains all fintie and all abelian groups and is closed under taking subgroups, quotients, extensions, and direct limits is called the class of *elementary amenable groups.*

We now return to considering amenable groups from the more geometric perspective of Folner sequences. We fix a group G generated by a finite set S for the rest of this section.

Lemma 5.15. *G is non-amenable if and only there exists constants $D \geq 2$ and $\beta > 1$ such for any non-empty finite set A,*

$$|A^{+D}| \geq \beta |A|$$

Proof. If G is non-amenable, then there exists $\varepsilon > 0$ such that for any finite set A, $|\partial A| \geq \varepsilon |A|$. Since $A^{+1} = A \sqcup \partial A$, it follows that $|A^{+1}| \geq (1+\varepsilon)|A|$.

Now suppose for any non-empty finite set A, $|A^{+D}| \geq \beta |A|$. After possibly increasing D, we can assume that $\beta \geq 2$. Note that $A^{+D} = A \cup \partial A^{+D-1}$, hence $|\partial A^{+D-1} \setminus A| \geq |A|$. If $m = |S|$, then $|\partial A^{+D-1}| \leq m^{D-1} |\partial A|$. In particular,

$$|\partial A \geq \frac{1}{m^{D-1}} |\partial A^{+D-1}| \geq \frac{1}{m^{D-1}} |A|$$

Thus $\frac{|\partial A|}{|A|} \geq \frac{1}{m^{C-1}}$ holds for all finite sets A, so G has no Folner sequence and hence is non-amenable.

□

Theorem 5.16. *Every finitely generated group of sub-exponential growth is amenable.*

Proof. Let G be a non-amenable group generated by a finite set S, let B_n denote the ball of radius n centered at 1 with respect to the metric d_S. Lemma 5.15 shows that there exists $\beta > 1$ such that for all n,

$$|B_{n+1}| \geq \beta |B_n|.$$

It follows inductively that $|B_{n+1}| \geq \beta^n |B_1|$, hence

$$\limsup_{n \to \infty} \sqrt[n]{|B_n|} \geq \beta > 1.$$

Hence G has exponential growth.

□

Note that the converse of this does not hold, because there are solvable (hence amenable) groups of exponential growth.

Theorem 5.17. *Let G and H be finitely generated groups. If G is amenable and $H \sim_{qi} G$, then H is amenable.*

Proof. Let $f \colon G \to H$ and $g \colon H \to G$ be $(\lambda, C, \varepsilon)$ quasi-isometries which are quasi-inverse to each other. Let K be a constant such that for all $x \in G$, $d(x, g(f(x)) \leq K$. Assume that G is non-amenable and D, β are given by Lemma 5.15. Let b be a constant such that f and g are both at most $b - -to - -1$. After possibly increasing D, we can assume that $\beta > b^2$.

Let A be a finite subset of G. By our choice of b, $|f(A)| \geq \frac{1}{b}|A|$. By Lemma 5.15, $|f(A)^{+D}| \geq \beta |f(A)|$. Now let $y \in f(A)^{+D}$, and let $y' \in f(A)$ such that $d(y, y') \leq D$. Then $d(g(y'), A) \leq K$, so $d(g(y), A) \leq d(g(y), g(y') + d(g(y), A) \leq \lambda D + C + K$. This shows that

$$g(f(A)^{+D}) \subseteq A^{\lambda D + C + K}.$$

Hence

$$|A^{\lambda D+C+K}| \geq |g(f(A)^{+D})| \geq \frac{1}{b}|f(A)^{+D}| \geq \frac{\beta}{b}|f(A)| \geq \frac{\beta}{b^2}|A|$$

Therefore, H is non-amenable by Lemma 5.15. □

5.2 Paradoxical decompositions

Example 5.18. F_2 is not amenable. Suppose for the sake of contradiction that μ is a left-invariant finitely additive probability measure on F_2. Let $\{a, b\}$ be a free generating set for F_2. Let $A_\pm$ be the set of reduced words in F_2 that start with $a^{\pm 1}$ and $B_\pm$ the set of reduced words that start with $b^{\pm 1}$. Note that $F_2 = a^{-1}A_+ \cup A_-$, and since these sets are disjoint $1 = \mu(F_2) = \mu(a^{-1}A_+ \cup A_-) = \mu(a^{-1}A_+) + \mu(A_-) = \mu(A_+) + \mu(A_-)$. Also, $F_2 = b^{-1}B_+ \cup B_-$, so similarly $1 = \mu(B_+) + \mu(B_-)$. But $A_\pm$ and $B_\pm$ are all disjoint, so $\mu(A_+ \cup A_- \cup B_+ \cup B_-) = \mu(A_+) + \mu(A_-) + \mu(B_+) + \mu(B_-) = 2$, which is a contradiction.

Corollary 5.19. *Any group containing F_2 is non-amenable. In particular, every non-elementary hyperbolic group is non-amenable and every linearly group is either virtually solvable or non-amenable.*

By embedding the free group into the group of rotations of the sphere, $SO(3)$, one can use this paradoxical decomposition of F_2 to build a paradoxical decomposition of a closed ball in $\mathbb{R}^3$. This is called the Banach-Tarski paradox, for details see [20].

6 Bass-Serre Theory

6.1 Graphs of groups

Bass-Serre theory gives a correspondence between splittings of a group and actions on trees. Here a "splitting" really refers to a graph of groups decomposition, the simplest examples of which are amalgamated products and HNN extensions.

Definition 6.1. Let G_1 be a group with presentation $\langle S_1 | R_1 \rangle$ and G_2 a group with presentation $\langle S_2 | R_2 \rangle$. Suppose $C_1 \leq G_1$, $C_2 \leq G_2$, and $\varphi\colon C_1 \to C_2$ is an isomorphism. Then the amalgamted product of G_1 and G_2 over $C_1 = C_2$ is the group given by the presentation

$$\langle S_1 \sqcup S_2 | R_1 \sqcup R_2 \sqcup \{c = \varphi(c) | \forall\ c \in C_1\}\rangle$$

For the sake of notation, we typically refer to the isomorphic subgroups C_1 and C_2 by a single letter C and denote the amalgamated produce of G_1 and G_2 over C by $G_1 {*_C} G_2$. Note, however, that this depends on the isomorphism φ even though this is supressed from the notation. Alternatively, one can consider the inclusion maps $C \to C_1 \leq G_1$ and $C \to C_2 \leq G_2$ and define the amalgamted product as the pushout of the corresponding diagram.

If $C = \{1\}$, then this is called the free product of G_1 and G_2 and denoted $G_1 * G_2$.

This construction occurs naturally in topology throught the well-konwn Van-Kampen Theorem

Theorem 6.2. *If a topological space X is the union of two open, path connected subspaces Y and Z such that $Y \cap Z$ is non-empty and path connected, then*

$$\pi_1(X) \cong \pi_1(Y) *_{\pi_1(Y\cap Z)} \pi_1(Z)$$

Example 6.3. Let γ be simple closed curve on S_2 which cuts the surface into two tori with boundary. Each of these tori has a free fundamental group and γ is the boundary curve.

$$\pi_1(S_2) \cong F_2 *_{\mathbb{Z}} F_2$$

Definition 6.4. Let G be a group given by a presentation $\langle S, |R\rangle$, and let C_1, C_2 be subgroups of G with $\varphi\colon C_1 \to C_2$ an isomorphism.The HNN-exention of G over $C_1 = C_2$ is given by

$$\langle S \cup \{t\} | R \cup \{t^{-1}ct = \varphi(c) | \forall\ c \in C_1\}\rangle$$

Topologically HNN exensions can be realized by glueing a cylinder onto a space X with the ends attaching to homeomorphic subspaces of X. Again, we typically identify the isomorphic group C_1 and C_2 and write them with a single letter C. We denote the HNN-extension by $G*_C$.

Example 6.5. $BS(n,m) = \langle a, t | t^{-1}a^n t = a^m\rangle$. These groups, called *Baumslag-Solitar groups*, are exatly the HNN-exentions of $\mathbb{Z}$. Note that $\mathbb{Z}^2 = BS(1,1)$.

A group G splits as an HNN-extension if $G \cong A*_C$ or as an amalgamated product if $G = A*_C B$. The splitting as an amalgamated produce is called *trivial* if either $C = A$ or $C = B$. Any other splitting of G is called *non-trivial*.

This is a simplified version of the main theorem of Bass-Serre Theory.

Theorem 6.6. *A group G acts on a tree with no global fixed point if and only if G splits non-trivially as an HNN-extension or an amalgamated product.*

In order to state the full version of the fundamental theorem we need to define a graph of groups. For this purpose, it is most convenient to word with Serre's definition of a graph.

Definition 6.7. A graph is composed of two sets, V and E, together with a function $\alpha\colon E \to V$ and fixed point free involution ${}^{-1}\colon E \to E$. We also define $\omega\colon E \to V$ by $\omega(e) = \alpha(e^{-1})$. For simplicity we denote a graph by the pair (V, E).

Definition 6.8. A graph of group $\mathcal{G}$ consists of a connected graph (V, E) and a collection of *vertex groups,* $\{G_v | v \in V\}$, and *edge groups*, $\{G_e | e \in E\}$ such that $G_e = G_{e^{-1}}$, together with injective homomorphisms $\alpha_e\colon G_e \to G_{\alpha(e)}$. We refer to (V, E) as the underlying graph of $\mathcal{G}$.

Given a graph of groups $\mathcal{G}$, a *path* in $\mathcal{G}$ is a sequence $(g_0, e_1, g_1, ..., e_n, g_n)$ such that $(e_1, ..., e_n)$ is an edge path in the underlying graph, and each $g_i \in G_{v_i}$ where $v_i = \alpha(e_{i+1})$. We define an equivalence relation on paths generated by the following two elementary equivalences:

1. $(..., g, e, 1, e^{-1}, h, ...) \sim (..., gh, ...)$
2. $(..., g, e, h, ...) \sim (..., g\alpha_e(x), e, \omega_e(x^{-1})h, ...)$ for any $x \in G_e$.

Definition 6.9. Let $\mathcal{G}$ be a graph of groups. The fundamenatal group of $\mathcal{G}$, $\pi_1(\mathcal{G})$, is defined as the set of equivalences class of closed paths based at a fixed vertex with multiplication given by concatenation.

It is straitforward to check that up the isomorphism $\pi_1(\mathcal{G})$ is independent of the choice of basepoint.

Examples 6.10.

1. If all vertex groups of $\mathcal{G}$ are trivial, then $\pi_1(\mathcal{G})$ is just the usual fundamenal group of the underlying graph (V, E). In particular, it is a free group.
2. If $\mathcal{G}$ has one vertex v and one edge e, then $\pi_1(\mathcal{G}) \cong G_v *_{G_e}$.
3. If $\mathcal{G}$ has two vertices u and v and one edge e, then $\pi_1(\mathcal{G}) \cong G_v *_{G_e} G_u$.

If $G = \pi_1(\mathcal{G})$, then $\mathcal{G}$ is called a *graph of groups decomposition of* G. If $\mathcal{G}$ has one edge, i.e. if it corresponds to an amalgamated product or HNN-extension, then it is called a *one-edge splittings* of G.

We now state the main theorem of Bass-Serre Theory:

Theorem 6.11. *Let G be a group and let $\mathcal{G}$ be a graph of groups decomposition of G. Then there exists a tree $T_{\mathcal{G}}$, called the* Bass-Serre *tree on which G acts by automorphisms with each vertex stabilizer conjugate into a vertex group of $\mathcal{G}$, each edge stabilizer conjugate into an edge group of $\mathcal{G}$, and $T_{\mathcal{G}}/G$ isomorphic to the underlying graph of $\mathcal{G}$.*

Conversely, if G acts by automorphisms on a tree T without inverting edges, then G has a graph of groups decomposition $\mathcal{G}_T$ with underlying graph T/G such that for each vertex v and each edge e of T/G, G_v is the stabilizer of a lift of v and G_e is the stabilizer of a lift of e.

Morever, $\mathcal{G}$ and $\mathcal{G}_{T_{\mathcal{G}}}$ are isomorphic as graphs of groups and T and $T_{\mathcal{G}_T}$ are isomorphic as G–spaces.

We give only a brief sketch of the proof and refer to [33] for details.

Given $\mathcal{G}$, define $T_{\mathcal{G}}$ to be the set of paths $(g_0, e_1, g_1, ..., e_n, 1)$ based at a fixed vertex v_0 modulo the same equivalenc relation as before. There is a natural (covering) map $T_{\mathcal{G}} \to (V, E)$ given by $(g_0, e_1, g_1, ..., e_n, 1) \to \omega(e_n)$. The action of $\pi_1(\mathcal{G})$ on $T_{\mathcal{G}}$ is given by $[p][q] = [qp]$, where p is a closed path based at v_0. It remains to show that $T_{\mathcal{G}}$ is, in fact, a tree.

Given G acting on a tree T, define (V, E) to be the graph T/G. To each vertex v, choose a lift $\tilde{v}$ and define $G_v = Stab_G(v)$. Note that for any two distinct lifts $\tilde{v}$ and $\tilde{v}'$, there exists $g \in G$ such that $g\tilde{v} = \tilde{v}'$, hence $g^{-1}Stab_G(\tilde{v})g = Stab_G(\tilde{v}')$. Hence the isomorphism type of G_v is independent of the choice of lift. For each edge e we define G_e similarly using a lift e, and if $v = \alpha(e)$ then there exists some $g \in G$ such that $g\tilde{v} = \alpha(\tilde{e})$. Hence $G_e = Stab_G(\tilde{e}) \subseteq Stab_G(\alpha(\tilde{e}) = gStab_G(\tilde{v})g^{-1} = gG_vg^{-1}$, and this allows us to define the inclusion map $\alpha_e \colon G_e \to G_v$. It remains to show that the map sending each $Stab_G(\tilde{v}) \to G_v$ extends to an isomorphism $G \cong \pi_1(\mathcal{G})$.

Here are some theorems which follow easily from the main theorem of Bass-Serre Theory. Corollaries:

Theorem 6.12. *G is a free group if and only if G has a free action on a tree.*

Corollary 6.13. *Every subgroup of a free group is a free group.*

Theorem 6.14 (Kurosh)**.** *Let H be a subgroup of a free product $*_{i\in I} G_i$. Then H is a free products of subgroups of conjugates of the G_i's and a free group.*

6.2 Groups acting on trees

Let T be a tree and let f be an isometry of T. f is called *elliptic* if it fixes a point of T, that is for some $x \in T$ $f(x) = x$. Let $\ell(f) = \inf_{x\in T} d(x, f(x))$. f is called *hyperbolic* if $\ell(f) > 0$. Clearly every elliptic isometry is not hyperbolic, but in general (hyperbolic) metric spaces there may be isometries which are not hyperbolic or elliptic (i.e. parabolic isometries). For trees, however, these do not occur.

Lemma 6.15. *Let f be an isometry of a tree T. Then either f is hyperbolic or elliptic. Morever, if f is hyperbolic then T contains a unique embedded line, called the axis of f on which f acts as a non-trivial translation.*

Proof. Let f be an isometry of a tree T, and let $x \in T$. Let m be the midpoint of the segment $[x, f(x)]$. Let o be the center of the tripod with vertices x, $f(x)$, and $f^2(x)$.

Case 1: $d(x, o) \leq d(x, m)$. In this case m is on the segment $[o, f(x)]$ and $f(x)$ is on the segment $[f(x), f^2(x)] = [f(x), o] \cup [o, f^2(x)]$. However

$$d(f(x), m) = d(x, m) = d(f(x), f(m))$$

Hence $f(m) = m$ and f is an elliptic isometry.

Case 2: $d(x, o) < d(x, m)$. In this case $o \in [m, f(m)]$ so $f(o) \in [f(m), f^2(m)]$. Then $d(o, f(o)) = d(f(x), f^2(x)) - 2d(o, f(x)) = d(x, f(x) - 2(\frac{1}{2}d(f(x), f^2(x)) - d(o, f(m)) = 2d(o, f(m)$ If follows that $f(m) \in [o, f(o)] \subseteq [m, f^2(m)]$. Hence, the f-translates of the segment $[m, f(m)]$ form a line that is invariant under f.

□

If f is a hyperbolic isometry, then it acts as a translation on its axisy by $\ell(f)$.

Lemma 6.16. *Suppose f, g are isometries of a tree T. If $Fix(f) \cap Fix(g) = \emptyset$, then fg is a hyperbolic isometry.*

Proof. Let $x \in Fix(f)$ and $y \in Fix(g)$ such that $d(x, y) = d(Fix(f), Fix(g))$. Observe that $d(y, fg(y)) = d(y, x) + d(x, fg(y))$, otherwise y would not be the closest point to x fixed by g. But $d(x, fg(y)) = d(f(x), fg(y)) = d(x, y)$. Hence $d(y, fg(y)) = 2d(x, y)$. Similarly $d(y, (fg)^2(y)) = 4d(x, y) = d(y, fg(y))$, hence fg is a hyperbolic isometry. □

Lemma 6.17. *If f and g are hyperbolic isometries of a tree with disjoint axis, then fg is a hyperbolic isometry.*

Theorem 6.18 (Helly's Theorem for trees)**.** *If $Y_1, ..., Y_n$ are closed subtrees of a tree T and $T_i \cap T_j \neq \emptyset$ for all i, j, then*

$$\bigcap_{i=1}^{n} Y_i \neq \emptyset$$

Proof. If $n = 3$, choose $x_k \in X_i \cap X_j$ and let o be the center of the tripod x_1, x_2, x_3. A standard induction extends this to the general case.

$\square$

A group G acting on a tree is *elliptic* if there exists $x \in T$ such that $gx = x$ for all $g \in G$. Before we refered to this type of action as *trivial*.

Corollary 6.19. *If $g_1, ..., g_n$ are isometries of a tree and $Fix(g_i) \cap Fix(g_j) \neq \emptyset$ for all i, j, then $\langle g_1, ..., g_n \rangle$ is elliptic.*

Corollary 6.20. *Suppose G is generated by a fintie set S and G acts on a tree T such that for all $s, t \in S$, s is elliptic and st is elliptic. Then G is elliptic.*

Corollary 6.21. *If G is finite, then every action of G on a tree has a global fixed point.*

Lemma 6.22. *Ping Pong Lemma.*

Lemma 6.23. *Suppose g and h are hyperbolic isometries of a tree T such that $Axis(g) \cap Axis(h)$ is compact. Then there exists $n \in \mathbb{N}$ such that $\langle g^n, h^n \rangle \cong F_2$.*

This argument can be generalized to groups acting on hyperbolic metric spaces, in particular the following is one of the main ingredients in the Tit's alternative for hyprebolic groups.

Theorem 6.24. *Let G be a hyperbolic group and let g and h be elements of infinite order. Then there exists $n \in \mathbb{N}$ such that $\langle g^n, h^n \rangle \cong F_2$.*

6.3 Stallings Theorem and Dunwoody's Theorem

Definition 6.25. Ends of a group.

Proposition 6.26. *Every 2-ended group is virtually $\mathbb{Z}$.*

Theorem 6.27 (Stallings)**.** *Let G be a finitely generated group. Then G splits over a finite subgroup if and only if $e(G) > 1$.*

Corollary 6.28. *If G splits over a finite subgroup and $G \sim_{qi} H$, then H splits over a finite subgroup.*

Definition 6.29. A group G is called *accessible* if G has a graph of groups decomposition with finite edge groups in which all vertex groups have at most one end. Such a graph of groups is called a *Dunwoody decomposition* of G.

Theorem 6.30 (Dunwoody Accessibility)**.** *Every finitely presented groupis accessible.*

Remark 6.31. Dunwoody also has an example of a finitely generated group which is not accessible.

We will sketch the main ideas in Dunwoody's proof of both Stalling's Theorem and Dunwoody's Accessibility Theorem; for details see [10]. We restrict to the case of a finitely presented group G. Associated to a finite presentation of such a group is a 2–complex, L, called the presentation complex whose univeral cover $K = \tilde{L}$ is called the Cayley complex (see Section 2.1)

Definition 6.32. A track τ in a 2-complex K is a connected subset which satisfies

1. $\tau \cap K^{(0)} = \emptyset$
2. For each edge e, $\tau \cap e$ is finite.
3. For each 2-cell σ, $\tau \cap \sigma$ consists of a finite collection of disjoint straight arcs connecting points on $\partial\sigma$.

Dunwoody's proof proceeds via the following steps:

1. If K has more then one end, then there exists a track τ which intersects finitely many edges of K and which separates K into two infinite components.
2. A collection of minimal such tracks which do not intersect on edges of K can be rearranged to gives a disjoint collection of minimal tracks.
3. Using such a rearrangment, one can find a minimal track τ such that for all $g \in G$, either $g\tau = \tau$ or $g\tau \cap \tau = \emptyset$.
4. Given such a τ, the connected components of $K \setminus \bigcup_{g\in G} g\tau$ naturally form the vertices of a tree T which two such vertices adjacent if they ar separated by a translate of τ.
5. If the stabilizer of one of these components is not one ended, a new set of tracks can be constructed in this componenet as above an arranged to be disjoint from all the original tracks. Note that the new set of tracks projects to a track on L disjoint from the projection of the original track τ.
6. Finally, there is a univeral upper bound on the number of disjoint (non-parallel) tracks in any finite 2-complex, so the above procedure cannot repreat indefinitely.

Papasoglu-Whyte showed that the one-ended vertex groups in a Dunwoody decomposition of a finitely generated group are invariant under quasi-isometry. From this we derive the following:

Theorem 6.33. *If $G \sim_{qi} F_k$, then G is virtually a free group.*

Proof. If $G \sim_{qi} F_k$, then G is finitely presented and hence accessible. F_k has no one-ended subgroups, hence the Dunwoody decomposition of G has only finite vertex groups. The result now follows from the fact that a graph of groups with all vertex groups finite is virtually free. □

Staillings showed that ends can be described in terms of cohomology, in particlar if G has *cohomolgical dimension* 1 then G is torsion-free and has at least 2 ends. Hence Stallings Theorem implies the following, which was generalized to arbitrary (non-finitely generated) groups by Swan.

Theorem 6.34 (Stallings-Swan)**.** *G is a free group if and only if $cd(G) = 1$.*

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