Gaussian fluid models; a survey

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Abstract

In this paper we survey problems concerning fluid models with a Gaussian input. We consider the transient fluid queue $Q(T)$ at time T and the steady state fluid queue Q^*. For both the cases we show theorems on asymptotics $\mathbb{P}(Q(T) > u)$ or $\mathbb{P}(Q^* > u)$, as $u \to \infty$. We include into our study Long and short range dependence cases.

1 Introduction

Fluid models have been used in modeling and analysis of high-speed telecommunication networks. For example in ATM networks informations flows in the form of 53-byte packets or cells. The high-speed (e.g. 155-622 Mbits/sec) of the ATM network implies that it can transmit millions of cells per second. This makes fluid models useful in describing the flow of cells. Suppose that $\xi(t)$ is the cumulative amount fluid (information, work) input to the system over $[0, t]$. Furthermore we assume that the service mechanism deterministically removes the fluid from the system at rate $\mu > 0$. Assuming that there is $Q(0)$ of the fluid in the system at $t = 0$, the amount of fluid at time t (or the *fluid queue*) is given by

$$Q(t) = \sup_{0 \le s \le t} \left(Q(0) + \xi(t) - \mu t, \xi(t) - \xi(s) - \mu(t-s)\right) . \tag{1.1}$$

Different assumptions have been made in literature concerning the input process $\xi(t)$, while studying the process $Q(t)$:

- $\xi(t) = \int_0^t Z(s)\, ds$, where $Z(t)$ is a finite CTMC; classical models are due to Kosten (1986), Anick, Mitra & Sondhi (1982),

- $\xi(t) = \int_0^t Z(s)\,ds$, where $Z(t)$ is a superposition of on-off processes, see Palmowski & Rolski [20],

- $\xi(t) = \int_0^t Z(s)\,ds$, where $Z(t)$ is a Gaussian process; see e.g. Simonian [22], Kulkarni & Rolski [14].

The process $Z(t)$ is the rate at which the informations flow into the system. If $Z(t)$ is a superposition of a big number of on-off process (or of the processes modelling the rate of flown information from one source), then the central limit theorem suggests a Gaussian approximation for $Z(t)$; formalized models for Gaussian approximation were studied in [14, 7]. Since in case of $Z(t)$ being Gaussian, the cumulative process $\xi(t)$ is Gaussian too, we can describe the input in terms of cumulative processes. This approach has some advantages because it allows also to study cumulative processes of processes with nowhere differentiable realizations. For example it was reported by Willinger, Taqqu, Sherman & Wilson [23] that $\xi(t)$ can be a fractional Brownian motion with index $H > 1/2$, and under such the assumption the steady state fluid queue Q^* was studied by Norros [19] and also by many other authors [18, 11]. For the theory of Gaussian processes we refer to books [1, 16, 21].

2 Transient and steady state cases

Suppose we can represent $\xi(t) = \zeta(t) + \tilde{d}(t)$, where $\zeta(t)$ is a centered, Gaussian process, with stationary increments (we always postulate continuous trajectories). Here $\tilde{d}(t)$ is a deterministic function. Typically $\tilde{d}(t) = \delta(t) + \lambda t$ and $\delta(t)$ represents some deterministic perturbations caused by nonstationarity of the input process. We assume that $\tilde{d}(t)$ is a continuous function.

Consider the fluid queue $Q(T)$ at T:

$$Q(T) = \sup_{0\le s\le T} \left(\zeta(T) - \zeta(s) + \tilde{d}(T) - \tilde{d}(s) - \mu(T-s)\right) . \tag{2.2}$$

We now introduce new time $v = T - s$ and let

$$\eta(v) = \zeta(t) - \zeta(t-v), \qquad 0 \le v \le T$$

The process $\zeta(s),\ 0 \le s \le T$ has the same distribution as $\eta(s),\ 0 \le s \le T$. Thus we can write

$$\begin{aligned} \psi_T(u) &= \mathbb{P}(Q(T) > u) \\ &= \mathbb{P}(\sup_{0\le v\le T} \zeta(v) - d_T(v) > u), \end{aligned}$$

where $d_T(s) = -(\tilde{d}(T) - \tilde{d}(T-s)) + \mu s$. Since T is fixed in this model we simply write $d(s)$. Notice that in the important case $d(t) = \lambda t$ we have $d_T(s) = (\mu - \lambda)s$.

Similarly like in the theory of queues, the steady state amount of fluid in the system is more frequently studied, instead from the transient process $Q(t)$. However for the steady state case some further assumptions on $\xi(t)$ are required. In the sequel we assume that

- $\xi(t)$ has stationary increments,
- it is ergodic, so $\xi(t)/t \to \lambda$, with probability 1,
- $\xi(t)$ has continuous realizations.

Then $\xi(t) = \zeta(t) + \lambda t$, where $\zeta(t)$ is a centered Gaussian process with continuous trajectories and $\zeta(0) = 0$. From now on this will be tacitly assumed on $\zeta(t)$. Thus the stationary fluid queue process $Q^*(t)$ can be represented by the formula

$$Q^*(t) = \sup_{s \le t}(\zeta(t) - \zeta(s) - c(t-s)), \tag{2.3}$$

where $c = \mu - \lambda$ and $c > 0$ for stability.

The process $\zeta(t)$ is ergodic if and only if $\sigma_\zeta^2(t)/t^2 \to 0$; see [24]. For the generic steady state random variable we simply write $Q^* = Q^*(0)$. Futher on we will denote the variance function of $\zeta(t)$ by

$$\sigma_\zeta^2(t) = \mathbb{D}^2\zeta(t), \qquad t \ge 0.$$

For the derivative of $\sigma_\zeta^2(t)$ we write $\dot{\sigma}_\zeta^2(t)$ and we tacitly assume that the variance function is differentiable. We also write $\sigma_\zeta(t) = \sqrt{\sigma_\zeta^2(t)}$.

3 Important examples

We now present a few important examples, which will serve as illustrations of the general theory presented below.

We say that $\zeta(t)$ is *integrated Gaussian* (IG) if for some process $Z(t)$ (it must be stationary and ergodic and of course centered Gaussian)

$$\zeta(t) = \int_0^\infty Z(s)\, ds, \qquad t \ge 0\ .$$

Notice that if $R(t) = \mathbb{E}Z(0)Z(t)$ is the covariance function of $Z(t)$, then
$\sigma_\zeta^2(t) = 2\int_0^t \int_0^s R(v)\, dv\, ds$.

We say that a process with stationary increments and a variance function $\sigma^2(t)$ has *Hurst index* H, if

$$H = \sup\{h > 0 : \limsup_{t\to\infty} t^{-2h}\sigma^2(t) = \infty\}.$$

Such the definition was adopted in the theory of point processes; see e.g. [2].

The following special cases are of interest.

- *Short range dependent* case (SRD) if $H \le 1/2$. For the IG case it means $\int_0^\infty |R(t)|\, dt < \infty$; however we will require more, that is $\int_0^\infty t^2|R(t)|\, dt < \infty$. In this case

$$\sigma^2(t) = 2t/G - 2B + o(t^{-1}), \qquad \text{for } t \to \infty,$$

  where $G = 1/\int_0^\infty R(t)\, dt$ and $B = \int_0^\infty tR(t)\, dt$.

- *Long range dependent* case (LRD) is typically considered when Hurst index $H > 1/2$. For the IG case it means that $\int_0^\infty |R(t)|\, dt = \infty$; a special case of interest is when $R(t) = t^{-2(1-H)}L(t)$ for $H > 1/2$, where $L(t)$ is a slowly varying function at ∞.

Gauss-Markov flow. Let $Z(t)$ be defined by $Z(t) = \sum_{j=1}^n X_j(t)$ where $\boldsymbol{X}(t) = (X_1(t), \dots, X_n(t))^T$ is a stationary *Gauss-Markov process.* That is $\boldsymbol{X}(t)$ is a Gaussian process in $\mathbb{R}^n$ with mean zero and covariance function

$$\boldsymbol{R}(t) = \boldsymbol{\Sigma} e^{t\boldsymbol{A}^T}, \tag{3.4}$$

where $\boldsymbol{\Sigma}$ is a covariance matrix and $\boldsymbol{A}$ has strictly negative real parts of its eigenvalues, fulfilling for some nonsingular matrix $\boldsymbol{B}$ the following equation

$$\boldsymbol{A\Sigma} + \boldsymbol{\Sigma A}^T = -\boldsymbol{BB}^T.$$

This process is also a diffusion process with infinitesimal drift $\boldsymbol{Ax}$ and infinitesimal variance $\boldsymbol{BB}^T$, or in terms of sde it fulfills

$$d\boldsymbol{X}(t) = \boldsymbol{AX}(t)\, dt + \boldsymbol{B}\, d\boldsymbol{W}(t), \qquad t \ge 0 \ ,$$

where components of $\boldsymbol{W}(t)$ are independent Brownian motions. In this note vectors and matrices are denoted by bold type characters and vectors are column vectors. A special case of interest for $n = 1$ is the *Ornstein-Uhlenbeck process* $X(t)$. In this case $\boldsymbol{A} = -a$, $\boldsymbol{BB}^T = 2r^2a$ and then $R(t) = r^2e^{-at}$.

FBM. A process $\zeta(t)$ is a fractional Brownian motion (FBM) with Hurst index $0 < H \le 1$ if $\sigma^2_\zeta(t) = t^{2H}$. This process cannot be of IG type because realizations are nowhere differentiable. Since for a process with stationary increments

$$\mathbf{Cov}(\zeta(s), \zeta(t)) = \frac{1}{2}(\sigma^2_\zeta(t) + \sigma^2_\zeta(s) - \sigma^2_\zeta(|t-s|),$$

for the FBM case

$$\mathbf{Cov}(\zeta(s),\zeta(t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \ .$$

Moreover for the covariance of increment sequence $\{\zeta_i\}$ defined by $\zeta_i = \zeta(hi) - \zeta(h(i-1))$ we have for $H < 1$

$$\rho_n = \mathbb{E}\zeta_1\zeta_{n+1} = 2H(2H-1)h^{2H}n^{2(H-1)} + o(n^{2(H-1)}).$$

Note that we have three cases: LRD for $H > 1/2$, the Brownian motion for $H = 1/2$ and SRD for $H < 1/2$.

4 Asymptotics of the $\mathbb{P}(Q^* > u)$

In principle the distributions of $\psi(u) = \mathbb{P}(Q^* > u)$ and $\psi_T(u) = \mathbb{P}(Q(T) > u)$ are unknown. Therefore the study of asymptotics for these functions as $u \to \infty$ attracted some attention. The following are possible results which we are interested in this study.

- *logarithmic asymptotics* – results like $(-\log\psi(u))/g(u) \to Const > 0$, for some $g(u)$ converging to ∞.
- exact asymptotics – results like $\psi(u) = f(u)\exp(-g(u))(1 + o(1))$, where $\log f(u) = o(g(u))$.

4.1 Generalized Pickands constants

For a centered Gaussian process $\zeta(t)$ with stationary increments we define

$$\mathcal{H}(\zeta) = \lim_{T\to\infty} \frac{\mathcal{H}(\zeta;T)}{T} \ ,$$

where

$$\mathcal{H}(\zeta;T) = \mathbb{E}\exp[\max_{t\in[0,T]}(\sqrt{2}\zeta(t) - \sigma_\zeta^2(t))] \ .$$

In [4, 5] it was shown that the constants are well defined and positive if
C1 $\sigma_\zeta^2(t) \in C_1([0,\infty))$ is strictly increasing and

$$\frac{\dot\sigma_\zeta^2(t)}{\sigma_\zeta^2(t)} = O(t^{-1}) \ .$$

C2 $\sigma_\zeta^2(t)$ is regularly varying at 0 with index $\alpha_0 \in (0,2]$ and at ∞ with index $\alpha_0 \in (0,2)$.

If $\zeta(t)$ is FBM with Hurst index $H = \alpha/2$, then $\mathcal{H}_\alpha = \mathcal{H}(\zeta)$ are called Pickands constants; see [21] and references therein. Even for Pickands constants little is known on theirs numerical values, except from $\mathcal{H}_1 = 1$ and $\mathcal{H}_2 = 1/\sqrt{\pi}$.

4.2 Function $m(t)$

The crucial role for the asymptotics plays the following function. For the steady state case we define

$$m(u) = \min_{t \geq 0} \frac{(u+ct)}{\sigma_\zeta(t)}. \tag{4.5}$$

The minimum of $m(u)$ is attained for t_u fulfilling $c\phi(t_u) = u$, where

$$\phi(x) = \frac{2\sigma_\zeta^2(t)}{\dot{\sigma}_\zeta^2(t)} - t.$$

For the uniqueness we need to assume that $\phi(x)$ is ultimately increasing, at least for big values of u. In most cases t_u increases at least linearly fast, as $u \to \infty$ and this will be tacitly assumed later. This is for example true when

- $\dot{\sigma}_\zeta^2(t)$ is converging to a positive constant,
- $\dot{\sigma}_\zeta^2(t) = t^\beta L(t)$, where $0 < \beta \leq 1$ and $L(t)$ is slowly varying function.

For the transient case we need to assume that

$$g_u(t) = \frac{u+ct}{\sigma_\zeta(t)}$$

is increasing (at least ultimately for u big) and then

$$m(u) = \frac{u+cT}{\sigma_\zeta(T)}\ .$$

4.3 Steady state; general case

For the short range dependent case consider the IG model under the following assumptions:

SRD.1 $R(t)$ is continuous and $\lim_{t\to\infty} tR(t) = 0$,

SRD.2 $\int_0^\infty t^2 R(t)\, dt < \infty$,

SRD.3 $\int_0^\infty R(t)\, dt > 0$.

Then we have

Theorem 4.1 ([4])

$$\psi(u) = \frac{\mathcal{H}(\frac{cG}{\sqrt{2}}\zeta)}{Gc^2} e^{-c^2 G^2 B} e^{-Gcu}(1+o(1)). \tag{4.6}$$

It was shown in [4, 5] that $\mathcal{H}(\eta) \leq \mathbb{D}^2 Z(t)/\sqrt{\pi}$. Hence we rediscover again the following old result:

Theorem 4.2 ([8])

$$\psi(u) \leq \sqrt{\frac{\mathbb{D}^2 Z(t)}{2c^2\pi}} e^{-c^2 G^2 B} e^{-Gcu}(1+o(1))$$

Some further estimations for asymptotics of $\psi(u)$ were given in [6].

For the FBM case we have the following result. Let

$$C(H) = \frac{\mathcal{H}_{2H}\sqrt{\pi}}{2^{(1-H)/2H}\sqrt{H}} \left(\frac{H}{c(1-H)}\right)^{H-1} \left(\frac{1}{1-H}\right)^{(2-H)/H}$$

and

$$a(H) = \left(\frac{H}{c(1-H)}\right)^{-H} \frac{1}{1-H} .$$

We also denote

$$\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{\frac{v^2}{2}}\, dv .$$

Theorem 4.3 ([18, 11])

$$\psi(u) = C(H) u^{(1-H)^2/H} \psi\left(a(H) u^{1-H}\right)(1+o(1)).$$

Using that $\Psi(u) \sim \exp(-u^2/2)/(u\sqrt{2\pi})$ we have that if $H > 1/2$, then for some $w > 0$ and $v < 1$ $\psi(u) \sim u^{-w}\exp(-u^v)$ and if $H < 1/2$, then for some $w > 0$ and $v > 1$ $\psi(u) \sim u^w \exp(-u^v)$.

4.4 Markovian case

The following model allows to study fluid flows with several traffic flows without independence assumptions between flows. Here $\zeta(t) = \int_0^t \sum_{j=1}^n X_j(s)\, ds$, where $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))^T$ is the stationary Gauss-Markov process defined in Section 3. The covariance function of process $Z(t) = \sum_{j=1}^n X_j(t)$ is $R(t) = \boldsymbol{e}^T \boldsymbol{\sigma} e^{\boldsymbol{A}^T t} \boldsymbol{e}$, where $\boldsymbol{e} = (1, \ldots, 1)^T$. Let

$$G = \frac{2}{\boldsymbol{e}^T \boldsymbol{A}^{-1} \boldsymbol{B}\boldsymbol{B}^T (\boldsymbol{A}^T)^{-1} \boldsymbol{e}}.$$

The following theorem extends results from [14, 15], where only one dimensional Ornstein-Uhlenbeck process or the sum of independent Ornstein-Uhlenbeck processes were studied.

Theorem 4.4

$$\psi(u) = \mathbb{P}(\sup_{t\geq 0}(\zeta(t) - ct) > u) = e^{-cGu}\tilde{\mathbb{E}}e^{\boldsymbol{\theta}_0(\boldsymbol{X}(0) - \boldsymbol{X}(\tau(u)))},$$

where $\tilde{\mathbb{E}}$ means that the expectation is computed under the Gaussian law with mean

$$\boldsymbol{m}(t) = G(\boldsymbol{I} - e^{-\boldsymbol{A}t})\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{B}^T(\boldsymbol{A}^T)^{-1}\boldsymbol{e}$$

and covariance function

$$\tilde{\boldsymbol{R}}(t) = \boldsymbol{\Sigma}e^{\boldsymbol{A}^T t},$$

$\boldsymbol{\theta}_0 = cG\boldsymbol{e}^T\boldsymbol{A}^{-1}$, *and* $\tau(u) = \min\{t > 0 : \zeta(t) + ct = u\ \}$.

Note that $G = 1/\int_0^\infty R(t)\,dt$. Thus using now (4.6) we have the following corollary:

Corollary 4.5

$$\lim_{u\to\infty} \tilde{\mathbb{E}}e^{\boldsymbol{\theta}_0(\boldsymbol{X}(0) - \boldsymbol{X}((\tau(u)))}$$

exists and is positive and finite.

Although a chance for a nice theoretical formula for the limit seems to be hopeless, we believe that this limit can be simulated.

If $Z(t)$ is a one dimensional Orstein-Uhlenbeck process, then from Theorem 4.4 we can conclude the following double sided exponential bounds:

Theorem 4.6 ([14]) *If $Z(t)$ is the Orstein-Uhlenbeck process, then*

$$\Psi\Big(\frac{c}{r}\Big)\exp\Big(-\frac{c^2}{r^2}\Big)\exp\Big(-\frac{\alpha c}{r^2}u\Big) \leq \psi(u) \leq \exp\Big(-\frac{c^2}{r^2}\Big)\exp\Big(-\frac{\alpha c}{r^2}u\Big), \qquad u \geq 0.$$

4.5 Transient queue; general case

For the transient fluid queue $Q(T)$ we have the following result about asymptotics of $\psi_T(u)$, for $u \to \infty$. Here $\zeta(t)$ is an IG process with a continuous covariance function $R(t)$ of $Z(t)$ and such that the variance function $\sigma_\zeta^2(t)$ is strictly positive or it is FBM with Hurst index $H > 1/2$. We also assume that $Q(0) = 0$ and $d(t)$ fulfilling $|d(s) - d(t)| \leq C_d|s - t|$, $0 \leq s, t \leq T$ for some $C_d > 0$.

Theorem 4.7 ([9]) *For $u \to \infty$*

$$\psi_T(u) = \Psi\left(\frac{u + d(T)}{\sigma_\zeta(T)}\right)(1 + o(1))\ .$$

From this theorem we can see that for the transient case the values of $\mathbf{Cov}(\zeta(t),\zeta(s))$ for $0\le t,s\le T$ and $d(t), 0\le t\le T$ are needed only, therefore the properties of SRD or LRD are not utilized. This confirms the observations from [13, 12].

We also have the following surprising result. In this case $d(t)=ct$.

Theorem 4.8 ([9])

$$\mathbb{P}(\sup_{0\le t\le T} Q(t)>u)=\Psi\left(\frac{u+cT}{\sigma_\zeta(T)}\right)(1+o(1))\ .$$

The case when the FBM is with hurst index $H<1/2$ is also studied in [9].

5 Existence of moments for the steady state case

Consider Q^* as defined in (2.3). Following Dębicki [4] or Duffield and O'Connell [10]

$$\lim_{u\to\infty}\log\mathbb{P}(Q^*>u)/m^2(u)=-\frac{1}{2}\ , \tag{5.7}$$

where $m(u)$ was defined in (4.5). Then from (5.7)

$$\mathbb{P}(Q^*>u)\le e^{-\frac{1-\delta}{2}m^2(u)},$$

for some $\delta>0$ Recall also that assumed that $\liminf_{u\to\infty} t_u/u>0$. If $H<1$ we can write for some $0<\epsilon<2-2H$

$$m^2(u)=u^{2-2H-\epsilon}\left(\frac{u+t_u}{t_u}\right)^2\frac{t_u^{2-2H-\epsilon}}{u^{2-2H-\epsilon}}\frac{t_u^{2H+\epsilon}}{\sigma_\zeta^2(t_u)}.$$

Hence $\int_0^\infty t^{p-1}\mathbb{P}(Q^*>t)\,dt<\infty$ which yields $\mathbb{E}[Q^*]^p<\infty$. This shows that all moments of Q^* are finite. The question is also how important is our assumption that $H<1$. The variance function $\sigma^2(t)$ has representation

$$\sigma^2(t)=\int_{\mathbb{R}}\frac{1-\cos(at)}{a^2}\,M(da),$$

where M is a measure such that $\int\frac{1}{1+a^2}\,M(da)<\infty$; see e.g. [24]. Representing $M(dx)=f(x)\,dx+M^{\mathrm{s}}(dx)$, where M^{s} is a singular part, we need that

$$\int_{\mathbb{R}}\frac{1}{1+a^2}f(a)\,da<\infty.$$

In the case when $f(x) = x^{-\alpha}$ for $0 < \alpha < 1$ we have $\sigma^2(t) = c(\alpha)t^{\alpha+1}$. We remark that one can find $f(x) \geq 0$ such that $\sigma^2(t)/t^2 \to 0$ but $\sigma^2(t)/t^{2-\epsilon} \to \infty$, for all $\epsilon > 0$. Namely take

$$f(a) = \sum_{i=1}^{\infty} \frac{b_i}{c(\alpha_i)} t^{-\alpha_i},$$

where $0 < \alpha_i \uparrow 1$, $b_i > 0$ and such that

$$\sum_{i=1}^{\infty} \frac{b_i}{(1-\alpha_i)c(\alpha_i)} < \infty .$$

Therefore we cannot conclude that always all moments of Q^* are finite because we do not know how fast $m^2(t)$ grows.

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