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On a Poisson hyperbolic staircase

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Abstract

We consider a process that starts at height y, stays there for a time $X_0 \sim \exp(y)$ when it drops to a level $Z_1 \sim U(0,y)$. Thereafter it stays at level Z_n for time $X_n \sim \exp(Z_n)$, then drops to a level $Z_{n+1} \sim U(0, Z_n)$. We investigate properties of this process, as well as the Poisson hyperbolic process which is obtained by randomizing the starting point y of the above process. This process is associated with a rate 1 Poisson process in the positive quadrant: Its path is the minimal RCLL decreasing step function through Poisson points in the positive quadrant. The finite dimensional distributions are then multivariate exponential in sense of Marshall-Olkin.

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1 Introduction

Consider the process $Z_z(t), t \geq 0$ which starts at time 0 at level z. The process stays at level z for a time $X_0 \sim \exp(z)$, and then it drops to a level $Z_1 \sim U(0,z)$. Thereafter, the process remains at the level Z_n for a time $X_n \sim \exp(Z_n)$, and then it drops to a level $Z_{n+1} \sim U(0, Z_n)$, for $n = 1, \ldots$. Realizations of this process form *decreasing step functions*, which we take as right continuous with left limits, RCLL. We denote the special case $Z_1(t) = Z(t)$.

There are two scenarios in which this process may arise. The first is associated with an epidemic: For a population of many items, all of which start off as being susceptible to some infectious disease, the time until an outbreak may be exponential, with a rate proportional to the fraction of susceptibles in the population. Once an outbreak occurs, it spreads through the population within a negligible time, and decimates a random uniformly distributed fraction of the susceptibles (through death or immunization).

The second scenario is based on a Poisson process on the plane, with rate 1. Consider the process in the strip $x > 0$, $0 < y < z$, and form a RCLL decreasing step function through points of the process, starting at time zero from z and choosing at each step the next lower point; see Fig 1.

Define $Z_{t_0,x_0}(t)$ as the level of the random decreasing step function at time $t \geq t_0$, given that the process started at level x_0 at time t_0, and proceeded as before. We prove that $\{Z_{t_0,x_0}(t),\ t \geq t_0\}$ is a Markov process, and moreover it has the following scaling behavior:

$$Z_{t_0,x_0}(t) \stackrel{D}{=} x_0 Z(x_0(t - t_0)), \qquad t \geq t_0. \tag{1.1}$$

In Section 2 we study the process $\{Z_z(t)\}$ in detail. In view of the scaling behavior it suffices to study $\{Z(t)\}$. First we derive the properties of the sequences X_n, Z_n. Next we show that $Z(t)$ is distributed as $\exp(t) \wedge 1$, that is an exponential random variable with parameter t truncated at 1. Finally we study the counting process $N(t)$ of the events (the downwards jumps) up to time t.

The process $Z_{t_0,x_0}(t),\ t \geq t_0$ is of course transient, and therefore has no stationary distribution. It does however have a canonical form, which is the main object of study in this paper. Thus in Section 3 we consider the process $Z^*(t),\ t > 0$ obtained from the random starting conditions $Z^*(1) \sim \exp(1)$ and such that $\{Z^*(t),\ t \geq 1\} \stackrel{D}{=} \{Z_{1,M}(t),\ t \geq 1\}$, where $\{Z_{1,x}(t),\ t \geq 0, x \geq 0\}$ and M are independent and $M \sim \exp(1)$. The reversed process is described constructively as follows: For all $t > 0$, if the process is at level x at time t, the previous event is uniformly distributed on $(0,t)$, and the previous level is $x+W$, where $W \sim \exp(t)$. By considering the original process for $t > 1$, and the reversed process for $0 < t < 1$, both with $Z^*(1) \sim exp(1)$, we obtain a canonical process $\{Z^*(t),\ t > 0\}$ defined for all $t > 0$. We call this process *the Poisson Hyperbolic Staircase.*

We also find that the point process N^* of jumps of Z^* is a non-homogeneous Poisson process, with rate $1/t$, so $N^*(1,t]$, the number of jumps in $(1,t]$, is a Poisson($\log(t)$) random variable. Next we reverse time and show that for $0 < t < 1$, $Z^*(t) \sim \exp(t)$, and $N^*(t,1] \sim$ Poisson($-\log(t)$).

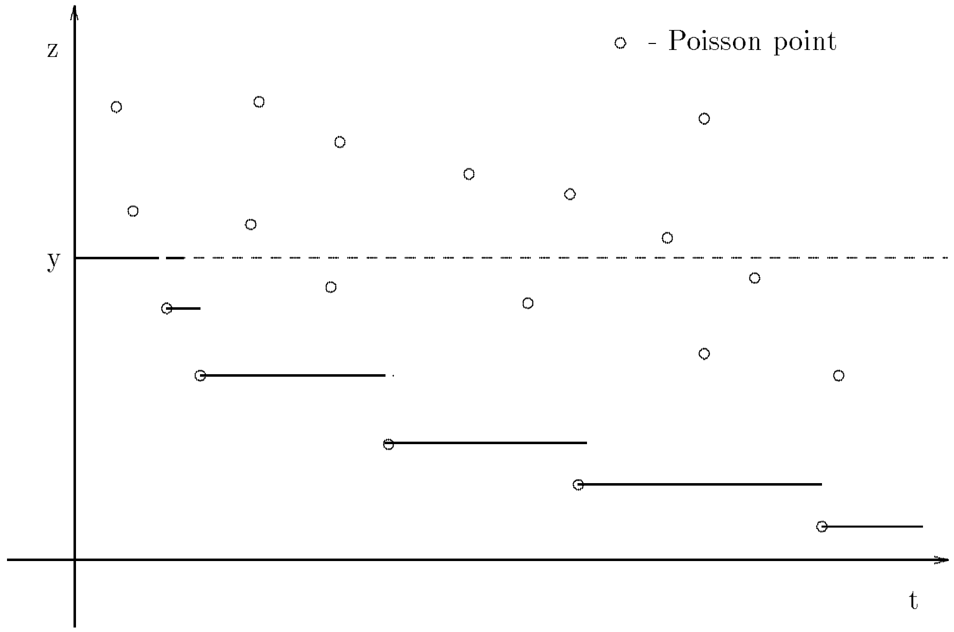

Figure 1.1: A typical realization of $Z_y(t)$

It turns out that the process $Z^*(t)$ can be taken as minimal RCLL decreasing step function on $(0, \infty)$ passing through points of the Poisson process on the quadrant $(0, \infty) \times (0, \infty)$; see Fig. 1.

The Poisson hyperbolic staircase has several remarkable properties:

1. The process is Markovian, with finite dimensional distributions which are multivariate exponential in the sense of Marshall-Olkin [3, 8], see also Kopocinski [10],

   in particular $Z^*(t) \sim \exp(t)$ and hence $E(Z^*(t)) = 1/t$,

2. The counting process of the jumps is a non-homogeneous Poisson process with rate $1/t$.

3. The process fulfills the distributional equation that $\{Z^*(\cdot)\} \stackrel{D}{=} \{aZ^*(a\cdot)\}$, where $a > 0$. Notice that functional equation $f(x) = af(ax)$ has the solution $f(x) = f(1)/x$ and hence the name hyperbolic.

4. The process $Z^*(\cdot)$ is symmetric around the line $x = t$: The inverse process $Z^{*-1}(\cdot)$ defined by $Z^{*-1}(y) = \inf\{x : \ Z(x) \leq y\}$ is equal in distribution to $Z^*(\cdot)$.

5. The canonical process is the limit of the process $Z_{0,1}(t)$ in the sense that $tZ_{0,1}(t)$ converges weakly to $tZ^*(t)$ outside the interval $(1/L, L)$ as $L \to \infty$.

In Section 4 we relate the process $Z^*(t) \sim \exp(t)$ to a planar Poisson process of rate 1 in the positive quadrant. Define the points of the process 'closest' to the positive axes as

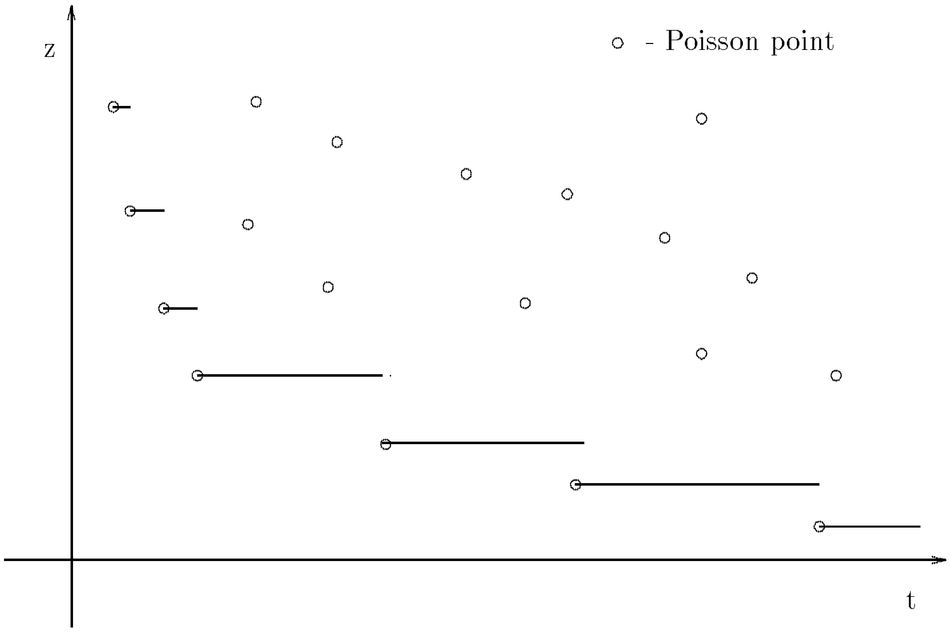

Figure 1.2: A typical realization of $Z^*(t)$

those points (x, y) such that the rectangle $(0, x) \times (0, y)$ has no other points of the Poisson process. Then these points form a doubly infinite sequence, increasing in x and decreasing in y, and 'hugging' the axes. The random step function through these points is exactly our process $Z^*(t)$. To show this let (X_0, Y_0) be the first point of the Poisson process reached by a square growing from $(0, 0)$. Conditional on $X_0 = x_0, Y_0 = y_0$ we look at $Z_{x_0,y_0}(t)$, and unconditioning we obtain $Z^*(t)$. A recent interest in decreasing step function defined by a planar Poisson process is also generated by the research on Hammersley problem; see [7, 2] and [5], Example 6.7.2.

We also consider the point processes generated by projecting the hyperbolic staircase onto the axes. Contrary to our expectation, these processes are not independent.

We believe that our derivation and detailed study of the Poisson hyperbolic staircase are new. At the same time though, the Poisson hyperbolic staircase turns out to be a special case of an extremal process as originated by Dwass [6, 1964] and described in [12, Resnick 1987]. We explain this connection in Section 5.

2 The random decreasing step function with fixed initial state

2.1 Embedded sequences.

In view of the scaling property (1.1) it suffices to study only the process $Z(t)$. The intervals $\{X_n\}_{n=0}^{\infty}$, and the levels $\{Z_n\}_{n=1}^{\infty}$ can be constructed from two independent sequences of i.i.d random variables, $\{V_n\}_{n=0}^{\infty}$, $V_n \sim \exp(1)$, and $\{U_n\}_{n=1}^{\infty}$, $U_n \sim U(0,1)$. With these we can write:

$$Z_n = \prod_{j=1}^{n} U_j, \quad X_n = V_n / \prod_{j=1}^{n} U_j, \quad n = 0, 1, \ldots$$

this also defines $Z_0 = 1$.

The probability density function of Z_n can be calculated recursively, $f_{Z_1}(u) = 1$, $0 < u < 1$,

$$f_{Z_n}(u) = \int_u^1 \frac{1}{v} f_{Z_{n-1}}(v) dv, \qquad 0 < u < 1$$

but an easier way is to consider $W_n = -\log(Z_n)$. As an independent sum of the $-\log U_j$, we have $W_n \sim \gamma(n, 1)$ (gamma density with shape parameter n and rate parameter 1). Use $f_{W_n}(w) = \frac{w^{n-1}}{(n-1)!} e^{-w}$ to get, for $n > 1$:

$$f_{Z_n}(u) = f_{W_n}(-\log u) \frac{1}{u} = \frac{(-\log u)^{n-1}}{(n-1)!}, \quad 0 < u < 1.$$

The expectation of Z_n is simply:

$$E(Z_n) = E(\prod_{j=1}^{n} U_j) = 1/2^n.$$

To study the properties of X_n, we note

$$P(X_n > x) = E(e^{-xZ_n}) \stackrel{\text{def}}{=} \phi_{Z_n}(x) = \int_0^1 \frac{(-\log u)^{n-1}}{(n-1)!} e^{-xu} du$$

which unfortunately is not very informative.

A surprise is the fact that for $n = 1, \ldots$, $E(X_n)$ is infinite:

$$E(X_n) = E(E(X_n \,|\, Z_n)) = E(\frac{1}{Z_n}) = \left(E(\frac{1}{U_1})\right)^n = \infty.$$

2.2 The continuous time level process

Let $Z(t)$ denote the level of the process at time $t \geq 0$. This is the value of Z_n for the last jump at or before t that is

$$Z(t) = Z_n, \qquad \text{for} \qquad \sum_{j=0}^{n-1} X_j \leq t < \sum_{j=0}^{n} X_j,$$

in particular, $Z(0) = 1$.

The following recursion holds: for each $t > 0$

$$\{(Z(t+s)|Z(t) = z), \ s \geq 0\} \stackrel{D}{=} \{z\breve{Z}(zs), \ s \geq 0\}$$

or equivalently:

$$Z(t+s) = Z(t)\breve{Z}(Z(t)s),$$

where $\{\breve{Z}(t)\}$ is an independent copy of $\{Z(t)\}$. In words, the continuation of the process beyond t evolves like the process itself starting from zero, with time and space scaled by the value of the process at time t, $Z(t)$. Note that $Z(t+s)|Z(t)=z) = (Z(t+s)|Z(t) = z, Z(r)\ 0<r<t)$, so $\{Z(t)\}$ is Markov.

In particular, we can condition on the first event, and obtain a renewal like relation. With $X_0 = \tau$ and $Z_1 = U_1 = u$,

$$Z(t) = \begin{cases} 1 & t<\tau, \\ u\breve{Z}(u(t-\tau)) & t \geq \tau. \end{cases} \tag{2.1}$$

From equation (2.1) we can calculate for example the distribution of $Z(t)$.

We denote by $\exp(t)\wedge x$ the distribution of $(V_0/t)\wedge x$, where we use notation $x\wedge y = \min(x,y)$. In the proof of the following proposition we use Banach's fixed point theorem which says: let $\mathcal{X}$ be a complete metric space with a metric ρ and $\Phi : \mathcal{X}\to\mathcal{X}$ is a mapping fulfilling the contraction principle that $\rho(\Phi(x),\Phi(y)) \leq K\rho(x,y)$, where $K<1$; then there exists a unique point x such that $x = \Phi(x)$. We apply this theorem with $\mathcal{X}$ being the space of bounded and continuous functions on $[0,\infty)\times[0,T]$, with the sup norm.

Proposition 2.1

$$Z(t) \sim \exp(t)\wedge 1.$$

Proof A similar application of the renewal argument (2.1) can be used for the Laplace transform of $Z(t)$, $\varphi(s,t) = E(e^{-sZ_{0,1}(t)})$,

$$\varphi(s,t) = e^{-(s+t)} + \int_0^t\int_0^1 e^{-(t-\tau)}\varphi(su,\tau u)\,dud\tau. \tag{2.2}$$

It is easy to check that this equation has a unique solution: The mapping

$$\Phi : \varphi(s,t) \to \int_0^t\int_0^1 e^{-(t-\tau)}\varphi(su,\tau u)\,dud\tau + e^{-(s+t)}$$

for φ a bounded function over $(s,t)\in[0,\infty)\times[0,T]$ is a contraction for every fixed $T>0$ and this proves uniqueness of $\varphi(s,t), 0<s,t<\infty$. Simple manipulation of (2.2) yields the integral-differential equation:

$$\frac{\partial}{\partial t}\varphi(s,t) + \varphi(s,t) = \int_0^1 \varphi(su,tu)du.$$

By inspection we check that

$$\varphi(s,t) = \frac{t+se^{-(s+t)}}{t+s} \tag{2.3}$$

is the desired solution. It is now immediate to see that $\varphi(s,t)$ is the Laplace transform of $1\wedge\exp(t)$. $\square$

An alternative approach to derive Proposition 2.1, is to write a set of differential equations for the moments of $Z(t)$, based on the renewal type argument and scaling property:

$$km_k(t) + (t+k+1)m_k'(t) + tm_k''(t) = 0, \quad m_k(0) = 1, \quad k = 1, \dots$$

which is solved by

$$m_k(t) = \frac{k!}{t^k}(1 - e^{-t}\sum_{j=0}^{k-1}\frac{t^j}{j!}).$$

¿From these we obtain the Laplace transform of $Z(t)$ as in (2.3). In particular we have:

$$m(t) = EZ(t) = \frac{1}{t}(1 - e^{-t}). \tag{2.4}$$

2.3 Scaling and joint distributions

The result of Section 2.2 says that apart from the unavoidable upper bound of 1, $Z(t)$ has an exponential distribution, with parameter t. Consider now the process $Z_z(t)$, with the same structure as $Z(t)$, but starting at time 0 with the initial value z, $Z(0) = z$. The process stays at the level z for a time $\exp(z)$ then drops to a lower level $\sim U(0, z)$, and so on. This process is a scaled version of the original process $Z(t)$, with level multiplied by z, while the time in the new process moves z time faster: $Z_z(t) \stackrel{D}{=} zZ(tz)$. Hence, $Z_z(t) \sim z \wedge \exp(tz/z) = z \wedge \exp(t)$.

Combining the Markov property and the scaling property, we can now write the joint density of $Z_z(t_1), \dots, Z_z(t_l)$, for any $t_1 < \cdots < t_l$:

$$f_{z;t_1,\dots,t_l}(z_1, \dots, z_l) =$$

$$t_1 e^{-t_1 z_1}(t_2 - t_1)e^{-(t_2-t_1)z_2} \cdots (t_l - t_{l-1})e^{-(t_l - t_{l-1})z_l}, z_l < z_{l-1} < \cdots < z_1 < z.$$

At the boundary of $z_l < z_{l-1} < \cdots < z_1 < z$ the distribution is singular, and outside the region there is no probability mass. Careful integration over the relevant regions yields a better description through the joint survival function:

$$\bar{F}_{z;t_1,\dots,t_l}(z_1, \dots, z_l) = P(Z_x(t_1) > z_1, \dots, Z_x(t_l) > z_l)$$

$$= \begin{cases} e^{-t_1 \max(z_1,\dots,z_l)} e^{-(t_2-t_1)\max(z_2,\dots,z_l)} \cdots e^{-(t_l - t_{l-1})z_l}, & 0 \le z_j < z,\ j = 1, \dots, l \\ 0, & \text{if } z_j \ge z \text{ for some } j. \end{cases} \tag{2.5}$$

We now explore the connection between the process $Z(t)$ and the planar Poisson process. Recall the definition of a planar Poisson process, homogeneous with rate 1: It is a process of random points in the plane $\mathcal{R}^2$, such that for any Borel set $A \subset \mathcal{R}^2$ the number of points in A, denoted $\mathcal{N}(A)$ has a Poisson distribution with parameter $\ell(A)$ where ℓ is Lebesgue measure (or area), and for any family of non-overlapping $\{A_\alpha\}$ the random variables $\mathcal{N}(A_\alpha)$ are independent. By its definition, $P(\mathcal{N}(A)=0)=e^{-\ell(A)}$. Similarly, one can define a planar Poisson process, homogeneous with rate 1, on a region of the plane, $D\subset\mathcal{R}^2$, by considering the points in $A\cap D$.

Consider now the planar Poisson process with rate 1 on the planar strip $0<x$, $0<y<1$. Consider the point $(x_0,y_0)=(0,1)$ and the sequence of points (x_n,y_n), $n=1,\ldots$ of the planar Poisson process in the strip such that (x_n,y_n) is the left-most point of the process with $x_n>x_{n-1}, y_n<y_{n-1}$. It follows from well known properties of the planar Poisson process that $x_{n+1}-x_n\sim\exp(y_n)$, and that $y_{n+1}\sim U(0,y_n)$. It is therefore immediately obvious that the RCLL decreasing step function through these points is $Z(x)$.

This leads to an alternative derivation of the distribution of the random variable $Z(t)$. For any $z\le 1$, the event $Z(t)>z$ is equivalent to the event that there are no points (x,y) of the Poisson process in the rectangle $(0,t)\times(0,z)$ and the probability of this event is e^{-zt}, as required. Similarly we can derive (2.5).

2.4 The counting process of events

Let $N_z(t)$ count the number of events (jumps of $Z_z(\cdot)$) in the interval $(0,t]$. For the special case $Z(0)=1$, we let $N(t)=N_1(t)$. We note the following scaling property: The process $\{N_z(t),\ t\ge 0\}$ is equal in distributions to $\{zN(zt),\ t\ge 0\}$.

If the first event happens at $V_0=\tau$, and the level falls to $U_1=u$, the conditional renewal argument for $N(t)$ is:

$$N(t)=\begin{cases}0 & t<\tau,\\ 1+\breve{N}(u(t-\tau)) & t\ge\tau.\end{cases}\tag{2.6}$$

where $\breve{N}$ is an independent copy of N. We denote

$$\mathrm{Ein}(t)=\int_0^t\frac{1-e^{-u}}{u}\,du,\qquad t>0.$$

Proposition 2.2 *The expected value of $N(t)$, is given by*

$$\nu(t)=E(N(t))=\int_0^t EZ(u)du=\mathrm{Ein}(t).\tag{2.7}$$

Proof Conditioning on the first event we obtain the renewal like equation:

$$\nu(t)=\int_0^t\left(1+\int_0^1\nu(u(t-\tau))du\right)e^{-\tau}d\tau.$$

Straightforward algebraic steps lead to the differential equation:

$$(1+t)\nu'(t)+t\nu''(t)=1,\qquad \nu(0)=1,\nu'(0)=1.$$

which is solved by (2.7). The last equation follows from (2.4). □

We can derive a similar differential equation and solve it for the probability generating function of $N(t)$

Proposition 2.3 *The probability generating function of $N(t)$ is given by*

$$M(s,t) \stackrel{\text{def}}{=} E(s^{N(t)}) = \frac{1}{\Gamma(s)\Gamma(1-s)} \int_0^1 \frac{e^{-tu}}{u^s(1-u)^{1-s}} du. \tag{2.8}$$

Proof Condition on a first event at τ, with level falling to u:

$$\begin{aligned} M(s,t) &= e^{-t} + s \int_0^t \int_0^1 e^{-\tau} M(s, u(t-\tau)) du d\tau \\ &= e^{-t} \left(1 + s \int_0^t \frac{1}{x} e^x \int_0^x M(s,w) dw dx\right). \end{aligned} \tag{2.9}$$

We multiply (2.9) by e^t, differentiate w.r.t. t, cancel e^t and multiply by t, and differentiate again w.r.t. t. We get eventually, with the notation $M'(s,t) = \frac{\partial}{\partial t} M(s,t)$, the equation:

$$(1-s)M(s,t) + (1+t)M'(s,t) + tM''(s,t) = 0,$$

with the boundary conditions $M(s,0) = 1$.

The general solution to this equation follows from [9, Kamke 1971, page 426, §C2.108], in the range $0 < s < 1$ and $t > 0$. It is:

$$M(s,t) = C_1(s) \int_0^1 \frac{e^{-tu}}{u^s(1-u)^{1-s}} du + C_2(s) \int_1^\infty \frac{e^{-tu}}{u^s(u-1)^{1-s}} du.$$

As $t \to 0$ the second integral becomes infinite, for all $0 \le s < 1$, as is seen from:

$$\begin{aligned} & \int_1^\infty \frac{e^{-tu}}{u^s(u-1)^{1-s}} du = \int_t^\infty \frac{e^{-w}}{w^s(w-t)^{1-s}} dw \\ \ge & \int_t^\infty \frac{e^{-w}}{w} dw = E_1(t) \to \infty \quad \text{as } t \to 0; \end{aligned}$$

where $E_1(t)$ is the exponential integral [1, Abramowitz and Stegun 1964 p. 228, §5.1.1, and Figure 5.1]. Hence, $C_2(s) = 0$, for $0 \le s < 1$.

Substitute $t = 0$ in the first integral, to obtain for $0 < s < 1$

$$\int_0^1 \frac{1}{u^s(1-u)^{1-s}} du = \Gamma(s)\Gamma(1-s),$$

and hence, by $M(s,0) = 1$,

$$C_1(s) = \frac{1}{\Gamma(s)\Gamma(1-s)}.$$

The proposition follows. $\square$

This form of the probability generating function does not look very useful. It is however the basis of further derivations in the next section. ¿From equation (2.9) we can also find a recursion for $a_n(t) = P(N(t) = n)$. Clearly $a_0(t) = \exp(-t)$. We now plug $\sum_{n=0}^{\infty} a_n(t)s^n$ into (2.9) to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(t)s^n &= e^{-t}(1 + s\int_0^t \frac{1}{x}e^x \int_0^x \sum_{n=0}^{\infty} a_n(w)s^n \, dw \, dx) \\ &= e^{-t} + e^{-t}\sum_{n=1}^{\infty} s^n \int_0^t \frac{1}{x}e^x \int_0^x a_{n-1}(w) \, dw \, dx.\end{aligned}$$

Hence

$$a_n(t) = e^{-t}\int_0^t \frac{1}{x}e^x \int_0^x a_{n-1}(w) \, dw \, dx, \qquad n = 1, 2, \ldots$$

In particular $a_1(t) = \mathrm{Ein}(t)$.

2.5 The area under $Z(t)$

We consider the line $Z_z(t)$ of Section 2, and study $\int_0^t Z_z(r)\, dr$. However by scaling $\{Z_z(t),\ t \geq 0\} \stackrel{D}{=} \{zZ(zt),\ t \geq 0\}$ so $\int_0^t Z_z(r)\, dr$ equals in distribution to $\int_0^{tz} Z(r)\, dr$. Hence we consider integrals of $\{Z(t)\}$ for which we derive the Laplace Stieltjes transform. Conditioning on the time of the first event, τ, and on the level u to which Z drops at τ:

$$\int_0^t Z(r)dr = \begin{cases} t & \tau > t \\ \tau + \int_0^{t-\tau} u\breve{Z}(ur)dr & 0 < \tau < t,\ 0 < u < 1. \end{cases}$$

The Laplace-Stieltjes transform of the integral satisfies:

$$\begin{aligned}\phi(s,t) &= E(e^{-s\int_0^t Z(r)dr}) \\ &= e^{-t}e^{-st} + \int_0^t e^{-\tau}\int_0^1 E(e^{-s\tau}e^{-s\int_0^{t-\tau} uZ(ur)dr})dud\tau \\ &= e^{-(1+s)t}(1 + \int_0^t e^{(1+s)\tau}\int_0^1 \phi(s,u\tau)dud\tau) \qquad (2.10)\end{aligned}$$

Multiply by $e^{(1+s)t}$, take derivative with respect to t, and cancel $e^{(1+s)t}$, to obtain:

$$(1+s)\phi(s,t) + \frac{d}{dt}\phi(s,t) = \int_0^1 \phi(s,ut)du = \frac{1}{t}\int_0^t \phi(s,r)dr.$$

Multiplying by t, and taking a second derivative with respect to t we obtain that $\phi(s,t)$ fulfills the second order differential equation:

$$t\phi'' + (1 + (1+s)t)\phi' + s\phi = 0 \qquad (2.11)$$

with initial conditions $\phi(s,0) = 1$ and $\phi^{(\prime)}(s,0) = -s$. In the following proposition we need the notation

$$k_n(s) = (-1)^n \prod_{k=0}^{n-1}((1+s)k + s).$$

In particular $k_n(-1) = 1$ and $k_n(1) = (-1)^n(2n-1)!!$. Although equation (2.11) can be reduced to equation type C2.113 from Kamke [9], we derive its solution directly.

Proposition 2.4 *The solution to (2.11) is*

$$\phi(s,t) = \sum_{n=0}^{\infty} \frac{k_n(s)t^n}{(n!)^2}.$$

Proof That $\phi(s,t)$ is indeed a solution we can check by inspection. Thus assuming that $\phi(s,t) = \sum_{n=0}^{\infty} a_n(s)t^n$, inserting it into (2.11) we deduce that

$$a_{n+1}(s) = -\frac{(1+s)n+s}{(n+1)^2} a_n(s).$$

¿From this we derive $a_n(s)$ inductively. $\square$

Note that in particular, when $s=-1$, the solution is expressed via the modified Bessel function of order 0, $I_0(x)$:

$$y(x) = E(e^{\int_0^t Z(r)dr}) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} = I_0(2\sqrt{t}) \ .$$

3 A Canonical Process

In the previous section we considered the random decreasing step function $Z(t)$ starting from $Z(0)=1$, and we saw that it had a marginal distribution $\sim \exp(t) \wedge 1$. Similarly, for $Z_x(0)=x$, the marginal distribution is $\sim \exp(t) \wedge x$. Clearly, for any initial condition the process $Z_z(t)$ is transient. However, it seems that the actual influence of the initial condition is spurious. It has no effect on the future of the process except as a truncation value. Can we get rid of this truncation? The answer is yes, and in so doing we obtain a much simpler and more elegant process $Z^*(t)$, which we call the the Poisson hyperbolic staircase. The key step is to assume that $Z^*(1) \sim \exp(1)$, or more generally, $Z^*(t) \sim$ exp(t), for some $t > 0$.

3.1 $Z(t)$ with randomized initial conditions.

Assume a random initial condition $Z(0) \sim \exp(\alpha)$. Conditional on $Z(0)=u$, $Z(t)$ is the value of the process t time units later, where the level now is multiplied by u, and time is scaled by u. As we saw in section 2.3 scaling of time and space cancels, so $Z(t)$ is a truncated exponential random variable with parameter t, and the value of the initial state appears only as the truncation value u. Hence for all $t > 0$:

$$P(Z(t) > x | Z(0) = u) = \begin{cases} e^{-tx} & x \le u, \\ 0 & x > u, \end{cases}$$

and so if we randomize:

$$\begin{aligned} P(Z(t) > x) &= \int_0^\infty P(Z(t) > x | Z(0) = u)\alpha e^{-\alpha u}\, du = \\ & \int_x^\infty \alpha e^{-\alpha u} e^{-tx} du = e^{-(t+\alpha)x}, \quad x > 0. \end{aligned}$$

Thus $Z(t) \sim \exp(t+\alpha)$, $t > 0$.

3.2 Definition of the Poisson hyperbolic staircase process.

Clearly in the same way we can start at all $s > 0$ from $Z(s) \sim \exp(s)$ and, conditional on that, obtain that $Z(t) \sim \exp(t)$, for all $t > s$.

Similar to (2.5), for $0 < t_1 < \cdots < t_l$, if $Z_{t_1} \sim \exp(t_1)$, the joint distribution at these time points is given by:

$$\begin{aligned} &\bar{F}_{t_1,\ldots,t_l}(z_1,\ldots,z_l) = P(Z(t_1) > z_1,\ldots,Z(t_l) > z_l) \\ = \quad & e^{-t_1 \max(z_1,\ldots,z_l)} e^{-(t_2-t_1)\max(z_2,\ldots,z_l)} \cdots e^{-(t_l-t_{l-1})z_l}, \quad 0 \le z_1,\ldots,z_l. \end{aligned} \tag{3.1}$$

The family of finite dimensional distributions

$$\{F_{t_1,\ldots,t_l}(z_1,\ldots,z_l), \quad t_1 < t_2 < \ldots < t_l, l = 1,2,\ldots\} \tag{3.2}$$

defined in (3.1) is a consistent family in the sense of Kolmogorov's consistency theorem and by the Kolmogorov theorem, there exist a stochastic process $\{Z^*(t),\ t > 0\}$ with such finite dimensional distributions. Note that if we let $W_1,\ldots,W_l$ be independent random variables with $W_i \sim \exp(t_i - t_{i-1})$ $(t_0 = 0)$ then we have:

$$(W_1, W_1 \wedge W_2, \ldots, W_1 \wedge W_2 \wedge \ldots \wedge W_l) \sim F_{t_1,\ldots,t_l}. \tag{3.3}$$

Each member of the family (3.2) of distributions is a multivariate exponential distribution in sense of Marshall-Olkin; see Barlow and Proschan, [3, p 127] or Johnson and Kotz [8]. By analogy with Gaussian processes the stochastic process $Z^*(t)$ belongs to the class of *exponential processes.*

Proposition 3.1 *The exponential process $Z^*(t)$ is Markov with transition density function $f_h(x,dy)$ being defined by exp$(h) \wedge x$. Moreover $Z^*(t) \sim \exp(t)$, for all $t > 0$.*

Proof By property (3.3) we easily check the Markovian property of $Z^*(t)$. $\square$

3.3 Time reversal.

Consider $t_1 < t_2$, and look at the joint distribution of $Z^*(t_1), Z^*(t_2)$, by (3.1):

$$P(Z^*(t_1) > z_1, z_2 < Z^*(t_2) \le z_2 + dz_2) = \begin{cases} e^{-t_1 z_1}(t_2 - t_1)e^{-(t_2-t_1)z_2}dz_2, & z_1 > z_2, \\ t_2 e^{-t_2 z_2}dz_2, & z_1 \le z_2. \end{cases}$$

Hence, conditioning on $Z^*(t_2) = z_2$, and recalling that $Z^*(t_2) \sim \exp(t_2)$,

$$P(Z^*(t_1) > z_1 | Z^*(t_2) = z_2) = \begin{cases} \frac{t_2-t_1}{t_2} e^{-t_1(z_1-z_2)}, & z_1 > z_2, \\ 1, & z_1 \le z_2. \end{cases} \tag{3.4}$$

Furthermore, for $t_1 < t_2 < \ldots < t_l$

$$P(Z^*(t_1) > z_1 | Z^*(t_2) = z_2, Z^*(t_3) > z_3, \ldots, Z^*(t_l) > z_l) = P(Z^*(t_1) > z_1 | Z^*(t_2) = z_2)$$

since all the terms relating to $Z^*(t_3),\ldots,Z^*(t_l)$ cancel. Of course this last property is always true for Markov processes. Hence we have shown:

Proposition 3.2 *The conditional distribution of* $(Z^*(t_1)|Z^*(s), s > t_2)$ *for* $t_1 < t_2$ *is equal to* $Z(t_2)$ *with probability* t_1/t_2*, and with probability* $1 - t_1/t_2$ *is distributed like a shifted exponential, starting from* $Z(t_2)$*, with parameter* t_1*; symbolically we write this as* $Z(t_1) \sim Z(t_2) + \text{Bernoulli}(1 - t_1/t_2)\exp(t_1)$

3.4 Construction of the reversed process.

We now discuss the sample paths of the Markov process in reversed time. For simplicity, we condition on the value of $Z^*(1)$.

Let $T = \sup\{t : 0 < t < 1, Z^*(t) > Z^*(1)\}$, then $Z^*(t) = Z^*(1)$ for $T < t \leq 1$, while $Z^*(t) > Z^*(1)$ for $0 < t < T$; this follows since the sample paths of $Z^*(t)$ are non-increasing and RCLL. Hence, T is the time of the jump prior to time 1 and the events $t \geq T$ and $Z^*(t) = Z^*(1)$ are equivalent. Hence, by Proposition 3.2, the distribution of T is:

$$P(T \leq t) = P(Z^*(t) = Z^*(1)) = t$$

and so T is uniform on the interval $(0, 1)$.

Furthermore, if at time $t < 1$, $Z^*(t) > Z^*(1)$ then by Proposition 3.2 $Z^*(t) - Z^*(1)$ is distributed exponentially with parameter t. Hence, $Z^*(T-) - Z^*(1) \sim \exp(T)$.

More generally, starting at any time s, and conditioning on the value of $Z^*(s)$, the reversed process keeps at this level for part of the distance from s to 0, in fact it moves back to a point T uniformly distributed on $(0, s)$. It then jumps up to a higher level, the height of the jump being $\sim \exp(T)$.

3.5 A constructive definition.

We show here a constructive definition of the staircase process $Z^*(t)$, showing in particular that there is a version of the process with RCLL sample paths on $(0, \infty)$. We have two Markovian mechanisms, for regular time and for reversed time: If the process is at level z at time t, then it moves forward by remaining at that level for an exponential time with parameter z, then drops to a level U uniform on $(0, z)$. Moving backwards, the process remains at level z from time t back to a time point T uniform on $(0, t)$, then jumps up to a higher level where the random jump has size $\exp(T)$.

We can now define a complete process, by specifying all its joint distributions, as follows: Start at time 1, with $Z^*(1) \sim \exp(1)$, and define the process for $t > 1$ by the forward Markovian mechanism, and for $0 < t < 1$ by the reversed Markovian mechanism. By the calculations of sections 3.1,3.3 the marginal distributions of the process for all $t > 0$ are $Z^*(t) \sim \exp(t)$ and, in general, the joint distributions are as in (3.1).

Alternatively, one can start from any time $s > 0$, with the assumption that $Z^*(s) \sim \exp(s)$, and define the process for $t > s$, and for $t < s$ using the forward and the reversed mechanism, and again for all $t > 0$, $Z(t) \sim \exp(t)$.

We have a family of constructions of Markov processes with the given transition mechanisms in forward and in reversed time, with fully determined joint distributions. Furthermore, no matter at what s we start the construction, we obtain the same joint distributions.

Now the processes are pure jump processes, and we take the versions with RCLL sample paths. Hence all these definitions construct the same unique process $Z^*(t), t > 0$, which is the canonical process.

3.6 A geometric approach.

Consider now the planar Poisson process with rate 1 on $\mathcal{R}^2$ and define $\{Z^*(t),\ t > 0\}$ as the maximal RCLL decreasing step function such that there are no points of the planar Poisson process below it. Alternatively, it is the minimal decreasing RCLL step function with the right limits at the jumps being points of the planar Poisson process. Note that $Z^*(t_1) > x_1, \ldots, Z^*(t_n) > x_n$, for $t_1 < t_2 < \ldots < t_n$, if and only if there are no points of the Poisson process in the rectangle $(0, t_1) \times (0, \max(x_1, \ldots, x_n)$, no points in the rectangle $(t_1, t_2), (0, \max(x_2, \ldots, x_n)$, etc. and hence we immediately get (3.1). We will say more about this approach in Section 4.

3.7 The canonical counting process $\{N^*(t)\}$.

In this part we consider a countning process $\{N^*(t)\}$ on $(0, \infty)$ counting events (jump points of the process $\{Z^*(t)\}$). However we first prove a lemma on a randomized version of the counting process N, which counts jumps of the process $\{Z(t)\}$.

Lemma 3.3 *If $Z(0) \sim \exp(\alpha)$, then $N(t) \sim \mathrm{Poisson}(\log\{(t+\alpha)/\alpha\})$.*

Proof Compute first, for $0 < z \leq 1$ and $a > 0$

$$\int_0^1 \frac{1}{u^z(1-u)^{1-z}(1+au)} du.$$

Substituting $v = \frac{1-u}{u}$, and then $w = \frac{v}{1+a}$ we get:

$$\begin{aligned} \int_0^1 \frac{1}{u^z(1-u)^{1-z}(1+au)} du &= \int_0^\infty \frac{v^{z-1}}{1+a+v} dv \\ &= (a+1)^{z-1} \int_0^\infty \frac{w^{z-1}}{1+w} dw \\ &= \Gamma(z)\Gamma(1-z)(a+1)^{z-1}, \end{aligned}$$

where the last equality follows from $\int_0^\infty \frac{w^{z-1}}{1+w} dw = \Gamma(z)\Gamma(1-z)$ (see Abramowitz and Stegun, [1, page 256, Equation 6.1.17]). Conditional on $Z(0) = x$, $N(t)$ is the number of events during an interval of length t, and time in that interval is scaled by a factor of x. Hence the probability generating function is

$$E(z^{N(t)} | Z(0) = x) = M(z, xt),$$

where $M(z, t)$ is given by (2.8). Taking expectation, w.r.t. $Z(0) \sim \exp(\alpha)$:

$$E(z^{N(t)}) = \alpha \int_0^\infty e^{-\alpha x} M(z, xt) dx$$

$$
\begin{aligned}
&= \alpha\frac{1}{\Gamma(z)\Gamma(1-z)}\int_0^\infty e^{-\alpha x}\int_0^1 \frac{e^{-xtu}}{u^z(1-u)^{1-z}}dudx \\
&= \frac{1}{\Gamma(z)\Gamma(1-z)}\int_0^1 \frac{1}{u^z(1-u)^{1-z}(1+\frac{t}{\alpha}u)}du \\
&= (\frac{t+\alpha}{\alpha})^{z-1}.
\end{aligned}
$$

But this is exactly the probability generating function of a Poisson random variable with parameter $\log\{(t+\alpha)/\alpha\}$. □

In the sequel we denote $N^*((s,t]) = N^*(t) - N^*(s)$. Clearly, applying Lemma 3.3 to the counting process $\{N^*(t)\}$ we get that $N^*((1,t]) \sim \text{Poisson}(\log(t))$ (Use Lemma 3.3 with $\alpha = 1$ and $t-1$). Similarly, $N^*(s,t] \sim \text{Poisson}(\log(t) - \log(s))$ (Use Lemma 3.3 with s and $t-s$).

Proposition 3.4 *The countning process $\{N^*(t)\}$ is a non-homogeneous Poisson process on $(0,\infty)$ with rate $1/t$.*

Proof In view of Theorem 7.3.II from Daley and Vere-Jones (1988) [4], it suffices to check that for $s_1 < t_1 < s_2 < t_2 < \ldots < s_n < t_n$

$$
P(N^*((s_1,t_1]) = 0, \ldots, N^*((s_n,t_n]) = 0) = \prod_{i=1}^n e^{(-\log(t_i) - \log(s_i))} = \prod_{i=1}^n (\frac{s_i}{t_i}). \tag{3.5}
$$

However, using the constructive method of Section 3.5

$$
\begin{aligned}
&P(N^*((s_1,t_1]) = 0, \ldots, N^*((s_n,t_n]) = 0) = \\
&= \int_0^\infty P(N^*((s_1,t_1]) = 0, \ldots, N^*((s_n,t_n]) = 0 | Z^*(t_1) = x) t_1 e^{-t_1 x}\, dx \\
&= \int_0^\infty P(N^*((s_1,t_1]) = 0 | Z^*(t_1) = x) P(N^*((s_2,t_2]) = 0, \ldots, N^*((s_n,t_n]) = 0 | Z^*(t_1) = x) t_1 e^{-t_1 x}\, dx \\
&= \frac{s_1}{t_1}\int_0^\infty P(N^*((s_2,t_2]) = 0, \ldots, N^*((s_n,t_n]) = 0 | Z^*(t_1) = x) t_1 e^{-t_1 x}\, dx \\
&= \frac{s_1}{t_1} P(N^*((s_2,t_2]) = 0, \ldots, N^*((s_n,t_n]) = 0).
\end{aligned}
$$

and so inductively we prove (3.5). □

3.8 Symmetry

Note that if we change the role of the t and z axes above, we retain the same structure of the process: The process whose realizations are the inverse functions of the paths of the process $Z^*(t)$ is identically distributed to the original process.

This enables us to analyze two more quantities associated with the original process. For the canonical process, let $T(z) = \sup\{t : Z^*(t) > z\} = \inf\{t : Z^*(t) \le z\}$. Then by the above, $T(z) \sim \exp(z)$. Also, if we let $W^*((1,z])$ denote the number of events for the process to drop from level 1 to level z, $W^*((1,z]) \sim \text{Poisson}(-\log(z))$.

Comment on the term staircase. A step function consists of horizontal line segments and vertical gaps. By considering both the process and its inverse, we are actually thinking about the continuous line consisting of horizontal and vertical line segments. We feel that it is appropriate to name such an object a staircase. The RCLL choice for both the process and the inverse process includes the lower left corner points, rather than the upper right corner points of the staircase.

3.9 Convergence to the Canonical Process

The following comment is perhaps not very deep but we feel it is worth making. If one considers an ergodic Markov process, and one conditions on an initial distribution at time t_0, and then runs the process forwards and backwards for $-\infty < t < \infty$ with the forward and backward transition kernels of the stationary version of the process, as $|t - t_0| \to \infty$ the process converges weakly to the stationary process. For our process $Z_{t_0,x_0}(t)$ we cannot expect such behavior, since the process is transient, and is only defined for $t > 0$. Furthermore, $Z_{t_0,x_0}(t)$ and $Z^*(t)$ both converge to 0 as $t \to \infty$ and to ∞ as $t \to 0$. Nevertheless we have:

Proposition 3.5 *The process $Z_{t_0,x_0}(t)$ which has value x_0 at time t_0 and which follows the forward and backward construction of Section 3.5, converges weakly to $Z^*(t)$ in the following sense: Consider at the two processes for $t < t_0/L, t > t_0 L$, where $L > 1$, and look at $tZ_{t_0,x_0}(t)$, $tZ^*(t)$. Then as $L \to \infty$, $tZ_{t_0,x_0}(t) \xrightarrow{D} tZ^*(t)$.*

Proof For $t > Lt_0$:

$$P(tZ_{t_0,x_0}(t) > z) = \begin{cases} e^{-z(t-t_0)/t} & z < tx_0 \\ 0 & z > tx_0 \end{cases}$$

and for $t < t_0/L$, by 3.4,

$$P(tZ_{t_0,x_0}(t) > z) = \begin{cases} \frac{t_0-t}{t_0} e^{-t(\frac{z}{t}-x_0)} & z > tx_0 \\ 1 & z \leq tx_0 \end{cases}$$

both of which converge to $e^{-z} = P(tZ^*(t) > z)$ uniformly as $L \to \infty$. $\square$

The scaled process $W(t) = tZ^*(t)$ may be of interest in its own right. It is a non-homogeneous Markov process, with all the marginal distributions exp(1). The sample paths of $W(t)$ are constructed as follows: Let $(t_n, z_n), n = \ldots, -1, 0, 1, \ldots, t_0 \leq 1 < t_1$ be the embedded sequence of jumps in the RCLL process $Z^*(t)$. Define the rays $l_n(t) = z_n t, t > 0$. Then

$$W(t) = l_n(t) \qquad t_n \leq t < t_{n+1}.$$

4 A Poisson staircase

In this section we consider a Poisson process of rate 1 on $(0, \infty) \times (0, \infty)$. We are interested in points (x, y) belonging to the Poisson process such that they are 'close' to the origin in the sense that the rectangle $(0,x)\times(0,y)$ has no other points of the process. Points of this form can be ordered by increasing x, in which case their y will be decreasing, and they will form a doubly infinite sequence. There will always be one special point, (x_0, y_0) which will be the point on the boundary of the smallest square with corner at $(0,0)$.

Let $R = \max(X_0, Y_0)$, in words this means that R is the side of the largest square which contains no points of the planar Poisson process. Then we have $P(R > r) = e^{-r^2}$, and the probability density of R is

$$f_R(r) = 2re^{-r^2}, \quad r > 0.$$

Furthermore, the conditional location of the point (X_0, Y_0) under the condition $R = r$ is uniform on the intervals $(r,0),(r,r)$ and $(r,r),(0,r)$. Thus

$$P(X_0 \in dx, Y_0 \in dy \mid R = r) = \frac{1}{2}1(r \in dx)\frac{1}{r}1(y \le r)\,dy + \frac{1}{2}1(r\in dy)\frac{1}{r}1(x\le r)\,dx$$

and hence the density of of (X_0, Y_0) is

$$f_{(X_0,Y_0)}(x,y) = e^{-x^2}1(y\le x)\,dx\,dy + e^{-y^2}1(x\le y)\,dy\,dx.$$

In particular

$$f_{X_0}(x) = \frac{1}{2}(xe^{-x^2} + \int_x^\infty e^{-y^2}\,dy).$$

It is easy to see that if we start from given (x_0, y_0), then conditional on those, the half of the sequence with $x > x_0$ will be generated like the forward random decreasing step function, while the half with $x < x_0$ is generated like the backwards process. We will show that if we randomize the above construction conditional on (x_0, y_0), with that distribution, the result is the canonical process.

We wish to calculate $P(Z^*(t) > z)$, conditional on the distribution of the point (x_0, y_0). We shall assume first that $t > z$. The event $Z^*(t) > z$ can happen in one of three ways:

(i) If $R > t$ then the staircase will be above the point (t,z). The probability for this is e^{-t^2}.

(ii) If $z < R = r < t$, and $X_0 < Y_0 = r$, then the rectangle $(X_0, r)\times(0,r)$ is also empty of Poisson points, and $Z^*(r) = r$. The event $Z^*(t) > z|(X_0,Y_0)$ is equivalent to the event that $r\breve{Z}(r(t-r)) > z$, where $\breve{Z}$ is an independent copy of the process starting with $\breve{Z}(0) = 1$. The conditional probability is then $e^{-(t-r)z}$, and this has to be integrated over the density of $0 < X_0 < R, Y_0 = R$ and of $z < R < t$.

(iii) If $z < R < t$, and $z < Y_0 = y < X_0 = R = r$, then the event $Z^*(t) > z|(X_0,Y_0)$ is equivalent to the event that $y\breve{Z}(y(t-r)) > z$, where $\breve{Z}$ is an independent copy of the process starting with $\breve{Z}(0) = 1$. The conditional probability is then $e^{-(t-r)z}$, and this has to be integrated over the density of $z < Y_0 < R, X_0 = R$ and of $z < R < t$.

Hence

$$\begin{aligned}
P(Z^*(t) > z) &= e^{-t^2} + \int_z^t 2re^{-r^2}\left(\int_0^r \frac{1}{2r}e^{-(t-r)z}dx + \int_z^r \frac{1}{2r}e^{-(t-r)z}dy\right)dr \\
&= e^{-t^2} + \int_z^t e^{-r^2}(2r-z)e^{-(t-r)z}dr \\
&= e^{-t^2} + e^{\frac{z^2}{4}-tz}\int_z^t (2r-z)e^{-(r-\frac{z}{2})^2}dr \\
&= e^{-t^2} + e^{\frac{z^2}{4}-tz}\int_{\frac{z}{2}}^{t-\frac{z}{2}} 2ue^{-u^2}du \\
&= e^{-t^2} + e^{\frac{z^2}{4}-tz}(e^{-\frac{z^2}{4}} - e^{-(t-\frac{z}{2})^2}) \\
&= e^{-t^2} + e^{-tz} - e^{-t^2} = e^{-tz}.
\end{aligned}$$

In other words, for $t > z$, $Z^*(t) \sim \exp(t)$. For $z > t$, one can look at the symmetric calculation exchanging the roles of x and y. Hence we immediately obtain that the finite dimensional distributions of the process $\{Z^*(t)\}$ are as in (3.1). Thus we have shown that the process that passes through all the Poisson points 'closest' to the axes is the canonical process of section 3.

Note: The counting process in the rectangle: Let y be a given level, and t a time. We are interested in the counting process that counts the number of Poisson points (X, Y) of the staircase such that $Y < y$, $X < t$, denoted by $N = N((0,0),(0,y),(t,0),(t,y))$. It is easy to see that the generating function of N is simply $M(z, yt)$, where M is given by equation (2.8).

4.1 Projection onto the axes

Consider the points of the planar Poisson process in the positive quadrant which have no other points below and to the left (equivalently the jumps of the hyperbolic staircase $Z^*(t)$), numbered as in the previous section (X_n, Y_n) where Y_n is decreasing, X_n increasing in n. We refer to these points as *the lower envelope of the planar Poisson process in the positive quadrant.* We now study the projections of these points on the x and y axes. By Section 3.7 these form the events of rate $1/x$, $1/y$ non-homogeneous Poisson processes on the axes, which we denote $X(t), Y(t)$. Clearly, the sequences X_n, Y_n, $n = 0, \pm 1, \pm 2, \ldots$ are not independent, in fact $X_{n+1} - X_n \sim \exp(Y_n)$. Let us now however 'lose' the pairing of the points: Retain the sequence X_n, but re-index the projection on the y axis, as Y_m where the sequence is now increasing in m, and we no longer know which m corresponds to the previous index n. We can no longer reconstruct the points of the planar process from the sequences X_n, Y_m, unless we are given a single 'pairing', i.e. a pair of indices n_0, m_0 such that (X_{n_0}, Y_{m_0}) is one of the original Poisson points.

We now pose the question whether these point processes are independent. If that should be the case, it would provide an elegant construction of the lower envelope (equivalently, the hyperbolic staircase): We would generate two independent Poisson processes of rates $1/x, 1/y$ on the axes, and all we will need to do then is to generate a single pairing. Contrary to our expectation, based on our belief in the inherent simplicity of the Poisson process, the processes are however not independent. To see this we calculate $P(X(t)-X(s)=Y(w)-Y(v)=0)$ and find that it is different from $P(X(t)-X(s)=0)P(Y(w)-Y(v)=0)$. By Section 3.7, $P(X(t)-X(s)=0)P(Y(w)-Y(v)=0)=\frac{sv}{tw}$.

Proposition 4.1

$$\begin{aligned}
& P(X(t)-X(s)=Y(w)-Y(v)=0)=\frac{sv}{tw}e^{-tw} \\
+ \quad & (1-\frac{s}{t})e^{-tv}+(1-\frac{v}{w})e^{-sw}+\frac{s}{t}+\frac{v}{w}-1-e^{sv-tv-sw} \\
+ \quad & \frac{t-s}{w-v}e^{-tw}\left\{\mathrm{Ei}(tw)-\mathrm{Ei}((t-s)w)-\mathrm{Ei}(t(w-v))+\mathrm{Ei}((t-s)(w-v))\right\} \\
\neq \quad & \frac{sv}{tw}
\end{aligned} \tag{4.1}$$

where $\mathrm{Ei}(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt$ *is the exponential integral [1, Abramowitz and Stegun 1964 p. 228, §5.1.2, and Figure 5.1].*

Proof The event $X(t)-X(s)=Y(w)-Y(v)=0$ can happen in two ways: Either the hyperbolic staircase process $Z^*(t)$ is above the rectangle $(s,t)\times(v,w)$, and there are no jumps of the process in (s,t) and no jumps of the reversed process in (v,w), which we denote as event A, or $Z^*(t)$ is below the rectangle, and there are no jumps of the reversed process in (v,w) and no jumps of the process in (s,t), which we denote as event B.

To obtain $P(A)$ condition on $Z^*(s)=z>w$, multiply by the probability that the process remains at level z until time t, and by the probability that from level z at time t the process reaches a level below v without staying at any level in (v,w). The latter probability is easily seen to be $\frac{v}{w}$. Uncondition to get:

$$P(A)=\int_w^\infty se^{-sz}e^{-z(t-s)}\frac{v}{w}dz=\frac{sv}{tw}e^{-tw},$$

which yields the first summand in (4.1).

To obtain $P(B)$ we condition on $Z^*(s)=z, Z^{*-1}(v)=x$. The conditional probability $P(B|Z^*(s)=z,Z^{*-1}(v)=x)=e^{-x(w-v)}e^{-z(t-s)}$, and one can show that the conditioning joint density is

$$f(Z^*(s)=z,Z^{*-1}(v)=x)=(1+(s-x)(v-z))e^{-sz-xv+xz}$$

¿From which

$$P(B)=\int_0^v\int_0^s(1+(s-x)(v-z))e^{-sz-xv+xz}e^{-x(w-v)}e^{-z(t-s)}dxdz$$

which yields the rest of the formula.

To check the inequality it suffices to calculate:

$$P(X(2)-X(1)=Y(2)-Y(1)=0)=0.3029\neq 0.25$$

□

5 Connection to Extremal Processes

Consider a distribution function F, let $X_j, j = 1, 2, \ldots$ be i.i.d from F, let $M_n = \min(X_1, \ldots, X_n)$, let $L(1) = 1, L(n) = \min\{j : j > L(n-1), M_j < M_{j-1}\}$, and $R(n) = M_{L(n)}$, we call M the sequence of minima, $L(n)$ the record times, and $R(n)$ the record values. Consider the process of minima. It is straight forward to see that for integers $0 < t_1 < t_2 < \ldots < t_l$, the joint distribution of minima at these times is given by

$$P(M_{t_1} > z_1, M_{t_2} > z_2, \ldots, M_{t_l} > z_l) = \bar{F}^{t_1}(\bigvee_{j=1}^{l} z_j)\bar{F}^{t_2-t_1}(\bigvee_{j=2}^{l} z_j)\cdots\bar{F}^{t_l-t_{l-1}}(z_l) \tag{5.1}$$

The extremal process for minima of F, $M(t), t > 0$ is defined by the family of finite dimensional joint distributions of this form, where one allows $0 < t_1 < t_2 < \ldots < t_l$ to be real. It plays two roles in describing the minima of the i.i.d. X_j: First, the sequence of minima is the embedded process of M at integer times. Second, asymptoticaly, if F belongs to the domain of attraction of an extreme value minima distribution G, with norming constants a_n, b_n, then $(M_{[nt]} - b_n)/a_n$ converges weakly to $M(t)$ as $n \to \infty$, where $M(t)$ is the extremal process for minima of G.

Extremal processes were first studied by Dwass [6, 1964] and by Lamperti [11, 1964], and are described in great detail in Resnick [12], mainly in the context of maxima. Of course the maxima of X_j are the minima of $-X_j$, so minima and maxima have the same theory. A special example that is much discussed is the maxima of the distriubtion Λ which is the Gumbel distribution, and is one of the limiting maximal extreme value distributions.

Our porcess $Z^*(t)$ is itself another example of an extremal process of minima, where the distiribution of the X_j is $\exp(1)$. This can be seen immediately by comparing (5.1) to (3.1). The exponential distribution itself is a minimal extreme value distribution, and so $Z^*(t)$ plays a dual role: The embedded process in discrete times is the process of the minima of X_j i.i.d. $\exp(1)$, and if we look at these minima, then asymptotically $M_{[nt]}/n \xrightarrow{W} Z^*(t)$. As far as we know this example of an extremal process has not been discussed previously in the literature.

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