Asymptotics of Palm-Stationary Buffer Content Distributions in Fluid Flow Queues

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Abstract

We study a fluid flow queueing system with m independent sources alternating between silence and activity periods; $m \geq 2$. The distribution function of the activity periods of one source, is supposed to be *intermediate regular varying*. We show that the distribution of the net increment of the buffer during an aggregate activity period (i.e. at least one source is active) is asymptotically tail-equivalent to the distribution of the net input during a single activity period with intermediate regular varying distribution function. In this way, we arrive at an asymptotical representation of the Palm-stationary tail-function of the buffer content at the beginning of aggregate activity periods. Our approach is probabilistic and extends recent results of Boxma (1996, 1997) who considered the special case of regular variation.

Keywords: Regular variation, Subexponential distribution, Aggregating, Tail asymptotics, Random walk, On-off fluid flow

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1 Introduction

Motivated by data flows in telecommunication networks, there is an increasing interest in fluid flow queues with merging on-off sources having heavy-tailed governing distributions, see Anantharam (1995), Boxma (1996, 1997), Choudhury and Whitt (1997), Heath, Resnick and Samorodnitsky (1996a,b), Heyman and Lakshman (1996), Jelenković and Lazar (1996), Whitt (1993). The present paper is devoted to the study of a fluid flow queueing system with m independent sources alternating between silence and activity periods; $m \geq 2$. The distribution function of the activity periods of one source, say number 1, is supposed to be intermediate regular varying, where we say that a distribution function F is *intermediate regular varying* ($F \in \mathcal{IR}$), if

$$\lim_{c \downarrow 1} \liminf_{x \to \infty} \frac{\overline{F}(cx)}{\overline{F}(x)} = \lim_{c \downarrow 1} \limsup_{x \to \infty} \frac{\overline{F}(cx)}{\overline{F}(x)} = 1. \tag{1}$$

We call this activity period a heavy-tailed activity period. The activitiy periods of the other sources and all silence periods are assumed to be exponentially distributed. The input rates of all sources are assumed to be greater than the output rate.

We show that the distribution of the net increment of the buffer during an aggregate activity period (i.e. at least one source is active) is asymptotically tail-equivalent to the distribution of the net input during a single heavy-tailed activity period. In this way, we receive an asymptotical representation of the Palm-stationary tail-function of the buffer content at the beginning of aggregate activity periods. Using Laplace-Stieltjes transforms, the same result has been obtained in Boxma (1996) for the case of two sources and extended in Boxma (1997) to $m \geq 2$ sources, one of them with regular varying distribution function of activity periods which is a stronger condition than (1). In return, Boxma (1997) considers more general distributions for the activity periods of sources $2, \ldots, m$, assuming that they have an exponentially bounded tail. However, in contrast to Boxma's proof, our approach is probabilistic and thus offers much more insight. Moreover, we conjecture that future research might show that our approach can be used for an even broader class of heavy-tailed distributions than that defined by (1).

We also remark that in Theorem 10 of Jelenković and Lazar (1996) the asymptotical tail behaviour of the time-stationary buffer content distribution has been studied for the considered model. However, it seems to be impossible to obtain the asymptotical tail behaviour of the Palm-stationary buffer content distribution directly from their result. Besides this we propose other proving techniques.

2 Model Description and Notation

We consider a fluid flow queueing system with infinite buffer size and constant unit output rate. The system is fed by m independent on-off sources, each of which has mutually independent alternating activity and silence periods. They are also called on and off periods respectively. Activity periods of source 1 are denoted by T_n and supposed to have an intermediate regular varying distribution F with finite mean $\mathbb{E}T$, whereas activity periods of source $k \geq 2$, denoted by $J_n^{(k)}$, as well as the silence periods of source 1, S_n, and source $k \geq 2$, $I_n^{(k)}$, $n = 1, 2, \ldots$ have an exponential distribution with parameter $\lambda_{\text{on}}^{(k)}, \nu$ and $\lambda_{\text{off}}^{(k)}$ respectively. The generic random variables of $I_n^{(k)}, J_n^{(k)}, S_n, T_n$ are denoted by $I^{(k)}, J^{(k)}, S, T$ respectively and they are assumed to be mutually independent. When active, the sources transmit at constant rate r_k with

$$r_k \geq 1, \qquad k = 1, \ldots, m \; . \tag{2}$$

Furthermore, we assume that the following *stability condition* holds:

$$L \stackrel{\text{def}}{=} \frac{r_1 \mathbb{E}T\nu}{1 + \mathbb{E}T\nu} + \sum_{k=2}^{m} \frac{r_k \lambda_{\text{off}}^{(k)}}{\lambda_{\text{on}}^{(k)} + \lambda_{\text{off}}^{(k)}} < 1 \; , \tag{3}$$

i.e. the expected total input into the system per unit time is less than the output rate. We also assume that the buffer is empty and all sources are off at time zero, that is we start with a silence period of the whole system. By $r_{\max} = \sum_{k=1}^{m} r_k - 1$ we denote the maximum net input rate.

We further use the following notation. Let C_n, $n = 1, 2, \ldots$, be the length of the nth aggregate activity period after zero followed by a silence period whose generic random variable is defined by $\min\{S, I^{(2)}, \ldots, I^{(m)}\}$. This silence period of the whole system is exponentially distributed with parameter $\Lambda \stackrel{\text{def}}{=} (\nu + \sum_{k=2}^{m} \lambda_{\text{off}}^{(k)})$. By B_n and W_n, $n = 1, 2, \ldots$, we denote the net increment of the buffer content during C_n and the buffer content at the beginning of the nth aggregate activity period respectively. Figure 1 shows a typical sample path of the on-off processes in the case of two sources as well as a typical sample path of the resulting buffer content process. Note that $B_1, B_2, \ldots$ are i.i.d. random variables with distribution, say, $B(\cdot)$ and

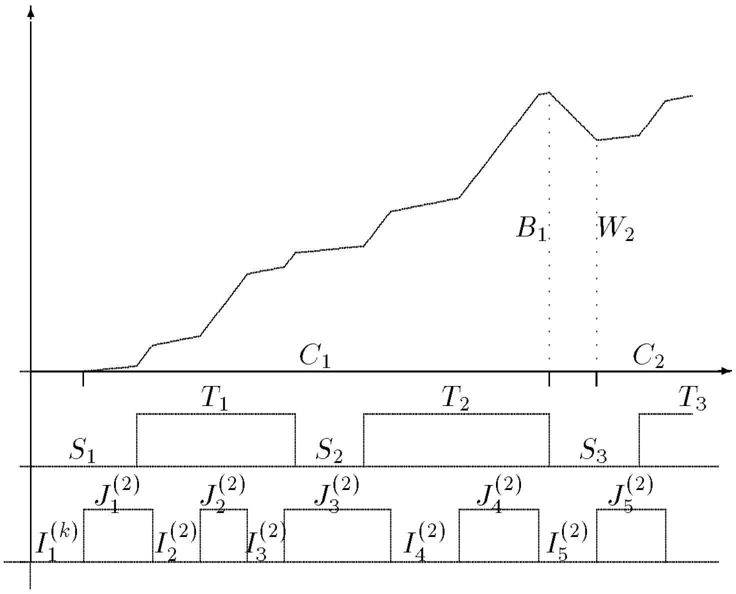

Figure 1: Aggregate Activity Period of Two Merging On-Off-Sources

that W_n, B_n and the following silence period of the system are independent, too. Thus we can reinterpret the development of the buffer content after successive aggregate activity and silence periods of the system as a random walk with generic increment $B - \min\{S, I^{(2)}, \ldots, I^{(m)}\}$, and W_n then corresponds to the finite-horizon maximum of this random walk up to step n, $n = 1, 2, \ldots$. The distribution of the infinite-horizon maximum of this random walk we denote by $W(\cdot)$, i.e. $W(x) = \lim_{n\to\infty} \mathbb{P}(W_n \leq x)$.

Under stability condition (3) this random walk has negative drift. The limit distribution $W(\cdot)$ is therefore well-defined and can be interpreted as Palm-stationary distribution of the buffer content at the beginning of aggregate activity periods. It turns out that $W(\cdot)$ is asymptotically tail-equivalent to the so-called tail-integrated distribution of $B(\cdot)$. Furthermore, $B(\cdot)$ is asymptotically tail-equivalent to the distribution of the net input during a single activity period of source 1. Thus, in Section 3 we begin with the investigation of the buffer content process during such a single activity period. For simplicity of notation, in Section 4 we first determine the asymptotical tail behaviour of $B(\cdot)$ for the case of two merging sources, i.e. $m = 2$. The investigation of the general case is postponed to Section 5. If $m = 2$, we write $I_n, J_n, \lambda_{\text{off}}, \lambda_{\text{on}}$ instead of $I_n^{(k)}, J_n^{(k)}, \lambda_{\text{off}}^{(k)}, \lambda_{\text{on}}^{(k)}$ respectively.

3 Net Input during a Single Activity Period of Source 1

In this section we prove two asymptotical properties of the net input during a single activity period of source 1 which are crucial for showing that the distribution $B(\cdot)$ asymptotically behaves like the distribution of the net input during a single activity period of source 1. However, we first state two auxiliary results for homogeneous Markov processes with finite state space.

3.1 Some Preliminaries

Let $\{\xi(t), t \geq 0\}$ be a homogeneous, irreducible Markov process with finite state space $E = \{0, 1, \ldots, \ell\}$. By $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_\ell)$ we denote the initial distribution of $\{\xi(t)\}$, by $\boldsymbol{P}(t) = (p_{ij}(t))$ its matrix transition function, and by $\boldsymbol{Q}$ the matrix of transition intensities. Let $\boldsymbol{\pi} = (\pi_0, \ldots, \pi_\ell)$ be the stationary (limit) distribution of $\{\xi(t)\}$, i.e. $\boldsymbol{\pi Q} = \mathbf{0}$ where $\mathbf{0} = (0, \ldots, 0)$, and let $\boldsymbol{a} = (a_0, \ldots, a_\ell)$ be an arbitrary $(\ell + 1)$-dimensional vector with real-valued components such that $\sum_{i=0}^{\ell} \pi_i a_i < 0$.

Lemma 3.1 *There exist positive constants $c, \gamma < \infty$ such that for all $x \geq 0$,*

$$\mathbb{P}\left(\sup_{t \geq 0} \int_0^t a_{\xi(s)}\, ds > x\right) \leq c\mathrm{e}^{-\gamma x} . \tag{4}$$

Lemma 3.2 *There exist positive constants $t_0, \delta < \infty$ such that for all $t \geq t_0$,*

$$\max_{i,j \in E} |p_{ij}(t) - \pi_j| \leq \mathrm{e}^{-\delta t} . \tag{5}$$

For a proof of Lemma 3.1 we refer to Palmowski and Rolski (1996). A proof of Lemma 3.2 can be found e.g. in Iosifescu (1980).

3.2 Basic Lemma

We apply Lemmas 3.1 and 3.2 to the following case. The random variable

$$r_1 - 1 + \sum_{k=2}^{m} r_k \mathbb{I}(\text{source } k \text{ is active at time } t) , \tag{6}$$

where $\mathbb{I}(A)$ denotes the indicator of the set A, describes the net input rate at time t provided that t belongs to an activity period of source 1. The generic representation of the resulting stochastic process during an activity period of source 1 is as follows. Consider the family E of all subsets of $\{2,\ldots,m\}$. Sometimes it is convenient to identify E with $\{0,1,\ldots,\ell\}$, where we relate the empty set to 0 and clearly $\ell = 2^{m-1}-1$. Let $\xi(t)$ be the random set of those sources with numbers from 2 to m which are active at time t. Under the assumptions of Section 2, $\{\xi(t), t\geq 0\}$ is a Markov process of the type considered in Section 3.1. Note that there and also in the present section $\{\xi(t)\}$ can have an arbitrary initial distribution $\boldsymbol{\alpha}$. Let T be a generic activity period of source 1 and define the process $\{\eta(t), t\geq 0\}$ by

$$\eta(t) = r_1 - 1 + \sum_{k=2}^{m} r_k \mathbb{I}(k\in\xi(t)) \ . \tag{7}$$

Then, supposing that the activity period of source 1 begins at $t=0$, the process $\{\eta(t)\}$ describes the net input rate in time interval $[0,T]$. Furthermore, define the mean rate during activity period of source 1 by $\mu = \lim_{t\to\infty}\mathbb{E}\eta(t)$ and note that (2) implies $\mu>0$. But it is evident that the converse implication is not true. However, in Sections 3.2 and 3.3 we only need that $\mu > 0$.

Let $\eta_T = \int_0^T \eta(s)ds$. The following result is an important tool for getting the other asymptotical equivalence properties derived in the present paper.

Lemma 3.3 *For $x\to\infty$,*

$$\mathbb{P}(\eta_T > x) \sim \mathbb{P}(\mu T > x) \ . \tag{8}$$

Proof Let $\varepsilon>0$ be fixed. Then,

$$\mathbb{P}(\eta_T > x) = \mathbb{P}(T > \frac{x}{r_{\max}}, \eta_T > x) = \varphi_1^+(\varepsilon,x) + \varphi_2^+(\varepsilon,x) \ ,$$

where $\varphi_1^+(\varepsilon,x) = \mathbb{P}(r_{\max}T > x, \eta_T > x, \eta_T - \mu T > \varepsilon T)$ and $\varphi_2^+(\varepsilon,x) = \mathbb{P}(r_{\max}T > x, \eta_T > x, \eta_T - \mu T \leq \varepsilon T)$. By Lemma 3.1 we receive

$$\begin{aligned}
\varphi_1^+(\varepsilon,x) \ &\leq\ \mathbb{P}\left(T > \frac{x}{r_{\max}}, \frac{1}{T}\int_0^T \left(\eta(s) - (\mu + \frac{\varepsilon}{2})\right) ds > \frac{\varepsilon}{2}\right) \\
&\leq\ \mathbb{P}\left(\sup_{t\geq 0}\int_0^t \left(\eta(s) - (\mu+\frac{\varepsilon}{2})\right) ds > x\frac{\varepsilon}{2r_{\max}}\right) \leq c\mathrm{e}^{-\tilde{\gamma}x} \ ,
\end{aligned}$$

where $\tilde{\gamma} = \gamma\varepsilon/(2r_{\max})$ and c and γ according to Lemma 3.1. For the term $\varphi_2^+(\varepsilon,x)$ we obtain the following upper bound

$$\varphi_2^+(\varepsilon,x) \leq \mathbb{P}(\eta_T > x, \eta_T \leq T(\mu+\varepsilon)) \leq \mathbb{P}(T(\mu+\varepsilon) > x) \, .$$

Combining this, we obtain

$$\limsup_{x\to\infty} \frac{\mathbb{P}\,(\eta_T > x)}{\mathbb{P}(\mu T > x)} \le \limsup_{x\to\infty} \frac{c\mathrm{e}^{-\tilde{\gamma} x}}{\mathbb{P}(\mu T > x)} + \limsup_{x\to\infty} \frac{\mathbb{P}(T(\mu+\varepsilon) > x)}{\mathbb{P}(\mu T > x)} \; .$$

Using that the distribution F_T of T is from $\mathcal{IR}$, letting $\varepsilon \downarrow 0$ finally yields

$$\limsup_{x\to\infty} \frac{\mathbb{P}\,(\eta_T > x)}{\mathbb{P}(\mu T > x)} \le 1 \; .$$

To get a lower bound, note that

$$\begin{aligned} \mathbb{P}(\eta_T > x) &\ge \mathbb{P}\Big(\eta_T > x, \eta_T \ge T(\mu-\varepsilon)\Big) \\ &\ge \mathbb{P}\Big(T(\mu-\varepsilon) > x, \int_0^T \Big(\eta(s) - (\mu - \frac{\varepsilon}{2})\Big)\, ds \ge -T\frac{\varepsilon}{2}\Big) \\ &\ge \varphi_1^-(\varepsilon, x) - \varphi_2^-(\varepsilon, x) \; , \end{aligned}$$

where $\varphi_1^-(\varepsilon, x) = \mathbb{P}(T(\mu-\varepsilon) > x)$ and

$$\varphi_2^-(\varepsilon, x) = \mathbb{P}\Big(T(\mu-\varepsilon) > x, \int_0^T \Big(\eta(s) - (\mu - \frac{\varepsilon}{2})\Big)\, ds < -x\frac{\varepsilon}{2(\mu-\varepsilon)}\Big) \; .$$

Application of Lemma 3.1 to the term $\varphi_2^-(\varepsilon, x)$ yields

$$\varphi_2^-(\varepsilon, x) \le \mathbb{P}\Big(\sup_{t\ge 0} \int_0^t \Big((\mu - \frac{\varepsilon}{2}) - \eta(s)\Big)\, ds > \frac{x\varepsilon}{2(\mu-\varepsilon)}\Big) \le c\mathrm{e}^{-\gamma^* x} \; ,$$

where $\gamma^* = \gamma\varepsilon/(2(\mu-\varepsilon))$ and c and γ are from Lemma 3.1, i.e.

$$\liminf_{x\to\infty} \frac{\mathbb{P}\,(\eta_T > x)}{\mathbb{P}(\mu T > x)} \ge \liminf_{x\to\infty} \left(\frac{\mathbb{P}(T(\mu-\varepsilon) > x)}{\mathbb{P}(\mu T > x)} - \frac{c\mathrm{e}^{-\gamma^* x}}{\mathbb{P}(\mu T > x)} \right) \; ,$$

and thus, since $F_T \in \mathcal{IR}$, $\varepsilon \downarrow 0$ yields

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left(\int_0^T \eta(s)\, ds > x\right)}{\mathbb{P}(\mu T > x)} \ge 1 \; . \qquad \square$$

Remark Recall that the distribution function F of the generic activity period T of source 1 is assumed to be intermediate regular varying, i.e. condition (1) is satisfied, which is slightly more general than regular variation. For details and examples we refer to Cline (1994) and Bingham, Goldie and Teugels (1987). We also mention that in the proof of Lemma 3.3 we did not use that T is independent of $\{\xi(t)\}$. Furthermore, we did not use that $\{\xi(t)\}$ is Markov. Thus each process $\{\xi(t)\}$ satisfying (4) and (5) is possible.

3.3 Refined Asymptotics

In Sections 4 and 5 we will use the following refinement of Lemma 3.3.

Lemma 3.4 *For all* $i \in \{0, 1, \ldots, \ell\}$,

$$\mathbb{P}\left(\eta_T > x, \xi(T) = i\right) \sim \pi_i \mathbb{P}(\mu T > x) \tag{9}$$

as $x \to \infty$, *where* $\boldsymbol{\pi} = (\pi_0, \ldots, \pi_\ell)$ *is the stationary distribution of* $\{\xi(t)\}$.

Proof We subdivide the proof into two steps showing that

$$\mathbb{P}(\mu T > x, \xi(T) = i) \sim \pi_i \mathbb{P}(\mu T > x)\ ,\ x \to \infty \tag{10}$$

and

$$\mathbb{P}(\eta_T > x, \xi(T) = i) \sim \mathbb{P}\left(\mu T > x, \xi(T) = i\right)\ ,\ x \to \infty\ . \tag{11}$$

In order to prove (10) we first derive an upper bound. Recall that $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_\ell)$ denotes the initial distribution of $\{\xi(t)\}$ and $\boldsymbol{P}(t) = (p_{ij}(t))$ its matrix transition function. Then, using the fact that T is independent of $\{\xi(t)\}$, Lemma 3.2 gives for some $\delta > 0$ and for all x sufficiently large,

$$\begin{aligned}
\mathbb{P}(\mu T > x, \xi(T) = i) &= \sum_{k=0}^{m} \int_{x/\mu}^{\infty} \left(p_{ki}(s) - \pi_i + \pi_i\right) dF(s) \alpha_k \\
\leq\ & \sum_{k=0}^{m} \Big(\int_{x/\mu}^{\infty} \left(|p_{ki}(s) - \pi_i| + \pi_i\right) dF(s) \Big) \alpha_k \leq \Big(\mathrm{e}^{-\delta \frac{x}{\mu}} + \pi_i \Big) \overline{F}_T\Big(\frac{x}{\mu}\Big)\ ,
\end{aligned}$$

and thus

$$\limsup_{x \to \infty} \frac{\mathbb{P}(\mu T > x, \xi(T) = i)}{\mathbb{P}(\mu T > x)} \leq \pi_i\ . \tag{12}$$

To get a lower bound note that

$$\begin{aligned}
\mathbb{P}(\mu T > x, \xi(T) = i) &= \sum_{k=0}^{m} \int_{x/\mu}^{\infty} \left(p_{ki}(s) - \pi_i + \pi_i\right) dF(s) \alpha_k \\
\geq\ & \sum_{k=0}^{m} \Big(- \int_{x/\mu}^{\infty} \mathrm{e}^{-\delta s} dF(s) + \pi_i \overline{F}_T\Big(\frac{x}{\mu}\Big) \Big) \alpha_k \geq -\mathrm{e}^{-\delta \frac{x}{\mu}} \overline{F}_T\Big(\frac{x}{\mu}\Big) + \pi_i \overline{F}_T\Big(\frac{x}{\mu}\Big)
\end{aligned}$$

for x sufficiently large, and thus

$$\liminf_{x \to \infty} \frac{\mathbb{P}(\mu T > x, \xi(T) = i)}{\mathbb{P}(\mu T > x)} \geq \liminf_{x \to \infty} \Big(-\mathrm{e}^{-\delta \frac{x}{\mu}} + \pi_i \Big) = \pi_i\ . \tag{13}$$

Combining (12) and (13) yields (10). In order to show (11) we again derive an upper and a lower bound. Let $\varepsilon > 0$ be fixed. Then,

$$\mathbb{P}(\eta_T > x, \xi(T) = i) = \mathbb{P}(T > \frac{x}{r_{\max}}, \eta_T > x, \xi(T) = i) = \varphi_1^+(\varepsilon, x) + \varphi_2^+(\varepsilon, x)$$

with $\varphi_1^+(\varepsilon, x) = \mathbb{P}(r_{\max}T > x, \eta_T > x, \eta_T - \mu T > \varepsilon T, \xi(T) = i)$ and $\varphi_2^+(\varepsilon, x) = \mathbb{P}(r_{\max}T > x, \eta_T > x, \eta_T - \mu T \leq \varepsilon T, \xi(T) = i)$. The term $\varphi_1^+(\varepsilon, x)$ can be bounded as follows:

$$\begin{aligned}
\varphi_1^+(\varepsilon, x) &\leq \mathbb{P}\Big(T > \frac{x}{r_{\max}}, \frac{1}{T}\int_0^T (\eta(s) - (\mu + \frac{\varepsilon}{2}))\, ds > \frac{\varepsilon}{2}\Big) \\
&\leq \mathbb{P}\Big(\sup_{t \geq 0} \int_0^t (\eta(s) - (\mu + \frac{\varepsilon}{2}))\, ds > x\frac{\varepsilon}{2r_{\max}}\Big) \leq c\mathrm{e}^{-\gamma x} ,
\end{aligned}$$

where c and $\gamma > 0$ are chosen according to Lemma 3.1. Furthermore,

$$\varphi_2^+(\varepsilon, x) \leq \mathbb{P}\Big(\eta_T > x, \eta_T \leq T(\mu+\varepsilon), \xi(T) = i\Big) \leq \mathbb{P}(T(\mu+\varepsilon) > x, \xi(T) = i).$$

Note that by (10) and the fact that $F \in \mathcal{IR}$,

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \limsup_{x \to \infty} \frac{\mathbb{P}(T(\mu+\varepsilon) > x, \xi(T) = i)}{\mathbb{P}(\mu T > x, \xi(T) = i)} \\
&\quad = \lim_{\varepsilon \downarrow 0} \limsup_{x \to \infty} \frac{\mathbb{P}(T(\mu+\varepsilon) > x, \xi(T) = i)}{\mathbb{P}(T(\mu+\varepsilon) > x)} \frac{\mathbb{P}((\mu+\varepsilon)T > x)}{\mathbb{P}(\mu T > x)} \\
&\qquad \times \frac{\mathbb{P}(\mu T > x)}{\mathbb{P}(\mu T > x, \xi(T) = i)} = 1
\end{aligned}$$

and therefore

$$\begin{aligned}
\limsup_{x \to \infty} \frac{\mathbb{P}\left(\eta_T > x, \xi(T) = i\right)}{\mathbb{P}\left(\mu T > x, \xi(T) = i\right)} &\leq \limsup_{x \to \infty} \frac{c\mathrm{e}^{-\gamma x}}{\mathbb{P}(\mu T > x)} \frac{\mathbb{P}(\mu T > x)}{\mathbb{P}(\mu T > x, \xi(T) = i)} \\
&\quad + \lim_{\varepsilon \downarrow 0} \limsup_{x \to \infty} \frac{\mathbb{P}(T(\mu+\varepsilon) > x, \xi(T) = i)}{\mathbb{P}(\mu T > x, \xi(T) = i)} \leq 1 .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}\Big(\eta_T > x, \xi(T) = i\Big) \geq \mathbb{P}\Big(\eta_T > x, \eta_T \geq T(\mu - \varepsilon), \xi(T) = i\Big) \\
&\geq \mathbb{P}\Big(T(\mu - \varepsilon) > x, \int_0^T \Big(\eta(s) - (\mu - \frac{\varepsilon}{2})\Big)\, ds \geq -T\frac{\varepsilon}{2}, \xi(T) = i\Big) \\
&= \varphi_1^-(\varepsilon, x) - \varphi_2^-(\varepsilon, x) ,
\end{aligned}$$

where $\varphi_1^-(\varepsilon,x) = \mathbb{P}(T(\mu-\varepsilon) > x, \xi(T) = i)$ and

$$\varphi_2^-(\varepsilon,x) = \mathbb{P}\Big(T(\mu-\varepsilon) > x, \xi(T) = i, \int_0^T \Big(\eta(s) - (\mu - \frac{\varepsilon}{2})\Big)\, ds < \frac{-x\varepsilon}{2(\mu-\varepsilon)}\Big) \,.$$

The term $\varphi_2^-(\varepsilon,x)$ is exponentially bounded since by Lemma 3.1

$$\varphi_2^-(\varepsilon,x) \le \mathbb{P}\Big(\int_0^T \Big((\mu - \frac{\varepsilon}{2}) - \eta(s)\Big)\, ds > x\frac{\varepsilon}{2(\mu-\varepsilon)}\Big) \le c\mathrm{e}^{-\gamma x}$$

for some $c, \gamma > 0$. Similar to the considerations on $\varphi_2^+(\varepsilon,x)$ above,

$$\lim_{\varepsilon\downarrow 0} \liminf_{x\to\infty} \frac{\varphi_1^-(\varepsilon,x)}{\mathbb{P}(\mu T > x, \xi(T) = i)} = 1 \,,$$

and hence

$$\begin{aligned} \liminf_{x\to\infty} \frac{\mathbb{P}(\eta_T > x, \xi(T) = i)}{\mathbb{P}(\mu T > x, \xi(T) = i)} &\ge \liminf_{x\to\infty} \frac{\mathbb{P}(T(\mu-\varepsilon) > x, \xi(T) = i)}{\mathbb{P}(\mu T > x, \xi(T) = i)} \\ &\quad - \limsup_{x\to\infty} \frac{c\mathrm{e}^{-\gamma x}}{\mathbb{P}(\mu T > x, \xi(T) = i)} \stackrel{\varepsilon\downarrow 0}{\longrightarrow} 1 \,. \end{aligned}$$

□

4 Net Input during Aggregate Activity Periods

Consider the Markov process $\{\xi(t)\}$ introduced in Section 3.2 and note that $\{\xi(t)\}$ can be represented in the following way. Suppose that the initial state $\xi(0)$ of $\{\xi(t)\}$ at time $t = 0$ is deterministic and let $\{\zeta_k^{i_k}(t)\}$ be the process which takes value 1 if source k is active and 0 otherwise. The superscript $i_k = 0, 1$ indicates, whether at $t = 0$ the source is active ($i_k = 1$) or silent ($i_k = 0$). Then, in addition to the set representation of $\xi(t)$ given in Section 3.2, we can write $\xi(t) = (\zeta_2^{i_2}(t), \ldots, \zeta_m^{i_m}(t))$. Under our assumptions on the model, the processes $\{\zeta_k^{i_k}(t)\}$ $(k = 2, \ldots, m)$ are Markovian and mutually independent. Let $\boldsymbol{i} = (i_2, \ldots, i_m)$ and in particular, $\mathbf{1} = (1, \ldots, 1)$. We now append the superscript $\boldsymbol{i}$ to $\xi(t)$, writing $\xi^{\boldsymbol{i}}(t)$ to emphazise that the initial state of the defining process $\{\zeta_k^{i_k}(t)\}$ is i_k; $k = 2, \ldots, m$. Similarly we append the superscript $\boldsymbol{i}$ to the process $\{\eta(t)\}$ writing $\eta^{\boldsymbol{i}}(t)$.

In the rest of the paper $\{\xi_n^{\boldsymbol{i}}(t), t \ge 0\}$ are independent copies of $\{\xi^{\boldsymbol{i}}(t), t \ge 0\}$ and $I_0^{(k)}, I_1^{(k)}, \ldots$ are i.i.d. exponentially distributed with parameter

$\lambda_{\text{off}}^{(k)}$, $J_0^{(k)}, J_1^{(k)}, \ldots$ are i.i.d. exponentially distributed with parameter $\lambda_{\text{on}}^{(k)}$, $S_0, S_1, \ldots$ are i.i.d. exponentially distributed with parameter ν. Besides we assume that $\{I_n^{(k)}\}$, $\{J_n^{(k)}\}$, $\{S_n\}$, $\{\xi_n^{\boldsymbol{i}}(t)\}$ ($\boldsymbol{i} \in E$) are mutually independent. Note that from now on we occasionally use a somewhat different numbering of activity and silence periods which does not always coincide with the numbering introduced at the beginning of the paper, i.e. in Section 2. Furthermore, by $\{\eta_1^{\boldsymbol{i}}(t)\}, \{\eta_2^{\boldsymbol{i}}(t)\}, \ldots$ we denote independent and identically distributed copies of the process $\{\eta^{\boldsymbol{i}}(t)\}$.

To make our consideration more transparent we first consider the case $m = 2$. We assume that an aggregate activity period of the whole system begins at time $t = 0$. Now, $E = \{0, 1\}$ and the random variable $\xi(t) \in \{0, 1\}$ indicates whether source 2 is off or on at time t. Recall that $\boldsymbol{\pi} = (\pi_0, \pi_1)$ denotes the stationary distribution of $\{\xi(t)\}$, and hence $\mu = r_1 - 1 + \pi_1 r_2 > 0$. Then we have the following asymptotical representation of the distribution of the generic net inflow B during this aggregate activity period.

Theorem 4.1 *For $x \to \infty$,*

$$\mathbb{P}(B > x) \sim \frac{\nu(\lambda_{\text{on}} + \lambda_{\text{off}})}{(\nu + \lambda_{\text{off}})\lambda_{\text{on}}} \mathbb{P}(\mu T > x) \ . \tag{14}$$

Remark We will see in Section 4.5 that (14) can be rewritten in the form

$$\mathbb{P}(B > x) \sim (\mathbb{P}(S < I) + \mathbb{P}(S > I)\mathbb{P}(S < J)) \sum_{n=1}^{\infty} (1-p)p^{n-1}\overline{G^{*n}}(x) \ , \tag{15}$$

where $G(x) = \mathbb{P}(\mu T \leq x)$ and $p = \pi_1 \nu(\nu + \lambda_{\text{on}})^{-1}$. Intuitively this asymptotical representation formula is clear. As the silence periods of source 1 are light-tailed and thus can asymptotically be neglected, the net inflow into the buffer during an aggregate activity period is governed by the inflow during activity periods of source 1 and therefore depends on the number of these activity periods. To have exactly n such activity periods each of the $n - 1$ silence periods inbetween must be bridged by an activity period of source 2 which is the case if at the end of an activity period of source 1 source 2 is active with the remaining time of this activity period lasting longer than the silence period of source 1. But, in steady state, this event has probability p and thus (15) is plausible.

4.1 A Representation Formula

The formal proof of Theorem 4.1 will be partitioned into steps. We first discuss some structural details and derive some generic asymptotic representation formulas. However, all these considerations are motivated by the fact that the random variable B can be represented in the following way.

Lemma 4.1

$$\begin{aligned}
B = {} & 1\!\!\mathrm{I}(S_0 > I_0)\,1\!\!\mathrm{I}(S_1 > J_0)(r_2 - 1)J_0 \\
+ {} & 1\!\!\mathrm{I}(S_0 > I_0)\,1\!\!\mathrm{I}(S_1 < J_0)\Big(\sum_{n=1}^{N^1}\Big((r_2-1)S_n + Y_n^1\Big) \\
& +(r_2-1)J_{N^1}1\!\!\mathrm{I}(\xi^1_{N^1}(T_{N^1}) = 1)\Big) \qquad (16)\\
+ {} & 1\!\!\mathrm{I}(S_0 < I_0)\Big\{Y_1^0 + (r_2-1)J_1 1\!\!\mathrm{I}(\xi_1^0(T_1) = 1)\,1\!\!\mathrm{I}(S_2 > J_1) \\
+ {} & 1\!\!\mathrm{I}(N^0 \geq 2)\Big(\sum_{n=2}^{N^0}\Big((r_2-1)S_n + Y_n^1\Big) + (r_2-1)J_{N^0}1\!\!\mathrm{I}(\xi^1_{N^0}(T_{N^0}) = 1)\Big)\Big\},
\end{aligned}$$

where

$$\begin{aligned}
Y_n^i &= \int_0^{T_n} \eta_n^i(s)\,ds \ \text{ for } n \geq 1\,, \quad \eta_n^i(s) = r_1 - 1 + r_2 1\!\!\mathrm{I}(\xi_n^i(s) = 1)\,, \\
N^0 &= \begin{cases} 1 & \xi_1^0(T_1) = 0 \text{ or } S_2 > J_1, \\ \min\{n \geq 2 : \xi_n^1(T_n) = 0 \text{ or } S_{n+1} > J_n\} & \text{otherwise,} \end{cases} \\
N^1 &= \min\Big\{n \geq 1 : \xi_n^1(T_n) = 0 \text{ or } S_{n+1} > J_n\Big\}\ .
\end{aligned}$$

The *proof* of Lemma 4.1 is omitted since it is evident.

We will use (16) in order to determine the asymptotical tail behaviour of B. In connection with this, it is useful to consider a class of distributions of nonnegative random variables which is larger than $\mathcal{IR}$ and which contains, besides regular varying distributions, for example also the lognormal and Weibull distributions.

4.2 Subexponential Distributions

Let $\mathcal{S}$ denote the class of all *subexponential distributions* on $\mathbb{R}_+$ where a distribution function F is said to be subexponential if

$$\lim_{x\to\infty} \frac{1 - F^{*2}(x)}{1 - F(x)} = 2. \qquad (17)$$

Lemma 4.2 *Each distribution function from $\mathcal{IR}$ belongs to $\mathcal{S}$.*

Proof Using the defining equations (1) and (17), the statement of this lemma can easily be concluded from Corollary 1.2 of Cline (1994). $\square$

Knowing that $\mathcal{IR} \subset \mathcal{S}$, the following properties of subexponential distributions are useful to prove certain tail behaviours of distributions from class $\mathcal{IR}$.

Lemma 4.3 *Let F, G be two distributions on $\mathbb{R}_+$ and assume that there exists a constant $c \in (0,\infty)$ with $\overline{G}(x) \sim c\overline{F}(x)$. Then, $F \in \mathcal{S}$ if and only if $G \in \mathcal{S}$.*

Lemma 4.4 *Let F, G, H be distributions on $\mathbb{R}_+$ such that $F \in \mathcal{S}$, $\overline{G}(x) \sim c_1\overline{F}(x)$ and $\overline{H}(x) \sim c_2\overline{F}(x)$, where $c_i \in (0,\infty)$ for $i = 1, 2$. Then, $\overline{G * H}(x) \sim (c_1 + c_2)\overline{F}(x)$ and in particular $\overline{F * H}(x) \sim (1 + c_2)\overline{F}(x)$.*

Lemma 4.5 *Let $F \in \mathcal{S}$ and let G be a distributions on $\mathbb{R}$ such that $\overline{G}(x) = o(\overline{F}(x))$ as $x \to \infty$. Then, $\overline{F * G}(x) \sim \overline{F}(x)$.*

Lemma 4.6 *If $F \in \mathcal{S}$, then, for each $\varepsilon > 0$, there exists a constant $c < \infty$ such that $\overline{F^{*n}}(x) \le c(1+\varepsilon)^n\overline{F}(x)$ for all $n \ge 2$, $x \ge 0$.*

Remark The results stated in Lemmas 4.3 to 4.6 are well-known. For a proof of Lemmas 4.3 and 4.4, see the papers of Cline (1986) and Embrechts and Goldie (1982), for Lemma 4.5 see Pitman (1980). Lemma 4.6 is due to Kesten and can be found in Athreya and Ney (1972).

4.3 Asymptotics of Single Summands

We begin the proof of Theorem 4.1 with investigating the asymptotic behaviour of single summands appearing in (16).

Lemma 4.7 *For $x \to \infty$,*

$$\mathbb{P}\left(Y_1^0 + (r_2 - 1)J_1 > x, \xi_1^0(T_1) = 1, S_2 > J_1\right) \sim \mathbb{P}(Y_1^0 > x, \xi_1^0(T_1) = 1)\mathbb{P}(S > J)\,, \tag{18}$$

$$\mathbb{P}\left((r_2 - 1)S_n + Y_n^1 > x, \xi_n^1(T_n) = 0\right) \sim \mathbb{P}(Y_n^1 > x, \xi_n^1(T_n) = 0)\,, \tag{19}$$

$$\mathbb{P}\left((r_2 - 1)S_n + Y_n^1 + (r_2 - 1)J_n > x, \xi_n^1(T_n) = 1, S_{n+1} > J_n\right) \sim \mathbb{P}(Y_n^1 > x, \xi_n^1(T_n) = 1)\mathbb{P}(S > J)\,. \tag{20}$$

Proof Note that

$$\begin{aligned}&\mathbb{P}\left(Y_1^0 + (r_2-1)J_1 > x, \xi_1^0(T_1) = 1, S_2 > J_1\right)\\ &= \mathbb{P}\left(Y_1^0 + (r_2-1)J_1 > x, \xi_1^0(T_1) = 1\right)\mathbb{P}(S > J)\ .\end{aligned}$$

On the other hand, Lemmas 3.4 and 4.3 imply that the tail function $\overline{F}(x) = \mathbb{P}\left(Y_1^0 > x \mid \xi_1^0(T_1) = 1\right)$ belongs to $\mathcal{S}$. Thus, having in mind that the tail of J_1 is exponentially bounded, (18) follows from Lemma 4.5. The proof of (19) and (20) is analogous. □

4.4 Asymptotics of Finite Sums

We now investigate the asymptotics of finite sums of terms appearing in (16). For $n \geq 1$, $i = 0, 1$, let $\mathcal{A}_n^i = \{\xi_n^i(T_n) = 1, S_{n+1} < J_n\}$, $\mathcal{B}_n^0 = \{\xi_n^1(T_n) = 0\}$ and $\mathcal{B}_n^1 = \{\xi_n^1(T_n) = 1, S_{n+1} > J_n\}$. For notational ease we sometimes drop the superscript i. We also write Y_n for Y_n^1.

Lemma 4.8 *For $x \to \infty$, $i = 0, 1$,*

(a) $\mathbb{P}(Y_1^i + Y_2^1 > x, \mathcal{A}_1^i, \mathcal{B}_2^0) \sim (\pi_1 \mathbb{P}(S < J)\mathbb{P}(\mathcal{B}_2^0) + \mathbb{P}(\mathcal{A}_1^i)\pi_0)\, \mathbb{P}(\mu T > x)$

(b) $\mathbb{P}(Y_1^i + Y_2^1 > x, \mathcal{A}_1^i, \mathcal{B}_2^1)$
$\sim (\pi_1 \mathbb{P}(S < J)\mathbb{P}(\mathcal{B}_2^1) + \mathbb{P}(\mathcal{A}_1^i)\pi_1 \mathbb{P}(S > J))\, \mathbb{P}(\mu T > x)$.

Proof Note that $\mathbb{P}(Y_1^i + Y_2^1 > x, \mathcal{A}_1^i, \mathcal{B}_2^0) = \mathbb{P}(\mathcal{A}_1^i)\mathbb{P}(\mathcal{B}_2^0)\overline{G_1 * G_2}(x)$, where

$$\overline{G}_1(x) = \frac{\mathbb{P}(Y_1^i > x, \mathcal{A}_1^i)}{\mathbb{P}(\mathcal{A}_1^i)}, \qquad \overline{G}_2(x) = \frac{\mathbb{P}(Y_2^1 > x, \mathcal{B}_2^0)}{\mathbb{P}(\mathcal{B}_2^0)}\ .$$

By the result of Lemma 3.4 we have

$$\overline{G}_1(x) \sim \frac{\pi_1 \mathbb{P}(S < J)}{\mathbb{P}(\mathcal{A}_1^i)}\mathbb{P}(\mu T > x), \qquad \overline{G}_2(x) \sim \frac{\pi_0}{\mathbb{P}(\mathcal{B}_2^0)}\mathbb{P}(\mu T > x)\ .$$

Thus, since T is subexponential, Lemma 4.4 implies that

$$\mathbb{P}(Y_1^i + Y_2^1 > x, \mathcal{A}_1^i, \mathcal{B}_2^0) \sim \left(\pi_1 \mathbb{P}(S < J)\mathbb{P}(\mathcal{B}_2^0) + \mathbb{P}(\mathcal{A}_1^i)\pi_0\right)\mathbb{P}(\mu T > x)$$

and (a) is shown. The proof of (b) is analogous. □

Lemma 4.9 *For $n \geq 2, i = 0,1$ and $x \to \infty$,*

$$\begin{aligned}
\text{(a)}\quad & \mathbb{P}(Y_1^i + Y_2 + \ldots + Y_n > x, \mathcal{A}_1^i, \mathcal{A}_2^1, \ldots, \mathcal{A}_{n-1}^1, \mathcal{B}_n^0) \\
& \sim \Big(\pi_1 \mathbb{P}(S < J) \mathbb{P}(\mathcal{A}_2^1)^{n-2} \mathbb{P}(\mathcal{B}_n^0) \\
& \quad + (n-2)\pi_1 \mathbb{P}(S < J) \mathbb{P}(\mathcal{A}_1^i) \mathbb{P}(\mathcal{A}_2^1)^{n-3} \mathbb{P}(\mathcal{B}_n^0) \\
& \quad + \mathbb{P}(\mathcal{A}_1^i) \mathbb{P}(\mathcal{A}_2^1)^{n-2} \pi_0 \Big) \mathbb{P}(\mu T > x),
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\quad & \mathbb{P}(Y_1^i + Y_2 + \ldots + Y_n > x, \mathcal{A}_1^i, \mathcal{A}_2^1, \ldots, \mathcal{A}_{n-1}^1, B_n^1) \\
& \sim \Big(\pi_1 \mathbb{P}(S < J) \mathbb{P}(\mathcal{A}_2^1)^{n-2} \mathbb{P}(\mathcal{B}_n^1) \\
& \quad + (n-2)\pi_1 \mathbb{P}(S < J) \mathbb{P}(\mathcal{A}_1^i) \mathbb{P}(\mathcal{A}_2^1)^{n-3} \mathbb{P}(\mathcal{B}_n^1) \\
& + \mathbb{P}(\mathcal{A}_1^i) \mathbb{P}(\mathcal{A}_2^1)^{n-2} \pi_1 \mathbb{P}(S > J) \Big) \mathbb{P}(\mu T > x).
\end{aligned}$$

Proof We prove the lemma by induction with respect to n. For $n = 2$ this is just Lemma 4.8. Suppose that (a) holds for some $n \geq 2$. Note that

$$\begin{aligned}
& \mathbb{P}(Y_1^i + Y_2 + \ldots + Y_{n+1} > x, \mathcal{A}_1^i, \mathcal{A}_2^1, \ldots, \mathcal{A}_n^1, \mathcal{B}_{n+1}^0) \\
& = \quad \mathbb{P}(\mathcal{A}_1^i) \mathbb{P}(\mathcal{A}_2^1)^{n-1} \mathbb{P}(\mathcal{B}_{n+1}^0) \overline{G_1 * G_2}(x) ,
\end{aligned}$$

where

$$\overline{G}_1(x) = \frac{\mathbb{P}(Y_1^i + Y_2 + \ldots + Y_{n-1} + Y_{n+1} > x, \mathcal{A}_1^i, \mathcal{A}_2^1, \ldots, \mathcal{A}_{n-1}^1, \mathcal{B}_{n+1}^0)}{\mathbb{P}(\mathcal{A}_1^i) \mathbb{P}(\mathcal{A}_2^1)^{n-2} \mathbb{P}(\mathcal{B}_{n+1}^0)}$$

and $\overline{G}_2(x) = \mathbb{P}(Y_n > x, \mathcal{A}_n^1)(\mathbb{P}(\mathcal{A}_n^1))^{-1}$. Applying our hypothesis to $\overline{G}_1(x)$ and the result of Lemma 3.4 to $\overline{G}_2(x)$, Lemma 4.4 yields (a). The proof of (b) is analogous. □

4.5 Proof of Theorem 4.1

¿From representation formula (16) we get

$$\begin{aligned}
& \mathbb{P}(B > x) = \mathbb{P}\left((r_2 - 1)J_0 > x, S_0 > I_0, S_1 > J_0\right) + \mathbb{P}(S < I) \times \\
& \times \Big[\mathbb{P}(Y_1^0 > x, \xi_1^0(T_1) = 0) + \mathbb{P}(Y_1^0 + (r_2 - 1)J_1 > x, \xi_1^0(T_1) = 1, S_2 > J_1) \\
& \qquad + \sum_{n=2}^{\infty} \Big\{ \mathbb{P}\Big(Y_1^0 + \sum_{j=2}^{n} \left((r_2 - 1)S_j + Y_j^1 \right) > x, \mathcal{A}_1^0, \mathcal{A}_2^1, \ldots, \mathcal{A}_{n-1}^1, \mathcal{B}_n^0 \Big)
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{P}\Big(Y_1^0+\sum_{j=2}^{n}\Big((r_2-1)S_j+Y_j^1\Big)+(r_2-1)J_n>x,\\
&\mathcal{A}_1^0,\mathcal{A}_2^1,\ldots,\mathcal{A}_{n-1}^1,\mathcal{B}_n^1)\Big\}\Big]\\
+\quad&\mathbb{P}(S>I)\mathbb{P}(S<J)\Big[\mathbb{P}\Big((r_2-1)S_1+Y_1^1>x,\xi_1^1(T_1)=0\Big)\\
&+\mathbb{P}\Big((r_2-1)S_1+Y_1^1+(r_2-1)J_1>x,\xi_1^1(T_1)=1,S_2>J_1\Big)\\
&+\sum_{n=2}^{\infty}\Big\{\mathbb{P}\Big(\sum_{j=1}^{n}\Big((r_2-1)S_j+Y_j^1\Big)>x,\mathcal{A}_1^1,\ldots,\mathcal{A}_{n-1}^1,\mathcal{B}_n^0\Big)\\
&+\mathbb{P}\Big(\sum_{j=1}^{n}\Big((r_2-1)S_j+Y_j^1\Big)+(r_2-1)J_n>x,\mathcal{A}_1^1,\ldots,\mathcal{A}_{n-1}^1,\mathcal{B}_n^1\Big)\Big\}\Big]
\end{aligned}
$$

Following the same argumentation as in the proof of Lemma 4.7 we can asymptotically neglect the S_j and J_j in all summands. Furthermore, using Lemma 4.6, we can pass with the limit $x\to\infty$ under the infinite sum $\sum_{n=2}^{\infty}$. With the notation $p=\mathbb{P}(S<J)$, $\alpha_i^j=\mathbb{P}(\xi^j(T)=i)$ for $i,j=0,1$, by Lemma 4.9 we thus obtain the following result. Due to the fact that $\mathbb{P}(\mathcal{B}_n^0)+\mathbb{P}(\mathcal{B}_n^1)=1-\mathbb{P}(\mathcal{A}_n^1)$, for $x\to\infty$,

$$
\begin{aligned}
\mathbb{P}(B>x)&\sim\Big\{\mathbb{P}(S<I)\Big[(1-\pi_1p)+\sum_{n=2}^{\infty}\Big(\pi_1p(\alpha_1^1p)^{n-2}(1-\alpha_1^1p)\\
&\quad+(n-2)\pi_1p\alpha_1^0p(\alpha_1^1p)^{n-3}(1-\alpha_1^1p)+\alpha_1^0p(\alpha_1^1p)^{n-2}(1-\pi_1p)\Big)\Big]\\
&\quad+\mathbb{P}(S>I)p\Big[(1-\pi_1p)+\sum_{n=2}^{\infty}\Big((n-1)\pi_1p(\alpha_1^1p)^{n-2}(1-\alpha_1^1p)\\
&\quad+(\alpha_1^1p)^{n-1}(1-\pi_1p)\Big)\Big]\Big\}\mathbb{P}(\mu T>x)\\
&=\Big(\mathbb{P}(S<I)\frac{1-\pi_1-\alpha_1^1p+2\pi_1p-p}{(1-\pi_1)(1-\alpha_1^1p)}+\frac{p}{1-\alpha_1^1p}\Big)\mathbb{P}(\mu T>x)\ ,
\end{aligned}
$$

where in the last equation we have used that $\pi_0\alpha_1^0+\pi_1\alpha_1^1=\pi_1$. Since $\mathbb{P}(S<I)=\nu/(\nu+\lambda_{\text{off}})$, $\pi_1=\lambda_{\text{off}}/(\lambda_{\text{on}}+\lambda_{\text{off}})$, $p=\nu/(\nu+\lambda_{\text{on}})$, we obtain

$$
\begin{aligned}
&\mathbb{P}(S<I)\frac{1-\pi_1-\alpha_1^1p+2\pi_1p-p}{(1-\pi_1)(1-\alpha_1^1p)}+\frac{p}{1-\alpha_1^1p}\\
&\quad=\frac{\mathbb{P}(S<I)}{1-\pi_1}+\Big(\mathbb{P}(S<I)\,\frac{2\pi_1p-\pi_1-p}{1-\pi_1}+p\Big)\,\frac{1}{1-\alpha_1^1p}
\end{aligned}
$$

$$= \frac{\mathbb{P}(S < I)}{1-\pi_1} = \frac{\nu(\lambda_{\rm on}+\lambda_{\rm off})}{(\nu+\lambda_{\rm off})\lambda_{\rm on}} \, .$$

Thus,

$$\lim_{x\to\infty} \frac{\mathbb{P}(B > x)}{\mathbb{P}(\mu T > x)} = \frac{\nu(\lambda_{\rm on}+\lambda_{\rm off})}{(\nu+\lambda_{\rm off})\lambda_{\rm on}} \, . \qquad \square$$

5 Fluid Model with m Sources

Suppose now that $m \geq 2$. In this more general case we can use similar arguments as in Section 4 since a representation formula analogous to (16) can be derived for the generic net inflow B during an aggregate activity period. Let $q \in \{2,\ldots,m\}$. The idea is to group together sources 1 to q to a single source with on-periods $C^{(q)}$, where $C^{(q)}$ denotes the aggregate activity period of sources 1 to q, and off-periods $Z^{(q)} = \min(S, I^{(2)},\ldots,I^{(q)})$, which are exponentially distributed with mean $1/(\nu+\sum_{k=2}^{q}\lambda_{\rm off}^{(k)})$. Let $B^{(q)}$ be the net inflow of *all* sources 1 to m during $C^{(q)}$. Then, with the notation

$$\begin{aligned}
\mathcal{A}^{i}_{q,n} &= \left\{\zeta^{i}_{q,n}(C^{(q-1)}_{n}) = 1, Z^{(q-1)}_{n+1} < J^{(q)}_{n}\right\} \, , \\
\mathcal{B}^{i,0}_{q,n} &= \left\{\zeta^{i}_{q,n}(C^{(q-1)}_{n}) = 0\right\} \, , \quad \mathcal{B}^{i,1}_{q,n} = \left\{\zeta^{i}_{q,n}(C^{(q-1)}_{n}) = 1, Z^{(q-1)}_{n+1} > J^{(q)}_{n}\right\} \, ,
\end{aligned}$$

where $\{\zeta^{i_q}_{q}(t)\}$ is the $(q-1)$-st component of $\{\xi^{\boldsymbol{i}}(t)\}$, indicating whether at time t source q is active or not, we have

$$\begin{aligned}
&\mathbb{P}\left(B^{(q)} > x\right) \\
&\sim \mathbb{P}\left(Z^{(q-1)} < I^{(q)}\right)\Big[\mathbb{P}\left(B^{(q-1)}_{1} > x, \mathcal{B}^{0,0}_{q,1}\right) + \mathbb{P}\left(B^{(q-1)}_{1} > x, \mathcal{B}^{0,1}_{q,1}\right) \\
&\quad + \sum_{n=2}^{\infty}\Big\{\mathbb{P}\Big(B^{(q-1)}_{1} + \ldots + B^{(q-1)}_{n} > x, \mathcal{A}^{0}_{q,1}, \mathcal{A}^{1}_{q,2},\ldots,\mathcal{A}^{1}_{q,n-1}, \mathcal{B}^{1,0}_{q,n}\Big) \\
&\quad + \mathbb{P}\left(B^{(q-1)}_{1} + \ldots + B^{(q-1)}_{n} > x, \mathcal{A}^{0}_{q,1}, \mathcal{A}^{1}_{q,2},\ldots,\mathcal{A}^{1}_{q,n-1}, \mathcal{B}^{1,1}_{q,n}\right)\Big\}\Big] \\
&+ \mathbb{P}\left(Z^{(q-1)} > I^{(q)}\right)\mathbb{P}\left(Z^{(q-1)} < J^{(q)}\right)\Big[\mathbb{P}\left(B^{(q-1)} > x, \mathcal{B}^{1,0}_{q,1}\right) \\
&\quad + \mathbb{P}\left(B^{(q-1)} > x, \mathcal{B}^{1,1}_{q,1}\right) \qquad (21) \\
&\quad + \sum_{n=2}^{\infty}\Big\{\mathbb{P}\Big(B^{(q-1)}_{1} + \ldots + \mathcal{B}^{(q-1)}_{n} > x, \mathcal{A}^{1}_{q,1},\ldots,\mathcal{A}^{1}_{q,n-1}, \mathcal{B}^{1,0}_{q,n}\Big)
\end{aligned}$$

$$+\mathbb{P}\Big(B_1^{(q-1)}+\ldots+B_n^{(q-1)}>x,\mathcal{A}^1_{q,1},\ldots,\mathcal{A}^1_{q,n-1},\mathcal{B}^{1,1}_{q,n}\Big)\Big\}\Big] .$$

¿From this we obtain the following result.

Theorem 5.1 *For each* $q\in\{2,\ldots,m\}$,

$$\mathbb{P}(B^{(q)}>x)\sim\frac{\nu}{\nu+\sum_{k=2}^q\lambda^{(k)}_{\rm off}}\prod_{k=2}^q\left(1+\frac{\lambda^{(k)}_{\rm off}}{\lambda^{(k)}_{\rm on}}\right)\mathbb{P}(\mu T>x)\ ,\quad x\to\infty\ . \quad (22)$$

We prove Theorem 5.1 by induction with respect to the number $q\in\{2,\ldots,m\}$ of grouped sources. Note that our considerations in Section 4 do not depend on the fact that for $m=2$ the process $\{\eta(t)\}$ admits only the two states r_1-1 and r_1-1+r_2. Thus, considering the process $\{\eta(t)\}$ given in (7), we also obtain

$$\mathbb{P}(B^{(2)}>x)\sim\frac{\nu}{\nu+\lambda^{(2)}_{\rm off}}\left(1+\frac{\lambda^{(2)}_{\rm off}}{\lambda^{(2)}_{\rm on}}\right)\mathbb{P}(\mu T>x)\ ,\quad x\to\infty\ ,$$

where $\mu=\lim_{t\to\infty}\mathbb{E}\left(r_1-1+\sum_{k=2}^m r_k\mathbb{I}(k\in\xi(t))\right)$. Now suppose that (22) holds for some $q\in\{2,\ldots,m-1\}$. Then, the distribution of $B^{(q)}$ is subexponential and thus the lemmas stated in Section 4.2 can be applied to this distribution. Theorem 5.1 is proved if we show that (22) holds for $q+1$ as well. For showing this we will make use of the representation formula (21) and therefore we need some auxiliary results on the asymptotical behaviour of the summands in (21).

5.1 Some Auxiliary Results

Let $\tilde{\boldsymbol{\pi}}_q=(\tilde\pi_{q,0},\tilde\pi_{q,1})$ be the stationary distribution of $\{\zeta^i_q(t)\}, i=0,1$.

Lemma 5.1 *For* $x\to\infty, i,j=0,1$,

$$\mathbb{P}\left(B^{(q-1)}>x,\zeta^i_q(C^{(q-1)})=j\right)\sim\tilde\pi_{q,j}\mathbb{P}\left(B^{(q-1)}>x\right)\ .$$

Proof Similar as in the proof of Lemma 3.4 we have, with the abbreviating notation $B=B^{(q-1)}$ and $C=C^{(q-1)}$, for some $\delta>0$ and for all x sufficiently large,

$$\begin{aligned}\mathbb{P}\Big(B>x,\zeta^i_q(C)=j\Big)&=\mathbb{P}\Big(B>x,C>\frac{x}{r_{\max}},\zeta^i_q(C)=j\Big)\\ &=\int_x^\infty\int_{\frac{x}{r_{\max}}}^\infty(p_{ij}(t)-\tilde\pi_{q,j}+\tilde\pi_{q,j})\,dF_{(B,C)}(s,t)\\ &\le\Big(e^{-\delta\frac{x}{r_{\max}}}+\tilde\pi_{q,j}\Big)\mathbb{P}\Big(B>x,C>\frac{x}{r_{\max}}\Big)\end{aligned}$$

and therefore

$$\limsup_{x\to\infty} \frac{\mathbb{P}\left(B > x, \zeta_q^i(C) = j\right)}{\mathbb{P}(B > x)} \le \tilde{\pi}_{q,j} \ .$$

On the other hand,

$$\begin{aligned}
\mathbb{P}\left(B > x, \zeta_q^i(C) = j\right) &= \int_x^\infty \int_{\frac{x}{r_{\max}}}^\infty \left(p_{ij}(t) - \tilde{\pi}_{q,j} + \tilde{\pi}_{q,j}\right) dF_{(B,C)}(s,t) \\
&\ge -\int_x^\infty \int_{\frac{x}{r_{\max}}}^\infty e^{-\delta t} dF_{(B,C)}(s,t) + \tilde{\pi}_{q,j}\mathbb{P}\left(B > x, C > \frac{x}{r_{\max}}\right) \\
&\ge -e^{-\delta\frac{x}{r_{\max}}}\mathbb{P}\left(B > x, C > \frac{x}{r_{\max}}\right) + \tilde{\pi}_{q,j}\mathbb{P}(B > x)
\end{aligned}$$

and thus

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left(B > x, \zeta_q^i(C) = j\right)}{\mathbb{P}(B > x)} \ge \tilde{\pi}_{q,j} \ . \qquad \square$$

The following two lemmas are similar to Lemmas 4.8 and 4.9.

Lemma 5.2 *For* $x \to \infty, i = 0, 1$,

(a) $\mathbb{P}\left(B_1^{(q-1)} + B_2^{(q-1)} > x, \mathcal{A}_{q,1}^i, \mathcal{B}_{q,2}^{1,0}\right)$
$\sim \left(\tilde{\pi}_{q,1}\mathbb{P}(Z^{(q-1)} < J^{(q)})\mathbb{P}(\mathcal{B}_{q,2}^{1,0}) + \mathbb{P}(\mathcal{A}_{q,1}^i)\tilde{\pi}_{q,0}\right)\mathbb{P}(B^{(q-1)} > x)$,

(b) $\mathbb{P}\left(B_1^{(q-1)} + B_2^{(q-1)} > x, \mathcal{A}_{q,1}^i, \mathcal{B}_{q,2}^{1,1}\right) \sim \left(\tilde{\pi}_{q,1}\mathbb{P}(Z^{(q-1)} < J^{(q)})\mathbb{P}(\mathcal{B}_{q,2}^{1,1})\right.$
$\left. + \mathbb{P}(\mathcal{A}_{q,1}^i)\tilde{\pi}_{q,1}\mathbb{P}(Z^{(q-1)} > J^{(q)})\right)\mathbb{P}(B^{(q-1)} > x)$.

Proof Note that

$$\mathbb{P}\left(B_1^{(q-1)} + B_2^{(q-1)} > x, \mathcal{A}_{q,1}^i, \mathcal{B}_{q,2}^{1,0}\right) = \mathbb{P}\left(\mathcal{A}_{q,1}^i\right)\mathbb{P}\left(\mathcal{B}_{q,2}^{1,0}\right)\overline{G_1 * G_2}(x)$$

where

$$\overline{G}_1(x) = \frac{\mathbb{P}\left(B_1^{(q-1)} > x, \mathcal{A}_{q,1}^i\right)}{\mathbb{P}\left(\mathcal{A}_{q,1}^i\right)}, \quad \overline{G}_2(x) = \frac{\mathbb{P}\left(B_2^{(q-1)} > x, \mathcal{B}_{q,2}^{1,0}\right)}{\mathbb{P}\left(\mathcal{B}_{q,2}^{1,0}\right)} .$$

By the result of Lemma 5.1 we have

$$\overline{G}_1(x) \sim \frac{\tilde{\pi}_{q,1}\mathbb{P}\left(Z^{(q-1)} < J^{(q)}\right)}{\mathbb{P}\left(\mathcal{A}_{q,1}^i\right)}\mathbb{P}\left(B^{(q-1)} > x\right)$$

and

$$\overline{G}_2(x) \sim \frac{\tilde{\pi}_{q,0}}{\mathbb{P}\left(\mathcal{B}_{q,2}^{1,0}\right)} \mathbb{P}\left(B^{(q-1)} > x\right) .$$

Thus, Lemma 4.4 implies that

$$\begin{aligned} &\mathbb{P}\left(B_1^{(q-1)} + B_2^{(q-1)} > x, \mathcal{A}_{q,1}^i, \mathcal{B}_{q,2}^{1,0}\right) \\ &\quad \sim \quad \left(\tilde{\pi}_{q,1}\mathbb{P}\left(Z^{(q-1)} < J^{(q)}\right) \mathbb{P}\left(\mathcal{B}_{q,2}^{1,0}\right) + \mathbb{P}\left(\mathcal{A}_{q,1}^i\right) \tilde{\pi}_{q,0}\right) \mathbb{P}\left(B^{(q-1)} > x\right) \end{aligned}$$

and (a) is shown. The proof of (b) is analogous. $\square$

Lemma 5.3 *For $n \geq 2, i = 0, 1$ and $x \to \infty$,*

(a) $$\begin{aligned} &\mathbb{P}(B_1^{(q-1)} + \ldots + B_n^{(q-1)} > x, \mathcal{A}_{q,1}^i, \mathcal{A}_{q,1}^2, \ldots, \mathcal{A}_{q,n-1}^1, \mathcal{B}_{q,n}^{1,0}) \\ &\sim \Big(\tilde{\pi}_{q,1}\mathbb{P}(Z^{(q-1)} < J^{(q)})\mathbb{P}(\mathcal{A}_{q,2}^1)^{n-2}\mathbb{P}(\mathcal{B}_{q,n}^{1,0}) \\ &\quad +(n-2)\tilde{\pi}_{q,1}\mathbb{P}(Z^{(q-1)} < J^{(q)})\mathbb{P}(\mathcal{A}_{q,1}^i)\mathbb{P}(\mathcal{A}_{q,2}^1)^{n-3}\mathbb{P}(\mathcal{B}_{q,n}^{1,0}) \\ &\quad +\mathbb{P}(\mathcal{A}_{q,1}^i)\mathbb{P}(\mathcal{A}_{q,2}^1)^{n-2}\tilde{\pi}_{q,0}\Big)\mathbb{P}(B^{(q-1)} > x), \end{aligned}$$

(b) $$\begin{aligned} &\mathbb{P}(B_1^{(q-1)} + B_2^{(q-1)} + \ldots + B_n^{(q-1)} > x, \mathcal{A}_{q,1}^i, \mathcal{A}_{q,2}^1, \ldots, \mathcal{A}_{q,n-1}^1, \mathcal{B}_{q,n}^{1,1}) \\ &\sim \Big(\tilde{\pi}_{q,1}\mathbb{P}(Z^{(q-1)} < J^{(q)})\mathbb{P}(\mathcal{A}_{q,2}^1)^{n-2}\mathbb{P}(\mathcal{B}_{q,n}^{1,1}) \\ &\quad +(n-2)\tilde{\pi}_{q,1}\mathbb{P}(Z^{(q-1)} < J^{(q)})\mathbb{P}(\mathcal{A}_{q,1}^i)\mathbb{P}(\mathcal{A}_{q,2}^1)^{n-3}\mathbb{P}(\mathcal{B}_{q,n}^{1,1}) \\ &\quad +\mathbb{P}(\mathcal{A}_{q,1}^i)\mathbb{P}(\mathcal{A}_{q,2}^1)^{n-2}\tilde{\pi}_{q,1}\mathbb{P}(Z^{(q-1)} > J^{(q)})\Big)\mathbb{P}(B^{(q-1)} > x). \end{aligned}$$

The *proof* of this lemma is analogous to the proof of Lemma 4.9.

5.2 Proof of Theorem 5.1

We are now in the position to perform the induction step. We use the notation $\tilde{p}_q = \mathbb{P}(Z^{(q-1)} < J^{(q)})$ and $\tilde{\alpha}_{q,i}^j = \mathbb{P}(\zeta_q^j(C^{(q-1)}) = i)$ for $i, j = 0, 1$. Then, due to the fact that $\mathbb{P}(\mathcal{B}_{q,n}^{1,0}) + \mathbb{P}(\mathcal{B}_{q,n}^{1,1}) = 1 - \mathbb{P}(\mathcal{A}_{q,n}^1)$, we obtain from representation formula (21) and Lemma 5.3, for $x \to \infty$,

$$\begin{aligned} \mathbb{P}(B^{(q)} > x) &\sim \left(\mathbb{P}\left(Z^{(q-1)} < I^{(q)}\right)\left(1 + \frac{\tilde{\alpha}_{q,1}^0 \tilde{p}_q}{1 - \tilde{\alpha}_{q,1}^1 \tilde{p}_q}\right)\right. \\ &\qquad \left. +\mathbb{P}\left(Z^{(q-1)} > I^{(q)}\right)\tilde{p}_q \frac{1}{1 - \tilde{\alpha}_{q,1}^1 \tilde{p}_q}\right)\mathbb{P}\left(B^{(q-1)} > x\right) \\ &= \quad \left(\mathbb{P}\left(Z^{(q-1)} < I^{(q)}\right)\frac{1 - \tilde{\pi}_{q,1} - \tilde{\alpha}_{q,1}^1 \tilde{p}_q + 2\tilde{\pi}_{q,1}\tilde{p}_q - \tilde{p}_q}{(1 - \tilde{\pi}_{q,1})(1 - \tilde{\alpha}_{q,1}^1 \tilde{p}_q)} + \frac{\tilde{p}_q}{1 - \tilde{\alpha}_{q,1}^1 \tilde{p}_q}\right) \times \\ &\qquad \times \mathbb{P}\left(B^{(q-1)} > x\right) , \end{aligned}$$

where in the last equation we have used that $\tilde{\pi}_{q,0}\tilde{\alpha}^0_{q,1}+\tilde{\pi}_{q,1}\tilde{\alpha}^1_{q,1}=\tilde{\pi}_{q,1}$. Since $\mathbb{P}\left(Z^{(q-1)}<I^{(q)}\right)=(\nu+\sum_{k=2}^{q-1}\lambda^{(k)}_{\text{off}})/(\nu+\sum_{k=2}^{q}\lambda^{(k)}_{\text{off}})$, $\tilde{\pi}_{q,1}=\lambda^{(q)}_{\text{off}}/(\lambda^{(q)}_{\text{on}}+\lambda^{(q)}_{\text{off}})$ and $\tilde{p}_q=(\nu+\sum_{k=2}^{q-1}\lambda^{(k)}_{\text{off}})/(\nu+\sum_{k=2}^{q-1}\lambda^{(k)}_{\text{off}}+\lambda^{(q)}_{\text{on}})$, we obtain

$$\begin{aligned}
&\mathbb{P}(Z^{(q-1)}<I^{(q)})\frac{1-\tilde{\pi}_{q,1}-\tilde{\alpha}^1_{q,1}\tilde{p}_q+2\tilde{\pi}_{q,1}\tilde{p}_q-\tilde{p}_q}{(1-\tilde{\pi}_{q,1})(1-\tilde{\alpha}^1_{q,1}\tilde{p}_q)}+\frac{\tilde{p}_q}{1-\tilde{\alpha}^1_{q,1}\tilde{p}_q}\\
&= \frac{\mathbb{P}(Z^{(q-1)}<I^{(q)})}{1-\tilde{\pi}_{q,1}}+\left(\mathbb{P}(Z^{(q-1)}<I^{(q)})\frac{2\tilde{\pi}_{q,1}\tilde{p}_q-\tilde{\pi}_{q,1}-\tilde{p}_q}{1-\tilde{\pi}_{q,1}}+\tilde{p}_q\right)\times\\
&\quad\times\frac{1}{1-\tilde{\alpha}^1_{q,1}\tilde{p}_q}=\frac{\mathbb{P}(Z^{(q-1)}<I^{(q)})}{1-\tilde{\pi}_{q,1}}=\frac{\nu+\sum_{k=2}^{q-1}\lambda^{(k)}_{\text{off}}}{\nu+\sum_{k=2}^{q}\lambda^{(k)}_{\text{off}}}\frac{\lambda^{(q)}_{\text{on}}+\lambda^{(q)}_{\text{off}}}{\lambda^{(q)}_{\text{on}}}\,.
\end{aligned}$$

Thus, by the hypothesis of our induction, for $x\to\infty$

$$\begin{aligned}
\mathbb{P}(B^{(q)}>x)&\sim\frac{\nu+\sum_{k=2}^{q-1}\lambda^{(k)}_{\text{off}}}{\nu+\sum_{k=2}^{q}\lambda^{(k)}_{\text{off}}}\frac{\lambda^{(q)}_{\text{on}}+\lambda^{(q)}_{\text{off}}}{\lambda^{(q)}_{\text{on}}}\mathbb{P}\left(B^{(q-1)}>x\right)\\
&\sim\frac{\nu+\sum_{k=2}^{q-1}\lambda^{(k)}_{\text{off}}}{\nu+\sum_{k=2}^{q}\lambda^{(k)}_{\text{off}}}\frac{\lambda^{(q)}_{\text{on}}+\lambda^{(q)}_{\text{off}}}{\lambda^{(q)}_{\text{on}}}\frac{\nu}{\nu+\sum_{k=2}^{q-1}\lambda^{(k)}_{\text{off}}}\prod_{k=2}^{q-1}\left(1+\frac{\lambda^{(k)}_{\text{off}}}{\lambda^{(k)}_{\text{on}}}\right)\mathbb{P}(\mu T>x)\\
&=\frac{\nu}{\nu+\sum_{k=2}^{q}\lambda^{(k)}_{\text{off}}}\prod_{k=2}^{q}\left(1+\frac{\lambda^{(k)}_{\text{off}}}{\lambda^{(k)}_{\text{on}}}\right)\mathbb{P}(\mu T>x)\,,
\end{aligned}$$

which means that (22) holds for $q+1$. This completes the proof. □

6 Palm-Stationary Buffer Content Distribution

Recall that the development of the fluid flow queueing system can be seen as a random walk with generic increment $B-\min\{S,I^{(2)},\ldots,I^{(m)}\}$. This enables us to determine the asymptotical behaviour of the tail of the steady-state distribution $W(x)=\lim_{n\to\infty}\mathbb{P}(W_n\le x)$ of the buffer content at the beginning of aggregate activity periods. However, in connection with this we have to determine the expected net input $\mathbb{E}B$ during aggregate activity periods. ¿From (22) we conclude that $\mathbb{E}B<\infty$ and hence $\mathbb{E}C<\infty$. The process $\{\xi(t)\}$ with $\xi(t)=\sum_{k=1}^{m}r_k\mathbb{I}(\text{source } k \text{ is active at time } t)$ is regenerative and hence the stochastic mean value theorem (see Cohen (1976)) implies that the expected total input L into the system per unit time can be given in the following way:

$$L = \frac{\mathbb{E}\int_0^{C+Z} \sum_{k=1}^m r_k \mathbb{1}(\text{source } k \text{ is active at time } t)\, dt}{\mathbb{E}(C+Z)} \ ,$$

where L is given in (3) and $Z = \min\{S, I^{(2)}, \ldots, I^{(m)}\}$. Thus,

$$L = \frac{\mathbb{E}B + \mathbb{E}C}{\mathbb{E}C + \mathbb{E}Z} \ . \tag{23}$$

The same argument applied to the alternating renewal process with generic alternating variables Z and C yields

$$\frac{1}{1+\mathbb{E}T\nu} \prod_{k=2}^{m} \frac{\lambda_{\text{on}}^{(k)}}{\lambda_{\text{on}}^{(k)} + \lambda_{\text{off}}^{(k)}} = \frac{\mathbb{E}Z}{\mathbb{E}C + \mathbb{E}Z} \ , \tag{24}$$

where $\mathbb{E}Z = 1/\Lambda = 1/(\nu + \sum_{k=2}^m \lambda_{\text{off}}^{(k)})$. Now, (23) and (24) imply that

$$\mathbb{E}B = \frac{1}{\Lambda}\left(1 - (1-L)(1+\mathbb{E}T\nu)\prod_{k=2}^{m}\left(1 + \frac{\lambda_{\text{off}}^{(k)}}{\lambda_{\text{on}}^{(k)}}\right)\right) \ , \tag{25}$$

see also Cohen (1994). This, together with Theorem 5.1, allows us to determine the asymptotical behaviour of $\overline{W}(x) = 1 - W(x)$ as $x \to \infty$. By Theorem 5.1 we have

$$\mathbb{P}(B > x) \sim \frac{\nu}{\Lambda}\prod_{k=2}^{m}\left(1 + \frac{\lambda_{\text{off}}^{(k)}}{\lambda_{\text{on}}^{(k)}}\right)\mathbb{P}(\mu T > x)\,, \quad x \to \infty \ , \tag{26}$$

i.e. the distribution F_B of the net input B during aggregate activity periods belongs to $\mathcal{IR}$ since $F \in \mathcal{IR}$. Furthermore, we need the following closure property of class $\mathcal{IR}$. Let F be the distribution of a nonnegative random variable with finite positive expectation μ. The distribution F^{s} given by

$$F^{\mathrm{s}}(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)\, dy, \qquad x \geq 0 \tag{27}$$

is called the *integrated tail distribution* of F.

Lemma 6.1 *If $F \in \mathcal{IR}$, then $F^{\mathrm{s}} \in \mathcal{IR}$.*

The *proof* easily follows from the defining equations (1) and (27).

Theorem 6.1 *Let L denote the quantity given in* (3). *If $F \in \mathcal{IR}$, then*

$$\overline{W}(x) \sim \frac{\nu}{(1-L)(1+\mathbb{E}T\nu)} \int_x^\infty \mathbb{P}(\mu T > u)du\,, \quad x \to \infty\,. \tag{28}$$

Proof ¿From Lemmas 4.2 and 6.1 we conclude that F_B^s is subexponential. Then, a well-known result on the asymptotical tail behaviour of the maximum of random walks with negative drift (see Embrechts and Veraverbeke (1982), Pakes (1975)) gives, together with (26),

$$\begin{aligned}\overline{W}(x) &\sim \frac{\nu + \sum_{k=2}^m \lambda_{\text{off}}^{(k)}}{1 - \left(\nu + \sum_{k=2}^m \lambda_{\text{off}}^{(k)}\right)\mathbb{E}B} \int_x^\infty \mathbb{P}(B > u)du \\ &\sim \frac{\Lambda}{1-\Lambda\mathbb{E}B}\frac{\nu}{\Lambda}\prod_{k=2}^m \left(1 + \frac{\lambda_{\text{off}}^{(k)}}{\lambda_{\text{on}}^{(k)}}\right) \int_x^\infty \mathbb{P}(\mu T > u)du\,.\end{aligned}$$

Now, (28) follows from (25). □

Corollary 6.1 *If $\overline{F}$ is regular varying at infinity, i.e. $\overline{F}(x) \sim (\alpha-1)(\mathbb{E}T)^\alpha$ $x^{-\alpha}l(x)$, $x \to \infty$, $\alpha > 1$ and $l(x)$ slowly varying at infinity, then*

$$\overline{W}(x) \sim \frac{\nu}{(1-L)(1+\nu\mathbb{E}T)} M^\alpha x^{-(\alpha-1)} l(x)\,, \quad x \to \infty\,, \tag{29}$$

where L is given in (3) *and* $M = \mathbb{E}T\left(r_1 - 1 + \sum_{k=2}^m (r_k\lambda_{\text{off}}^{(k)})/(\lambda_{\text{on}}^{(k)} + \lambda_{\text{off}}^{(k)})\right)$.

Proof By Karamata's theorem (see Bingham, Goldie and Teugels (1987))

$$\begin{aligned}\int_x^\infty \mathbb{P}(\mu T > u)du &= \mu \frac{\int_{x/\mu}^\infty \mathbb{P}(T > u)du}{(x/\mu)\mathbb{P}(T > (x/\mu))} \frac{x}{\mu} \mathbb{P}(T > (x/\mu)) \\ &\sim 1/(\alpha-1)x(\alpha-1)(\mathbb{E}T)^\alpha (x/\mu)^{-\alpha} l(x/\mu) \sim (\mathbb{E}T\mu)^\alpha x^{-(\alpha-1)} l(x)\,,\end{aligned}$$

and thus (29) follows from (28). □

Remark Note that Corollary 6.1 is a special case of a result which has been recently obtained in Boxma (1997). As mentioned before, the result in Boxma (1997) concerns more general activity period distributions of sources $2, \ldots, m$, namely such with an exponentially bounded tail.

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