On the supremum from Gaussian processes over infinite horizon

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Abstract

In the paper we study the asymptotic of the tail of distribution function $\mathbb{P}(A(X,c) > x)$ for $x \to \infty$, where $A(X,c)$ is the supremum of $X(t) - ct$ over $[0,\infty)$. In particular $X(t)$ is the fractional Brownian motion, a nonlinearly scaled Brownian motion or some integrated stationary Gaussian processes. For the fractional Brownian motion we give a stronger result than a recent one of Duffield and O'Connell [5].

Keywords: fractional Brownian motion, scaled Brownian motion, Gaussian process, fluid model, exponential bound.

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1 Introduction

In this paper we study the asymptotic behaviour of the tail of distribution function of

$$A(X,c) = \sup\{X(t) - ct : \ t \geq 0\},$$

where $X(t)$ is a mean zero Gaussian process. Thus X is:
FBM: a fractional Brownian motion with parameter H $(0 < H \leq 1)$, which we denote by $B_H(t)$,
SBM: a scaled Brownian motion $S_H(t) = B(t^{2H})$ with parameter $H > 0$,
FiBM: a filtered Brownian motion

$$B^\circ(t) = \sqrt{2H}\int_0^t (t-s)^{H-1/2}\, dB(s)$$

with parameter $H > 0$,
DG: a degenerated Gaussian

$$N_H(t) = t^H \mathcal{N}$$

with parameter $H > 0$ where $\mathcal{N}$ is the standard normal random variable and $H > 0$,
GI: the integral $\int_0^t Z(s)\, ds$ where Z is a stationary short range dependent stationary process.

The common feature of all considered above processes, except from the **GI** case, is that the variance $\mathbb{D}^2 X(t) = t^{2H}$. For the **GI** case the variance is an asymptotically linear function of t.

For $H = 1/2$ the **FBM** is the Brownian motion $B_{1/2}(t) = B(t)$ and it is well known that $\mathbb{P}(A(B_{1/2}, c) > u) = \exp(-2cu)$. Dębicki and Rolski [4] studied some **GI** processes of the form $X(t) = \int_0^t Z(s)\, ds$, where $Z(t)$ is a short range dependent stationary Gaussian process with mean 0, for which they proved that $\mathbb{P}(A(X,c) > u) \leq \text{const}\exp(-\gamma u) + o(\exp(-\gamma u))$. The short range dependence means $0 < \int_0^\infty R(t)\, dt < \infty$ and then $\gamma = c/\int_0^\infty R(t)\, dt$, however for their result Dębicki and Rolski needed more assumptions. In Section 5, under the condition that $R(t) \geq 0$, we show some improvements of bounds and their proofs from [4]. However recently more often are required models with $X(t)$ having higher irregularity than Brownian motion and also dependent increments. In class of Gaussian process there is a lot of interest in studying of the **FBM**. Note that if $H \geq 1/2$, then the process has positively correlated increments. We would like to point out that **FiBM** and **DG** are not processes with stationary increments.

The tool to study $\psi(u) = \mathbb{P}(A(B_H, c) > u)$ is the inequality

$$\mathbb{P}(A(B_H, c) > u) \leq \mathbb{P}(A(S_H, c) > u),$$

that follows from Slepian's theorem and we recall this theorem in Section 4. The **FiBM** appeared in Mandelbrot & Van Ness [17], who referred for this process to Levy [16].

Since

$$\mathbb{P}(\sup_{t\geq 0}\{B(t^{2H}) - ct\} > u) = \mathbb{P}(\sup_{t \geq 0}\{B(t) - ct^{\frac{1}{2H}}\} > u),$$

the problem for the **SBM** can be reduced to study a Brownian motion with the nonlinear drift $B(t) - ct^{\frac{1}{2H}}$. The supremum of the Brownian motion with nonlinear drift $B(t) - t^{1/2}$ was studied by Klüppelberg and T. Mikosch [11]. Other papers with nonlinear drift are Ferebee [7], [8] and Jennen [10].

There is a vast literature dealing with the distribution of the supremum of centered Gaussian processes on a compact interval, with the most celebrated Borell inequality see e.g. Adler [1]. Berman [2] obtained bounds for the supremum of Gaussian processes with stationary increments over finite intervals by the use Slepian inequality. Michna [18] recently studied bounds for the supremum of the FBM with drift over a finite interval. In contrast to such results, the tail of the distribution function of the supremum over unbounded sets (in our case $\mathbb{R}_+$) of a Gaussian process with negative drift usually has a different asymptotic. For $X(t)$ being the **FBM** the first result in this direction seem to be of Norros [19] who showed a lower bound for $\Psi(b)$ and we recover his bounds while considering **DG** case. Duffield & O'Connell [5] studied asymptotic

$$\log(\mathbb{P}(A(B_H, c) > u)) \sim \alpha u^{2-2H} \qquad \text{for} \qquad u \to \infty. \tag{1.1}$$

It turns out that their result is typical for a larger class of Gaussian processes and that the variance function is responsible for the asymptotics of (1.1). The papers [5] and [19] studied the buffer content in a stationary fluid model, where for the net-input process $X(t) = B_H(t) - ct$, the buffer content process was given by $Y(t) = \sup_{s \leq t}(X(t) - X(s))$. This study is motivated by problems of queue theory, risk theory and other applications. For example recent measurements and statistical analyses of traffic data are best explained by traffic models possessing long-range dependence, in particular by the **FBM**; see for instance Leland et al. [15] and Willinger et al. [23].

The study of the case **SBM** is pithier. In Section 4 we use Slepian's theorem to compare different models.

2 An application of local times for scaled Brownian motion

An important role plays the function

$$\ell_{H,c}(u) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{t^H} \exp\left(-\frac{(ct+u)^2}{2t^{2H}}\right) dt,$$

where $0 < H < 1$. In the special case we get $\ell_{1/2,c}(u) = \frac{1}{c}\exp(-2xu)$; see the table of Laplace transforms in [14]. The following result will be useful.

Lemma 2.1 *As $u \to \infty$*

$$\ell_{H,c}(u) \sim \frac{1}{c}\left(\frac{H}{1-H}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\left(\frac{c}{H}\right)^{2H}\left(\frac{1}{1-H}\right)^{2-2H} u^{2-2H}\right).$$

Proof. The proof uses Theorem 2.3 and Corollary 2.1 from Fedoryuk [6]. The details how to check assumptions of this theorem are given in the Appendix.

□

The following considerations are valid for stochastic process $X(t)$, where $X(t) = B_H(t) - ct$ or $S_H(t) - ct$. Let

$$L(u; X) = L(u) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^\infty \mathbb{I}[X(t) \in (u-\epsilon, u+\epsilon)]\, dt$$

be the local time of the process $X(t)$ at level u, where the limit exists almost surely. However for this paper we take another approach to define the local time following papers by Geman & Horowitz [9], Berman [3]. Let $X(t)$ be a stochastic process fulfilling:

- $X(t)$ has continuous trajectories,

- for all $t \geq 0$, the distribution of $X(t)$ is absolutely continuous with respect the Lebesgue measure m with density $g_{X(t)}$,
- $\int_0^\infty g_{X(t)}(x)\, dt < \infty$, for all $x \in \mathbb{R}$.

Define random measure M_ω by

$$\begin{aligned} M_\omega(A) &= m(\{t \geq 0 : \; X(t) \in A\}) \\ &= \int_0^\infty \mathbb{I}(X_\omega(t) \in A)\, dt, \qquad A \in \mathcal{B}(\mathbb{R}_+) \end{aligned}$$

We have

$$\mathbb{E}M_\omega(A) = \int_A \left(\int_0^\infty g_{X(t)}(x)\, dt \right) dx.$$

Since the measure $\mathbb{E}M$ is absolutely continuous with respect m, we have that for almost all ω, the measure M_ω is absolutely continuous with respect m. Therefore there exists $L_\omega(u)$ such that

$$\int_A L_\omega(u)\, du = M_\omega(A), \qquad A \in \mathcal{B}(\mathbb{R}_+).$$

Thus

$$\int_A \mathbb{E}L(u)\, du = \mathbb{E}M(A) = \int_A \left(\int_0^\infty g_{X(t)}(u)\, dt \right) du$$

for all $A \in \mathcal{B}(\mathbb{R}_+)$ and hence

$$\mathbb{E}L(u) = \ell_{H,c}(u) = \int_0^\infty g_{X(t)}(u)\, dt, \qquad \mathrm{m-a.e.}$$

If we choose a continuous version of $\mathbb{E}L(u)$, then we have:

Lemma 2.2

$$\mathbb{E}L(u; \{B_H(s) - cs\}) = \mathbb{E}L(u; \{S_H(s) - cs\}) = \ell_{H,c}(u).$$

Let

$$\tau(u) = \inf\{S_H(t) - ct > u : \; t \geq 0\}$$

be the crossing time of the level u by $S_H(t) - ct$ and $\mu_u(\cdot)$ its distribution. Note that μ_u is defective for $H < 1$ (use the law of iterated logarithm). By $\mu_u^\circ(\cdot)$ we denote the conditional distribution of $\tau(u)$ under the condition that $\tau(u) < \infty$. Consider jointly $(L(u), \tau(u))$ under the condition that $\tau(u) < \infty$. Let $\mathbb{E}(L(u)|\tau(u) = t)$ denotes a version of the conditional probability $\mathbb{E}(L(u)|\tau(u))$. The idea is to consider

$$\begin{aligned} \frac{\mathbb{E}L(u)}{\mathbb{P}(\sup_{t\geq 0}\{S_H(t) - ct\} > u)} &= \frac{\int_0^\infty \mathbb{E}(L(u)|\tau(u) = t)\, \mu_u(dt)}{\mathbb{P}(\tau(u) < \infty)} \\ &= \int_0^\infty \mathbb{E}(L(u)|\tau(u) = t)\, \mu_u^\circ(dt). \end{aligned} \tag{2.1}$$

Now using that the process $S_H(t) - ct$ has independent increments (but not stationary)

$$\begin{aligned} \mathbb{E}(L(u)|\tau(u) = t) &= \mathbb{E}(L(0; \{B((t+s)^{2H} - t^{2H}) - cs\}) \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{(t+x)^{2H} - t^{2H}}} \exp\left(- \frac{c^2 x^2}{2[(x+t)^{2H} - t^{2H}]} \right) dx \end{aligned} \tag{2.2}$$

Lemma 2.3 *(i) The function $y(z) = z/\sqrt{(z+1)^{2H} - 1}$ is strictly increasing from 0 to ∞ and similarly its inverse $z(t)$.*
(ii) For $0 < H < 1/2$ the function

$$f^*(z) = \frac{(z+1)^{2H} - 1}{(z+1)^{2H} - 1 - Hz(z+1)^{2H-1}} \tag{2.3}$$

is decreasing
(iii) For $1/2 < H < 1$ the function

$$f^*(z) = \frac{(z+1)^{2H} - 1}{(z+1)^{2H} - 1 - Hz(z+1)^{2H-1}} \tag{2.4}$$

is increasing
(iv) For $0 < H < 1$

$$\lim_{z\to 0} f^*(z) = \lim_{y\to 0} f(y) = 2 \tag{2.5}$$

$$\lim_{z\to\infty} f^*(z) = \lim_{y\to\infty} f(y) = \frac{1}{1-H}. \tag{2.6}$$

Proof. The proof goes by standard calculations. We only show that $f^*(z)$ is increasing. Since for all $z > 0$

$$\frac{d}{dz} f^*(z) \quad = \quad \frac{H(z+1)^{2H-2}[(z+1)^{2H} - 1 - 2Hz]}{[(z+1)^{2H} - 1 - Hz(z+1)^{2H-1}]^2} \geq 0$$

the proof is completed.

□

The key idea is given in the following lemma.

Lemma 2.4 *If $1/2 \leq H < 1$, then for all $u \geq 0$*

$$\frac{1}{2-2H}\frac{1}{c} \geq \mathbb{E}(L(u)|\tau(u) = t) \geq \frac{1}{c}. \tag{2.7}$$

Proof. Substituting $z = x/t$ in 2.2 we receive

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{(x+t)^{2H} - t^{2H}}} \exp\left(-\frac{c^2x^2}{2[(x+t)^{2H} - t^{2H}]}\right) dx = \\ &= \frac{1}{\sqrt{2\pi}} t^{1-H} \int_0^\infty \frac{1}{\sqrt{(z+1)^{2H} - 1}} \exp\left(-\frac{c^2 t^{2-2H} z^2}{2[(z+1)^{2H} - 1]}\right) dz. \end{aligned}$$

We now make the next substitution $y = z/\sqrt{(z+1)^{2H} - 1}$. Note that from Lemma 2.3 $y(z) = z/\sqrt{(z+1)^{2H} - 1}$ is strictly increasing from 0 to ∞ and therefore there exists the inverse $z(y)$. Thus

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} t^{1-H} \int_0^\infty \frac{1}{\sqrt{(z+1)^{2H} - 1}} \exp\left(-\frac{c^2 t^{2-2H} z^2}{2[(z+1)^{2H} - 1]}\right) dz = \\ &= \frac{1}{\sqrt{2\pi}} t^{1-H} \int_0^\infty \frac{(z+1)^{2H} - 1}{(z+1)^{2H} - 1 - Hz(z+1)^{2H-1}} \exp\left(-\frac{c^2 t^{2-2H} y^2}{2}\right) dy \\ &= \frac{1}{\sqrt{2\pi}} t^{1-H} \int_0^\infty f^*(z(y)) \exp\left(-\frac{c^2 t^{2-2H} y^2}{2}\right) dy, \end{aligned}$$

where $f^*(z)$ was defined in (2.3). Note that by Lemma 2.3 the function $f(y) = f^*(z(y))$ is increasing. Since $f(y)$ is increasing and (2.5)

$$
\begin{aligned}
E(L(u)|\tau(u)=t) &= \frac{t^{1-H}}{\sqrt{2\pi}} \int_0^\infty f(y) \exp(-\frac{c^2 t^{2-2H} y^2}{2})\ dy \\
&\geq \frac{2t^{1-H}}{\sqrt{2\pi}} \int_0^\infty \exp(-\frac{c^2 t^{2-2H} y^2}{2})\ dy \\
&= \frac{1}{c}. \qquad (2.8)
\end{aligned}
$$

Analogously from (2.6)

$$
\begin{aligned}
E(L(u)|\tau(u)=t) &= \frac{t^{1-H}}{\sqrt{2\pi}} \int_0^\infty f(y) \exp(-\frac{c^2 t^{2-2H} y^2}{2})\ dy \\
&\leq \frac{t^{1-H}}{\sqrt{2\pi}(1-H)} \int_0^\infty \exp(-\frac{c^2 t^{2-2H} y^2}{2})\ dy \\
&= \frac{1}{2(1-H)c}. \qquad (2.9)
\end{aligned}
$$

The proof is completed.

□

Remark 2.1 *Note that if $0 < H < 1/2$, then after analogous calculations we can obtain:*

$$\frac{1}{2-2H}\frac{1}{c} \leq \mathbb{E}(L(u)|\tau(u)=t) \leq \frac{1}{c}. \qquad (2.10)$$

The only difference is that for $0 < H < 1/2$ from Lemma 2.3 the function $f^*(x)$ given by (2.3) is decreasing.

3 Suprema from scaled Brownian motion with drift

In this section we study asymptotic for $\mathbb{P}(A(S_H, c) > u)$. We denote

$$\alpha = \frac{1}{2}\left(\frac{c}{H}\right)^{2H}\left(\frac{1}{1-H}\right)^{2-2H}. \qquad (3.1)$$

Theorem 3.1 *Let $1/2 \leq H < 1$.*
(i) *For all $u \geq 0$:*

$$2(1-H)c\ell_{H,c}(u) \leq \mathbb{P}(A(S_H, c) > u) \leq c\ell_{H,c}(u)\ . \qquad (3.2)$$

(ii) *For all $u \geq 0$:*

$$\mathbb{P}(A(S_H, c) > u) \geq \exp(-\alpha u^{2-2H}) \qquad (3.3)$$

(iii) *For $u \to \infty$*

$$
\begin{aligned}
2\sqrt{H(1-H)}\exp(-\alpha u^{2-2H}) \quad &+ \quad o(\exp(-\alpha u^{2-2H}) \leq \mathbb{P}(A(S_H, c) > u) \\
&\leq \sqrt{\frac{H}{1-H}}\exp(-\alpha u^{2-2H}) + o(\exp(-\alpha u^{2-2H}). \qquad (3.4)
\end{aligned}
$$

Proof. We get (3.2) inserting the inequality (2.7) to (2.1). Now (3.4) follows from the result of Lemma (2.1).

The proof of (3.3) is based on the observation that $k_x(t) = \frac{c}{2H} x^{\frac{1-2H}{2H}} t + c\frac{1-2H}{2H} x^{\frac{1}{2H}}$ is the tangent function to the function $m(t) = ct^{\frac{1}{2H}}$ at point x $(x > 0)$ and

$$\begin{aligned}
\mathbb{P}(A(S_H, c) > u) &= \mathbb{P}(\sup_{t\geq 0}(B(t) - ct^{\frac{1}{2H}}) > u) \\
&\geq \mathbb{P}(\sup_{t\geq 0}(B(t) - k_x(t)) > u) \qquad (3.5)\\
&= \mathbb{P}(\sup_{t\geq 0}(B(t) - \frac{c}{2H} x^{\frac{1-2H}{2H}} t - c\frac{2H-1}{2H} x^{\frac{1}{2H}}) > u) \\
&= \mathbb{P}(\sup_{t\geq 0}(B(t) - \frac{c}{2H} x^{\frac{1-2H}{2H}} t) > c\frac{2H-1}{2H} x^{\frac{1}{2H}} + u) \\
&= \exp(-\frac{c}{H} x^{\frac{1-2H}{2H}} (c\frac{2H-1}{2H} x^{\frac{1}{2H}}) + u)) \\
&\geq \exp(-\frac{c}{H} (\frac{uH}{c(1-H)})^{1-2H} (\frac{2H-1}{2(1-H)} u + u)) \qquad (3.6)\\
&= \exp(-\alpha u^{2-2H})
\end{aligned}$$

for all $x > 0$. Inequality (3.5) is a consequence of the concavity of $m(t)$. Inequality (3.6) follows from the fact that the function $-\frac{c}{H} x^{\frac{1-2H}{2H}} (c\frac{2H-1}{2H} x^{\frac{1}{2H}} + u)$ takes its maximum at point $x = (\frac{uH}{c(1-H)})^{2H}$.

The following result gives the asymptotic of $\mathbb{P}(A(S_H, c) > u)$.

Theorem 3.2 *For* $0 < H < 1$ *and* $u \to \infty$

$$\mathbb{P}(A(S_H, c) > u) \sim \sqrt{\frac{H}{1-H}} \exp(-\alpha u^{2-2H}).$$

Proof. ¿From the proof of Lemma 2.4 we have

$$\begin{aligned}
&\mathbb{E}(L(u; \{S_H(t) - ct\}) | \tau(u) = t) = \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{c} \int_0^\infty f^*(z(y)) \frac{1}{\sqrt{c^{-2} t^{2H-2}}} \exp(-\frac{y^2}{2c^{-2} t^{2H-2}})\, dy, \qquad (3.7)
\end{aligned}$$

where $f^*(z)$ was defined in (2.3) and the function $z(y)$ is the inverse to $y = z/\sqrt{(z+1)^{2H} - 1}$. Now (3.7) can be written as $\mathbb{E} f(z(N_t))/(2c)$, where N_t has the distribution as the conditional $c^{-1} t^{H-1} N$ under the condition that $N \geq 0$ and N is the standard normal random variable. By simple calculations we see that for all $x \geq 0$, $\mathbb{P}(N_t \geq x) \to 0$ as $t \to \infty$, which means that $N_t \to 0$ in distribution. Since $f^*(z(y))$ is bounded and continuous,

$$\mathbb{E}(L(u; \{S_H(t) - ct\}) | \tau(u) = t) \to \frac{1}{c}.$$

Now the result follows from (2.1) and Lemma 2.1.

□

Remark 3.1 *Note that from the law of iterated logarithm we have for* $H \geq 1$ *and* $u \geq 0$ $\mathbb{P}(A(S_H, c) > u) = \mathbb{P}(\sup_{t\geq 0}\{B(t) - ct^{\frac{1}{2H}}\} > u) = 1.$

Remark 3.2 *The right hand side of (3.4) can be derived using a result for the first passage density for the Brownian motion to a barrier. Thus using the fact that $sB(1/s)$ is a Brownian motion we get*

$$\begin{aligned}
\mathbb{P}(A(S_H,c)>u) &= \mathbb{P}(\inf_{s>0}(u+cs^{\frac{1}{2H}}-B(s))<0)\\
&= \mathbb{P}(\inf_{s>0}(u+cs^{-\frac{1}{2H}}-B(1/s))<0)\\
&= \mathbb{P}(\inf_{s>0}(us+cs^{1-\frac{1}{2H}}-B(s))<0).
\end{aligned}$$

To compute the last probability we need the first passage density for the Brownian motion to the barrier $f(s)=us+cs^{1-\frac{1}{2H}}$. From the integral equation for the first passage density for Brownian motion to the barrier f (see e.g. Ferebee [7]) it follows that

$$\mathbb{P}(\inf_{s>0}(us+cs^{1-\frac{1}{2H}}-B(s))<0)\le\frac{c}{H\sqrt{8\pi}}\int_0^\infty t^{-\frac{1}{2}-\frac{1}{2H}}\exp\{-\tfrac{1}{2}(ut^{\frac{1}{2}}+ct^{\frac{1}{2}-\frac{1}{2H}})^2\}\,dt.$$

Substituting $s=\frac{2H}{1-H}t^{\frac{1}{2}-\frac{1}{2H}}$ we obtain

$$\begin{aligned}
&\mathbb{P}(\inf_{s>0}(us+cs^{1-\frac{1}{2H}}-B(s))<0)\le\\
&\quad\frac{c}{H\sqrt{8\pi}}\int_0^\infty\exp\{-\tfrac{1}{2}(u(\tfrac{2H}{1-H})^{\frac{H}{1-H}}s^{\frac{-H}{1-H}}+c\tfrac{1-H}{2H}s)^2\}\,ds.
\end{aligned}$$

We have to treat the above integral for large u. In order to conclude the upper bound in (3.4) it is enough to make an appropriate use of Theorem 2.3. and Corollary 2.1 from Fedoryuk [6] which shows the behavior of such integrals. The details how to check the assumptions of this theorem are given in the Appendix.

4 Application of Slepian theorem

We use the following result of Slepian [22] in the form presented in Piterbarg [20], page 6.

Theorem 4.1 *Let Gaussian processes $\{X_1(t),\ t\ge 0\}$ and $\{X_2(t),\ t\ge 0\}$ with covariance functions $r_{X_1}(s,t)$ and $r_{X_2}(s,t)$ respectively be separable (in particular continuous). If for all $s,t\ge 0$*

$$\begin{aligned}
r_{X_1}(t,t) &\equiv r_{X_2}(t,t) && (4.8)\\
r_{X_1}(s,t) &\ge r_{X_2}(s,t), && (4.9)\\
\mathbb{E}X_1(t) &= \mathbb{E}X_2(t), && (4.10)
\end{aligned}$$

then for all $u\ge 0$

$$\mathbb{P}(\sup_{t\ge 0}\{X_1(t)\}\ge u)\le\mathbb{P}(\sup_{t\ge 0}\{X_2(t)\}\ge u)$$

We use Theorem 4.1 to compare $\mathbb{P}(A(X,c)>x)$ for different Gaussian processes X. Recall that the covariance function of the **FBM** with parameter H is $r_{B_H}(s,t)=(1/2)(t^{2H}+s^{2H}-|t-s|^{2H})$ and in the **SBM** $S_H(t)$ the covariance function is $r_{S_H}(s,t)=\min(s^{2H},t^{2H})$. Michna [18] showed that $r_{S_H}(s,t)\le r_{B_H}(s,t)$ for $1/2\le H\le 1$ and in the following lemma we find furhter relationships between useful covariance functions.

Lemma 4.1 *The covariance function of the* **FiBM** B_H° *is*

$$r_{B_H^\circ}(s,t) = 2H\int_0^{s\wedge t}((s-x)(t-x))^{H-1/2}\,dx$$

and of the process **DG** N_H *is* $r_{N_H}(s,t) = (st)^H$. *For all* $t \geq 0$

$$r_{S_H}(t,t) = r_{B_H^\circ}(t,t) = r_{B_H}(s,t) = r_{N_H}(s,t) = t^{2H}, \qquad t \geq 0$$

Moreover for $1/2 \leq H \leq 1$

$$r_{S_H}(s,t) \leq r_{B_H^\circ}(s,t) \leq r_{B_H}(s,t) \leq r_{N_H}(s,t), \qquad s,t \geq 0$$

and for $0 < H \leq 1/2$

$$r_{B_H^\circ}(s,t) \leq r_{B_H}(s,t) \leq r_{S_H}(s,t) \leq r_{N_H}(s,t), \qquad s,t \geq 0 \ .$$

Proof. In the proof $s \leq t$. We have

$$\begin{aligned}
r_{B_H^\circ}(s,t) &= \\
&= \mathbb{E}2H\int_0^t((s-x)_+^{H-1/2}\,dB(x)\int_0^t((t-x)^{H-1/2}\,dB(x) \\
&= 2H\int_0^t((s-x)_+^{H-1/2}(t-x)^{H-1/2}\,dx && (4.11)\\
&= 2H\int_0^s v^{H-1/2}(t-s+v)^{H-1/2}\,dv && (4.12)
\end{aligned}$$

and

$$r_{N_H}(s,t) = \mathbb{E}s^H\mathcal{N}t^H\mathcal{N} = (st)^H \ .$$

If $1/2 \leq H \leq 1$, then

$$\begin{aligned}
r_{S_H}(s,t) = s^{2H} &= 2H\int_0^s(s-x)^{2H-1}\,dx \\
&\leq 2H\int_0^s(s-x)^{H-1/2}(t-x)^{H-1/2}\,dx = r_{B_H^\circ}(s,t). && (4.13)
\end{aligned}$$

We now represent

$$\begin{aligned}
r_{B_H}(s,t) &= \frac{1}{2}(t^{2H}+s^{2H}-(t-s)^{2H}) \\
&= H\int_0^s v^{2H-1}\,dv + H\int_{t-s}^t v^{2H-1}\,dv \\
&= H\int_0^s((x^{H-1/2})^2+((t-s+x)^{H-1/2})^2\,dx \\
&\geq 2H\int_0^s x^{H-1/2}(t-s+x)^{H-1/2}\,dx = r_{B_H^\circ}(s,t).
\end{aligned}$$

Since

$$t^H \leq (t-s)^H + s^H$$

we have

$$(t^H - s^H)^2 \leq (t-s)^{2H}$$

and hence

$$(t^H)^2 + (s^H)^2 - ((t-s)^H)^2 \leq 2s^Ht^H$$

which proves $r_{B_H}(s,t) \le r_{N_H}(s,t)$. If $0 < H \le 1/2$, then the inequality (4.13) is reversed. $\square$

¿From Lemma 4.1 and Slepian's Theorem we get the following chain of inequalities:

$$\mathbb{P}(A(N_H,c) > u) \le \mathbb{P}(A(B_H,c) > u) \le \mathbb{P}(A(B_H^\circ,c) > u) \le \mathbb{P}(A(S_H,c) > u), \qquad u \ge 0$$

provided $1/2 \le H < 1$, otherwise

$$\mathbb{P}(A(N_H,c) > u) \le \mathbb{P}(A(S_H,c) > u) \le \mathbb{P}(A(B_H,c) > u) \le \mathbb{P}(A(B_H^\circ,c) > u), \qquad u \ge 0$$

The following result was given by Norros [19] but his proof is different.

Lemma 4.2

$$\begin{aligned}\mathbb{P}(A(N_H,c) > u) \quad &\ge \quad \frac{1}{\sqrt{2\pi}} \frac{2\alpha u^{2-2H} - 1}{(2\alpha)^{3/2} u^{3-3H}} e^{\alpha u^{2-2H}} \\ &\sim \frac{1}{2\pi} \frac{1}{2\alpha u^{1-H}} e^{\alpha u^{2-2H}}, \qquad u \to \infty\end{aligned}$$

Proof. Function $h(x,t) = \sup\{xt^H - ct\}$ achieves its maximum with respect to t for

$$t_x = \left(\frac{xH}{c}\right)^{\frac{1}{1-H}}$$

and then

$$h(x) = h(x,t_x) = cx^{\frac{1}{1-H}} \left(\frac{H}{c}\right)^{\frac{1}{1-H}} \left(\frac{1-H}{H}\right).$$

Thus

$$\begin{aligned}\mathbb{P}(h(\mathcal{N}) > u) \quad &= \quad \mathbb{P}\left(\mathcal{N} > \left(\frac{H}{1-H}\frac{1}{c}\right)^{1-H} \frac{c}{H} u^{1-H}\right) \\ &\ge \quad \frac{1}{\sqrt{2\pi}} \frac{2\alpha u^{2-2H} - 1}{(2\alpha)^{3/2} u^{3-3H}} e^{\alpha u^{2-2H}} \\ &\sim \frac{1}{2\pi} \frac{1}{2\alpha u^{1-H}} e^{\alpha u^{2-2H}}, \qquad u \to \infty\end{aligned}$$

$\square$

In the following corollary the first equation is due to Duffield and O'Connel [5]

Corollary 4.1 *For* $1/2 \le H \le 1$

$$\lim_{t\to\infty} \frac{1}{t^{2-2H}} \log \mathbb{P}(A(B_H,c) > u) = -\alpha$$

and

$$\lim_{t\to\infty} \frac{1}{t^{2-2H}} \log \mathbb{P}(A(B_H^\circ,c) > u) = -\alpha$$

5 Exponential bounds

Using Slepian inequality we can obtain improvements of results from Dębicki & Rolski [4]. Consider a stationary, centered Gaussian process $\{Z(t); t \geq 0\}$ with covariance function $R(t) = EZ(0)Z(t)$ such that

(i) R is continuous;
(ii) $0 < R(t) < \infty$ for all $t \geq 0$;
(iii) $\int_0^\infty R(s)ds < \infty$.

Define

$$X(t) = \int_0^t Z(s)ds.$$

Let

$$\gamma = \frac{c}{\int_0^\infty R(s)ds}.$$

We have

Theorem 5.1 *Under assumptions $(i)-(iii)$ and $c > 0$, for each $u \geq 0$*

$$\mathbb{P}(A(X,c) > u) \leq e^{-\gamma u}.$$

Proof. Let $s < t$. We have

$$\mathrm{Cov}(X(t), X(s)) = \mathbb{D}^2(X(s)) + \int_0^s dw \int_{s-w}^{t-w} R(v)dv \tag{5.1}$$

and

$$\mathbb{D}^2(X(t)) = 2\int_0^t ds \int_0^s R(v)dv \leq \frac{2c}{\gamma}t\ . \tag{5.2}$$

From Theorem 4.1 and (5.1)

$$\mathbb{P}(A(X,c) > u) \quad \leq \quad \mathbb{P}(\sup_{t\geq 0}(B(\mathbb{D}^2X(t)) - ct) > u)$$

Now using (5.2)

$$\begin{aligned}
\mathbb{P}(\sup_{t\geq 0}(B(\mathbb{D}^2X(t)) - ct) > u) &\leq \mathbb{P}(\sup_{t\geq 0}(B(\frac{2c}{\gamma}t)) - ct) > u) \\
&= \mathbb{P}(\sup_{t\geq 0}(B(t)) - \frac{\gamma}{2}t) > u) \\
&= e^{-\gamma t}.
\end{aligned}$$

□

Remark 5.1 *Note that if the increments of the process $\{X(t); t \geq 0\}$ are positively correlated and covariance function $R(t)$ is continuous, then $R(t) > 0$ for all $t \geq 0$.*

Remark 5.2 *Kulkarni & Rolski [12] studied process $Z(t) = \sum_{i=1}^n \rho_i Z_i(t)$ where $Z_i(t)$ are independent stationary Orstein-Uhlenbeck processes with covariance function $R_i(t) = e^{-\alpha_i t}$*

$(i=1,...,n)$ and ρ_i $(i=1,...,n)$ are positive constants. It is easy to check that assumptions $(i)-(iii)$ hold for $Z(t)$. From Theorem 5.1 we have

$$\mathbb{P}(A(X,c)>u) \le \exp\left(-\frac{cu}{\sum_{i=1}^{n}\frac{\rho_i^2}{\alpha_i}}\right)$$

for all $u \ge 0$. Note that Kulkarni & Rolski [13] obtained

$$\mathbb{P}(A(X,c)>u) \le e^{-\frac{c^2}{2\rho^2}} e^{-\gamma u}, \qquad u \ge 0$$

for the special case of $Z(t)$ being the pure Orstein-Uhlenbeck process with $R(t)=\rho^2 e^{-\alpha t}$.

If we additionally assume that

(iv) $\qquad\qquad \int_0^\infty t^2 R(t)dt < \infty,$

then we can proof the theorem:

Theorem 5.2 *Under assumptions $(i)-(iv)$ and $c>0$, for $u \to \infty$*

$$\mathbb{P}(A(X,c)>u) \le e^{-\gamma^2\beta} e^{-\gamma u} + o(e^{-\gamma u}),$$

where

$$\beta = \int_0^\infty tR(t)dt.$$

Proof. The proof of Theorem 5.2 is analogous to the proof of Theorem 5.1. It is based on the observation that if $(i)-(iv)$ hold, then

$$\mathbb{D}^2(X(t)) = \frac{2c}{\gamma}t - 2\beta + r(t) \tag{5.3}$$

where $r(t) = 2\int_t^\infty (s-t)R(s)ds = o(t^{-1})$ (see [4]).

□

The following theory gives a lower bound for the supremum $\mathbb{P}(A(X,c)>u)$. The idea of the proof is analogous to the proof of lower bound for $X(t)=B_H(t)$ given by Norros [19]. However we think that similarly as in the Norros paper the prefactor $u^{-1/2}$ is redundant.

Theorem 5.3 *Under assumptions $(i)-(iv)$, for $u \to \infty$*

$$\mathbb{P}(A(X,c)>u) \ge \frac{1}{2\sqrt{\pi\gamma}} u^{-\frac{1}{2}} \exp(-\gamma u) + o(u^{-\frac{1}{2}}\exp(-\gamma u)).$$

Proof. Let $\mathcal{N}$ be a standard Gaussian random variable. We have

$$\begin{aligned}
\mathbb{P}(A(X,c)>u) &\ge \sup_{t\ge 0} \mathbb{P}((X(t)-ct)>u) \\
&= \sup_{t\ge 0} \mathbb{P}\left(\mathcal{N} > \frac{u+ct}{\sqrt{\mathbb{D}^2(X(t))}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{P}\left(\mathcal{N} > \frac{2u}{\sqrt{\mathbb{D}^2(X(\frac{u}{c}))}}\right) && (5.4)\\
&\geq \frac{1}{\sqrt{2\pi}}\left(\frac{4u^2}{\mathbb{D}^2(X(\frac{u}{c}))}-1\right)\left(\frac{\sqrt{\mathbb{D}^2(X(\frac{u}{c}))}}{2u}\right)^3 \exp\left(-\frac{2u^2}{\mathbb{D}^2(X(\frac{u}{c}))}\right)\\
&= \frac{1}{\sqrt{2\pi}}[4u^2-\mathbb{D}^2(X(\frac{u}{c}))]\frac{\sqrt{\mathbb{D}^2(X(\frac{u}{c}))}}{(2u)^3}\exp\left(-\frac{2u^2}{\mathbb{D}^2(X(\frac{u}{c}))}\right)
\end{aligned}
$$

where in (5.4) we take $t = \frac{u}{c}$.

¿From $(i) - (iv)$ we have the expansion (5.3) and

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}}[4u^2-\mathbb{D}^2(X(\frac{u}{c}))]\frac{\sqrt{\mathbb{D}^2(X(\frac{u}{c}))}}{(2u)^3}\exp(-\frac{2u^2}{\mathbb{D}^2(X(\frac{u}{c}))}) &= \frac{1}{2\sqrt{\pi\gamma}}u^{-\frac{1}{2}}\exp(-\gamma u)\\
&+ o(u^{-\frac{1}{2}}\exp(-\gamma u)).
\end{aligned}
$$

for $u \to \infty$.

□

Remark 5.3 Note that from Theorem 5.2 and 5.3 we have that if assumptions $(i) - (iv)$ hold then the constant γ in the exponent is asymptotically the best:

$$\lim_{u\to\infty} \frac{\log \mathbb{P}(A(X,c) > u)}{u} = -\gamma.$$

A Appendix

In the appendix we study the asymptotic for $u \to \infty$ of the integral

$$\int_0^\infty \exp(S(x,u))\,dx$$

for particular forms of $S(x,u)$. We need the following notation. Let $x_0(u)$ denotes the point at which the function $S(x,u)$ of x achieves its maximum over $[0,\infty)$. Denote for some suitable chosen function $q(u)$

$$U(x_0(u)) = \{x : |x - x_0(u)| \leq q(u)|S''_{x,x}(x_0(u),u)|^{-1/2}\}.$$

We state now the result of Theorem 2.3 and Corollary 2.1 from Fedoryuk [6].

Theorem A.1 *Suppose that*
(a) *there exists a function $q(u) \to \infty$ as $u \to \infty$ such that*

$$S''_{x,x}(x,u) = S''_{x,x}(x_0(u),u)[1+o(1)]$$

as $u \to \infty$ uniformly for $x \in U(x_0(u))$,
(b) $S''_{x,x}(x,u) < 0$ *for all x, u,*
(c)

$$\lim_{u\to\infty} x_0(u)\sqrt{|S''_{x,x}(x_0(u),u)|} = \infty.$$

Then as $u \to \infty$

$$\int_0^\infty \exp[S(x,u)]\,dx \sim \sqrt{-\frac{2\pi}{S''_{x,x}(x_0(u),u)}}\exp[S(x_0(u),u)].$$

We now apply Theorem A.1 to get the following asymptotic:

Lemma A.1 *For* $0 < H < 1$ *and* $u \to \infty$

$$\int_0^\infty \frac{1}{t^H}\exp\left(-\frac{(ct+u)^2}{2t^{2H}}\right)\,dt \sim \frac{\sqrt{2\pi}}{c}\left(\frac{H}{1-H}\right)^{1/2}\exp\left(-\left(\frac{c}{H}\right)^{2H}\left(\frac{1}{1-H}\right)^{2-2H}u^{2-2H}\right).$$

Proof. We apply Theorem A.1 to

$$\int_0^\infty \frac{1}{t^H}\exp\left(-\frac{(ct+u)^2}{2t^{2H}}\right)\,dt = \frac{1}{1-H}\int_0^\infty \exp\left(-\frac{(ct^{1/(1-H)}+u)^2}{2t^{2H/(1-H)}}\right)\,dt,$$

where $0 \le H < 1$. We take

$$S(x,u) = -\frac{(cx^{1/(1-H)}+u)^2}{2t^{2H/(1-H)}}$$

By inspection

- $x_0(u) = \left[\frac{H}{c(1-H)}\right]^{1-H} u^{1-H}$
- $S''_{x,x}(x_0(u),u) = -\frac{c^2}{H(1-H)}$.

Hence we have that condition (c) of Theorem A.1 hold.

To check condition (a) we choose

$$q(u) = \frac{c}{\sqrt{H(1-H)}}u^{H(1-H)}.$$

We show now that

$$S''_{x,x}(x,u) - S''_{x,x}(x_0(\lambda),\lambda) \to 0$$

as $u \to \infty$ uniformly in

$$x \in \{y : |y - x_0(u)| \le u^{H(1-H)}\}. \tag{1.1}$$

It suffices to check that (1.1) holds for $x = x_0(u) \pm u^{H(1-H)}$ because in this case the difference in (1.1) is the biggest. Computing we get

$$\begin{aligned} S''_{x,x}(x,u) - S''_{x,x}(x_0(u),u) &= \\ &= -c^2 - \frac{(2H-1)H}{(1-H)^2}c\left[\left(\frac{H}{c(1-H)}\right)^{1-H} + u^{(H-1)(1-H)}\right]^{-\frac{1}{1-H}} - \\ &\quad -\frac{H(H+1)}{(1-H)^2}\left[\left(\frac{H}{c(1-H)}\right)^{1-H} + u^{(H-1)(1-H)}\right]^{-\frac{2}{1-H}} + \frac{c^2}{H(1-H)} \to 0 \end{aligned}$$

as $u \to \infty$.

To check condition (b) we compute

$$S''_{x,x}(x,u) = -c^2 - \frac{(2H-1)H}{(1-H)^2}cux^{-\frac{1}{1-H}} - \frac{(H+1)H}{(1-H)^2}u^2x^{-\frac{2}{1-H}}.$$

After substitution $z = x^{-\frac{1}{1-H}}$ we obtain the square function $-c^2 - \frac{(2H-1)H}{(1-H)^2} cuz - \frac{(H+1)H}{(1-H)^2} u^2 z^2$ of variable z. By standard calculation it can be checked that for all u and z (so all x)

$$S''_{x,x}(x,u) = -c^2 - \frac{(2H-1)H}{(1-H)^2} cuz - \frac{(H+1)H}{(1-H)^2} u^2 z^2 < 0$$

and the proof is completed.

□

Analogously we can prove:

Lemma A.2 *For $1/2 \le H < 1$ and $u \to \infty$*

$$\frac{c}{H\sqrt{8\pi}} \int_0^\infty \exp\{-\tfrac{1}{2}(u(\tfrac{2H}{1-H})^{\frac{H}{1-H}} s^{\frac{-H}{1-H}} + c\tfrac{1-H}{2H} s)^2\}\, ds \sim$$

$$\sim \left(\frac{H}{1-H}\right)^{1/2} \exp\left(-\left(\frac{c}{H}\right)^{2H} \left(\frac{1}{1-H}\right)^{2-2H} u^{2-2H}\right).$$

Proof. We need to study the following function

$$S(x,u) = -\tfrac{1}{2}(u(\tfrac{2H}{1-H})^{\frac{H}{1-H}} x^{\frac{-H}{1-H}} + c\tfrac{1-H}{2H} x)^2 \tag{1.2}$$

The first and second derivative of the function S have the following form

$$S'_x(x,u) = -(u(\tfrac{2H}{1-H})^{\frac{H}{1-H}} x^{\frac{-H}{1-H}} + c\tfrac{1-H}{2H} x)(-u(\tfrac{2H}{1-H})^{\frac{H}{1-H}} \tfrac{H}{1-H} x^{\frac{-1}{1-H}} + c\tfrac{1-H}{2H}) \tag{1.3}$$

$$\begin{aligned} S''_{xx}(x,u) &= -(-u(\tfrac{2H}{1-H})^{\frac{H}{1-H}} \tfrac{H}{1-H} x^{\frac{-1}{1-H}} + c\tfrac{1-H}{2H})^2 - \\ &\quad (u^2(\tfrac{2H}{1-H})^{\frac{2H}{1-H}} \tfrac{H}{(1-H)^2} x^{\frac{-2}{1-H}} + u\tfrac{c}{2(1-H)}(\tfrac{2H}{1-H})^{\frac{H}{1-H}} x^{\frac{-1}{1-H}}) \end{aligned} \tag{1.4}$$

For fixed u the function $S(x,u)$ takes its maximum at point

$$x_0(u) = 2(\tfrac{u}{c})^{1-H}(\tfrac{H}{1-H})^{2-H} \tag{1.5}$$

and then

$$S''_{xx}(x_0(u),u) = -c^2 \frac{1-H}{4H^3}. \tag{1.6}$$

Define the function

$$q(u) = u^{H(1-H)} c\sqrt{\frac{1-H}{4H^3}}. \tag{1.7}$$

Using (1.2), (1.3), (1.4), (1.5), (1.6) and (1.7) we can check the following conditions

- the function $q(u) \to \infty$ as $u \to \infty$ and

  $$S''_{xx}(x,u) - S''_{xx}(x_0(u),u) \to 0$$

  uniformly on the interval $(x_0(u) - u^{H(1-H)},\ x_0(u) + u^{H(1-H)})$ as $u \to \infty$,
- for all x and u $S''_{xx}(x,u) < 0$,
- $$\lim_{u\to\infty} x_0(u)\sqrt{|S''_{xx}(x_0(u),u|} = \infty.$$

Now it is sufficient to apply Theorem A.1 in order to complete the proof.

□

References

[1] Adler, R.J., *An Introduction to continuity, extrema, and related topics for general Gaussian processes* Inst. Math. Statist. Lecture Notes -Monograph Series, vol. 12, Inst. Math. Statist., Hayward, CA, 1990.

[2] Berman, S.M. (1985a) An asymptotic formula for the distribution of the maximum of Gaussian process with stationary increments. *Journal of Applied Probability* **22**, 454–460.

[3] Berman, S.M. (1985b) Joint continuity of the local times of Markov processes. *Z. Wahrscheinlichkeitstheorie verw. Gebeite* **69**, 37–46.

[4] Dębicki, K. & Rolski, T. (1995) A Gaussian fluid model. *Queueing Systems* **20**, 433–452.

[5] Duffield, N. & O'Connell, N. (1995) Large deviations and overflow probabilities for general single-server queue, with applications. *Mathematical Proceedings of the Cambridge Philosophical Society* **118**, 363–374.

[6] Fedoryuk, M. (1977) *The saddle-point method*, Nauka, Moscow (in Russian).

[7] Ferebee, B. (1982) The tangent approximation to one-sided Brownian exit-densities *Z. Wahrscheinlichkeitstheorie verw. Gebeite* **61**, 309–366.

[8] Ferebee,B. (1983) An Asymptotic Expansion for One-Sided Brownian Exit Densities *Z. Wahrscheinlichkeitstheorie verw. Gebeite* **63**, 1–15.

[9] Geman, D. and Horowitz, J. (1980) Occupation Densities. *The Annals of Probability* **8**, 1–67.

[10] Jennen, Ch. (1985) Second-order approximations to the density, mean and variance of Brownian first-exit times. *The Annals of Probability* **13**, 126–144.

[11] Klüppelberg, K. & Mikosch, T. (1996) Explosive Poisson shot noise processes with applications to risk reserves. *Bernoulli*

[12] Kulkarni, V. & Rolski, T. (1993) Fluid model driven by a multidimensional Ornstein-Uhlenbeck process. *Technical Report* No. UNC/OR/TR/93-7.

[13] Kulkarni, V. & Rolski, T. (1994) Fluid model driven by an Ornstein-Uhlenbeck process. *Prob. Eng. Inf. Sci.* **8**, 403–417.

[14] Korn, G.A. and Korn, T.M. (1968) *Mathematical Handbook for Scientists and Engineers*, Mc Graw-Hill, New York.

[15] Leland, W.E., Taqqu, M.S., Willinger, W. and Wilson, D.V. (1994) On the self-similar nature of Ethernet traffic. *IEEE/ACM Trans. Netw.*, **2**, 1–15.

[16] Levy, P. (1953) Random functions: General theory with special reference to Laplacian random functions. *Univ. California Publ. Statist.*, **1**, 331–390.

[17] Mandelbrot, B.B. and Van Ness, J.W. (1968) Fractional Brownian motions, fractional noises and applications. *SIAM Review*, **10**, 422–437.

[18] Michna, Z. (1996) Self-Similar Processes in Collective Risk Theory. *Technical Report.* University of Lund. Dept. of Math. Statistics.

[19] Norros, I. (1994) A storage model with self-similar input. *Queueing Systems*, **16**, 387–396.

[20] Piterbarg, V.I. (1996) *Asymptotic methods in the theory of Gaussian processes and fields.* Translations of Mathematical Monographs 148, AMS, Providence.

[21] Rice, J.R. (1983) *Matrix Computations and Mathematical Software.* McGraw-Hill, Tokyo.

[22] Slepian, D. (1962) The one-sided barrier problem for Gaussian noise. *Bell System Techn. J.*, **41**, 463–501.

[23] Willinger, W., Taqqu, M.S., Sherman, R. and Wilson, D.V. (1995) Self-similarity through high-variability: statistical analysis of Ethernet LAN traffic at the source level. *Proc. ACM/Sigcomm '95*.