PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 9, Fasc. 2 (1988), p. 51–66

EXTREME VALUE THEORY FOR ASYMPTOTIC STATIONARY SEQUENCES

BY

WŁADYSŁAW SZCZOTKA (WROCŁAW)

Abstract. The problem of behaviour of $a_n(\max_{1\leqslant k\leqslant n} X_k - b_n)$ is considered when $a_n > 0$, $|b_n| < \infty$ and the sequence $X = \{X_k, k \geqslant 1\}$ is asymptotically stationary in variation.

X is said to be *asymptotically stationary in variation* if $\|\mathscr{L}(X_n) - \mathscr{L}(X^0)\| \to 0$, where $X_n = \{X_{n+k}, k \geqslant 1\}$, while $\mathscr{L}(X_n)$ and $\mathscr{L}(X^0)$ denote the distributions of the sequences X_n and $X^0 = \{X_k^0, k \geqslant 1\}$, respectively. The sequence X^0 of random variables X_k^0 is stationary and it is said to be a *stationary representation of* X.

The main result states: *under* $\|\mathscr{L}(X_n) - \mathscr{L}(X^0)\| \to 0$ *and some natural conditions concerned* X *and* X^0, *the sequence of distributions* $\mathscr{L}(a_n(\max_{1\leqslant k\leqslant n} X_k - b_n))$ *weakly converges provided the sequence of* $\mathscr{L}(a_n(\max_{1\leqslant k\leqslant n} X_k^0 - b_n))$ *weakly converges and the limits are the same.*

An analogous result is also formulated for the processes of exceedances.

1. INTRODUCTION

Let $Y = \{Y_k, k \geqslant 1\}$ be an S-valued discrete-time process and $Y_n = \{Y_{n,k} \stackrel{\text{df}}{=} Y_{n+k}, k \geqslant 1\}$, $n \geqslant 1$. S is assumed to be a Polish metric space. For the Borel σ-field of subsets of a space we write $\mathscr{B}$ before the symbol denoting the space, for the distribution of a random element (r.e.) we put $\mathscr{L}$ before the symbol denoting the r.e., for the total variation of the subtraction of probability measures μ and ν on a measurable space $(\Omega, \mathscr{F})$ we write $\|\mu - \nu\|$, i.e.

$$\|\mu - \nu\| = 2\sup_{B\in\mathscr{F}} |\mu(B) - \nu(B)|$$

and for the weak convergence of probability measures or distribution functions (d.f's) we write $\Rightarrow$. Further Y is said to be *asymptotically stationary in variation* if there exists an S-valued discrete-time process $Y^0 = \{Y_k^0, k \geqslant 1\}$ such that $\|\mathscr{L}(Y_n)-\mathscr{L}(Y^0)\| \to 0$. The process Y^0 is stationary and it is called a *stationary representation in variation* of Y.

Let $X = \{X_k, k \geqslant 1\}$ be a real-valued discrete-time process, $X^0 = \{X_k^0, k \geqslant 1\}$ its stationary representation in variation, $M_n = \max X_k$ and $M_n^0 = \max X_k^0$ $(1 \leqslant k \leqslant n)$. The main purpose of this paper is:

(1) to give conditions under which $\{\mathscr{L}(a_n(M_n-b_n))\}$ weakly converges provided $\{\mathscr{L}(a_n(M_n^0-n_n))\}$ weakly converges for some constants $a_n > 0$ and b_n;

(2) to give sufficient conditions under which the behaviour of exceedances processes defined for X and X^0 is similar.

The main results, solving the stated problems, are given in Theorems 1–4. A simplified version of the answer to problem (1) states:

If $\|\mathscr{L}(X_n)-\mathscr{L}(X^0)\| \to 0$ and there exist constants $a_n > 0$ and $b_n \in R$ such that $\mathscr{L}(a_n(M_n^0-b_n)) \Rightarrow \nu$, where ν is max-stable and, further, there exists a nondecreasing sequence of positive integers k_n such that $k_n \to \infty$, $k_n/n \to 0$ and $\mathrm{P}\{a_n(M_{k_n}-b_n) > x\} \to 0$ for $x > \inf\{y\colon \nu(-\infty, y] > 0\}$, then $\mathscr{L}(a_n(M_n-b_n)) \Rightarrow \nu$.

In the Extreme Value Theory sufficient conditions for the weak convergence of $\mathscr{L}(a_n(M_n-b_n))$ are known also in the case where X is not necessary stationary ([2], [8], [6], [1]). In papers [1], [6], and [8] this problem was considered in the situation when X is a homogeneous Markov chain or it is chain dependent. But then, under some additional natural conditions, X is asymptotically stationary in variation (see Section 4). Thus $\{\mathscr{L}(a_n(M_n-b_n))\}$ weakly converges provided $\{\mathscr{L}(a_n(M_n^0-b_n))\}$ does (see Example 6).

The similar fact, i.e. the asymptotic stationarity in variation, is true for the following processes:

(a) a regenerative process with the aperiodic distribution of the regenerative period and with the finite expectation of this period;

(b) the waiting time process if the generic sequence is asymptotically stationary in variation [10];

(c) $X = \{f(Y_k), k \geqslant 1\}$, where Y is asymptotically stationary in variation and f is a measurable mapping of S^∞ into R.

The main results (Theorems 1–4) are proved by the method based on the following

PROPOSITION 1. *Let μ and μ_n $(n \geqslant 1)$ be probability measures on $(S, \mathscr{B}(S))$, h_n $(n \geqslant 1)$ measurable mappings of S into a Polish metric space S', and ν a probability measure on $(S', \mathscr{B}(S'))$. Then the following implications hold:*

(i) *If $\|\mu_n-\mu\| \to 0$, then $\|\mu_n h_n^{-1}-\mu h_n^{-1}\| \to 0$.*

(ii) *If $\|\mu_n-\mu\| \to 0$, $\|\mu h_n^{-1}-\nu\| \to 0$ (or $\mu h_n^{-1} \Rightarrow \nu$), then $\|\mu_n h_n^{-1}-\nu\| \to 0$ (or $\mu_n h_n^{-1} \Rightarrow \nu$).*

Implication (i) follows from the inequality $\|\mu_n h_n^{-1} - \mu h_n^{-1}\| \leqslant \|\mu_n - \mu\|$, and (ii) is implied by the relations $|\mu_n h_n^{-1}(B) - \nu(B)| \leqslant |\mu_n h_n^{-1}(B) - \mu h_n^{-1}(B)| + |\mu h_n^{-1}(B) - \nu(B)| \leqslant \|\mu_n - \mu\| + |\mu h_n^{-1}(B) - \nu(B)|$.

It may be worth noticing here that Proposition 1 has also other consequences. The following one concerns a continuity problem in the Extreme Value Theory:

PROPOSITION 2. *If, for each* $n \geqslant 1$, $X(n) = \{X_{n,k}, k \geqslant 1\}$ *is an r.e. of* R^∞ *such that* $\|\mathscr{L}(X(n)) - \mathscr{L}(X)\| \to 0$ *and* $\mathscr{L}(a_n(M_n - b_n)) \Rightarrow \nu$, *then*

$$\mathscr{L}\big(a_n(\max_{1 \leqslant k \leqslant n} X_{n,k} - b_n)\big) \Rightarrow \nu.$$

Notice that if $\|\mathscr{L}(X(n)) - \mathscr{L}(X)\| \to 0$, then Proposition 2 may be also viewed as an other approach to the investigation of $\max X_{n,k}$ $(1 \leqslant k \leqslant n)$. Similarly, Proposition 1 and the convergence $\|\mathscr{L}(X(n)) - \mathscr{L}(X)\| \to 0$ allow us to find convergences of other than $\max X_{n,k}$ $(1 \leqslant k \leqslant n)$ functions of $X(n)$. E.g., in view of Serfozo [9], we may formulate an analogue of Proposition 2 for the extremal process or the process of exceedances.

2. MAIN RESULTS

To complete the set of notations from the previous section, let us introduce the following ones. Let $\{k_n\}$ denote a nondecreasing sequence of integers tending to infinity in such a way that $k_n/n \to 0$ as $n \to \infty$, F_k the d.f. of X_k $(k \geqslant 1)$, F the d.f. of X_1^0, $x(\nu) = \inf\{x \in R\colon \nu(-\infty, x] > 0\}$, where ν is a probability measure on $(R, \mathscr{B}(R))$, I_A the indicator of a set A and $\boldsymbol{x} = \{x_k, k \geqslant 1\}$ a point of R^∞, where $x_k \in R$. Further, for a sequence $\{u_n\}$ of real numbers let N_n and N_n^0 $(n \geqslant 1)$ be point processes defined by

$$N_n(B) = \sum_{\substack{1 \leqslant k \leqslant n \\ k/n \in B}} I_{B_{n,k}}(X), \qquad N_n^0(B) = \sum_{\substack{1 \leqslant k \leqslant n \\ k/n \in B}} I_{B_{n,k}}(X^0),$$

where B belongs to $\mathscr{B}((0, 1])$ and $B_{n,k} = \{\boldsymbol{x} \in R^\infty\colon x_k > u_n\}$. Obviously, N_n and N_n^0 are processes of exceedances of the level u_n by the processes X and X^0, respectively. Let, finally, $\mathscr{N}$ denote the space of all measures on $(0, 1]$ with values in the set of nonnegative integers. This space is considered with the vague topology (see e.g. [3], p. 11).

Now let us formulate the following conditions:

A$_1$. $\|\mathscr{L}(X_n) - \mathscr{L}(X^0)\| \to 0$.

A$_2$. There exist sequences $\{a_n\}$ and $\{b_n\}$ $(a_n > 0, b_n \in R)$ such that $\mathscr{L}(a_n(M_n^0 - b_n)) \Rightarrow \nu$.

A$_3$. There exist $\{k_n\}$ for which $\{a_n\}$ and $\{b_n\}$ from A$_2$ satisfy $a_n/a_{n-k_n} \to 1$ and $a_{n-k_n}(b_n - b_{n-k_n}) \to 0$.

A$_4$. There exists a $\{k_n\}$ such that, for each $x > x(\nu)$, $\mathrm{P}\{a_n(M_{k_n} - b_n) > x\} \to 0$, where $\{a_n\}$, $\{b_n\}$ and ν satisfy A$_2$.

A_5. There exists a $\{k_n\}$ such that, for each $x > x(\nu)$, $P\{a_n(M^0_{k_n} - b_n) > x\} \to 0$, where $\{a_n\}$, $\{b_n\}$ and ν satisfy A_2.

Behaviour of $\{M_n\}$. For a real-valued process $Z = \{Z_k, k \geqslant 1\}$ we have

$$(1) \quad P\{\max_{1 \leqslant k \leqslant n} Z_k \leqslant x\} = P\{\max_{i < k \leqslant n} Z_k \leqslant x\} - P\{\max_{1 \leqslant k \leqslant i} Z_k > x, \max_{i < k \leqslant n} Z_k \leqslant x\},$$

where $1 \leqslant i < n$, $n \geqslant 1$, $x \in R$.

THEOREM 1. *Let* A_1 *be satisfied and let* $\{k_n\}$ *and* $\{u_n\}$ *be such that*

$$(2) \qquad P\{M^0_{k_n} > u_n\} \to 0 \quad \textit{and} \quad P\{M_{k_n} > u_n\} \to 0.$$

Then

$$(3) \qquad P\{M^0_n > u_n\} - P\{M_n > u_n\} \to 0.$$

Proof. Define mappings $h_n \colon R^\infty \to R$ $(n \geqslant 1)$ by

$$h_n(x) = \max_{1 \leqslant k \leqslant n - k_n} x_k - u_n.$$

These mappings are measurable.

Rewriting relation (1) for X and X^0, we obtain

$$P\{M_n \leqslant u_n\} = P\{h_n(X_{k_n}) \leqslant 0\} - P\{M_{k_n} > u_n, h_n(X_{k_n}) \leqslant 0\}$$

and

$$P\{M^0_n \leqslant u_n\} = P\{h_n(X^0) \leqslant 0\} - P\{M^0_{k_n} > u_n, h_n(X^0_{k_n}) \leqslant 0\}.$$

But in view of the first implication of Proposition 1, we have

$$P\{h_n(X_{k_n}) \leqslant 0\} - P\{h_n(X^0) \leqslant 0\} \to 0,$$

which together with (2) gives (3).

In the case of linear normalization of M_n and M^0_n we obtain the following analogue of Theorem 1:

THEOREM 2. *Let conditions* A_1–A_4 *be satisfied, where* A_3 *and* A_4 *hold with the same* $\{k_n\}$. *Then*

$$(4) \qquad \mathscr{L}(a_n(M_n - b_n)) \Rightarrow \nu.$$

Proof. Define mappings $h_n \colon R^\infty \to R$ $(n \geqslant 1)$ by

$$h_n(x) = a_n(\max_{1 \leqslant k \leqslant n} x_k - b_n).$$

These mappings are measurable.

Rewriting relation (1) for X we obtain

$$\begin{aligned}
&\mathrm{P}\{a_n(M_n-b_n)\leqslant x\}\\
&\quad=\mathrm{P}\{a_n/a_{n-k_n}(h_{n-k_n}(X_{k_n})-a_{n-k_n}(b_n-b_{n-k_n}))\leqslant x\}-\\
&\qquad-\mathrm{P}\{a_n(M_{k_n}-b_n)>x,\ a_n(\max_{1\leqslant k\leqslant n-k_n}X_{k_n+k}-b_n)\leqslant x\}.
\end{aligned}$$

Now, by conditions A_1–A_4 and the second implication in Proposition 1, we find

$$\mathrm{P}\{a_n(M_n-b_n)\leqslant x\}\to\nu(-\infty,x]$$

if $x>x(\nu)$ and x is a continuity point of ν. Otherwise, i.e. if $x<x(\nu)$, it is obvious that $\mathrm{P}\{a_n(M_n-b_n)\leqslant x\}\to 0$. Thus the proof is complete.

In view of this proof we can state something about the necessarity of A_3 and A_4 in Theorem 2.

Remark 2.1. (i) Condition A_3 holds provided A_1, A_2, A_4 and (4) hold, where $\{k_n\}$ in A_3 is the same as in A_4.

(ii) If $X_1, X_2, \ldots$ are mutually independent, then condition A_4 holds, provided A_1–A_3 and (4) hold, where $\{k_n\}$ in A_4 is the same as in A_3.

Notice that condition A_3 ought to depend only on X^0. In the following it is proved that A_2 and A_5 are sufficient for A_3.

LEMMA 2.1. *Let* A_2 *and* A_5 *be satisfied. Then* A_3 *holds with the same* $\{k_n\}$ *as in* A_5.

Proof. Rewriting relation (1) for X^0 we obtain

$$\begin{aligned}
(5)\quad &\mathrm{P}\{a_n(M_n^0-b_n)\leqslant x\}\\
&\quad=\mathrm{P}\{a_n/a_{n-k_n}(a_{n-k_n}(M^0_{n-k_n}-b_{n-k_n})-a_{n-k_n}(b_n-b_{n-k_n}))\leqslant x\}-\\
&\qquad-\mathrm{P}\{a_n(M^0_{k_n}-b_n)>x,\ a_n(\max_{1\leqslant k\leqslant n-k_n}X^0_{k_n+k}-b_n)\leqslant x\}.
\end{aligned}$$

In view of A_2 the left-hand side of (5) converges to $\nu(-\infty,x]$ if $\nu\{x\}=0$, while $\mathscr{L}(a_{n-k_n}(M^0_{n-k_n}-b_{n-k_n}))\Rightarrow\nu$. Hence and from A_5 we have $a_n/a_{n-k_n}\to 1$ and $a_{n-k_n}(b_n-b_{n-k_n})\to 0$, which completes the proof.

As an immediate consequence of Theorem 2 and Lemma 2.1 we obtain

THEOREM 3. *Let conditions* A_1 *and* A_2 *be satisfied. Furthermore, let* A_4 *and* A_5 *hold with the same sequence* $\{k_n\}$. *Then* (4) *holds.*

Behaviour of $\{N_n\}$. We now prove

THEOREM 4. *Let* A_1 *hold and* $\mathscr{L}(N_n^0)\Rightarrow\mathscr{L}(N)$, *where* N *is a point process on* $(0, 1]$. *Furthermore, let a* $\{k_n\}$ *exist such that*

$$(6)\qquad \mathrm{P}\{M_{k_n}>u_n\}\to 0\quad\textit{and}\quad \mathrm{P}\{M^0_{k_n}>u_n\}\to 0,$$

where u_n *is the same as in the definitions of* N_n *and* N_n^0. *Then* $\mathscr{L}(N_n)\Rightarrow\mathscr{L}(N)$.

Proof. Let us define mappings $g_n\colon R^\infty \to R^\infty$, $h_n\colon R^\infty \to \mathcal{N}$ and $H_n\colon \mathcal{N} \to \mathcal{N}$ $(n \geqslant 1)$ as

$$g_n(x) = (u_{n-k_n}/u_n)\,x, \qquad h_n(x)(B) = \sum_{\substack{1 \leqslant k \leqslant n \\ k/n \in B}} I_{B_{n,k}}(x),$$

$$(H_n \gamma)(B) = \gamma\big(n/(n-k_n)B - k_n/(n-k_n)\big),$$

where $ax = \{ax_k,\ k \geqslant 1\}$ for $a \in R$, $\gamma \in \mathcal{N}$, and $aB - b = \{\min(ax-b,\ 1);\ x \in B\}$ for $B \in \mathcal{B}(0, 1]$ and $a, b > 0$.

Notice that g_n, h_n and H_n are measurable and

$$H_n \circ h_{n-k_n} \circ g_n(x_{k_n})(B) = \sum_{\substack{k_n \leqslant k \leqslant n \\ k/n \in B}} I_{B_{n,k}}(x)$$

for any $B \in \mathcal{B}((0,1])$ and $x \in R^\infty$, where $x_{k_n} = \{x_{k_n+k},\ k \geqslant 1\}$. Hence

$$\sum_{\substack{k_n < k \leqslant n \\ k/n \in B}} I_{B_{n,k}}(X^0) = H_n \circ h_{n-k_n} \circ g_n(X^0_{k_n})(B),$$

which, in view of the stationarity of X^0, gives

$$\mathcal{L}\big(H_n \circ h_{n-k_n} \circ g_n(X^0_{k_n})\big) = \mathcal{L}\big(H_n \circ h_{n-k_n} \circ g_n(X^0)\big).$$

Now define point processes $\tilde{N}_n$ and $\tilde{N}^0_n$ $(n \leqslant 1)$ as

$$\tilde{N}_n(B) = \sum_{\substack{1 \leqslant k \leqslant k_n \\ k/n \in B}} I_{B_{n,k}}(X)$$

and

$$\tilde{N}^0_n(B) = \sum_{\substack{1 \leqslant k \leqslant k_n \\ k/n \in B}} I_{B_{n,k}}(X^0).$$

In view of (6) the distributions $\mathcal{L}(\tilde{N}_n)$ and $\mathcal{L}(\tilde{N}^0_n)$ weakly converge to the distribution concentrated on the measure from $\mathcal{N}$ which is zero for each Borel subset of (0, 1]. But $N^0_n = \tilde{N}^0_n + H_n \circ h_{n-k_n} \circ g_n(X^0_{k_n})$. Hence and since $\mathcal{L}(N^0_n) \Rightarrow \mathcal{L}(N)$, we have

$$\mathcal{L}(H_n \circ h_{n-k_n} \circ g_n(X^0)) \Rightarrow \mathcal{L}(N),$$

which by A_1 and Proposition 1 gives

$$\mathcal{L}\big(H_n \circ h_{n-k_n} \circ g_n(X_{k_n})\big) \Rightarrow \mathcal{L}(N).$$

Hence, in view of the relation $N_n = \tilde{N}_n + H_n \circ h_{n-k_n} \circ g_n(X_{k_n})$ we find the assertion.

Theorem 4 allows us to formulate analogues of Theorems 5.3.1 and 5.3.4 of [3] or a behaviour of $M_n^{(k)}$ as $n \to \infty$, where $M_n^{(k)}$ is the k-th largest of $X_1, X_2, \ldots, X_n$.

3. EXAMINATION OF A_4 AND A_5

The following obvious fact is basic for the examination of A_4 and A_5:

Remark 3.1. Let $\{c_{n,k},\ k,\ n \geqslant 1\}$ be an array of real numbers such that, for each $k \geqslant 1$, $c_{n,k} \to c_k$ as $n \to \infty$ and $c_k \to 1$ as $k \to \infty$. Then there exists a $\{k_n\}$ such that $c_{n,k_n} \to 1$ as $n \to \infty$. Moreover, if $k'_n \leqslant k_n$, then $c_{n,k'_n} \to 1$.

LEMMA 3.1. *Let X be such that $X_1, X_2, \ldots$ are mutually independent and such that for some constant x_0 and for each k and each x, $x > x_0$, we have*

$$F_k(x/a_n+b_n) \to 1 \quad \text{as} \quad n \to \infty. \tag{1}$$

Then there exists a $\{k_n\}$ such that $\mathrm{P}\{a_n(M_{k_n}-b_n) > x\} \to 0$ for each $x > x_0$.

Proof. Let $\{x_k\}$ be any decreasing sequence tending to x_0. For each $n,\ k \geqslant 1$ and $x \in R$ define

$$A_{n,k}(x) = \prod_{i=1}^{k} F_i(x/a_n+b_n) \quad \text{and} \quad A_{n,k} = A_{n,k}(x_k).$$

Then, by (1), $A_{n,k} \to 1$ as $n \to \infty$ for all k. By Remark 3.1, $A_{n,k_n} \to 1$ as $n \to \infty$ for some $\{k_n\}$. But, for any x such that $x > x_0$, there exists an n_0 such that $x > x_{k_n}$ for $n \geqslant n_0$. Hence, for $x > x_0$, we have $A_{n,k_n} = A_{n,k_n}(x_{k_n}) \leqslant A_{n,k_n}(x) \leqslant 1$. This and the convergence $A_{n,k_n} \to 1$ yield $A_{n,k_n}(x) \to 1$ for each $x > x_0$, which completes the proof.

LEMMA 3.2. *Let A_2 be satisfied and the d.f. G, corresponding to v, be max-stable. Then A_5 is satisfied. Moreover, if the convergence in A_5 holds with $\{k_n\}$, then it holds with any $\{k'_n\}$ such that $k'_n \leqslant k_n$, $n \geqslant 1$.*

Proof. Let $\{x_k\}$ be any decreasing sequence of real numbers tending to $x(v)$ and such that $G^{1/k}(x_k) \to 1$ as $k \to \infty$. Write

$$A_{n,k}(x) = \mathrm{P}\{M^0_{[n/k]} \leqslant x/a_n+b_n\} \quad \text{and} \quad A_{n,k} = A_{n,k}(x_k).$$

Since G is max-stable, $A_{n,k} \to G^{1/k}(x_k)$ for each k. Hence and by Remark 3.1, there exists a $\{k_n\}$ such that $A_{n,k_n} \to 1$. Now, for any $x > x(v)$, there exists an' n_0 such that $x > x_{k_n}$ for $n > n_0$. Hence, for any x, $x > x(v)$, $A_{n,k_n}(x) \geqslant A_{n,k_n}(x_{k_n})$ which, in turn, for $x > x(v)$, yields $A_{n,k_n}(x) \to 1$ as $n \to \infty$. This completes the proof.

The following lemma admits more sequences $\{k_n\}$ in A_5:

LEMMA 3.3. *Let X^0 be such that $X^0_1, X^0_2, \ldots$ are mutually independent and let A_2 be satisfied. Then A_5 holds with each $\{k_n\}$.*

Proof. Notice that $\mathrm{P}\{a_n(M^0_{k_n}-b_n) > x\} = 1-\big(F(x/a_n+b_n)\big)^{k_n}$. But the limit of $n(1-F(x/a_n+b_n))$, as $n \to \infty$, is finite for each $x > x(v)$. Hence $k_n(1-F(x/a_n+b_n)) \to 0$, which completes the proof.

It follows from Loynes work [4] that the assumption of Lemma 3.2 holds if X^0 is uniformly strongly mixing (see also [3], p. 55). The following lemma states that X^0 inherits this property from X.

LEMMA 3.4. *Let* A_1 *be satisfied. Then the property of the uniform strong mixing of* X *implies the same for* X^0.

Proof. Denote the vectors $(X_k, X_{k+1}, \ldots, X_m)$ and $(X_k^0, X_{k+1}^0, \ldots, X_m^0)$ by $\bar{X}_{k,m}$ and $\bar{X}_{k,m}^0$, respectively. The fact that X is uniformly strongly mixing means that

$$\alpha(m) \stackrel{\text{df}}{=} \sup |\mathrm{P}\{\bar{X}_{1,k} \in A,\ X_{k+m} \in B\} - \mathrm{P}\{\bar{X}_{1,k} \in A\}\,\mathrm{P}\{X_{k+m} \in B\}| \to 0 \quad \text{as } m \to \infty,$$

where the supremum is taken over all k, all $A \in \mathscr{B}(R^k)$ and all $B \in \mathscr{B}(R^\infty)$.

But by A_1 we have

$$\mathrm{P}\{\bar{X}_{1,k} \in A,\ X_m^0 \in B\} = \lim_n \mathrm{P}\{\bar{X}_{k_n+1,k_n+k} \in A,\ X_{k_n+m} \in B\}$$

for all k and m, $A \in \mathscr{B}(R^k)$ and all $B \in \mathscr{B}(R^\infty)$. Hence

$$\begin{aligned}\sup |\mathrm{P}\{\bar{X}_{1,k}^0 \in A,\ X_{k+m}^0 \in B\} - \mathrm{P}\{\bar{X}_{1,k}^0 \in A\}\,\mathrm{P}\{X_{k+m}^0 \in B\}| \\ \leqslant \overline{\lim_n} \sup |\mathrm{P}\{\bar{X}_{k_n+1,k_n+k} \in A,\ X_{k_n+k+m} \in B\} - \\ - \mathrm{P}\{\bar{X}_{k_n+1,k_n+k} \in A\}\,\mathrm{P}\{X_{k_n+k+m} \in B\}| = \alpha(m),\end{aligned}$$

where the supremum is taken over all k, all $A \in \mathscr{B}(R^k)$ and all $B \in \mathscr{B}(R^\infty)$. Thus the proof is completed.

Now we show that if X is a Markov chain or X is chain dependent, then, under some natural additional conditions, A_5 implies A_4.

X is said to be *chain dependent* with respect to a homogeneous Markov chain $J = \{J_k, k \geqslant 1\}$ with a state space I being a Polish metric space if $X_1 = a \in R$ and

$$\mathrm{P}\{J_{n+1} \in A,\ X_{n+1} \in B \mid J_1, X_1, \ldots, J_n, X_n\} = \mathrm{P}\{J_{n+1} \in A,\ X_{n+1} \in B \mid J_n\} \text{ a.e.}$$

for $A \in \mathscr{B}(I)$, $B \in \mathscr{B}(R)$, $n \geqslant 1$.

Obviously, (J, X) is a Markov chain.

LEMMA 3.5. *Let* A_1 *and* A_5 *be satisfied if* X *is either* (i) *a homogeneous Markov chain such that* $\|\mathscr{L}(X_n) - \pi^0\| \to 0$ *or* (ii) *chain dependent with respect to a homogeneous Markov chain* J *such that* $\|\mathscr{L}(J_n) - \pi^0\| \to 0$, *where* π^0 *is a probability measure on* $(R, \mathscr{B}(R))$ *in case* (i), *and on* $(I, \mathscr{B}(I))$ *in case* (ii). *Then* A_4 *is satisfied with the same* $\{k_n\}$ *as in* A_5.

Proof. The proof is carried out parallelly in both cases.

Write, in case (i),

$$g_k(x, y) = \mathrm{P}\{\max_{1<j<k+1} X_j > x \mid X_1 = y\}$$

and, in case (ii),

$$g'_k(x, y) = \mathrm{P}\{\max_{1<j\leqslant k+1} X_j > x \mid J_1 = y\}.$$

Then, in (i),

$$\mathrm{P}\{\max_{1<j\leqslant k+1} X_j^0 > x \mid X_1^0 = y\} = g_k(x, y) \text{ a.e.}$$

and, in (ii),

$$\mathrm{P}\{\max_{1<j\leqslant k+1} X_j^0 > x \mid J_1^0 = y\} = g'_k(x, y) \text{ a.e.}$$

Now, rewriting relation (2.1) for X, we obtain

$$\mathrm{P}\{M_k > x\} = \mathrm{P}\{\max_{i<j\leqslant k} X_j > x\} - \mathrm{P}\{M_i > x, \max_{i<j\leqslant k} X_j \leqslant x\}. \tag{7}$$

Moreover, in (i),

$$\mathrm{P}\{\max_{i<j\leqslant k} X_j > x\} = \int_R \mathrm{P}\{\max_{i<j\leqslant k} X_j > x \mid X_i = y\}\, \mathrm{P}\{X_i \in dy\}$$

$$= \int_R \mathrm{P}\{\max_{1<j\leqslant k-i+1} X_j > x \mid X_1 = y\}\, \mathrm{P}\{X_i \in dy\} = \int_R g_{k-i}(x, y)\, \mathrm{P}\{X_i \in dy\}$$

and, in (ii),

$$\mathrm{P}\{\max_{i<i\leqslant k} X_j > x\} = \int_I \mathrm{P}\{\max_{i<j\leqslant k} X_j > x \mid J_i = y\}\, \mathrm{P}\{J_i \in dy\}$$

$$= \int_I g'_{k-i}(x, y)\, \mathrm{P}\{J_i \in dy\}.$$

In a similar way we find, in (i),

$$\mathrm{P}\{M_k^0 > x\} = \mathrm{P}\{\max_{i<j\leqslant k} X_j^0 > x\} + \mathrm{P}\{M_i^0 > x, \max_{i<j\leqslant k} X_j^0 \leqslant x\}, \tag{8}$$

$$\mathrm{P}\{\max_{i<j\leqslant k} X_j^0 > x\} = \int_R g_{k-i}(x, y)\, P\{X_i^0 \in dy\}$$

and, in (ii),

$$\mathrm{P}\{\max_{i<j\leqslant k} X_j^0 > x\} = \int_I g'_{k-i}(x, y)\, \mathrm{P}\{J_1^0 \in dy\}.$$

Define a measure μ as $\frac{1}{2}\mathscr{L}(X_1^0) + \frac{1}{2}\sum_{i=1}^{\infty} 2^{-i}\mathscr{L}(X_i)$ in case (i) and as $\frac{1}{2}\mathscr{L}(J_1^0) + \frac{1}{2}\sum_{i=1}^{\infty} 2^{-i}\mathscr{L}(J_i)$ in case (ii).

Obviously, the measures $\mathscr{L}(X_i^0)$, $\mathscr{L}(X_i)$ and $\mathscr{L}(J_i^0)$, $\mathscr{L}(J_i)$ are absolutely continuous with respect to μ in both cases.

Let p^0 and p_i denote the probability density functions (p.d.f.) of $\mathscr{L}(X_1^0)$ and $\mathscr{L}(X_i)$ with respect to μ defined in case (i), while q^0 and q_i denote the p.d.f.'s of $\mathscr{L}(J_1^0)$ and $\mathscr{L}(J_i)$ with respect to μ defined in case (ii). Then, in (i),

$$|\mathrm{P}\{\max_{i<j\leqslant k} X_j^0 > x\} - \mathrm{P}\{\max_{i<j\leqslant k} X_j > x\}| \leqslant \int_R |p^0(y)-p_i(y)|\,\mu(dy)$$
$$= \|\mathscr{L}(X_1^0)-\mathscr{L}(X_i)\|$$

and, in (ii),

$$|\mathrm{P}\{\max_{i<j\leqslant k} X_j^0 > x\} - \mathrm{P}\{\max_{i<j\leqslant k} X_j > x\}| \leqslant \int_R |q^0(y)-q_i(y)|\,\mu(dy)$$
$$= \|\mathscr{L}(J_1^0)-\mathscr{L}(J_i)\|.$$

The latter, next the convergences $\|\mathscr{L}(X_i)-\mathscr{L}(X_1^0)\| \to 0$ in (i) and $\|\mathscr{L}(J_i) - \mathscr{L}(J_1^0)\| \to 0$ in (ii), as $i\to\infty$, and finally (1), (2) and A_5 imply A_4 with the same $\{k_n\}$ as in A_5. This completes the proof.

4. ASYMPTOTIC STATIONARITY IN VARIATION

Conditions under which X is asymptotically stationary in variation are here investigated in three cases:

(a) X is a nonhomogeneous Markov chain,

(b) X is a function of a homogeneous Markov chain,

(c) X is disturbed by fading process.

4.1. Case (a). Let X be a nonhomogeneous Markov chain and $P_{n,n+1}(x, A) = \mathrm{P}\{X_{n+1}\in A \mid X_n = x\}$ for $x\in R$, $A\in\mathscr{B}(R)$, $n\geqslant 1$. Further, let $\tilde{\mu}$ be a convex combination of the Lebesgue measure on R and a discrete measure on R. By $\tilde{\mu}^k$ the k-multiple product of the measure $\tilde{\mu}$ is denoted.

Let us introduce the following conditions:

(i) For each $n\geqslant 1$, $\mathscr{L}(X_n)$ has the p.d.f. f_n with respect to the measure $\tilde{\mu}$.

(ii) For $\tilde{\mu}$-almost all x and $n\geqslant 1$, $P_{n,n+1}(x,\cdot)$ has the p.d.f. $f_{n,n+1}(x,\cdot)$ with respect to the measure $\tilde{\mu}$ and $f_{n,n+1}(x, y)$ as a function of (x, y) is jointly measurable.

(iii) $f_n \to f^0$ $\tilde{\mu}$-a.e., where f^0 is a p.d.f. with respect to $\tilde{\mu}$.

(iv) $f_{n,n+1} \to f$ $\tilde{\mu}^2$-a.e., where, for $\tilde{\mu}$-almost all x, $f(x, y)$ as a function of y is a p.d.f. with respect to $\tilde{\mu}$.

Set

$$f_{1,k}^0(x_1, x_2, \ldots, x_k) = f^0(x_1)\prod_{i=2}^{k} f(x_{i-1}, x_i)$$

and

$$Z_{n,k}(x_1, x_2, \ldots, x_k) = f_{n+1}(x_1)\prod_{i=2}^{k} f_{n-1+i,n+i}(x_{i-1}, x_i)/f_{1,k}^0(x_1, x_2, \ldots, x_k)$$

if $f_{1,k}^0 > 0$, and zero otherwise.

LEMMA 4.1. *Let conditions* (i)–(iv) *be satisfied and*

$$\sup_{1 \leqslant k < \infty} \int_{R^k} |1 - Z_{n,k}| f_{1,k}^0 \, d\tilde{\mu}^k \to 0. \tag{1}$$

Then $\|\mathscr{L}(X_n) - \mathscr{L}(X^0)\| \to 0$, *where* X^0 *is a stationary homogeneous Markov chain with the transition p.d.f.* $f(x, y)$, $x, y \in R$, *such that*

$$f^0(y) = \int_R f^0(x) f(x, y) \tilde{\mu}(dx) \qquad \tilde{\mu}\text{-a.e.}$$

Proof. The proof follows by part (a) of Theorem 1 from Vostrikova [11]. Indeed, for each $n \geqslant 1$ define a measurable space $(\Omega^n, \mathscr{F}^n)$, a filter F^n as well as probability measures P^n and $\tilde{P}^n$ on the measurable space $(\Omega^n, \mathscr{F}^n)$. Thus let $\Omega^n = R^\infty$, $\mathscr{F}^n = \mathscr{B}(R^\infty)$ and $F^n = \{\mathscr{F}_k^n, k \geqslant 0\}$, where $\mathscr{F}_0^n = \{\emptyset, \Omega^n\}$, while $\mathscr{F}_k^n = \mathscr{B}(R^k)$, $n, k \geqslant 1$. Furthermore, let $P^n = P = \mathscr{L}(X^0)$ and $\tilde{P}^n = \mathscr{L}(X_n)$, $n \geqslant 1$.

In these notations, Z_k^n occurring in [11] is equal to $Z_{n,k}$ and $\{Z_k^n\}$ satisfies the conditions from part (a) of Theorem 1 from [11]. Thus we find that $\|\tilde{P}^n - P\| \to 0$, which completes the proof.

Now let us consider the case of X where $X_1, X_2, \ldots$ are mutually independent. Then, by Lemma 4.1, we have

COROLLARY 4.1. *Let* X *be such that* $X_1, X_2, \ldots$ *are mutually independent with p.d.f.'s* $f_1, f_2, \ldots$ *with respect to* $\tilde{\mu}$. *Furthermore, let* $f_n \to f^0$ $\tilde{\mu}$*-a.e. and* f^0 *be a p.d.f. with respect to* $\tilde{\mu}$ *such that*

$$\sup_{1 \leqslant k < \infty} \int_{R^k} \Big|1 - \prod_{i=1}^{k} f_{n+i}(x_i)/f^0(x_i)\Big| \prod_{i=1}^{k} f^0(x_i) \tilde{\mu}^k(dx_1, dx_2, \ldots, dx_k) \to 0, \tag{2}$$

Then $\|\mathscr{L}(X_n) - \mathscr{L}(X^0)\| \to 0$, *where* X^0 *is such that* $X_1^0, X_2^0, \ldots$ *are i.i.d. r.v.'s with p.d.f.* f^0 *with respect to* $\tilde{\mu}$.

It is easy to note the following

Remark 4.1. If $\sum_{k=1}^{\infty} \int_R |1 - f_k(x)/f^0(x)| f^0(x) \tilde{\mu}(dx) < \infty$, then (2) holds.

4.2. Case (b). Now we prove the following

LEMMA 4.2. *Let* X *be chain dependent with respect to a homogeneous Markov chain* J *which has values in a Polish metric space* I. *Furthermore, let* $\|\mathscr{L}(J_n) - \pi^0\| \to 0$, *where* π^0 *is a probability measure on* $(I, \mathscr{B}(I))$. *Then* $\|\mathscr{L}(J_n, X_n) - \mathscr{L}(J^0, X^0)\| \to 0$, *where* (J^0, X^0) *is a stationary homogeneous Markov chain with the same transition probabilities as* (J, X).

Proof. Let (J^0, X^0) be a homogeneous Markov chain with the same transition probabilities as (J, X) and such that $\mathscr{L}(J_1^0) = \pi^0$. Then (J^0, X^0) is a stationary homogeneous Markov chain. Write $g(B, x) = \mathrm{P}\{(J_1, X_1) \in B \mid J_1 = x\}$ for B belonging to the product σ-field in $(I \times R)^\infty$ and $x \in I$. Then

$$\mathrm{P}\{(J_n, X_n) \in B \mid J_n = x\} = \mathrm{P}\{(J_n^0, X_n^0) \in B \mid J_n^0 = x\} = g(B, x) \text{ a.e.}$$

Hence

$$\mathrm{P}\{(J_n, X_n) \in B\} = \int_I g(B, x)\, \mathrm{P}\{J_n \in dx\},$$

$$\mathrm{P}\{(J_n^0, X_n^0) \in B\} = \int_I g(B, x)\, \pi^0(dx).$$

Set

$$\mu = \frac{1}{2}\pi^0 + \frac{1}{2}\sum_{k=1}^{\infty} 2^{-k} \mathscr{L}(J_k).$$

Hence μ is a finite measure on $(I, \mathscr{B}(I))$. Moreover, π^0 and $\mathscr{L}(J_n)$, $n \geqslant 1$, are absolutely continuous with respect to μ. Thus, denoting by f^0 and f_n the p.d.f.'s of π^0 and $\mathscr{L}(J_n)$, respectively, with respect to μ, we find

$$\|\mathscr{L}(J_n, X_n) - \mathscr{L}(J^0, X^0)\| = 2\sup_B \Big|\int_I g(B, x)\big(f_n(x) - f^0(x)\big)\mu(dx)\Big|$$
$$\leqslant 2\int_I |f_n(x) - f^0(x)|\, \mu(dx) = 2\|\mathscr{L}(J_n) - \pi^0\|,$$

where sup is taken over all B from the product σ-field in $(I \times R)^\infty$. This completes the proof.

Corollary 4.2. *Under assumptions of Lemma 4.2,* $\|\mathscr{L}(X_n) - \mathscr{L}(X^0)\| \to 0$, *where* X^0 *is the second component in the Markov chain* (J^0, X^0) *defined in Lemma 4.2.*

Corollary 4.3. *Let X be a homogeneous Markov chain such that* $\|\mathscr{L}(X_n) - \pi^0\| \to 0$, *where* π^0 *is a probability measure on* $(R, \mathscr{B}(R))$. *Then* $\|\mathscr{L}(X_n) - \mathscr{L}(X^0)\| \to 0$, *where* X^0 *is a stationary homogeneous Markov chain with the same transition probabilities as* X.

Sufficient conditions for the convergence in variation of $\{\mathscr{L}(J_n)\}$ with J being any homogeneous Markov chain gives Theorem 7.1 from Orey [7].

4.3. Case (c). Let μ and μ_n, $n \geqslant 1$, be probability measures on $(S, \mathscr{B}(S))$ while ν and ν_n, $n \geqslant 1$, probability measures on $(S', \mathscr{B}(S'))$. Further let $\mu \times \nu$ denote the product measure of μ and ν.

Lemma 4.3. *If* $\|\mu_n - \mu\| \to 0$ *and* $\|\nu_n - \nu\| \to 0$, *then* $\|\mu_n \times \nu_n - \mu \times \nu\| \to 0$.

Proof. For $A \in \mathscr{B}(S \times S')$ let $A_x = \{y \in S': (x, y) \in A\}$ and $A^y = \{x \in S: (x, y) \in A\}$, where $x \in S$, $y \in S'$. Note that

$$|\mu_n \times \nu_n(A) - \mu \times \nu(A)| \leqslant |\mu_n \times \nu_n(A) - \mu \times \nu_n(A)| + |\mu \times \nu_n(A) - \mu \times \nu(A)|$$
$$\leqslant \int_{S'} |\mu_n(A^y) - \mu(A^y)| \, \nu_n(dy) + \int_S |\nu_n(A_x) - \nu(A_x)| \, \mu(dx) \leqslant \|\mu_n - \mu\| + \|\nu_n - \nu\|,$$

which completes the proof.

As an immediate consequence of Lemma 4.3 and Proposition 1 we have

COROLLARY 4.4. *Let X and Y be independent sequences of r.v.'s such that $\|\mathscr{L}(X_n) - \mathscr{L}(X^0)\| \to 0$ and $\|\mathscr{L}(Y_n) - \mathscr{L}(Y^0)\| \to 0$, where X^0 and Y^0 are mutually independent.*

Then $\|\mathscr{L}(X_n, Y_n) - \mathscr{L}(X^0, Y^0)\| \to 0$ and $\|\mathscr{L}(X_n + Y_n) - \mathscr{L}(X^0 + Y^0)\| \to 0$.

To see the usefulness of Corollary 4.4 let us consider the following model of disturbance described by Y. Let $\{\nu_n\}$ be any sequence of probability measures on $(R, \mathscr{B}(R))$ which are absolutely continuous with respect to the Lebesgue measure l, and δ_0 the probability measure concentrated at zero. Denote by g_k, $k \geqslant 1$, the p.d.f. of ν_k with respect to l. Further, let Y be a sequence of independent r.v.'s $Y_1, Y_2, \ldots$ such that Y_k has the distribution $p_k \delta_0 + (1 - p_k) \nu_k$, $k \geqslant 1$, where $0 \leqslant p_k \leqslant 1$.

LEMMA 4.4. *If $\sum_{k=1}^{\infty} (1 - p_k)$ is finite, then $\|\mathscr{L}(Y_n) - \mathscr{L}(Y^0)\| \to 0$, where $Y^0 = (0, 0, \ldots)$.*

Proof. Writing $\tilde{\mu} = \frac{1}{2}\delta_0 + \frac{1}{2} l$, we see that δ_0 and ν_k are absolutely continuous with respect to $\tilde{\mu}$. Moreover, their p.d.f.'s with respect to $\tilde{\mu}$ are equal to $2\chi_0$ and $2(1 - \chi_0) g_k$, respectively, where $\chi_0(x) = 1$ if $x = 0$, and zero otherwise. Hence the p.d.f.'s of $\mathscr{L}(Y_k)$ and $\mathscr{L}(Y_1^0)$ with respect to $\tilde{\mu}$ are equal to $f_k = 2p_k \chi_0 + 2(1 - p_k)(1 - \chi_0) g_k$ and $f^0 = 2\chi_0$, respectively. Therefore $\int_R |1 - f_k/f^0| f^0 \, d\tilde{\mu} = 1 - p_k$.

Hence and in view of Remark 4.1 and the assumed condition we obtain the assertion.

5. EXAMPLES OF X FOR WHICH THEOREMS 1–4 HOLD

5.1. Dependence case. Let us present some examples.

Example 1. Let $Y = \{Y_k, k \geqslant 1\}$ be an asymptotic stationary in variation sequence of r.v.'s (it is not assumed the mutual independence of $Y_1, Y_2, \ldots$) such that $Y_1 + Y_2 + \ldots + Y_k \to -\infty$ in probability. Furthermore, let its stationary representation Y^0 be such that $Y_{-1}^* + Y_{-2}^* + \ldots + Y_{-k}^* \to -\infty$ a.s., where $\{Y_k^*, -\infty < k < \infty\}$ is a stationary sequence of r.v.'s such that $\mathscr{L}(\{Y_k^*, k \geqslant 1\}) = \mathscr{L}(Y^0)$. Define r.v.'s X_k, $k \geqslant 1$, by

$$X_{k+1} = \max(0, X_k + Y_k), \; k \geqslant 1,$$

where X_1 is any nonnegative r.v.

In Queueing Theory the sequence X is well known as the process of waiting time and it is denoted by $w = \{w_k, k \geqslant 1\}$. We have shown ([10], Theorem 3a) that under above assumptions this process is asymptotically stationary in variation. Thus, if $\{k_n\}$ and $\{u_n\}$ are such that $P\{M^0_{k_n} > u_n\} \to 0$ and $P\{M_{k_n} > u_n\} \to 0$, then by Theorem 1 we have $P\{M^0_n > u_n\} - P\{M_n > u_n\} \to 0$.

Example 2. Let $X = Z + Y$, where Z and Y are mutually independent sequences of r.v.'s such that Z is stationary and Y is a sequence of mutually independent r.v.'s $Y_1, Y_2, \ldots$ Assume that $\mathscr{L}(Y_k) = p_k \delta_0 + (1-p_k)\nu_k$, $k \geqslant 1$, where probability measures ν_k, $k \geqslant 1$, are absolutely continuous with respect to the Lebesgue measure and $\sum(1-p_k) < \infty$ $(k = 1, 2, \ldots)$. Then

(a) If Z is such that, for some $\{u_n\}$ and $\{k_n\}$,

$$P\{2 \max_{1 \leqslant k \leqslant k_n} Z_k > u_n\} \to 0 \quad \text{and} \quad P\{2 \max_{1 \leqslant k \leqslant k_n} Y_k > u_n\} \to 0,$$

then

$$P\{\max_{1 \leqslant k \leqslant n} Z_k > u_n\} - P\{\max_{1 \leqslant k \leqslant n} X_k > u_n\} \to 0.$$

(b) If Z satisfies condition A_2 and, for some $\{k_n\}$ and each $x > x(\nu)$,

$$P\{2a_n(\max_{1 \leqslant k \leqslant k_n} Z_k - b_n) > x\} \to 0 \quad \text{and} \quad P\{2a_n(\max_{1 \leqslant k \leqslant k_n} Y_k - b_n) > x\} \to 0,$$

then $\mathscr{L}(a_n(M_n - b_n)) \Rightarrow \nu$.

Indeed, in view of Lemma 4.4 and Corollary 4.4, Z is a stationary representation in variation of X, i.e. $\|\mathscr{L}(X_n) - \mathscr{L}(Z)\| \to 0$. Furthermore,

$$P\{M_{k_n} > x\} \leqslant P\left\{\max_{1 \leqslant k \leqslant k_n} Z_k > \frac{x}{2}\right\} + P\left\{\max_{1 \leqslant k \leqslant k_n} Y_k > \frac{x}{2}\right\}.$$

This and Theorems 1 and 2 give implications (a) and (b).

Example 3. Let X be either (i) a homogeneous Markov chain such that $\|\mathscr{L}(X_n) - \pi^0\| \to 0$ or (ii) a chain dependent with respect to a homogeneous Markov chain J such that $\|\mathscr{L}(J_n) - \pi^0\| \to 0$, where π^0 is a probability measure on $(R, \mathscr{B}(R))$ in case (i) and on $(I, \mathscr{B}(I))$ in case (ii). Then, in view of Lemma 4.2 and Corollary 4.3, X is asymptotically stationary in variation in both cases. Moreover,

(a) If, for some $\{u_n\}$ and $\{k_n\}$, $P\{M^0_{k_n} > u_n\} \to 0$, then

$$P\{M^0_n > u_n\} - P\{M_n > u_n\} \to 0.$$

(b) If A_2 and A_5 hold, then $\mathscr{L}(a_n(M_n - b_n)) \Rightarrow \nu$.

Indeed, implication (a) follows from Lemma 3.5 and Theorem 1, while (b) follows from Lemma 3.5 and Theorem 3.

5.2. Independence case. Let X be such that $X_1, X_2, \ldots$ are mutually independent with p.d.f.'s f_k, $k \geqslant 1$, with respect to the Lebesgue measure.

Example 4 (exponential distribution disturbed by a normal distribution). Let f^0 be the exponential p.d.f. with the parameter λ and $f_k = f^0 * g_k$, $k \geqslant 1$, where g_k is the normal p. d.f. with the mean zero and the variance $\sigma_k^2 > 0$, while $*$ denotes the convolution. Furthermore, suppose that for some α, $\alpha < 1$,

$$\sum_k \sigma_k^\alpha < \infty. \tag{1}$$

Then $\|\mathscr{L}(X_n) - \mathscr{L}(X^0)\| \to 0$. Moreover, with $a_n = \lambda$ and $b_n = \lambda^{-1} \log n$ the limiting d.f.'s of $a_n(M_n - b_n)$ and $a_n(M_n^0 - b_n)$ are equal to the d.f. G which is $G(x) = \exp(-e^{-x})$ for $x \in R$.

Indeed, note that

$$f^0 * g_k(x) = \lambda \exp(-\lambda x + \lambda^2 \sigma_k^2/2)\, \Phi(x/\sigma_k - \lambda\sigma_k),$$

where Φ denotes the standard normal d.f. Hence

$$\int_{-\infty}^{\infty} |1 - f_k(x)/f^0(x)| f^0(x)\,dx = \int_{-\infty}^{\infty} |1 - \exp(\lambda^2\sigma_k^2/2)\, \Phi(x/\sigma_k - \lambda\sigma_k)| f^0(x)\,dx \tag{2}$$

$$\leqslant \exp(\lambda^2 \sigma_k^2/2) - 1 + \int_{-\infty}^{\infty} |1 - \Phi(x/\sigma_k - \lambda\sigma_k)| f^0(x)\,dx.$$

Denote the integral part of the right-hand side of (2) by B and consider it for such k that $\sigma_k < 1$. Then decomposing $[0, \infty)$ in $[0, c_k]$ and (c_k, ∞), where $c_k = \lambda\sigma_k^2 + \sigma_k^\alpha$, $k \geqslant 1$, we have

$$B \leqslant 2\int_0^{c_k} f^0(x)\,dx + 1 - \Phi(\sigma_k^{\alpha-1}) \leqslant 2\big(1 - \exp(-\lambda(\lambda\sigma_k^2 + \sigma_k^\alpha))\big) + 1 - \Phi(\sigma_k^{\alpha-1})$$

$$\leqslant 2\lambda(\lambda+1)\sigma_k^\alpha + m_{2i}\,\sigma_k^{2i(1-\alpha)} \leqslant c\sigma_k^\alpha,$$

where m_i is the i-th moment of Φ, $2i > \alpha/(1-\alpha)$, and c is some constant depending on k and λ. Hence the right-hand side of (2) does not exceed $\big(c + \exp(a)\big)\sigma_k^\alpha$, where $a = \lambda^2 \sup \sigma_k^2/2$. This, in view of (1), Remark 4.1 and Corollary 4.1, implies $\|\mathscr{L}(X_n) - \mathscr{L}(X^0)\| \to 0$.

Now notice that $F_k(x/a_n + b_n) = F_k\big((x + \log n)/\lambda\big)$ for each x and n, $k \geqslant 1$. Hence, for each k and $x \in R$, $F_k(x/a_n + b_n) \to 1$ as $n \to \infty$. This and Lemmas 3.1 and 3.3 as well as Theorem 3 and Example 1.7.2 from [3] prove the correctness of the example.

I wish to express my thanks to Professor B. Kopociński for his valuable comments. I am also grateful to Dr W. Dziubdziela for his helpful remarks.

REFERENCES

[1] E. G. Denzel and G. L. O'Brien, *Limit theorems for extreme values of chain dependent process*, Ann. Prob. 3 (1975), p. 773–780.

[2] R. Durret and S. I. Resnick, *Functional limit theorems for dependent variables*, ibidem 6 (1978), p. 829–846.

[3] M. R. Leadbetter, G. Lindgren and H. Rootzen, *Extremes and related properties of random sequences and processes*, Springer-Verlag, New York 1982.

[4] M. R. Loynes, *Extreme values in uniformly mixing stationary stochastic processes*, Ann. Math. Statist. 36 (1965), p. 993–999.

[5] G. L. O'Brien, *Limit theorems for sums of chain dependent processes*, J. Appl. Prob. 11 (1974), p. 582–587.

[6] — *The limiting distribution of maxima of random variables defined on a denumerable Markov chain*, Ann. Prob. 2 (1974), p. 103–111.

[7] S. Orey, *Lecture notes on limit theorems for Markov chain transition probabilities*, Van Nostrand and Reinhold, London 1971.

[8] S. I. Resnick and M. F. Neuts, *Limit laws for maxima of sequence of random variables defined on a Markov chain*, Advances Appl. Prob. 2 (1970), p. 323–343.

[9] R. Serfozo, *Functional limit theorems for extreme values of arrays of independent random variables*, Ann. Prob. 10 (1982), p. 172–177.

[10] W. Szczotka, *Stationary representation of queues, I*, Advances Appl. 18 (1986), p. 815–848.

[11] J. Yu. Vostrikova, *On necessary and sufficient conditions for convergence of probability measures in variation*, Stoch. Processes and their Appl. 18 (1984), p. 99–112.

Wrocław University
Institute of Mathematics
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland

Received on 8. 12. 1986