PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 8 (1987), p. 63–69

CONDITIONED LIMIT THEOREMS FOR FUNCTIONS OF THE AVERAGE OF I.I.D. RANDOM VARIABLES

BY

A. SZUBARGA AND D. SZYNAL (LUBLIN)

Abstract. Let $\{\xi_k, k \geqslant 1\}$ be a sequence of i.i.d. random variables with $E\xi_1 = 0$, $0 < E\xi_1^2 = \sigma^2 < \infty$. Form the random walk $\{S_n, n \geqslant 0\}$ by setting $S_0 = 0$, $S_n = \xi_1 + \ldots + \xi_n$, $n \geqslant 1$. Let T denote the hitting time of the set $(-\infty, 0]$ by the random walk. Put $X_n(t) = S_{[nt]}/\sigma\sqrt{n}$, $0 \leqslant t \leqslant 1$. Let h be a real-valued, right-continuous function on R, having left limits, with $h(0) = 1$, and continuous at 0. For $\beta > 0$ we define the map H_n: $D[0, 1] \to D[0, 1]$ by $H_n(f) = fh(n^{-\beta} f)$, $f \in D[0, 1]$, $n \geqslant 1$. Put $Y_n = H_n(X_n)$. This note deals with the asymptotic behaviour of Y_n conditioned on $[T > n]$. Moreover, we investigate the asymptotic behaviour in the question when n is replaced by N_n, where $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables.

1. Introduction. Let $\{\xi_k, k \geqslant 1\}$ be a sequence of independent, identically distributed random variables with $E\xi_1 = 0$, $0 < E\xi_1^2 = \sigma^2 < \infty$, and let $\{N_m, m \geqslant 0\}$, $N_0 = 0$ a.s., be a sequence of positive integer-valued random variables. Form the random walk $\{S_n, n \geqslant 0\}$ by setting $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$, $n \geqslant 1$. Define the random function X_n by

$$X_n(t) = S_{[nt]}/\sigma\sqrt{n}, \quad 0 \leqslant t \leqslant 1,$$

where $[x]$ is the greatest integer in x. Next let T be the hitting time of the set $(-\infty, 0]$ by the random walk,

$$T = \inf\{n > 0;\, S_n \leqslant 0\},$$

where the infimum of the empty set is taken to be $+\infty$. Let h be a real-valued, right-continuous function on R, having left limits, with $h(0) = 1$ and continuous at 0. Let $D \equiv D[0, 1]$ be the space of real-valued, right-continuous functions on [0, 1] having left limits. For $\beta > 0$ we define the map H_n: $D[0, 1] \to D[0, 1]$ by

$$H_n(f) = fh(n^{-\beta} f), \quad f \in D[0, 1],\ n \geqslant 1.$$

Put $Y_n = H_n(X_n)$. The aim of this note is to give a functional central limit theorem for the random function Y_n, conditioned on $[T > n]$.

To be more specific we assume that $\{\xi_k, k \geqslant 1\}$ are the coordinate functions defined on the product space $(\Omega, \mathscr{A}, \mathrm{P})$. If $\Lambda_n = \{T > n\}$, then we let $(\Lambda_n, \Lambda_n \cap \mathscr{A}, \mathrm{P}_n)$ be the trace of $(\Omega, \mathscr{A}, \mathrm{P})$ on Λ_n, where $\Lambda_n \cap \mathscr{A} = \{\Lambda_n \cap F, F \in \mathscr{A}\}$ and $\mathrm{P}_n[A] = \mathrm{P}[A]/\mathrm{P}[\Lambda_n]$ for $A \in \Lambda_n \cap \mathscr{A}$. Let $\mathscr{D}$ be the σ-field of Borel sets on D, generated by the open sets of the Skorohod $\mathscr{J}_1$ topology. Let $D_+ = \{x \in D;\ x \geqslant 0\}$, and $\mathscr{D}_+ = D_+ \cap \mathscr{D}$. The measurable mappings X_n^+, Y_n^+: $(\Lambda_n, \Lambda_n \cap \mathscr{A}) \to (D_+, \mathscr{D}_+)$ are defined by

$$X_n^+(\cdot, \omega) = S_{[n\cdot]}(\omega)/\sigma\sqrt{n}, \quad \omega \in \Lambda_n,$$

and

$$Y_n^+(\cdot, \omega) = H_n(X_n^+(\cdot, \omega)), \quad \omega \in \Lambda_n.$$

The random function X_n^+ induces a probability measure (p. m.) μ_n^+ on $\mathscr{D}_+$: for $A \in \mathscr{D}_+$ we have

$$\mu_n^+(A) = \mathrm{P}_n[X_n^+ \in A] = \mathrm{P}[X_n^+ \in A]/\mathrm{P}[\Lambda_n] = \mathrm{P}[X_n \in A | \Lambda_n].$$

Iglehart [5], Theorem (3.4) (see also for this result Bolthausen [2]), has proved that $X_n^+ \Rightarrow W^+$, $n \to \infty$, i.e. $\mu_n^+ \Rightarrow \mu^+$, the p.m. of a Brownian meander W^+, the symbol $\Rightarrow$ means weak convergence. Alternatively, we write $(X_n | \Lambda_n) \Rightarrow W^+$, $n \to \infty$, for this result. The random function Y_n^+ induces a p.m. on $\mathscr{D}_+$: for $A \in D_+$ we have

$$\bar{\mu}_n^+(A) = \mathrm{P}_n[Y_n^+ \in A] = \mathrm{P}[Y_n^+ \in A]/\mathrm{P}[\Lambda_n].$$

The main result of this note is that $\bar{\mu}_n^+ \Rightarrow \mu^+$, $n \to \infty$. Moreover, we investigate the asymptotic behaviour in the question when n is replaced by N_n, where $\{N_n, n \geqslant 0\}$, $N_0 = 0$ a.s., is a sequence of positive integer-valued random variables.

We shall apply the following result of Doney [3]:

LEMMA 1. *For* $v > 0$

$$\lim_{n\to\infty} \mathrm{P}[\max_{0 \leqslant k \leqslant T} S_k/\sigma\sqrt{n} \leqslant v | \Lambda_n] = 1 - v^{-1}\sqrt{\frac{\pi}{2}} + 2\sum_{k=1}^{\infty} \exp(-2k^2 v^2). \tag{1}$$

We need in the sequel the following lemmas:

LEMMA 2. *Let* $\beta > 0$. *For* $\delta > 0$ *we have*

$$\limsup_{n\to\infty} \mathrm{P}[\sup_{0 \leqslant t \leqslant 1} |X_n(t)| > \delta | \Lambda_n] \leqslant \sqrt{\frac{\pi}{2}}\,\delta^{-1}, \tag{2}$$

and

(3) $$\lim_{n\to\infty} \mathrm{P}[\sup_{0\leqslant t\leqslant 1} |n^{-\beta} X_n(t)| > \delta | \Lambda_n] = 0.$$

Proof. We have

$$\mathrm{P}[\sup_{0\leqslant t\leqslant 1} |X_n(t)| > \delta | \Lambda_n] = \mathrm{P}[\sup_{0\leqslant t\leqslant 1} S_{[nt]}/\sqrt{n}\sigma > \delta | \Lambda_n]$$
$$\leqslant \mathrm{P}[\max_{0\leqslant k\leqslant T} S_k > \sigma\sqrt{n}\delta | \Lambda_n].$$

Hence, (2) follows from Lemma 1.

We now prove (3). For any given $\varepsilon > 0$ and $\delta > 0$ we have

$$\limsup_{n\to\infty} \mathrm{P}[\sup_{0\leqslant t\leqslant 1} |X_n(t)| > \delta n^{\beta} | \Lambda_n]$$
$$\leqslant \limsup_{n\to\infty} \mathrm{P}\left[\sup_{0\leqslant t\leqslant 1} |X_n(t)| > \frac{\delta}{\varepsilon} \middle| \Lambda_n\right] \leqslant \sqrt{\frac{\pi}{2}}\frac{\varepsilon}{\delta},$$

which implies

$$\lim_{n\to\infty} \mathrm{P}[\sup_{0\leqslant t\leqslant 1} |n^{-\beta} X_n(t)| > \delta | \Lambda_n] = 0$$

and completes the proof of Lemma 2.

LEMMA 3. *Let $\beta > 0$. For any $\delta > 0$*

(4) $$\lim_{n\to\infty} \mathrm{P}[\sup_{0\leqslant t\leqslant 1} |Y_n(t) - X_n(t)| > \delta | \Lambda_n] = 0.$$

Proof. Let $\varepsilon > 0$. Since h is continuous at the point $x = 0$, it follows that there exists an $\eta > 0$ such that

(5) $$|x| \leqslant \eta \Rightarrow |h(x) - 1| < \varepsilon.$$

Then, for any given $\delta > 0$, we have

$$\mathrm{P}[\sup_{0\leqslant t\leqslant 1} |Y_n(t) - X_n(t)| > \delta | \Lambda_n]$$
$$\leqslant \mathrm{P}[\sup_{0\leqslant t\leqslant 1} |X_n(t)|\,|1 - h(n^{-\beta} X_n(t))| > \delta,\ \sup_{0\leqslant t\leqslant 1} |n^{-\beta} X_n(t)| \leqslant \eta | \Lambda_n]$$
$$+ \mathrm{P}[\sup_{0\leqslant t\leqslant 1} |n^{-\beta} X_n(t)| > \eta | \Lambda_n]$$
$$\leqslant \mathrm{P}\left[\sup_{0\leqslant t\leqslant 1} |X_n(t)| > \frac{\delta}{\varepsilon} \middle| \Lambda_n\right] + \mathrm{P}[\sup_{0\leqslant t\leqslant 1} |n^{-\beta} X_n(t)| > \eta | \Lambda_n].$$

Hence, by Lemma 2, we get

$$\limsup_{n\to\infty} \mathrm{P}[\sup_{0\leqslant t\leqslant 1} |Y_n(t) - X_n(t)| > \delta | \Lambda_n] \leqslant \sqrt{\frac{\pi}{2}}\frac{\varepsilon}{\delta},$$

which proves (4).

Let d denote the metric on D ([1], p. 111). We can rewrite Lemma 3 in the following form:

(6) $$\lim_{n\to\infty} \mathrm{P}[d(X_n, Y_n) > \delta | A_n] = 0.$$

THEOREM 1. *Under the assumptions of this note we have*

(7) $$Y_n^+ = (Y_n | A_n) \Rightarrow W^+, \quad n\to\infty.$$

Proof. Assertion (7) follows from (5) and a result of Iglehart ($X_n^+ \Rightarrow W^+$, $n\to\infty$).

Denote by $\mathscr{G}$ the class of all continuous functions g differentiable at the point $x=0$ with $g'(0)\neq 0$.

COROLLARY 1. *For every* $g\in\mathscr{G}$

(8) $$\left(\left(\sqrt{n}/g'(0)\sigma\right)\left(g(S_{[n\cdot]}/n)-g(0)\right) | A_n\right) \Rightarrow W^+, \quad n\to\infty,$$

holds true.

Proof. Putting

$$h(x) = \begin{cases} \dfrac{g(x)-g(0)}{xg'(0)} & \text{if } x\neq 0,\\ 1 & \text{if } x=0,\end{cases}$$

we can write the left-hand side of (8) in the following form:

$$\left(\sqrt{n}/g'(0)\sigma\right)\left(g(S_{[n\cdot]}/n)-g(0)\right) = X_n h(n^{-1/2}X_n).$$

It is easy to verify that h satisfies our assumptions. Putting in Theorem 1 that $\beta = 1/2$, we obtain (8).

2. Random partial sum processes. In this section we are interested in the asymptotic behaviour of

$$Y_{N_m}^+ = (Y_{N_m} | A_{N_m}), \quad m\geqslant 1,$$

where $\{N_m, m\geqslant 0\}$, $N_0 = 0$ a.s., is a sequence of positive integer-valued random variables.

We now need the following extension of Lemma 1:

LEMMA 4. *Let* $\{\xi_k, k\geqslant 1\}$ *be a sequence of i.i.d. random variables with* $\mathrm{E}\xi_1 = 0$, $0<\mathrm{E}\xi_1^2 = \sigma^2<\infty$. *Suppose that* $\{N_m, m\geqslant 1\}$ *is a sequence of positive integer-valued random variables, and* $\{\alpha_m, m\geqslant 1\}$ *is a sequence of positive numbers with* $\alpha_m\to\infty$, $m\to\infty$, *such that, for any given* $\varepsilon>0$,

(9) $$\mathrm{P}[|N_m/\alpha_m - 1| \geqslant \varepsilon] = o(1/\sqrt{\alpha_m}).$$

Then, for $v > 0$,

$$(10)\quad \mathrm{P}\left[\max_{0\leqslant k\leqslant T}\frac{S_k}{\sigma\sqrt{N_m}}\leqslant v\,|\,T > N_m\right] = 1-v^{-1}\sqrt{\frac{\pi}{2}}+2\sum_{k=1}^{\infty}\exp(-2k^2v^2).$$

Proof. From Theorem 3.7, [6], we have

$$\mathrm{P}[T > n] = \mathrm{P}[S_1 > 0, \ldots, S_n > 0] \sim c/\sqrt{n}, \qquad n\to\infty,$$

where

$$c = \exp\Big\{\sum_{k=1}^{\infty}(1/k)(1/2-\mathrm{P}[S_k > 0])\Big\}.$$

Hence, by (9), for $1 > \varepsilon > 0$ we have

$$(11)\quad \frac{c}{\sqrt{[(1-\varepsilon)\alpha_m]}}-o\left(\frac{1}{\sqrt{\alpha_m}}\right)\leqslant \mathrm{P}[T > N_m]$$

$$\leqslant \frac{c}{\sqrt{[(1+\varepsilon)\alpha_m]}}+o\left(\frac{1}{\sqrt{\alpha_m}}\right), \qquad m\to\infty.$$

Put $A_m = \{k; (1-\varepsilon)\alpha_m \leqslant k \leqslant (1+\varepsilon)\alpha_m\}$, and let A_m^c denote the complement of A_m. Then, by (9) and (11), we have

$$\mathrm{P}[N_m \in A_m^c]/\mathrm{P}[T > N_m] \to 0, \qquad m\to\infty.$$

Hence, we can write

$$\mathrm{P}[\max_{0\leqslant k\leqslant T} S_k/\sigma\sqrt{N_m}\leqslant v\,|\,T > N_m]$$

$$= \mathrm{P}[\max_{0\leqslant k\leqslant T} S_k/\sigma\sqrt{N_m}\leqslant v,\ T > N_m,\ N_m\in A_m]/\mathrm{P}[T > N_m]+$$

$$+\mathrm{P}[\max_{0\leqslant k\leqslant T} S_k/\sigma\sqrt{N_m}\leqslant v,\ T > N_m,\ N_m\in A_m^c]/\mathrm{P}[T > N_m].$$

We get thus the following estimate:

$$\mathrm{P}[\max_{0\leqslant k\leqslant T} S_k/\sigma\sqrt{[(1-\varepsilon)\alpha_m]}\leqslant v,\ T >$$

$$[(1+\varepsilon)\alpha_m]]/\mathrm{P}[T > N_m]-\mathrm{P}[N_m\in A_m^c]/\mathrm{P}[T > N_m]$$

$$\leqslant \mathrm{P}[\max_{0\leqslant k\leqslant T} S_k/\sigma\sqrt{N_m}\leqslant v\,|\,T > N_m]\leqslant \mathrm{P}[\max_{0\leqslant k\leqslant T} S_k/\sigma\sqrt{[(1+\varepsilon)\alpha_m]}$$

$$\leqslant v,\ T > [(1-\varepsilon)\alpha_m]]/\mathrm{P}[T > N_m]+\mathrm{P}[N_m\in A_m^c]/\mathrm{P}[T > N_m].$$

Therefore, by (1) and letting $m\to\infty$ and next $\varepsilon\to 0$, we get (10).
From Lemmas 4 and 2 one can get the following

LEMMA 5. *Let* $\{\xi_k, k \geq 1\}$ *and* $\{N_m, m \geq 0\}$ *be as in Lemma* 4 *and let* $\beta > 0$. *For* $\delta > 0$ *we have*

$$\limsup_{m\to\infty} \mathrm{P}[\sup_{0\leq t\leq 1} |X_{N_m}(t)| > \delta \,|\, T > N_m] \leq \sqrt{\frac{\pi}{2}}\,\delta^{-1} \tag{12}$$

and

$$\lim_{m\to\infty} \mathrm{P}[\sup_{0\leq t\leq 1} |N_m^{-\beta} X_{N_m}(t)| > \delta \,|\, T > N_m] = 0. \tag{13}$$

LEMMA 6. *Under the assumptions of this note, for any* $\delta > 0$,

$$\lim_{m\to\infty} \mathrm{P}[\sup_{0\leq t\leq 1} |Y_{N_m}(t) - X_{N_m}(t)| > \delta \,|\, T > N_m] = 0, \tag{14}$$

where $Y_{N_m}(t) = X_{N_m}(t)\, h(N_m^{-\beta} X_{N_m}(t))$.

Note now that Theorem 3 of [7] implies $(X_{N_m} | T > N_m) \Rightarrow W^+$, $m \to \infty$. Thus, by (14), we get the following theorem:

THEOREM 2. *Let* $\{\xi_k, k \geq 1\}$ *be a sequence of i.i.d. random variables with* $\mathrm{E}\xi_1 = 0$, $0 < \mathrm{E}\xi_1^2 = \sigma^2 < \infty$. *Suppose that* $\{N_m, m \geq 1\}$ *is a sequence of positive integer-valued random variables, and* $\{\alpha_m, m \geq 1\}$ *is a sequence of positive numbers with* $\alpha_m \to \infty$, $m \to \infty$, *such that, for any given* $\varepsilon > 0$,

$$\mathrm{P}[|N_m/\alpha_m - 1| \geq \varepsilon] = o(1/\sqrt{\alpha_m}). \tag{15}$$

Then

$$(Y_{N_m} | T > N_m) \Rightarrow W^+, \quad m \to \infty. \tag{16}$$

COROLLARY 2. *For every* $g \in \mathcal{G}$,

$$\big((\sqrt{N_m}/g'(0)\sigma)(g(S_{[N_m]}/N_m) - g(0)) \,|\, T > N_m\big) \Rightarrow W^+, \quad m \to \infty. \tag{17}$$

Acknowledgement. The authors are very grateful to the referee for his valuable remarks and comments improving the previous version of this paper.

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Mathematical Institute
Maria Curie-Skłodowska University
ul. Nowotki 10
20-031 Lublin, Poland

Received on 30. 4. 1985;
revised version on 7. 1. 1986