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RANDOM WALKS WITH RANDOM INDICES AND NEGATIVE DRIFT CONDITIONED TO STAY POSITIVE

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Abstract. Let $\{X_k, k \geqslant 1\}$ be a sequence of independent, identically distributed random variables with $E|X_1| = \mu < 0$, and let $\{N_n, n \geqslant 0\}$, $N_0 = 0$ a.s., be a sequence of positive integer-valued random variables. Form the random walk $\{S_{N_n}, n \geqslant 0\}$ by setting $S_0 = 0$, $S_{N_n} = X_1 + \ldots + X_{N_n}$, $n \geqslant 1$.

The main result in this paper shows (under appropriate conditions on $\{N_n, n \geqslant 0\}$ and $\{X_k, k \geqslant 1\}$) that S_{N_n} conditioned on $[S_1 > 0, \ldots, S_{N_n} > 0]$ converges weakly to a random variable S^* considered by Iglehart [4].

1. Introduction. We assume that $\{X_k, k \geqslant 1\}$ are the coordinate functions defined on the product space

$$(\Omega, \mathscr{A}, P) = \mathop{\mathrm{X}}_{k=1}^{\infty} (\boldsymbol{R}, \mathscr{B}, \pi),$$

where $\boldsymbol{R} = (-\infty, \infty)$, $\mathscr{B}$ is the σ-field of Borel sets of $\boldsymbol{R}$, and π is the common probability measure of the X_k's. If $\Lambda_n = [S_1 > 0, \ldots, S_n > 0]$, then we let $(\Lambda_n, \Lambda_n \cap \mathscr{A}, P_n)$ be the trace of $(\Omega, \mathscr{A}, P)$ on Λ_n, where $\Lambda_n \cap \mathscr{A} = \{\Lambda_n \cap A, A \in \mathscr{A}\}$ and $P_n[A] = P[A]/P[\Lambda_n]$ for $A \in \Lambda_n \cap \mathscr{A}$. The expectation with respect to P_n is denoted by $E_n\{\cdot\}$. Let S_n^* denote the restriction of S_n to Λ_n, let

$$r_n = P[S_1 > 0, \ldots, S_n > 0]$$

and, for $u \geqslant 0$, set

$$f_n(u) = E_n\{\exp(-uS_n^*)\} = E\{\exp(-uS_n)|I[S_1 > 0, \ldots, S_n > 0]\},$$

where $I[\cdot]$ denotes the indicator function; in the same way, for $\{N_n, n \geqslant 0\}$, define

$$\hat{r}_n = P[S_1 > 0, \ldots, S_{N_n} > 0]$$

and, for $u \geqslant 0$,

$$\hat{f}_n(u) = E\{\exp(-uS_{N_n})|I[S_1 > 0, \ldots, S_{N_n} > 0]\},$$

where $\{N_n, n \geqslant 0\}$, $N_0 = 0$ a.s., is a sequence of positive integer-valued random variables.

We suppose that $\{X_k, k \geqslant 1\}$ is a sequence of independent, identically distibuted random variables and that the distribution of X_1 satisfies the following conditions:

(1) $-\infty \leqslant E\{X_1\} = \mu < 0$;

(2) $\Theta(s) = E\{\exp(sX_1)\}$ converges for real $s \in [0, a)$ for some $a > 0$;

(3) $\Theta(s)$ attains its infimum at a point τ, $0 < \tau < a$, where $\Theta(\tau) = \gamma < 1$ and $\Theta'(\tau) = 0$;

(4) if X_1 is a lattice, then $P[X_1 = 0] > 0$.

It has been proved by Bahadur and Rao [1] Theorem 1 that conditions (1)-(4) imply

$$P[S_n > 0] \sim (2\pi n)^{-1/2}\gamma^n(\alpha\tau)^{-1}, \quad n \to \infty, \tag{5}$$

where $\alpha = \Theta''(\tau)/\gamma$, $0 < \alpha < \infty$.

In the same way we have

$$E\{\exp(-uS_n)\colon S_n > 0\} \sim (2\pi\alpha)^{-1/2}\gamma^n\big(\alpha(\tau+u)\big)^{-1}, \quad n \to \infty. \tag{6}$$

Put now

$$M = [(2\pi)^{1/2}\alpha\tau]^{-1}\exp\{\sum_{n=1}^{\infty}(\gamma^{-n}/n^{3/2})P[S_n > 0]\}$$

which is finite by (5). Under assumptions (1)-(4) Iglehart [4] has proved that

$$r_n \sim (\gamma^n/n^{3/2})M \tag{7}$$

and, for $u \geqslant 0$

$$\lim_{n\to\infty} f_n(u) = [\tau/(\tau+u)]\exp\{\sum_{n=1}^{\infty}[\gamma^{-n}/n^{3/2}][E\{\exp(-uS_n^+)\}-1]\} \equiv f(u). \tag{8}$$

2. Results. Proofs of theorems 1-3 are given in Section 3.

THEOREM 1. *Suppose that conditions* (1)-(4) *are satisfied. If* $\{N_n, n \geqslant 0\}$, $N_0 = 0$ *a.s., is a sequence of positive integer-valued random variables independent of* $\{X_k, k \geqslant 1\}$ *and* $\{\alpha_n, n \geqslant 1\}$ *is a sequence of positive real numbers such that for any given* $\varepsilon > 0$

$$P[|N_n/\alpha_n - \lambda| \geqslant \varepsilon] = o\big(E(\gamma^{N_n}/N_n^{3/2})\big) \tag{9}$$

with $\alpha_n \to \infty$, $n \to \infty$, *and* λ *is a random variable such that*

$$P[\lambda \geqslant a] = 1 \quad \textit{for a constant } a > 0, \tag{10}$$

then for $u \geqslant 0$

$$\lim_{n\to\infty} \hat{f}_n(u) = f(u). \tag{11}$$

Remarks. Note that if λ is a degenerate random variable at $a > 0$, then (10) is trivially satisfied. Moreover, we note that in general (9) cannot be replaced by the weaker condition $N_n/\alpha_n \xrightarrow{P.} a$, $n \to \infty$ (P. – in probability) which is used in the random central limit theorem. This fact is established by the following

Example 1. Let $\{X_k, k \geqslant 1\}$ be a sequence of random variables which satisfies (1)-(4) with $\gamma = 1/2$ and independent of $\{N_n, n \geqslant 1\}$, where N_n is as follows:

$$P[N_n = 1] = 1/2^n n^{3/2}, \quad P[N_n = n] = 1 - 1/2^n n^{3/2}.$$

Then for any given $\varepsilon > 0$

$$P[|N_n/n - 1| \geqslant \varepsilon] = 1/2^n n^{3/2} \to 0, \quad n \to \infty,$$

i.e. $N_n/n \xrightarrow{P.} 1$, $n \to \infty$.

In this case we have

$$\hat{r}_n = r_1 P[N_n = 1] + r_n P[N_n = n]$$
$$\sim P[X_1 > 0]/2^{n}{}^{3/2} + M(1 - 1/2^n n^{3/2})/2^n n^{3/2}, \quad n \to \infty.$$

Hence, for $u \geqslant 0$ we have

$$\hat{f}_n(u) = \frac{f_1(u) r_1/2^n n^{3/2} + f_n(u) M(1 - 1/2^n n^{3/2})/2^n n^{3/2}}{r_1/2^n n^{3/2} + M(1 - 1/2^n n^{3/2})/2^n n^{3/2}}$$
$$\to \frac{f_1(u) r_1 + Mf(u)}{r_1 + M} \neq f(u), \quad n \to \infty.$$

Furthermore, we shall see that, in case where λ is nondegenerated random variable, condition (10) cannot be replaced by $P[\lambda > 0] = 1$ without changing (9).

Example 2. Let $(\langle 0, 1\rangle, \mathscr{B}(\langle 0, 1\rangle), P)$ be a probability space, where P is the Lebesgue measure and $\mathscr{B}(\langle 0, 1\rangle)$ is the σ-field of Borel subsets of $\langle 0, 1\rangle$. Assume that $\{X_k, k \geqslant 1\}$ is a sequence of random variables defined on $(\langle 0, 1\rangle, \mathscr{B}(\langle 0, 1\rangle))$ and satisfying (1)-(4). Let $\{N_n, n \geqslant 1\}$ be a sequence of random variables independent of X_k, $k \geqslant 1$, defined as follows:

$$N_n(\omega) = \begin{cases} 1, & \omega \in \langle 0, 1/n\rangle, \\ k, & \omega \in \langle (k-1)/n^4, k/n^4\rangle, \ k = n^3+1, \ldots, n^4. \end{cases}$$

We see that, for any given $\varepsilon > 0$,

$$P[|N_n/n^4 - \lambda| \geqslant \varepsilon] = 0$$

for n sufficiently large, where λ is the random variable, uniformly distributed on $\langle 0, 1\rangle$.

Iglehart [4] has proved that, for $u \geqslant 0$,

$$r_n f_n(u) \sim (\gamma^n/n^{3/2}) M_1(u),$$

where

$$M_1(u) = [(2\pi)^{1/2}\alpha(\tau+u)]^{-1}\exp\{\sum_{n=1}^{\infty}(\gamma^{-n}/n)E\{\exp(-uS_n)\colon S_n > 0\}\} < \infty .$$

In this case we have, for $u \geqslant 0$,

$$\hat{f}_n(u) = \big(f_1(u)r_1/n + \sum_{k=n^3+1}^{n^4} f_k(u)r_k/n^4\big)/\hat{r}_n$$

$$= \frac{f_1(u)r_1 + \sum_{k=n^3+1}^{n^4} f_k(u)r_k/n^3}{r_1 + \sum_{k=n^3+1}^{n^4} r_k/n^3} \to f_1(u) \neq f(u), \quad n\to\infty,$$

since

$$\sum_{k=n^3+1}^{n^4} r_k f_k(u)/n^3 \sim (1/n^3)\sum_{k=n^3+1}^{n^4}(\gamma^k/k^{3/2})M_1(u) \to 0,$$

and

$$\sum_{k=n^3+1}^{n^4}(r_k/n^3) \sim (1/n^3)\sum_{k=n^3+1}^{n^4}(\gamma^k/k^{3/2})M \to 0, \ n\to\infty .$$

In the case where λ is a nondegenerated random variable which satisfies only $P[\lambda > 0] = 1$, we have

THEOREM 2. *Suppose that conditions* (1)-(4) *are satisfied. If* $\{N_n, n \geqslant 0\}$, $N_0 = 0$ *a.s., is a sequence of positive integer-valued random variables independent of* $\{X_k, k \geqslant 1\}$ *and* $\{\alpha_n, n \geqslant 1\}$ *is a sequence of positive real numbers such that* $\lim_{n\to\infty}\alpha_n = \infty$, *and*

$$P[|N_n/\alpha_n - \lambda| \geqslant \varepsilon_n] = o\big(E(\gamma^{N_n}/N_n^{3/2})\big), \tag{12}$$

$$P[\lambda < 2\varepsilon_n] = o\big(E(\gamma^{N_n}/N_n^{3/2})\big), \tag{13}$$

then (11) *holds, where* λ *is a positive random variable and* $\{\varepsilon_n, n \geqslant 1\}$ *is a sequence of positive numbers such that* $0 < \varepsilon_n \to 0$, $\alpha_n\varepsilon_n \to \infty$, $n\to\infty$.

We now establish (11) without the assumption of independence $\{X_k, k \geqslant 1\}$ and $\{N_n, n \geqslant 0\}$. First we shall give an example which shows that in this case assumptions of type (9) and (12) are not sufficient for (11).

Example 3. Let $\{X_k, k \geqslant 1\}$ be a sequence of independent, identically distributed random variables such that X_1 is uniformly distributed on $\langle -2, 1\rangle$. It can be verified that X_1 satisfies conditions (1)-(4). Assume that

$\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables such that

$$[N_n = n] = [X_{n+1} \in \langle -2, 0)], \quad [N_n = n+1] = [X_{n+1} \in \langle 0, 1\rangle].$$

Note that for any given $\varepsilon > 0$

$$P[|N_n/n - 1| \geqslant \varepsilon] = 0,$$

whenever $n > n_0 = [1/\varepsilon]$.

Moreover, we see that, for $u \geqslant 0$,

$$\begin{aligned} \hat{f}_n(u) &= \{E\exp(-uS_n)I[S_1 > 0, \ldots, S_n > 0]I[X_{n+1} \in \langle -2, 0)] + \\ &\quad + E\exp(-uS_n)I[S_1 > 0, \ldots, S_n > 0]\exp(-uX_{n+1}) \times \\ &\quad \times I[X_{n+1} \in \langle 0, 1\rangle]\}/\{P[S_1 > 0, \ldots, S_n > 0, X_{n+1} \in \langle -2, 0)] + \\ &\quad + P[S_1 > 0, \ldots, S_{n+1} > 0, X_{n+1} \in \langle 0, 1\rangle]\} \\ &= E(\exp(-uS_n)|I[S_1 > 0, \ldots, S_n > 0])(2/3 + (1-e^{-u})/(3u)) \\ &= f_n(u)(2/3 + (1-e^{-u})/(3u)) \to (2/3 + (1-e^{-u})/(3u))f(u), \quad n \to \infty, \end{aligned}$$

which proves that assumptions of type (9) and (10) are not sufficient for (11).

When λ is a degenerated random variable, we can prove in the considerated case the following theorem which is in some sense the strongest:

THEOREM 3. *Suppose that conditions* (1)-(4) *hold and that* $\{N_n, n \geqslant 0\}$, $N_0 = 0$ *a.s., is a sequence of positive integer-valued random variables and* $\{\alpha_n, n \geqslant 1\}$ *is a sequence of positive integer numbers such that* $\lim_{n\to\infty} \alpha_n = \infty$. *If*

$$P[N_n \neq \alpha_n] = o(\gamma^{\alpha n}/\alpha_n^{3/2}), \tag{14}$$

then (11) *holds.*

3. Proofs of the results.

Proof of Theorem 1. Let $\varepsilon, 0 < \varepsilon < a$, be fixed and put $a_n = [(a-\varepsilon)\alpha_n]$. By (9), (10) and the assumption $\alpha_n \to \infty$, $n \to \infty$, we can choose n sufficiently large such that

$$\begin{aligned} 0 &\leqslant \sum_{k=1}^{a_n} P[S_1 > 0, \ldots, S_k > 0]P[N_n = k] \leqslant \sum_{k=1}^{a_n} P[N_n = k] \\ &\leqslant P[|N_n/\alpha_n - \lambda| \geqslant \varepsilon] = o(E(\gamma^{N_n}/N_n^{3/2})), \end{aligned}$$

and at the same time, by (7),

$$\begin{aligned} \sum_{k=a_n+1}^{\infty} P[S_1 > 0, \ldots, S_k > 0]P[N_n = k] &\approx \sum_{k=a_n+1}^{\infty} (M\gamma^k/k^{3/2})P[N_n = k] \\ &= M \cdot E(\gamma^{N_n}/N_n^{3/2}) - M\sum_{k=1}^{a_n} P[N_n = k]. \end{aligned}$$

But

$$0 \leqslant M \sum_{k=1}^{a_n} (\gamma^k/k^{3/2}) P[N_n = k] \leqslant M \sum_{k=1}^{a_n} P[N_n = k] = o\big(E(\gamma^{N_n}/N_n^{3/2})\big).$$

Hence

(15) $$\hat{r}_n \sim M \cdot E(\gamma^{N_n}/N_n^{3/2}), \quad n \to \infty.$$

Put now

(16) $$C_{n,k} = r_k P[N_n = k]/\hat{r}_n, \quad k \geqslant 1,\ n \geqslant 1.$$

We see that $\sum_{k=1}^{\infty} C_{n,k} = 1$ and, for fixed k, by (9) and (15)

$$0 \leqslant C_{n,k} \leqslant \frac{\sum_{k=1}^{a_n} P[N_n = k]}{\hat{r}_n} = \frac{o\big(E(\gamma^{N_n}/N_n^{3/2})\big)}{M \cdot E(\gamma^{N_n}/N_n^{3/2})} \to 0, \quad n \to \infty,$$

which proves that $[C_{n,k}]_{k=1,\ldots;n=1,\ldots}$ is a Toeplitz mátrix. Therefore, by [5], p. 472, for $u \geqslant 0$ we have

$$\hat{f}_n(u) = \sum_{k=1}^{\infty} f_k(u)\, C_{n,k} \to f(u), \quad n \to \infty,$$

which completes the proof of Theorem 1.

Proof of Theorem 2. By (12) and (13) we have, for sufficiently large n,

(17) $$\hat{r}_n = P[S_1 > 0, \ldots, S_{N_n} > 0]$$
$$= \sum_{k=1}^{[\alpha_n \varepsilon_n]} r_k P[N_n = k] + \sum_{k=[\alpha_n \varepsilon_n]}^{\infty} r_k P[N_n = k] \sim M \cdot E(\gamma^{N_n}/N_n^{3/2}).$$

One can see that

$$C_{n,j} = P[S_1 > 0, \ldots, S_j > 0]\, P[N_n = j]/\hat{r}_n, \quad n \geqslant 1, j \geqslant 1$$

is a Toeplitz matrix. Indeed, we have $C_{n,j} \geqslant 0$, $\sum_{j=1}^{\infty} C_{n,j} = 1$, and, by (18), (19) and (17), we get

$$0 \leqslant C_{n,j} \leqslant \frac{\sum_{k=n}^{[\alpha_n \varepsilon_n]} P[N_n = k]}{M \cdot E(\gamma^{N_n}/N_n^{3/2}} \sim \frac{P[|N_n/\alpha_n - \lambda| \geqslant \varepsilon_n] + P[\lambda \leqslant 2\varepsilon_n]}{M \cdot E(\gamma^{N_n}/N_n^{3/2})} \to 0,$$

$n \to \infty$, as $j \leqslant [\varepsilon_n \alpha_n]$ for sufficiently large n. Following the considerations of the proof of Theorem 1 we obtain (11).

Proof of Theorem 3. From (14) we have

$$\hat{r}_n = P[S_1 > 0, \ldots, S_{N_n} > 0] = \sum_{k=1}^{\infty} P[S_1 > 0, \ldots, S_k > 0, N_n = k]$$

$$= P[S_1 > 0, \ldots, S_{\alpha_n} > 0, N_n = \alpha_n] + P[S_1 > 0, \ldots, S_{N_n} > 0, N_n \neq \alpha_n].$$

Hence, by (7), we get

$$M(\gamma^{\alpha_n}/\alpha_n^{3/2}) - o(\gamma^{\alpha_n}/\alpha_n^{3/2}) \leqslant \hat{r}_n \leqslant M(\gamma^{\alpha_n}/\alpha_n^{3/2}) + o(\gamma^{\alpha_n}/\alpha_n^{3/2}). \tag{18}$$

Taking into account that, for $u \geqslant 0$,

$$\hat{r}_n \hat{f}_n(u) = E\{\exp(-uS_{\alpha_n}) I[S_1 > 0, \ldots, S_{\alpha_n} > 0] I[N_n = \alpha_n]\} +$$
$$+ E\{\exp(-uS_{N_n}) I[S_1 > 0, \ldots, S_{N_n} > 0] I[N_n \neq \alpha_n]\}$$
$$= E\{\exp(-uS_{\alpha_n}) I[S_1 > 0, \ldots, S_{\alpha_n} > 0]\} -$$
$$- E\{\exp(-uS_{\alpha_n}) I[S_1 > 0, \ldots, S_{\alpha_n} > 0] I[N_n \neq \alpha_n]\} +$$
$$+ E\{\exp(-uS_{N_n}) I[S_1 > 0, \ldots, S_{N_n} > 0] I[N_n \neq \alpha_n]\},$$

and (14), we have

$$r_{\alpha_n} f_{\alpha_n}(u) - o(\gamma^{\alpha_n}/\alpha_n^{3/2}) \leqslant \hat{r}_n \hat{f}_n(u) \leqslant r_{\alpha_n} f_{\alpha_n}(u) + o(\gamma^{\alpha_n}/\alpha_n^{3/2}).$$

Therefore, by (18), we get (11).

Note. The problem here considered, in the case where $E\{X_1\} = \mu = 0$, was treated in [7].

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