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HARNACK'S INEQUALITIES FOR DIRICHLET FORMS AND THEIR APPLICATIONS TO DIFFUSION PROCESSES

BY

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Abstract. Consider the Dirichlet space associated with a direct product diffusion process. Dirichlet forms having the same domain as it can be expressed by integro-differential forms [7]. We establish two estimates for harmonic functions with respect to such Dirichlet forms, which correspond to Harnack's inequalities in the theory of partial differential equations. Further we show the continuity of such harmonic functions. Then we apply those results to study some properties of diffusion processes associated with Dirichlet forms as above.

0. INTRODUCTION

Let m_i, $1 \leqslant i \leqslant d$ $(d \geqslant 2)$, be nonnegative Radon measures on $\boldsymbol{R}^1$ such that

$$m_i(dt) \geqslant A\,dt, \qquad 1 \leqslant i \leqslant d,\ t \in \boldsymbol{R}^1, \tag{0.1}$$

$$m_i([j, j+1]) \leqslant M, \qquad 1 \leqslant i \leqslant d,\ j = 0,\ \pm 1,\ \pm 2, \ldots, \tag{0.2}$$

for some $0 < A \leqslant 1 \leqslant M < \infty$, where dt is the one-dimensional Lebesgue measure.

Let us define d-dimensional Radon measures m, ν_{ij}, $1 \leqslant i, j \leqslant d$, and ν as follows:

$$m(dx) = m(dx_1 \ldots dx_d) = \prod_{1 \leqslant k \leqslant d} m_k(dx_k),$$

$$\nu_{ij}(dx) = \nu_{ij}(dx_1 \ldots dx_d) = \begin{cases} \prod_{k \neq i} m_k(dx_k)\,dx_i & \text{if } i = j, \\ \prod_{k \neq i,j} m_k(dx_k)\,dx_i\,dx_j & \text{if } i \neq j, \end{cases}$$

$$\nu = \sum_{1 \leqslant i \leqslant d} \nu_{ii}.$$

Let Ω be a domain in $\boldsymbol{R}^d$ and $\mathscr{F}_0^2(\Omega; \{\nu_{ii}\})$, $\mathscr{F}^2(\Omega; \{\nu_{ii}\})$ be the function spaces defined in [7], that is, $\mathscr{F}_0^2(\Omega; \{\nu_{ii}\})$ and $\mathscr{F}^2(\Omega; \{\nu_{ii}\})$ are the completions of $C_0^\infty(\Omega)$ and $C_0^\infty(\bar{\Omega})$, respectively, with respect to the norm $|||\ |||_{2,\Omega,\{\nu_{ii}\}}$,

$$|||u|||_{2,\Omega,\{\nu_{ii}\}} = \sum_{i=1}^{d} \left(\int_{\Omega} \left(\frac{\partial u}{\partial x_i} \right)^2 d\nu_{ii} \right)^{1/2} + \left(\int_{\Omega} u^2 \, dm \right)^{1/2},$$

where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions with compact support in Ω and $C_0^\infty(\bar{\Omega})$ stands for the space of all restrictions to $\bar{\Omega}$ of functions in $C_0^\infty(\boldsymbol{R}^d)$. Let E be a subdomain of Ω. We define a bilinear form $\mathscr{E}_E$ on $\mathscr{F}^2(E; \{\nu_{ii}\}) \times \mathscr{F}^2(E; \{\nu_{ii}\})$ by

$$\mathscr{E}_E(u, v) = \sum_{i,j=1}^{d} \int_E D_i u(x) D_j v(x) a_{ij}(x) \nu_{ij}(dx), \tag{0.3}$$

where each $D_i u$ is the weak derivative defined in [7], Section 1, and a_{ij}, $1 \leqslant i, j \leqslant d$, are measurable functions on Ω satisfying

$$a_{ij} = a_{ji}, \quad 1 \leqslant i, j \leqslant d, \tag{0.4}$$

$$\begin{aligned} \gamma^{-1} \sum_{i=1}^{d} \xi_i^2 \nu_{ii}(dx) &\leqslant \sum_{i,j=1}^{d} \xi_i \xi_j a_{ij}(x) \nu_{ij}(dx) \\ &\leqslant \gamma \sum_{i=1}^{d} \xi_i^2 \nu_{ii}(dx), \qquad \xi \in \boldsymbol{R}^d,\ x \in \Omega, \end{aligned} \tag{0.5}$$

for some $\gamma \geqslant 1$.

A function u is called a *solution* of $[\mathscr{E}_E, \mathscr{F}^2(E; \{\nu_{ii}\})]$ if it belongs to $\mathscr{F}^2(E; \{\nu_{ii}\})$ and satisfies $\mathscr{E}_E(u, \varphi) = 0$ for every $\varphi \in C_0^\infty(E)$. $|E|$ stands for the Lebesgue measure of E.

In Section 1 we show the following

THEOREM 1. *Assume* (0.1) *and* (0.2). *Let E be a domain with $E \cap \Omega \neq \emptyset$ and u be a nonnegative solution of $[\mathscr{E}_{E\cap\Omega}; \mathscr{F}^2(E \cap \Omega; \{\nu_{ii}\})]$ such that $|E \cap \{u > 0\}| > 0$ in case of $E \subset \Omega$, or such that $u = const > 0$ on $E \cap \partial\Omega$ and $|E - \Omega| > 0$ in case of $E \cap \partial\Omega \neq \emptyset$. Then*

$$u(x) > 0 \quad \textit{for } \nu\textit{-a.e. } x \in E \cap \Omega.$$

This fact is obtained by Moser [3] for $m_i(dt) = dt$, $1 \leqslant i \leqslant d$. We know [7] that some inequalities of Sobolev type hold for functions belonging to $\mathscr{F}_0^2(\Omega; \{\nu_{ii}\})$ or $\mathscr{F}^2(\Omega; \{\nu_{ii}\})$. Therefore we can employ his methods in our case.

In Section 2 we are concerned with continuity of solutions of $[\mathscr{E}_E, \mathscr{F}^2(E; \{\nu_{ii}\})]$. In case of $m_i(dt) = dt$, $1 \leqslant i \leqslant d$, Moser [3] showed that Harnack's inequality leads us to Hölder continuity of solutions of

$[\mathscr{E}_E, \mathscr{F}^2(E; \{v_{ii}\})]$. Though it may be hopeless to obtain such results in our general case, we can extend his results in the following direction:

THEOREM 2. *Assume* (0.1) *and* (0.2) *as well as*

(0.6) $$\lim_{\varrho \downarrow 0} m_i\big((a_i-\varrho, a_i+\varrho)\big)/\varrho(\log|\log \varrho|)^{1/(d-1)} = 0,$$
$$a=(a_1, \ldots, a_d)\in \bar{\Omega}, \quad 1 \leqslant i \leqslant d.$$

Let E be a domain with $E\cap\Omega \neq \emptyset$ and u be a solution of $[\mathscr{E}_{E\cap\Omega}, \mathscr{F}^2(E\cap\Omega; \{v_{ii}\})]$. Further, if $E\cap\partial\Omega \neq \emptyset$, suppose that for every $a\in E\cap\partial\Omega$ there is a cone included in $E-\Omega$ with vertex at a and $u=\mathrm{const}$ on $E\cap\partial\Omega$. Then u is continuous on $E\cap\bar{\Omega}$.

In Section 3 we show another type Harnack's inequality under the additional condition:

(0.7) $d=2$. For any rectangle $K=I^1\times I^2$ there are a real number $\delta = \delta(K) \geqslant 2$ and sequences of partitions of I^i's,

$$\Delta_n^i\colon I^i = \bigcup_{1\leqslant k\leqslant N_n^i} J_{n,k}^i, \quad i=1, 2, \quad n=0, 1, 2, \ldots,$$

where $1=N_0^i \leqslant N_1^i \leqslant \ldots \uparrow \infty$ and $J_{n,k}^i$'s fulfil the following properties for every i, j, k, l, n:

(i) $$m_i(J_{n,k}^i) > 0,$$

(ii) $$J_{n,k}^i \cap J_{n,l}^i = \emptyset \quad \text{if} \quad k\neq l,$$

(iii) $$|J_{n,k}^i| = \sup J_{n,k}^i - \inf J_{n,k}^i \quad \text{and} \quad \lim_{n\uparrow\infty} \max_{1\leqslant k\leqslant N_n^i} |J_{n,k}^i| = 0,$$

(iv) $$0 \leqslant |J_{n,k}^i| \leqslant \delta m_j(J_{n,l}^j),$$

(v) $$J_{n,k}^i = \bigcup_{p\in P_{n,k}^i} J_{n+1,p}^i \quad \text{and} \quad m_i(J_{n,k}^i) \leqslant \delta m_i(J_{n+1,p}^i), \quad p\in P_{n,k}^i,$$

where $\{P_{n,k}^i\colon 1\leqslant k\leqslant N_n^i\}$ is an appropriate partition of $\{1, 2, \ldots, N_{n+1}^i\}$.

THEOREM 3. *Suppose* (0.1), (0.2) *and* (0.7). *Let E be a subdomain of Ω and u be a nonnegative solution of $[\mathscr{E}_E, \mathscr{F}^2(E; \{v_{ii}\})]$. Then for any compact set K $(\subset E)$ there is a positive constant C independent of u such that*

$$m\text{-ess}\max_K u \leqslant C\, m\text{-ess}\min_K u.$$

Moser [4] proved this theorem for the case $m_i(dt)=dt$, $1\leqslant i\leqslant d$. He made good use of an important estimate on functions of bounded mean oscillations due to John and Nirenberg [2]. Under condition (0.7) we can give an analogous estimate on functions of bounded mean oscillations with respect to m. Thus we can utilize ideas in [4]. We will discuss on condition (0.7) in Section 3.

Now let n be a Radon measure on $\mathbf{R}^d$ such that

$$\Lambda^{-1}\,dx \leqslant n(dx) \leqslant \Lambda m(dx), \qquad x\in \mathbf{R}^d, \tag{0.8}$$

for some $1 \leqslant \Lambda < \infty$, dx being the d-dimensional Lebesgue measure. Let $(\mathscr{F}, \mathscr{E})$ be a Dirichlet space relative to $L^2(\Omega; n)$ such that $\mathscr{F} = \mathscr{F}_0^2(\Omega; \{\nu_{ii}\})$ and $\mathscr{E}(u, v) = 0$ if $u = \text{const}$ on Supp $[v]$. Then $\mathscr{E}$ can be expressed by

$$\mathscr{E}(u, v) = \sum_{i,j=1}^{d} \int_{\Omega} D_i u(x) D_j v(x) a_{ij}(x) \nu_{ij}(dx), \tag{0.9}$$

where a_{ij}, $1 \leqslant i, j \leqslant d$ fulfil (0.4) and (0.5) (see [7]). It is known [1] that there is a diffusion process on Ω associated with $(\mathscr{F}, \mathscr{E})$. This diffusion process has the resolvent density [7]. The estimate in Theorem 1 implies the positivity of the resolvent density, Theorem 2 leads us to the continuity and the estimate in Theorem 3 gives us a comparison theorem of the resolvent density with that of a direct product diffusion. These facts and some other properties will be proved in Section 4.

Finally, I would like to thank Professor Yukio Ogura for his valuable suggestions.

1. PROOF OF THEOREM 1

Throughout this section we assume (0.1) and (0.2). Given $E \subset \Omega$, we call a function u a *subsolution* of $[\mathscr{E}_E, \mathscr{F}^2(E; \{\nu_{ii}\})]$ if $u \in \mathscr{F}^2(E; \{\nu_{ii}\})$ and $\mathscr{E}_E(u, \varphi) \leqslant 0$ for every nonnegative $\varphi \in \mathscr{F}_0^2(E; \{\nu_{ii}\})$.

Put

$$Q(a, \varrho) = \{x = (x_1, \ldots, x_d) \in \mathbf{R}^d\colon\ |x_i - a_i| < \varrho,\ 1 \leqslant i \leqslant d\},$$

$$\Omega(a, \varrho) = \Omega \cap Q(a, \varrho),$$

$$\mu(a, \varrho) = \max_{1\leqslant i\leqslant d} m_i\big((a_i - \varrho, a_i + \varrho)\big)/2\varrho \qquad \text{for } a = (a_1, \ldots, a_d) \text{ and } \varrho > 0.$$

In order to prove Theorem 1 we need two lemmas. The first one is a small modification of [7], Theorem 2.5.

Lemma 1.1. *Let $Q(a, \varrho) \subset \Omega$ or $a \in \partial\Omega$. Let u be a subsolution of $[\mathscr{E}_{\Omega(a,\varrho)}, \mathscr{F}^2(\Omega(a, \varrho); \{\nu_{ii}\})]$ such that $u \leqslant \Phi$ $(= \text{const})$ on $\partial\Omega \cap Q(a, \varrho)$ if $a \in \partial\Omega$. Then*

$$\nu\text{-ess}\max_{\Omega(a,r)} u \leqslant \tilde{\Phi} + C_1 (\varrho - r)^{-d/2} \Big(\int_{\Omega(a,\varrho)} (u - \tilde{\Phi})^2\, d\nu \Big)^{1/2}, \qquad 0 < r < \varrho,$$

for a positive constant $C_1 = C_1(\Lambda, d, \gamma)$, where $\tilde{\Phi} = 0$ if $a \in \Omega$, $\tilde{\Phi} = \Phi \vee 0$ if $a \in \partial\Omega$.

Proof. Put $Q(r) = Q(a, r)$, $\Omega(r) = \Omega(a, r)$ and $E = \Omega(a, \varrho)$. For all $t \geqslant \tilde{\Phi}$, $v \equiv (u - t) \vee 0$ is nonnegative subsolution of $[\mathscr{E}_E, \mathscr{F}^2(E; \{\nu_{ii}\})]$ and, in particular, if $a \in \partial\Omega$, then v vanishes on $\partial\Omega \cap Q(\varrho)$. For each $0 < \varrho_2 < \varrho_1 \leqslant \varrho$ we choose a function $\varphi \in C_0^\infty(\overline{Q(\varrho)})$ such that $\varphi = 1$ on $Q(\varrho_2)$, $\varphi = 0$ outside $Q(\varrho_1)$ and $|\varphi| \leqslant 1$, $|D_i \varphi| \leqslant 2/(\varrho_1 - \varrho_2)$. Since $\varphi^2 v$ belongs to $\mathscr{F}_0^2(E; \{v_{ii}\})$ and is nonnegative, we have $\mathscr{E}_E(v, \varphi^2 v) \leqslant 0$. Hence

$$\sum_{i,j} \int_E \varphi^2 D_i v D_j v a_{ij}\, dv_{ij} \leqslant -2 \sum_{i,j} \int_E \varphi v D_i v D_j \varphi a_{ij}\, dv_{ij}$$
$$\leqslant 2\Big(\sum_{i,j} \int_E \varphi^2 D_i v D_j v a_{ij}\, dv_{ij}\Big)^{1/2} \Big(\sum_{i,j} \int_E v^2 D_i \varphi D_j \varphi a_{ij}\, dv_{ij}\Big)^{1/2},$$

that is,

$$\sum_{i,j} \int_E \varphi^2 D_i v D_j v a_{ij}\, dv_{ij} \leqslant 4 \sum_{i,j} \int_E v^2 D_i \varphi D_j \varphi a_{ij}\, dv_{ij}.$$

By condition (0.5)

$$\sum_i \int_E (\varphi D_i v)^2\, dv_{ii} \leqslant 4\gamma^2 \sum_i \int_E (v D_i \varphi)^2\, dv_{ii}.$$

Therefore, by using Hölder's inequality and [7], Proposition 1.4, we obtain

$$\int_{\Omega(\varrho_2) \cap \{u > t\}} (u-t)^2\, dv$$
$$\leqslant \int_E (\varphi v)^2\, dv$$
$$\leqslant \Big(\int_E |\varphi v|^q\, dv\Big)^{2/q} v(E \cap \{\varphi v \neq 0\})^{1-2/q}$$
$$\leqslant C_2\, v(E \cap \{\varphi v \neq 0\})^{2/d} \sum_i \int_E (v D_i \varphi + \varphi D_i v)^2\, dv_{ii}$$
$$\leqslant 2C_2 (1 + 4\gamma^2)\, v(E \cap \{\varphi v \neq 0\})^{2/d} \sum_i \int_E (v D_i \varphi)^2\, dv_{ii}$$
$$\leqslant 8C_2 (1 + 4\gamma^2)(\varrho_1 - \varrho_2)^{-2}\, v(E \cap \{\varphi v \neq 0\})^{2/d} \int_{\Omega(\varrho_1) \cap \{u > t\}} (u-t)^2\, dv$$
$$\leqslant 8C_2 (1 + 4\gamma^2)(\varrho_1 - \varrho_2)^{-2} (t-s)^{-4/d} \Big(\int_{\Omega(\varrho_1) \cap \{u > s\}} (u-s)^2\, dv\Big)^{1+2/d},$$

for $\tilde{\Phi} \leqslant s < t$, where $2 < q < 2d/(d-2)$ and $C_2 = C_2(A, d)$. Taking

$$\int_{\Omega(r) \cap \{u > t\}} (u-t)^2\, dv$$

as $\varphi(t, r)$ in [5], Lemma 5.1, we get the assertion, q.e.d.

LEMMA 1.2. *Assume $Q(a, \varrho_1) \subset \Omega$ or $a \in \partial\Omega$. Let $\varrho_1 > \varrho_2 > \varrho_3 > 0$ and let u be a nonnegative solution of $[\mathscr{E}_{\Omega(a,\varrho_1)}, \mathscr{F}^2(\Omega(a, \varrho_1); \{v_{ii}\})]$ such that $|\Omega(a, \varrho_2) \cap \{u \geqslant 1\}| \geqslant \varepsilon |\Omega(a, \varrho_2)|$ for some $\varepsilon > 0$ in case of $Q(a, \varrho_1) \subset \Omega$, or such that $u = const \geqslant 1$ on $\partial\Omega \cap Q(a, \varrho_1)$ and $|Q(a, \varrho_2) - \Omega| \geqslant \varepsilon |Q(a, \varrho_2)|$ for some $\varepsilon > 0$ in case of $a \in \partial\Omega$. Then*

$$\nu\text{-ess}\min_{\Omega(a,\varrho_3)} u$$
$$\geqslant \exp[-C_3\{\varrho_1^d\varrho_2^2(\varrho_1-\varrho_2)^{-2}(\varrho_2-\varrho_3)^{-d}\mu(a,\varrho_1)^{d-1}\mu(a,\varrho_2)^{d-1}\}^{1/2}]$$

for some positive constant $C_3 = C_3(A, d, \gamma, \varepsilon)$.

Remark. When $m_i(dt) = dt$, $1 \leqslant i \leqslant d$, taking $\varrho = \varrho_1 = 2\varrho_2 = 4\varrho_3$ shows

$$\operatorname*{ess\,min}_{\Omega(a,\varrho/4)} u \geqslant \exp(-2^d C_3),$$

which is the result due to Moser [3].

Proof. Put $E = \Omega(a, \varrho_1)$. Fix an arbitrary $\eta \in (0, 1)$ and let $f(x) = \{-\log(x+\eta)\} \vee 0$. In spite of $f(0) > 0$, the proof of Proposition 1.3 in [7] works in showing $f(u) \in \mathscr{F}^2(E; \{\nu_{ii}\})$. Since f is convex, $\mathscr{E}_E(f(u), \varphi) \leqslant \mathscr{E}_E(u, f'(u)\varphi) = 0$ for every nonnegative $\varphi \in C_0^\infty(E)$, that is, $f(u)$ is a subsolution of $[\mathscr{E}_E, \mathscr{F}^2(E; \{\nu_{ii}\})]$. Noting $f(u) = 0$ on $\partial\Omega \cap Q(a, \varrho_2)$ if $a \in \partial\Omega$, we have, by Lemma 1.1,

$$\nu\text{-ess}\max_{\Omega(a,\varrho_3)} f(u) \leqslant C_1(\varrho_2-\varrho_3)^{-d/2}\Big(\int_{\Omega(a,\varrho_2)} f^2(u)\,d\nu\Big)^{1/2}. \tag{1.1}$$

In the same way as in the proof of Proposition 1.2 in [7], we see that

$$\int_{\Omega(a,\varrho_2)} f^2(u)\,d\nu \leqslant C_4\,\mu(a,\varrho_2)^{d-1}\Big(\int_{\Omega(a,\varrho_2)} f^2(u)\,dx + \varrho_2^2\sum_i \int_{\Omega(a,\varrho_2)} (D_i f(u))^2\,d\nu_{ii}\Big)$$

for some positive $C_4 = C_4(A, d)$.

Set $N = Q(a, \varrho_2) \cap \{u \geqslant 1\}$ or $N = Q(a, \varrho_2) - \Omega$ according to $Q(a, \varrho_1) \subset \Omega$ or $a \in \partial\Omega$. By virtue of [3], Lemma 2, we obtain

$$\int_{\Omega(a,\varrho_2)} f^2(u)\,dx \leqslant C_5\,\varrho_2^2\sum_i \int_{\Omega(a,\varrho_2)} (D_i f(u))^2\,d\nu_{ii}$$

with $C_5 = C_5(A, d, \varepsilon)$. Therefore

$$\int_{\Omega(a,\varrho_2)} f^2(u)\,d\nu \leqslant C_4(C_5+1)\,\varrho_2^2\,\mu(a,\varrho_2)^{d-1}\sum_i \int_{\Omega(a,\varrho_2)} (D_i f(u))^2\,d\nu_{ii}. \tag{1.2}$$

Moreover, we have

$$\sum_i \int_{\Omega(a,\varrho_2)} (D_i f(u))^2\,d\nu_{ii} \leqslant 16\gamma^2(\varrho_1-\varrho_2)^{-2}\,\nu(\Omega(a,\varrho_1)). \tag{1.3}$$

Because $\mathscr{E}_E(u, f'(u)\varphi^2) = 0$, $\varphi \in C_0^\infty(Q(a, \varrho_1))$ and $f'' \geqslant (f')^2$, and hence

$$\sum_i \int_E (\varphi D_i f(u))^2\,d\nu_{ii} \leqslant 4\gamma^2 \sum_i \int_E (D_i\varphi)^2\,d\nu_{ii}, \quad \varphi \in C_0^\infty(Q(a, \varrho_1)). \tag{1.4}$$

In order to get (1.3) we may take a $\varphi \in C_0^\infty(Q(a, \varrho_1))$ with $\varphi = 1$ on $Q(a, \varrho_2)$, $|D_i\varphi| \leqslant 2/(\varrho_1-\varrho_2)$, $1 \leqslant i \leqslant d$. It follows from (1.1)-(1.3) that

$$\nu\text{-ess}\max_{\Omega(a,\varrho_3)} \{-\log(u+\eta)\}$$

$$\leqslant C_1 \{C_4(C_5+1)16\gamma^2(\varrho_1-\varrho_2)^{-2}(\varrho_2-\varrho_3)^{-d}\varrho_2^2\}^{1/2} \times$$
$$\times \{\mu(a, \varrho_2)^{d-1} \nu(\Omega(a, \varrho_1))\}^{1/2}$$
$$\leqslant C_1 \{C_4(C_5+1)16\gamma^2(\varrho_1-\varrho_2)^{-2}(\varrho_2-\varrho_3)^{-d}\varrho_2^2\}^{1/2} \times$$
$$\times \{\mu(a, \varrho_2)^{d-1} d(2\varrho_1)^d \mu(a, \varrho_1)^{d-1}\}^{1/2}.$$

Since η is arbitrary, we get the desired estimate, q.e.d.

Proof of Theorem 1. We can take a cube $Q(a, 3\varrho) \subset E$ and a positive constant C_6 such that $\varepsilon \equiv |Q(a, 2\varrho) \cap \{u \geqslant C_6\}|/|Q(a, 2\varrho)| > 0$ in case of $E \subset \Omega$, or such that $u = C_6$ on $E \cap \partial\Omega$ and $\varepsilon \equiv |Q(a, 2\varrho) - \Omega|/|Q(a, 2\varrho)| > 0$ otherwise. Namely, u/C_6 satisfies the conditions in Lemma 1.2. Therefore $u(x) \geqslant C_7(A, d, \gamma, \varepsilon, \varrho) > 0$ for ν-a.e. $x \in \Omega(a, \varrho)$. Repeating this argument, we obtain the conclusion, q.e.d.

As an immediate consequence of Theorem 1 we get the following

COROLLARY 1.3. *Let E be a subdomain of Ω. If u is a nonnegative continuous solution of $[\mathscr{E}_E, \mathscr{F}^2(E; \{\nu_{ii}\})]$, then u is positive or identically zero in E.*

2. CONTINUITY OF SOLUTIONS

First of all we give

Proof of Theorem 2. Fix an $a \in E \cap \bar{\Omega}$. If $a \in \partial\Omega$, by the assumption of the theorem there is an ε, $0 < \varepsilon \leqslant 1/2$, such that

$$\inf_{r>0} |Q(a, r) - \Omega|/|Q(a, r)| \geqslant \varepsilon.$$

Put $C_8 = C_3(A, d, \gamma, \varepsilon) 2^{d+1}$ and fix an $\alpha \in (0, C_8^{-1/(d-1)})$ and a $\beta \in (0, 1)$ such that $\alpha^{d-1} C_8 \leqslant (1-\beta)^{1+d/2} (< 1)$. Since $\mu(a, \varrho)^{d-1} \leqslant \alpha^{d-1} \log|\log \varrho|$ for sufficiently small ϱ, putting $\varrho_n = \beta^n$ gives us

$$\exp\{-C_8(1-\varrho_{n+1}/\varrho_n)^{-d/2-1} \mu(a, \varrho_n)^{d-1}\}$$
$$\geqslant \exp\{-C_8(1-\beta)^{-d/2-1} \alpha^{d-1} \log(n|\log \beta|)\}$$
$$\geqslant 1/n|\log \beta|$$

for $n \geqslant n_0$, n_0 being a certain number depending only on α and C_8.

Now set

$$S = \nu\text{-ess}\max_{\Omega(a,\varrho_n)} u \quad \text{and} \quad I = \nu\text{-ess}\min_{\Omega(a,\varrho_n)} u.$$

By Lemma 1.1 we have $-\infty < I \leqslant S < +\infty$. If $I < S$, we put $u_1 = 2(u-I)/(S-I)$, $u_2 = 2(S-u)/(S-I)$, which are nonnegative solutions of $[\mathscr{E}_{\Omega(a,\varrho_n)}, \mathscr{F}^2(\Omega(a, \varrho_n); \{\nu_{ii}\})]$.

Let us take $r=(\varrho_n+\varrho_{n+1})/2$. Obviously $|\Omega(a, r)\cap\{u_i\geqslant 1\}|\geqslant|\Omega(a, r)|/2 \geqslant \varepsilon|\Omega(a, r)|$ for $i=1$ or 2 and if $a\in\partial\Omega$, then $u_i=\text{const}\geqslant 1$ on $\partial\Omega\cap Q(a, r)$ and $|Q(a, r)-\Omega|\geqslant\varepsilon|Q(a, r)|$ for $i=1$ or 2. In view of Lemma 1.2 we obtain

$$\begin{aligned}
&\nu\text{-ess}\min_{\Omega(a,\varrho_{n+1})} u_i\\
&\geqslant \exp\left[-C_3(A, d, \gamma, \varepsilon)\{\varrho_n^d r^2(\varrho_n-r)^{-2}(r-\varrho_{n+1})^{-d}\mu(a, \varrho_n)^{d-1}\mu(a, r)^{d-1}\}^{1/2}\right]\\
&=\exp\{-C_3(A, d, \gamma, \varepsilon)2^{d/2}(1+\beta)(1-\varrho_{n+1}/\varrho_n)^{-d/2-1}\mu(a, \varrho_n)^{(d-1)/2}\mu(a, r)^{(d-1)/2}\}\\
&\geqslant \exp\{-C_8(1-\varrho_{n+1}/\varrho_n)^{-d/2-1}\mu(a, \varrho_n)^{d-1}\}\\
&\geqslant 1/n|\log\beta|,
\end{aligned}$$

where $i=1$ or 2.

Therefore $S\geqslant u\geqslant I+(S-I)/2n|\log\beta|$ or $I\leqslant u\leqslant S-(S-I)/2n|\log\beta|$ ν-a.e. on $\Omega(a, \varrho_{n+1})$. This implies

$$\begin{aligned}
\omega(\varrho_{n+1}) &\equiv \nu\text{-ess}\max_{\Omega(a,\varrho_{n+1})} u-\nu\text{-ess}\min_{\Omega(a,\varrho_{n+1})} u\leqslant(1-1/2n|\log\beta|)\,\omega(\varrho_n)\\
&\leqslant\prod_{k=n_0}^{n}(1-1/2k|\log\beta|)\,\omega(\varrho_{n_0})\leqslant\exp\{-\sum_{k=n_0}^{n}1/2k|\log\beta|\}\,\omega(\varrho_{n_0}),
\end{aligned}$$

whence

$$\lim_{n\uparrow\infty}\omega(\varrho_{n+1})=0.$$

Thus we can get a continuous version of u, q.e.d.

Now let n satisfy (0.8) and put $\mathscr{E}_{E,\lambda}(u, v)=\mathscr{E}_E(u, v)+\lambda(u, v)_n$ for $E\subset\Omega$ and $\lambda\geqslant 0$, $(\ ,\ )_n$ being the inner product in $L^2(E; n)$. A function u is called a *solution* of $[\mathscr{E}_{E,\lambda}, f, \mathscr{F}_0^2(E; \{\nu_{ii}\})]$ $([\mathscr{E}_{E,\lambda}, f, \mathscr{F}^2(E; \{\nu_{ii}\})])$ if u belongs to $\mathscr{F}_0^2(E; \{\nu_{ii}\})$ (resp. $\mathscr{F}^2(E; \{\nu_{ii}\})$) and satisfies $\mathscr{E}_{E,\lambda}(u, \varphi)=(f, \varphi)_n$ for every $\varphi\in\mathscr{F}_0^2(E; \{\nu_{ii}\})$.

PROPOSITION 2.1. *Assume* (0.1) *and* (0.2). *Let E be a bounded domain of Ω and u be a solution of $[\mathscr{E}_{E,\lambda}, f, \mathscr{F}_0^2(E; \{\nu_{ii}\})]$, where $f\in L^p(E; n)$ with $p>d$ and $\lambda\geqslant 0$. Then*

$$m\text{-ess}\max_E |u|\leqslant C_9\, m(E)^{1/d-1/p}\Big(\int_E|f|^p dn\Big)^{1/p}$$

with some $C_9=C_9(A, d, M, p, \gamma, \Lambda)$.

Noting Sobolev's inequalities ([7], Proposition 1.4), we can use the same methods as in [5], Theorem 4.2. So we omit the proof.

THEOREM 2.2. *Suppose* (0.1), (0.2) *and* (0.6). *Let E be a domain with $E\cap\Omega\neq\emptyset$ and u be a bounded solution of $[\mathscr{E}_{E\cap\Omega,\lambda}, f, \mathscr{F}^2(E\cap\Omega; \{\nu_{ii}\})]$ with $f\in L^p(E\cap\Omega; n)$, $p>d$ and $\lambda\geqslant 0$. Moreover, if $E\cap\partial\Omega\neq\emptyset$, suppose that for every $a\in E\cap\partial\Omega$ there is a cone included in $E-\Omega$ with vertex at a and $u=\text{const}$ on $E\cap\partial\Omega$. Then u is continuous on $E\cap\bar{\Omega}$.*

Proof. Fix a $p > d$, an $f \in L^p(E \cap \Omega; n)$ and a $\lambda \geqslant 0$. Let u be a bounded solution of $[\mathscr{E}_{E\cap\Omega,\lambda}, f, \mathscr{F}^2(E \cap \Omega; \{v_{ii}\})]$. Put $g = f - \lambda u$. Since m is a continuous measure, for any $\varepsilon > 0$ and for any $a \in E \cap \bar{\Omega}$ we have a $\varrho > 0$ for which $\Omega(a, \varrho) \subset E$ and

$$C_9\, m(\Omega(a, \varrho))^{1/d - 1/p}\Big(\int_{\Omega(a,\varrho)} |g|^p\, dn\Big)^{1/p} < \varepsilon/4.$$

It follows from Riesz theorem and [7], (1.11), that there is a unique solution w of $[\mathscr{E}_{\Omega(a,\varrho),0}, g, \mathscr{F}_0^2(\Omega(a, \varrho); \{v_{ii}\})]$. By Proposition 2.1

$$m\text{-}\operatorname*{ess\,max}_{\Omega(a,\varrho)} |w| < \varepsilon/4.$$

Set $v = u - w$. Then v is a solution of $[\mathscr{E}_{\Omega(a,\varrho)}, \mathscr{F}^2(\Omega(a, \varrho); \{v_{ii}\})]$ such that $v = \text{const}$ on $Q(a, \varrho) \cap \partial\Omega$ if $a \in \partial\Omega$.

By Theorem 2, v is continuous at a. We thus have an $r \in (0, \varrho)$ such that the oscillation of v in $\Omega(a, r)$ is smaller than $\varepsilon/2$ and hence that of u in $\Omega(a, r)$ is smaller than ε, which shows the assertion of the theorem, q.e.d.

3. PROOF OF THEOREM 3

We can obtain Theorem 3 in case of $d \geqslant 3$ if (0.7.iv) is replaced by

(iv)′ $$|J^i_{n,k}| \geqslant \delta^{-1} m_j(J^j_{n,l}).$$

However (0.1), (0.2) and (0.7) with (iv)′ in place of (iv) are satisfied if and only if $m_i(dt) = m_i^0(t)\, dt$, $A \leqslant m_i^0(t) \leqslant M$ a.e., $1 \leqslant i \leqslant d$. Putting $b_{ij} = a_{ii} \prod_{k \neq i} m_k^0$ if $i = j$, $b_{ij} = a_{ij} \prod_{k \neq i,j} m_k^0$ if $i \neq j$, we get

$$\mathscr{E}_E(u, v) = \sum_{i,j=1}^{d} \int_E D_i u(x)\, D_j v(x)\, b_{ij}(x)\, dx.$$

The matrix (b_{ij}) is symmetric and positive definite. Moreover, $\mathscr{F}_0^2(E; \{v_{ii}\})$ and $\mathscr{F}^2(E; \{v_{ii}\})$ coincide with $H_0^1(E)$ and $H^1(E)$ (= Sobolev spaces), respectively, and $D_i u$'s are the distributional derivatives. Thus the case where $d \geqslant 3$ and (0.1), (0.2) and (0.7) with (iv)′ in place of (iv) are satisfied is reduced to the case treated in [4]. Therefore we restrict ourselves to the case of $d = 2$. At the end of this section we will find some examples satisfying (0.1), (0.2) and (0.7), which contain every continuous measure.

Now Theorem 3 can be easily deduced from the following Theorem 3.1, so that we omit the proof of Theorem 3 itself.

THEOREM 3.1. *Let $Q(a, \varrho_1) \subset \Omega$ and $\varrho_1 > \varrho_2 > \varrho_3 > \varrho_4 > 0$. If u is a solution of $[\mathscr{E}_{Q(a,\varrho_1)}, \mathscr{F}^2(Q(a, \varrho_1); \{v_{ii}\})]$ and it is positive on $Q(a, \varrho_2)$, then*

$$m\text{-}\operatorname*{ess\,max}_{Q(a,\varrho_4)} u \leqslant \{C_{10}(1 - \varrho_4/\varrho_3)^{-1}\, \mu(a, \varrho_3)\}^{C_{11}}\, m\text{-}\operatorname*{ess\,min}_{Q(a,\varrho_4)} u$$

for $C_{10}=C_{10}(A, \gamma)$ *and* $C_{11}=C'_{11}(\gamma)\delta^{5/2}(1-\varrho_3/\varrho_2)^{-1}\mu(a, \varrho_2)^{1/2}$, *where* δ *is a positive number as in* (0.7) *for* $K=Q(a, \varrho_3)$.

To prove this we have to prepare some lemmas. Following (0.7) we take a positive number $\delta=\delta(Q(a, \varrho_3))$ and sequences $\{\Delta_n^i\}_{n=0}^{\infty}$,

$$\Delta_n^i\colon (a_i-\varrho_3, a_i+\varrho_3)=\bigcup_{1\leqslant k\leqslant N_n^i} J_{n,k}^i, \qquad i=1, 2.$$

Put $\mathscr{H}=\{J_{n,k_1}^1\otimes J_{n,k_2}^2;\ 1\leqslant k_i\leqslant N_n^i,\ i=1, 2,\ n\geqslant 0\}$. We may assume that u is strictly positive on $Q(a, \varrho_2)$ without loss of generality.

LEMMA 3.2. *Set* $v=-\log u$ *and* $v_H=\int_H v\,dm/m(H)$. *Then*

$$\int_H |v-v_H|\,dm\leqslant 2^4\gamma\delta^{1/2}(1-\varrho_3/\varrho_2)^{-1}\mu(a, \varrho_2)^{1/2}m(H)$$

for every $H\in\mathscr{H}$.

Proof. By Lemma 1.1, u is bounded on $Q(a, \varrho_2)$ and hence v belongs to $\mathscr{F}^2(Q(a, \varrho_2);\{v_{ii}\})$. Let $H=J_{n,k_1}^1\times J_{n,k_2}^2\in\mathscr{H}$. Hölder's inequality gives us

$$\begin{aligned}\Big(\int_H|v-v_H|\,dm\Big)^2 &\leqslant m(H)\int_H(v-v_H)^2\,dm=\tfrac{1}{2}\int_H\int_H(v(x)-v(y))^2\,m(dx)\,m(dy)\\ &\leqslant m(H)\sum_i |J_{n,k_i}^i|\,m_i(J_{n,k_i}^i)\int_H (D_i v)^2\,dv_{ii}\\ &\leqslant 4\gamma^2 m(H)\sum_i |J_{n,k_i}^i|\,m_i(J_{n,k_i}^i)\sum_j\int_{Q(a,\varrho_2)}(D_j\varphi)^2\,dv_{jj}\end{aligned}$$

for $\varphi\in C_0^\infty(Q(a, \varrho_2))$ with $\varphi=1$ on H, where the last inequality is shown in the same way as for (1.4).

Choose a $\varphi\in C_0^\infty(Q(a, \varrho_2))$ such that $\varphi=1$ on the set $Q(a, \varrho_3)$, $|D_i\varphi|\leqslant 2/(\varrho_2-\varrho_3)$. Since

$$\sum_i |J_{n,k_i}^i|\,m_i(J_{n,k_i}^i)\leqslant 2\delta m(H)$$

by (0.7.iv), we have

$$\begin{aligned}\Big(\int_H|v-v_H|\,dm\Big)^2 &\leqslant 2^5\gamma^2\delta m(H)^2(\varrho_2-\varrho_3)^{-2}\nu(Q(a, \varrho_2))\\ &\leqslant 2^8\gamma^2\delta m(H)^2(1-\varrho_3/\varrho_2)^{-2}\mu(a, \varrho_2),\end{aligned}$$

q.e.d.

The following lemma is a modification of the one in [2]. For completeness we give the proof at the last section.

LEMMA 3.3. *If there is a constant* C_{12} *such that*

$$\int_H|v-v_H|\,dm\leqslant C_{12}m(H), \qquad H\in\mathscr{H},$$

then

$$\int_H e^{C_{13}t|v-v_H|}\,dm \leqslant \frac{1+2t}{1-t}\,m(H), \qquad H \in \mathscr{H},$$

where $C_{13} = 1/e\delta^2 C_{12}$ *and* $0 < t < 1$.

Since

$$\int_Q u^{\pm C_{13}t}\,dm = \int_Q e^{\pm C_{13}tv\,dm} \leqslant e^{\pm C_{13}tv_Q}\int_Q e^{C_{13}t|v-v_Q|}\,dm,$$

from Lemmas 3.2 and 3.3 we obtain immediately

LEMMA 3.4.

$$\int_Q u^{C_{14}t}\,dm \int_Q u^{-C_{14}t}\,dm \leqslant \left(\frac{1+2t}{1-t}\,m(Q)\right)^2,$$

where $Q = Q(a, \varrho_3)$, $0 < t < 1$, *and* $C_{14} = C'_{14}(\gamma)\delta^{-5/2}(1-\varrho_3/\varrho_2)\,\mu(a, \varrho_2)^{-1/2}$.

LEMMA 3.5. *Set* $Q = Q(a, \varrho_2)$ *and* $w = u^k$ *with* $k \in \boldsymbol{R}^1$. *If* $k \neq 1/2$, *then*

$$\sum_{i=1,2}\int_Q (\varphi D_i w)^2\,dv_{ii} \leqslant \gamma^2\,(2k/(2k-1))^2 \sum_{i=1,2}\int_Q (wD_i\varphi)^2\,dv_{ii}$$

for every $\varphi \in C_0^\infty(Q)$.

Noting that u^{2k-1} belongs to $\mathscr{F}^2(Q; \{v_{ii}\})$ and $\mathscr{E}_Q(u, u^{2k-1}\varphi^2) = 0$, $\varphi \in C_0^\infty(Q)$, we can get above lemma. So we omit the proof.

Proof of Theorem 3.1. We may assume $C_{14} \leqslant 1$. Fix a positive integer N such that $6C_{14}^{-1} \leqslant 3^N \leqslant 18C_{14}^{-1}$. Set $t = 2(3^N C_{14})^{-1}$ and $\alpha = \pm C_{14}\,t = \pm 2/3^N$. Then $9^{-1} \leqslant t \leqslant 3^{-1}$ and $3^n\alpha \neq 1$, $n = 0, 1, 2, \ldots$

Put $w_n = u^{3^n\alpha/2}$, $r_n = \varrho_4 + (\varrho_3-\varrho_4)/(n+1)$, $Q_n = Q(a, r_n)$, $n - 0, 1, 2, \ldots$ For each n we take a function $\varphi_n \in C_0^\infty(Q_n)$ with $\varphi_n = 1$ on Q_{n+1}, $|D_i\varphi_n| \leqslant 2(r_n - r_{n+1})^{-1}$. Appealing to Proposition 1.4 of [7] and Lemma 3.5 we get

$$\Big(\int_{Q_{n+1}} w_n^6\,dm\Big)^{1/3} \leqslant \Big(\int_{Q_n} (\varphi_n w_n)^6\,dm\Big)^{1/3} \leqslant C_{15}\,m(Q_n)^{1/3}\sum_i \int_{Q_n} (D_i(\varphi_n w_n))^2\,dv_{ii}$$
$$\leqslant 4C_{15}\,m(Q_0)^{1/3}\,\{1+\gamma^2(3^n\alpha/(3^n\alpha-1))^2\}(r_n-r_{n+1})^{-2}\int_{Q_n} w_n^2\,dv$$

for some constant $C_{15} = C_{15}(A)$.

For all n, $(3^n\alpha/(3^n\alpha-1))^2 \leqslant 1$ if $\alpha = -C_{14}t$ or $\leqslant 4$ if $\alpha = C_{14}t$, and $r_n - r_{n+1} = (\varrho_3-\varrho_4)/(n+1)(n+2) \geqslant (\varrho_3-\varrho_4)/(n+2)^2$. Therefore we have

$$\Big(\int_{Q_{n+1}} w_n^6\,dm\Big)^{1/3} \leqslant 8A^{-1}C_{15}(1+4\gamma^2)(\varrho_3-\varrho_4)^{-2}\,m(Q_0)^{1/3}(n+2)^4\int_{Q_n} w_n^2\,dm.$$

Putting $U_n = \int_{Q_n} u^{3^n\alpha}\, dm$ leads us to

$$U_n \leqslant (C_{16}(n+1)^4 U_{n-1})^3 \leqslant C_{16}^{3+3^2+\dots+3^n}((n+1)^3 n^{3^2} \dots 2^{3^n})^4 U_0^{3^n},$$

where $C_{16} = 8A^{-1} C_{15}(1+4\gamma^2)(\varrho_3 - \varrho_4)^{-2} m(Q_0)^{1/3}$. Hence

$$m\text{-ess}\max_{Q(a,\varrho_4)} u^\alpha = \lim_{n\uparrow\infty} U_n^{3^{-n}} \leqslant C_{17}(\varrho_3-\varrho_4)^{-3} m(Q_0)^{1/2} \int_{Q(a,\varrho_3)} u^\alpha\, dm$$

for an appropriate $C_{17} = C_{17}(A, \gamma)$. Combining Lemma 3.4 with this and noting $1/9 \leqslant t \leqslant 1/3$, we obtain

$$\begin{aligned} m\text{-ess}\max_{Q(a,\varrho_4)} u &\leqslant \left(C_{17}(1+2t)(1-t)^{-1}(\varrho_3-\varrho_4)^{-3} m(Q_0)^{3/2}\right)^{2/C_{14}t} m\text{-ess}\min_{Q(a,\varrho_4)} u \\ &\leqslant \left(3C_{17}(\varrho_3-\varrho_4)^{-3} m(Q_0)^{3/2}\right)^{18/C_{14}} m\text{-ess}\min_{Q(a,\varrho_4)} u \\ &\leqslant \left(24C_{17}(1-\varrho_4/\varrho_3)^{-3}\mu(a,\varrho_3)^3\right)^{18/C_{14}} m\text{-ess}\min_{Q(a,\varrho_4)} u, \end{aligned}$$

which proves the theorem, q.e.d.

COROLLARY 3.6. *Let E be a subdomain of Ω and u a continuous solution of* $[\mathscr{E}_E, \mathscr{F}^2(E; \{v_{ii}\})]$. *If u attains the maximum (minimum) inside E, then u is constant.*

Proof. Assume

$$u(a) = \max_E u = \mu.$$

Choose a compact set $K(\subset E)$ containing a. For any $\varepsilon > 0$ let $v = \mu - u + \varepsilon$; then v is a positive solution of $[\mathscr{E}_E, \mathscr{F}^2(E; \{v_{ii}\})]$ and

$$\min_K v = \varepsilon.$$

Therefore

$$\max_K (\mu - u + \varepsilon) \leqslant C_{18}\,\varepsilon, \qquad \text{that is,} \qquad \mu \leqslant \varepsilon(C_{18}-1) + \min_K u.$$

Letting $\varepsilon \downarrow 0$, we get

$$\min_K u = \mu.$$

Since K is arbitrary, we have $u = \mu$ inside E, q.e.d.

Now we exhibit examples satisfying our assumptions.

PROPOSITION 3.7. *Let m_i, $i = 1, 2$, be continuous measures and assume* (0.1) *and* (0.2). *Then* (0.7) *follows automatically.*

Remark. Measures m_i's are not necessarily absolutely continuous with respect to dt.

Proof. Let $I^i = (a^i, b^i)$, $i = 1, 2$. By the continuity of m_i, for each i, n there are the points

$$a^i = c^i_{n,0} < c^i_{n,1} < \ldots < c^i_{n,2^n} = b^i$$

such that $m_i((c^i_{n,k-1}, c^i_{n,k})) = 2^{-n} m_i(I^i)$, $1 \leqslant k \leqslant 2^n$. Put $I^i_{n,k} = [c^i_{n,k-1}, c^i_{n,k}) - \{a^i\}$, $1 \leqslant k \leqslant 2^n$; (0.7) is satisfied by the following:

$$\Delta^i_n\colon I^i = \bigcup_{1 \leqslant k \leqslant N^i_n} I^i_{n,k}, \quad N^i_n = 2^n, \quad n \geqslant 0, \quad i = 1, 2,$$

$$\delta = 2 \vee [A^{-1} \max_{1 \leqslant i,j \leqslant 2} m_i(I^i)/m_j(I^j)],$$

q.e.d.

PROPOSITION 3.8. *Let*

$$m_i(dt) = m^c_i(dt) + \sum_{l=1}^{\infty} \alpha^i_l \delta_{\{d^i_l\}}(dt), \quad \alpha^i_l \geqslant 0, \quad \sum_{l \geqslant 1} \alpha^i_l < \infty, \quad i = 1, 2,$$

where each m^c_i is a continuous measure.

Assume (0.1) *and* (0.2). *Then* (0.7) *follows if for given intervals I^i, $i = 1, 2$, there is a positive number R such that*

$$(3.1) \qquad \Big(\sum_{l=1}^{\infty} \alpha^i_l\Big) \vee \Big(2^s \sum_{l \in L^i_{s,k}} \alpha^i_l\Big) \leqslant R, \quad i = 1, 2, \quad 1 \leqslant k \leqslant 2^s, \quad s = 1, 2, \ldots,$$

where $L^i_{s,k} = \{l\colon d^i_l \in I^i_{s,k},\ \alpha^i_l < 2^{-s+1} m^c_i(I^i)\}$ and $\{I^i_{s,k}\colon 1 \leqslant k \leqslant 2^s\}$ is a partition of I^i with $m^c_i(I^i_{s,k}) = 2^{-s} m^c_i(I^i)$, $1 \leqslant k \leqslant 2^s$, as mentioned above.

Remark. (3.1) is satisfied if one of the following holds for $i = 1$ or 2:

$$(3.2) \qquad \# \{l\colon \alpha^i_l > 0\} < \infty;$$

$$(3.3) \qquad \sup_{s \geqslant 1} \# \{l\colon \alpha^s \leqslant \alpha^i_l < \alpha^{s-1}\} < \infty \quad \text{for an } \alpha \in (0, 1);$$

$$(3.4) \qquad m^c_i(dt) = dt, \quad \alpha^i_l = l^{-\beta}, \quad d^i_l = l^{-\beta+1} \in I^i = (0, 1), \quad l \geqslant 2, \text{ with } \beta > 1.$$

Indeed, (3.1) is trivial in case of (3.2). If (3.3) holds,

$$\varkappa = \sup_{s \geqslant 1} \# \{l\colon 2^{-s} m^c_i(I^i) \leqslant \alpha^i_l < 2^{-s+1} m^c_i(I^i)\} < \infty,$$

whence

$$\sum_{l \in L^i_{s,k}} \alpha^i_l \leqslant \varkappa m^c_i(I^i) \sum_{t=s}^{\infty} 2^{-t+1} = 4\varkappa m^c_i(I^i) 2^{-s},$$

which implies (3.1).

If (3.4) is satisfied, then

$$I^i_{s,k} = [(k-1)2^{-s},\ k2^{-s}) - \{0\}, \qquad 1 \leqslant k \leqslant 2^s, \quad s \geqslant 1,$$

$$\sum_{l \in L^i_{s,1}} \alpha^i_l \leqslant \sum_{l > 2^{s/(\beta-1)}} l^{-\beta} \leqslant 2(\beta-1)^{-1}(2^{1/(\beta-1)}-1)^{-\beta+1}2^{-s},$$

$$\sum_{l \in L^i_{s,k}} \alpha^i_l \leqslant (\beta-1)^{-1}\{(k^{-1/(\beta-1)} - 2^{-s/(\beta-1)})^{-\beta+1} - \\ -((k-1)^{-1/(\beta-1)} + 2^{-s/(\beta-1)})^{-\beta+1}\}2^{-s}$$

$$\leqslant C_{19}2^{-s}, \qquad 2 \leqslant k \leqslant [2^{(s+\beta-1)/\beta}], \quad s \geqslant 2,$$

$$\sum_{l \in L^i_{s,k}} \alpha^i_l \leqslant (\beta-1)^{-1}\{2^s(2^{(s-1)/\beta}-1)^{-\beta+1} - \\ -((k-1)^{-1/(\beta-1)} + 2^{-s/(\beta-1)})^{-\beta+1}\}2^{-s}$$

$$\leqslant C_{20}2^{-s}, \qquad [2^{(s+\beta-1)/\beta}]+1 \leqslant k \leqslant 2^s, \quad s \geqslant 2,$$

where C_{19}, C_{20} depend only on β and $[a]$ denotes the smallest integer not exceeding a. Therefore (3.1) follows.

Proof of Proposition 3.8. Let

$$L^i_0 = \{l\colon\ \alpha^i_l \geqslant m^c_i(I^i)\}, \quad L^i_s = \{l\colon\ 2^{-s}m^c_i(I^i) \leqslant \alpha^i_l < 2^{-s+1}m^c_i(I^i)\}, \quad s = 1, 2, \ldots,$$

$$L^i_s = \{l^i_{s,1}, \ldots, l^i_{s,\varkappa^i_s}\}, \qquad \varkappa^i_s = \# L^i_s,$$

$$n_{-1} = 0, \qquad n_s = \sum_{j=0}^{s} \varkappa^1_j \vee \varkappa^2_j + s + 1,$$

$$d^i(s, j) = d^i_{l^i_{s,j}}, \qquad 1 \leqslant j \leqslant \varkappa^i_s, \quad s \geqslant 0.$$

We define sequences of partitions $\{\Delta^i_n\}_{n=0}^{\infty}$, $i = 1, 2$, as follows. Set

$$\Delta^i_n\colon\ I^i = \bigcup_{1 \leqslant k \leqslant N^i_n} J^i_{n,k}, \qquad \text{where } N^i_0 = 1.$$

(3.5) Case of $n_{s-1}+1 \leqslant n \leqslant n_s - 1$, $s = 0, 1, 2, \ldots$:

$$N^i_n = n \wedge \Big(\sum_{j=0}^{s} \varkappa^i_j\Big) + 2^s,$$

$$J^i_{n,k} = \begin{cases} \{d^i(t, k - \sum_{j=0}^{t-1} \varkappa^i_j)\} & \text{if } \ \sum_{j=0}^{t-1} \varkappa^i_j < k \leqslant \sum_{j=0}^{t} \varkappa^i_j, \quad 0 \leqslant t \leqslant s-1; \\ \{d^i(s, k - \sum_{j=0}^{s-1} \varkappa^i_j)\} & \text{if } \ \sum_{j=0}^{s-1} \varkappa^i_j < k \leqslant N^i_n - 2^s, \\ I^i_{s,k-N^i_n+2^s} - \bigcup_{1 \leqslant k \leqslant N^i_n - 2^s} J^i_{n,k} & \text{if } \ N^i_n - 2^s < k \leqslant N^i_n, \end{cases}$$

where $I^i_{s,k}$'s are intervals mentioned in the proof of Proposition 3.8, that is, $\{I^i_{s,k}\colon 1 \leqslant k \leqslant 2^s\}$ is a partition of I^i and $m^c_i(I^i_{s,k}) = 2^{-s} m^c_i(I^i)$.

(3.6) Case of $n = n_s$, $s = 0, 1, 2, \ldots$:

$$N^i_n = N^i_{n-1} + 2^s = \sum_{j=0}^{s} \varkappa^i_j + 2^{s+1},$$

$$J^i_{n,k} = \begin{cases} J^i_{n-1,k} = \{d^i(t, k - \sum_{j=0}^{t-1} \varkappa^i_j)\} & \text{if } \sum_{j=0}^{t-1} \varkappa^i_j < k \leqslant \sum_{j=0}^{t} \varkappa^i_j, \quad 0 \leqslant t \leqslant s, \\ I^i_{s+1,k-N^i_{n-1}} - \bigcup_{1 \leqslant k \leqslant N^i_{n-1}} J^i_{n-1,k} & \text{if } N^i_{n-1} - 2^s < k \leqslant N^i_n. \end{cases}$$

Obviously $N^i_n \to \infty$ $(n \to \infty)$ and (0.7.i)-(0.7.iii) are fulfiled. We prove that there is a positive number δ satisfying (0.7.iv) and (0.7.v).

By (0.1) and the definitions of $I^i_{s,k}$ and L^i_s we have

$$|J^i_{n,k}| \leqslant A^{-1} m^c_i(J^i_{n,k}) \leqslant A^{-1} 2^{-s} m^c_i(I^i) \leqslant A^{-1} m_i(J^i_{n,k}), \qquad 1 \leqslant k \leqslant N^i_n,$$

in case (3.5). By the same reason

$$|J^i_{n,k}| \leqslant A^{-1} 2^{-s-1} m^c_i(I^i) \leqslant A^{-1} m_i(J^i_{n,k}), \qquad 1 \leqslant k \leqslant N^i_n,$$

in case (3.6). Therefore

$$0 \leqslant |J^i_{n,k}|/m_j(J^j_{n,l}) \leqslant A^{-1} \max_{1 \leqslant i,j \leqslant 2} m^c_i(I^i)/m^c_j(I^j)$$

for $i, j = 1, 2,$ $\quad 1 \leqslant k,\ l \leqslant N^i_n,\ n \geqslant 0$.

This means that (0.7.iv) is satisfied if

$$\delta \geqslant A^{-1} \max_{i,j} m^c_i(I^i)/m^c_j(I^j).$$

In order to show (0.7.v) let

$$J^i_{n,k} = \bigcup_{p \in P^i_{n,k}} J^i_{n+1,p}$$

as in (0.7). Clearly $P^i_{n,k}$ is a single-point set or a two-points set. The latter case occurs if and only if either of the following is satisfied:

(3.7) $$n_{s-1} \leqslant n \leqslant n_s - 2, \quad J^i_{n,k} \supsetneqq J^i_{n,k} \cap \{d^i_l\colon l \in L^i_s\} \neq \emptyset,$$

(3.8) $$n = n_s - 1, \quad J^i_{n,k} \text{ is not a single-point.}$$

In case of (3.7), putting $P^i_{n,k} = \{p, q\}$, $p < q$, we actually see that

$$J^i_{n+1,p} = \{d^i_l\}, \quad J^i_{n+1,q} = I^i_{s,k} - \bigcup_{1 \leqslant k \leqslant N^i_{n+1} - 2^s} J^i_{n+1,k}$$

for some $l \in L_s^i$, $1 \leqslant k \leqslant 2^s$, and hence, in case of $s=0$, we have

$$m_i^c(I^i) \leqslant m_i(J_{n+1,p}^i) \leqslant \max_{l \in L_0^i} \alpha_l^i \leqslant R,$$

$$m_i^c(I^i) \leqslant m_i(J_{n+1,q}^i) \leqslant m_i^c(I^i) + \sum_{l=1}^{\infty} \alpha_l^i \leqslant m_i^c(I^i) + R;$$

for $s \geqslant 1$ we have

$$2^{-s} m_i^c(I^i) \leqslant m_i(J_{n+1,p}^i) < 2^{-s+1} m_i^c(I^i),$$

$$2^{-s} m_i^c(I^i) = m_i^c(J_{n+1,q}^i) \leqslant m_i(J_{n+1,q}^i) \leqslant m_i^c(J_{n+1,q}^i) + \sum_{l \in L_{s,k}^i}' \alpha_l^i \leqslant 2^{-s} m_i^c(I^i) + R \cdot 2^{-s}.$$

Therefore

$$m_i(J_{n,k}^i)/\{m_i(J_{n+1,p}^i) \wedge m_i(J_{n+1,q}^i)\} \leqslant 3 \vee \{2 + R/m_i^c(I^i)\}.$$

It is obvious that $m_i(J_{n,k}^i)/m_i(J_{n+1,p}^i) = 2$, $p \in P_{n,k}^i$ in case of (3.8). We thus take

$$\delta = \max\{A^{-1} \max_{1 \leqslant i,j \leqslant 2} m_i^c(I^i)/m_j^c(I^j),\ 3,\ 2 + R/\{m_1^c(I^1) \wedge m_2^c(I^2)\}\},$$

q.e.d.

4. APPLICATIONS TO DIFFUSION PROCESSES

Let Ω be a bounded rectangle whose faces are parallel to coordinate axes. We consider the Dirichlet space relative to $L^2(\Omega; n)$ such that $\mathscr{F} = \mathscr{F}_0^2(\Omega; \{v_{ii}\})$ and $\mathscr{E}(u, v) = 0$ if $u = \text{const}$ on $\text{Supp}[v]$. Note that $\mathscr{E}$ is given by (0.9). Further we assume (5.4) in [7]. Namely,

(4.1) $(\mathscr{F}, \mathscr{E})$ can be extended to a Dirichlet space $(\tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ relative to $L^2(\boldsymbol{R}^d; n)$ which is given by (0.9) with $\boldsymbol{R}^d$, $\tilde{a}_{ij}$ in place of Ω, a_{ij}, respectively, where $\tilde{a}_{ij}$, $1 \leqslant i, j \leqslant d$, are measurable, satisfy (0.4) and (0.5) for $\Omega = \boldsymbol{R}^d$ and for another positive constant $\tilde{\gamma}$, and $\tilde{a}_{ij} = a_{ij}$ on Ω. Moreover, for any bounded domain E, $f \in C_0^\infty(\bar{E})$ and $\lambda \geqslant 0$, every solution of $[\tilde{\mathscr{E}}_\lambda, f, E]$ belongs to $C(E)$. (See [7] for significations of above symbols.)

Let $\{G_\lambda: \lambda \geqslant 0\}$ be the resolvent associated with $(\mathscr{F}, \mathscr{E})$: $\mathscr{E}_\lambda(G_\lambda f, \varphi) = (f, \varphi)$, $f \in L^2(\Omega; n)$, $\varphi \in \mathscr{F}$. Under assumptions (0.1), (0.2) and (4.1) we have a unique diffusion process $X = [x_t, \zeta, P_x]$, $x \in \Omega$, such that

$$G_\lambda f(x) = E_x\Big[\int_0^\zeta e^{-\lambda t} f(x_t)\,dt\Big], \qquad \lambda \geqslant 0, \qquad f \in C_\infty(\Omega)$$

(cf. [7], Theorem 5.1). This diffusion X has the resolvent density $g_\lambda(x, y)$ such that for each $\lambda \geqslant 0$

(4.2) $$G_\lambda f(x) = \int_\Omega g_\lambda(x, y) f(y)\, n(dy), \qquad x \in \Omega, \quad f \in \mathscr{B}_b,$$

(4.3) $$g_\lambda(x, \cdot) = g_\lambda(\cdot, x) \in C(\Omega - \{x\}), \qquad x \in \Omega$$

(cf. [7], Theorems 5.6 and 5.8). Moreover, in view of [7], (5.22), and Corollary 1.3, we find

(4.4) $$g_\lambda(x, y) > 0, \qquad x, y \in \Omega, \qquad x \neq y.$$

From now on we write $g(x, y)$ instead of $g_0(x, y)$.

For an open set $E \subset \Omega$ let

$$\mathscr{L}_E = \{u \in \mathscr{F}\colon\ u \geqslant 1 \ \ n\text{-a.e. on } E\},$$

$$\operatorname{Cap}(E) = \begin{cases} \inf\limits_{u \in \mathscr{L}_E} \mathscr{E}(u, u) & \text{if } \mathscr{L}_E \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

For any set $E \subset \Omega$ let

$$\operatorname{Cap}(E) = \inf \{\operatorname{Cap}(G)\colon\ G \text{ is open, } G \supset E\}.$$

σ_E stands for the hitting time for E of X: $\sigma_E = \inf\{t > 0\colon\ x_t \in E\}$.

For $1 \leqslant i \leqslant d$, $a \in \mathbf{R}^1$, put

$$U_a^i(t) = \int_0^t m_i\big((a-s, a+s)\big)\, ds$$

and denote by Φ_a^i the inverse function of $t \mapsto U_a^i(t)$.

THEOREM 4.1. *Assume* (0.1), (0.2) *and* (4.1). *Then for a given point* $a = (a_1, \ldots, a_d) \in \Omega$, *the following* (i)-(iv) *are equivalent*:

(i) $\operatorname{Cap}(\{a\}) > 0$,

(ii) $\bar{g}(a, a) \equiv \lim\limits_{\varrho \downarrow 0} \max\limits_{|x-a| = \varrho} g(x, a) < \infty$,

(iii) $P_x(\sigma_{\{a\}} < \zeta) > 0, \quad x \in \Omega$,

(iv) $\int_0^r \Big\{\prod_{i=1}^d m_i\big((a_i - \Phi_{a_i}^i(t),\ a_i + \Phi_{a_i}^i(t))\big)\Big\}^{-1} dt$ *converges for some* $r > 0$.

Further, $\bar{g}(a, a) = 1/\operatorname{Cap}(\{a\})$ *whenever one of conditions* (i)-(iv) *is satisfied.*

Proof. By virtue of (4.4) and [7], Theorems 5.5 and 5.10, we obtain the equivalences: (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Let $B(a, \varrho)$ be a closed ball with center a and radius ϱ, where $0 < \varrho < \operatorname{dist}(a, \partial\Omega)$. It follows from [7], Theorem 5.10, that

(4.5) $$\min_{|x-a| = \varrho} g(a, x) \leqslant 1/\operatorname{Cap}\big(B(a, \varrho)\big) \leqslant \max_{|x-a| = \varrho} g(a, x),$$

which yields $1/\mathrm{Cap}(\{a\}) \leqslant \bar{g}(a, a)$ and the implication (ii) $\Rightarrow$ (i). Conversely, when (i) is satisfied, by using [7], (5.27), we see that

$$g(x, a)\,\mathrm{Cap}(\{a\}) = P_x(\sigma_{\{a\}} < \zeta) \leqslant 1, \quad x \in \Omega,$$

and hence we get (ii) as well as $\bar{g}(a, a) = 1/\mathrm{Cap}(\{a\})$, q.e.d.

We proceed our arguments under condition (0.7). Given two data $\{a_{ij}^{(1)}\}$, $\{a_{ij}^{(2)}\}$ satisfying (0.4) and (0.5), let $X^{(1)}$, $X^{(2)}$ be the corresponding diffusions as above. For each i we mark the characteristics of $X^{(i)}$ with (i): $g^{(i)}(x, y)$, $\mathrm{Cap}^{(i)}(E)$, etc.

THEOREM 4.2. *Assume* (0.1), (0.2), (0.7) *and* (4.1). *For any* $a \in \Omega$ *and any* $0 < \varrho < \mathrm{dist}(a, \partial\Omega)/2$, *there is a positive constant* C_{21} *such that*

$$C_{21}^{-1} g^{(2)}(a, b) \leqslant g^{(1)}(a, b) \leqslant C_{21} g^{(2)}(a, b), \quad |a-b| = \varrho.$$

Remark. (i) In the case of $d \geqslant 3$, mentioned at the beginning of Section 3, the same inequality is given by [5], Theorem 8.5.

(ii) C_{21} depends on A, γ and the behavior of m near $Q(a, \varrho)$.

(iii) Let $a_{ij}^{(2)} = \delta_{ij}/2$. Then $X^{(2)}$ is no other than a direct product diffusion (cf. [7]). Therefore $g^{(1)}(x, y)$ can be compared with that of the direct product diffusion.

Proof. By Theorem 3 we have

$$\max_{|x-a|=\varrho} g^{(i)}(x, a) \leqslant C_{22} \min_{|x-a|=\varrho} g^{(i)}(x, a), \quad i = 1, 2.$$

On the other hand, noting (0.5) and (0.9), we find

$$\gamma^{-2}\,\mathrm{Cap}^{(2)}(B(a, \varrho)) \leqslant \mathrm{Cap}^{(1)}(B(a, \varrho)) \leqslant \gamma^2\,\mathrm{Cap}^{(2)}(B(a, \varrho)).$$

Combining these with (4.5) shows the conclution, q.e.d.

Finally we observe the following examples.

Example 4.3. Let m_1 be a nonnegative measure on $\boldsymbol{R}^1$ with (0.1) and (0.2), and $m_i(dt) = dt$, $2 \leqslant i \leqslant d$. Let $n(dx) = dx$ and consider a Dirichlet space $(\mathscr{F}, \mathscr{E})$ as above. We assume on (0.9) expressing $\mathscr{E}$ that each a_{ij} belongs to C^{d_0}-class, where $d_0 = [(d-1)/2]+1$. (Of course (0.4) and (0.5) are fulfiled.) Then (4.1) is satisfied as it is claimed in [6].

Thus we have a unique diffusion process on Ω associated with $(\mathscr{F}, \mathscr{E})$, whose resolvent is continuous and whose resolvent density has properties (4.2)-(4.4). Moreover, $\mathrm{Cap}(\{a\}) > 0$ if and only if

$$\int_0^r U_{a_1}^1(t)^{-(d-1)/2}\,dt < \infty$$

for some $r > 0$ or, equivalently, $d = 2$ and

$$\int_0^r U_{a_1}^1(t)^{-1/2}\,dt < \infty \qquad \text{for some } r > 0.$$

Further, if m_1 satisfies the condition of Proposition 3.8, then (0.7) follows. In this case let $X^{(1)}$, $X^{(2)}$ be the diffusions corresponding to $\{a_{ij}^{(1)}\}$, $\{a_{ij}^{(2)}\}$ of C^1-class with (0.4) and (0.5). Then we have

$$\max_{\substack{1 \leqslant i,j \leqslant 2 \\ x:|a-x|=\varrho}} g^{(i)}(a, x)/g^{(j)}(a, x) \leqslant \left(C_{23}\,\mu(a, \varrho)\right)^{C_{24}\delta(Q(a,2\varrho))^{5/2}\mu(a,\varrho)^{1/2}},$$

where C_{23}, C_{24} are positive constants depending only on A and γ.

Example 4.4. Suppose (0.1), (0.2) and (0.6). In view of Theorem 2.2, we then get that $G_\lambda f$ belongs to $C_0(\Omega)$ for every bounded function f and $\lambda \geqslant 0$, where $C_0(\Omega)$ denotes the set of all continuous functions vanishing on $\partial\Omega$. Therefore we obtain the diffusion process uniquely associated with $(\mathscr{F}, \mathscr{E})$, whose resolvent density $g_\lambda(x, y)$ possesses (4.2)-(4.4) and, moreover,

$$\lim_{y \to a} g_\lambda(x, y) = 0 \qquad \text{for } x \in \Omega,\ a \in \partial\Omega.$$

Since the integral in (iv) of Theorem 4.1 always diverges in this case, $\mathrm{Cap}(\{a\}) = 0$, $a \in \Omega$. Each m_i is a continuous measure and hence (0.7) is satisfied in case of $d = 2$. Writing

$$\delta(Q(a, \varrho)) = 2 \vee A^{-1} \max_{1 \leqslant i,j \leqslant 2} m_i((a_i - \varrho,\ a_i + \varrho))/m_j((a_j - \varrho,\ a_j + \varrho)) \leqslant C_{25} \log|\log \varrho|$$

for some $C_{25} = C_{25}(A)$, we find that for the diffusions $X^{(k)}$, $k = 1, 2$, corresponding to $\{a_{ij}^{(k)}\}$ with (0.4) and (0.5),

$$\max_{\substack{1 \leqslant i,j \leqslant 2 \\ |x-y|=\varrho}} g^{(i)}(x, y)/g^{(j)}(x, y) \leqslant (C_{26} \log|\log \varrho|)^{C_{27}(\log|\log\varrho|)^3},$$

where C_{26}, C_{27} depend only on A, γ.

5. PROOF OF LEMMA 3.3

It suffices to show the following

PROPOSITION. *Under the same assumption as in Lemma* 3.3

$$m(\{x \in H:\ |v(x) - v_H| > \sigma\}) \leqslant 3e^{-C_{13}\sigma} m(H), \qquad \sigma > 0, \quad H \in \mathscr{H}.$$

In fact, putting $f(s) = \exp(C_{13}\,ts)$ and $\mu(s) = m(\{x \in H:\ |v(x) - v_H| > s\})$, we find

$$\int_H e^{C_{13}t|v-v_H|}\,dm = \int_H f(|v-v_H|)\,dm = C_{13}\,t\int_0^\infty \mu(s)\,e^{C_{13}ts}\,ds+\mu(0)$$

$$\leqslant \left\{3C_{13}\,t\int_0^\infty e^{-C_{13}(1-t)s}\,ds+1\right\}m(H) = \frac{1+2t}{1-t}\,m(H),$$

which proves Lemma 3.3.

Now we give

Proof of Proposition. We divide it into three steps.

Step 1. Let w be a function satisfying

$$\int_H |w|\,dm \leqslant sm(H)$$

for some $s>0$ and some $H\in\mathscr{H}$. Then we have a sequence $\{H_k\}_{k=1}^\infty$ such that

(i) $\qquad$ for each k, $H_k \subsetneqq H$ and $H_k\in\mathscr{H}$ whenever $H_k\neq\emptyset$,

(ii) $\qquad H_k\cap H_l=\emptyset \quad$ if $k\neq l$,

(iii) $\qquad |w(x)|\leqslant s\;\; m\text{-a.e. } x\in H-\bigcup_k H_k,$

(iv) $\qquad \int_{H_k}|w|\,dm\leqslant \delta^2 sm(H_k),\quad k\geqslant 1,$

(v) $$\sum_k m(H_k)\leqslant \frac{1}{s}\int_H |w|\,dm.$$

Indeed, we may set $H=J^1_{n_0,k_{01}}\times J^2_{n_0,k_{02}}$ with some $n_0\geqslant 0$, $1\leqslant k_{0i}\leqslant N^i_{n_0}$, $i=1, 2$. Let

$$\mathscr{H}_1=\{J^1_{n_0+1,k_1}\times J^2_{n_0+1,k_2}\subset H\colon\ 1\leqslant k_i\leqslant N^i_{n_0+1},\ i=1, 2\},$$

$$\mathscr{H}_1^0=\{H'\in\mathscr{H}_1\colon\ H'\subsetneqq H,\ sm(H')\leqslant \int_{H'}|w|\,dm\}.$$

Since

$$\mathscr{H}_1=\{J^1_{n_0+1,p_1}\times J^2_{n_0+1,p_2}\colon\ p_i\in P^i_{n_0,k_{0i}},\ i=1, 2\},$$

we have by means of (0.7v)

$$sm(H')\leqslant \int_{H'}|w|\,dm\leqslant \int_H |w|\,dm\leqslant sm(H)\leqslant \delta^2 sm(H'),\qquad H'\in\mathscr{H}_1. \tag{5.1}$$

Next let

$$\mathscr{H}_2 = \bigcup_{H' \in \mathscr{H}_1 - \mathscr{H}_1^0} \{J^1_{n_0+2,k_1} \times J^2_{n_0+2,k_2} \subset H' \colon 1 \leqslant k_i \leqslant N^i_{n_0+2},\ i = 1, 2\},$$

$$\mathscr{H}_2^0 = \bigcup_{H' \in \mathscr{H}_1 - \mathscr{H}_1^0} \{H'' \in \mathscr{H}_2 \colon H'' \subsetneqq H',\ sm(H'') \leqslant \int_{H''} |w|\, dm\}.$$

Then

$$sm(H'') \leqslant \int_{H''} |w|\, dm \leqslant \delta^2 sm(H''), \qquad H'' \in \mathscr{H}_2. \tag{5.2}$$

Repeating this argument, we obtain a denumerable set $\mathscr{H}_1^0 \cup \mathscr{H}_2^0 \cup \ldots$ whose elements are denoted by $H_1, H_2, \ldots$ Then properties (i)-(ii) are trivial. In the same way as (5.1) and (5.2), we see

$$sm(H_k) \leqslant \int_{H_k} |w|\, dm \leqslant \delta^2 sm(H_k), \qquad k \geqslant 1.$$

Hence (iv) follows and

$$\sum_k m(H_k) \leqslant \frac{1}{s} \sum_k \int_{H_k} |w|\, dm \leqslant \frac{1}{s} \int_H |w|\, dm,$$

which is (v). Further, by virture of (0.7.iii) for any $x \in H - \bigcup_k H_k$ we have a sequence $\{H_n'''\}_{n \geqslant 1}$ such that

$$\bigcap_{n \geqslant 1} H_n''' = \{x\}, \qquad H_n''' = J^1_{n_0+n,k_{n1}} \times J^2_{n_0+n,k_{n2}} \subset H$$

with some $1 \leqslant k_{ni} \leqslant N^i_{n_0+n}$, $i = 1, 2$, and

$$\int_{H_n'''} |w|\, dm < sm(H_n''').$$

Therefore (iii) is valid.

Step 2. Let $F(\sigma)$ be the supremum of

$$m(\{x \in H \colon |u - u_H| > \sigma\}) / \int_H |u - u_H|\, dm$$

which is taken over the set of all $H \in \mathscr{H}$ and all u such that

$$\int_H |u - u_H|\, dm \leqslant m(H), \qquad H \in \mathscr{H}, \tag{5.3}$$

where we put $0/0 = 0$. If $\sigma \geqslant \delta^2$, then

$$F(\sigma) \leqslant s^{-1} F(\sigma - \delta^2 s), \qquad 1 \leqslant s \leqslant \delta^{-2} \sigma. \tag{5.4}$$

Indeed, fix a set $H \in \mathscr{H}$ and a function u satisfying (5.3). Let $w = u - u_H$; then w fulfils (5.3), $w_H = 0$ and

$$\int_H |w|\, dm \leqslant m(H) \leqslant sm(H).$$

Since by the last inequality there is a sequence $\{H_k\}_{k \geqslant 1}$ with (i)-(v) in Step 1, we see

$$\begin{aligned} &m(\{x \in H\colon |w(x)| > \sigma\}) \leqslant \sum_k m(\{x \in H_k\colon |w(x)| > \sigma\}) \\ &\leqslant \sum_k m(\{x \in H_k\colon |w(x) - w_{H_k}| > \sigma - \delta^2 s\}) \leqslant F(\sigma - \delta^2 s) \sum_k \int_{H_k} |w - w_{H_k}|\, dm \\ &\leqslant F(\sigma - \delta^2 s) \sum_k m(H_k) \leqslant s^{-1} F(\sigma - \delta^2 s) \int_H |w|\, dm. \end{aligned}$$

Since H and u are arbitrary, we get (5.4).

Step 3. Let $\alpha = 1/\delta^2 e$ and $\beta = \delta^2 e/(e-1)$. An easy calculation shows

$$F(\sigma) \leqslant \sigma^{-1} \leqslant \beta^{-1} \leqslant e^{-(\beta + \delta^2 e)} \leqslant e^{-\alpha\sigma}, \qquad \beta \leqslant \sigma \leqslant \beta + \delta^2 e.$$

Hence, by (5.4) with $s = e$,

$$F(\sigma) \leqslant e^{-1} F(\sigma - \delta^2 e) \leqslant e^{-\alpha\sigma}, \qquad \beta + \delta^2 e \leqslant \sigma \leqslant \beta + 2\delta^2 e.$$

Iteration of this argument leads us to $F(\sigma) \leqslant \exp(-\alpha\sigma)$ for $\beta \leqslant \sigma$. Setting $u = v/C_{12}$, we find

$$m(\{x \in H\colon |v - v_H| > \sigma\}) \leqslant F(\sigma/C_{12}) C_{12}^{-1} \int_H |v - v_H|\, dm \leqslant e^{-\alpha\sigma/C_{12}} m(H)$$

for $H \in \mathscr{H}$, $\sigma \geqslant C_{12}\beta$. Obviously,

$$m(\{x \in H\colon |v - v_H| > \sigma\}) \leqslant m(H) \leqslant 3e^{-\alpha\sigma/C_{12}} m(H), \qquad 0 < \sigma \leqslant C_{12}\beta.$$

Thus we complete the proof, q.e.d.

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