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HOW TO SOLVE THE INEQUALITY $U_t m \leqslant m$ FOR EVERY t $(0 < t < 1)$?

BY

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Abstract. Let $\{U_t\colon 0 < t < 1\}$ be a semi-group of measurable transformations on a measurable space $(X, \mathscr{M})$. In this paper we characterize a σ-finite measures m on $\mathscr{M}$ satisfying the inequality $U_t m \leqslant m$ for every t $(0 < t < 1)$. Some applications are given for operator-selfdecomposable, V-decomposable, s-selfdecomposable, and multiply s-selfdecomposable measures.

1. Introduction. In the theory of limit distributions, for partial sums of sequences of X-valued independent random elements, we often meet the problem of finding a general form of measures m satisfying the inequality

$$U_t m \leqslant m \quad \text{for every } t \ (0 < t < 1).$$

More precisely, we ask about a structure of measures m, on a measurable space $(X, \mathscr{M})$, such that for arbitrary $A \in \mathscr{M}$ and arbitrary t $(0 < t < 1)$

$$m(\{x \in X\colon U_t x \in A\}) \leqslant m(A),$$

where $\{U_t\colon 0 < t < 1\}$ is a semi-group of measurable transformations on X. In the case where X can be considered as a compact topological space, the U_t's preserve the weak convergence of measures, and the m's are Borel probability measures on X, i.e. $m \in \mathscr{P}(X)$, K. Urbanik noticed that it is more convenient to investigate the algebraic structure of the set

$$C := \{m \in \mathscr{P}(X)\colon U_t m \leqslant m \text{ for } 0 < t < 1\},$$

which is convex and compact. Namely, we look for extreme points of C (they are usually concentrated on the orbits $\{U_t x\colon 0 < t < \infty\}$, $x \in X$), and then using Choquet's theorem we represent each element of C as a barycenter of the extreme points. This Urbanik's approach was applied by himself in [18]-

[24], by Jajte in [2], Jurek in [3], [5], Kucharczak in [9], Kumar and Schreiber in [10]-[12], and by Nguyen Van Thu in [13], [14]. The aim of this paper is to solve the inequality in question in a more general situation and in a more direct way. Our considerations are analogous to those in [6] where m has been a generalized Poisson exponent and $\{U_t\colon 0 < t < \infty\}$ a one-parameter strongly continuous group of bounded linear operators on a Banach space (cf. also [17]). In Section 3 some applications are given for the classes of operator-selfdecomposable, V-decomposable, s-selfdecomposable, and multiply s-decomposable probability measures. By our results we are able to obtain a new form of the known descriptions of these classes in terms of the characteristic functionals. But the contents of subsection (e), concerning the multiply s-selfdecomposability, is fairly new.

2. Main results. Let $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ be measurable spaces. If m is a measure on $\mathscr{M}$ and f is a measurable function from X into Y, then by fm we denote a measure on $\mathscr{N}$ defined by the formula

$$(fm)(A) := m\big(f^{-1}(A)\big) = m\big(\{x \in X\colon f(x) \in A\}\big).$$

In the sequel we assume that $U := \{U_t\colon t \in \boldsymbol{R}^+\}$ is a semi-group of measurable transformations on X, i.e. $U_t\colon X \to X$ is an $\mathscr{M}$-measurable mapping and $U_t(U_s x) = U_{ts} x$ for each $s, t \in \boldsymbol{R}^+$ and $x \in X$. $\boldsymbol{R}^+$ denotes the set of strictly positive real numbers. Further, assume that a measure m on $\mathscr{M}$ satisfies the following conditions:

(I) *There exist $X_0 \in \mathscr{M}$ and a measurable function $f\colon X_0 \to \boldsymbol{R}^+$ such that*

$$m(X \setminus X_0) = 0 \quad \text{and} \quad 0 < \int_{X_0} f\,dm < \infty.$$

(II) *For every t $(0 < t < 1)$ and every $A \in \mathscr{M}_0 := \mathscr{M} \cap X_0$,*

$$(U_t m)(A) \leqslant m(A).$$

(III) *X_0 admits polar coordinates with respect to U, i.e. there exist a measurable space $(Z, \mathscr{Z})$ and a mapping $\psi\colon X_0 \to Z \times \boldsymbol{R}^+$ such that*

(i) *ψ is an isomorphism, i.e. ψ is 1-1, onto, and ψ, ψ^{-1} are measurable;*

(ii) *$U_t\big(\psi^{-1}(z, s)\big) = \psi^{-1}(z, ts)$ for every $z \in Z$ and $s, t \in \boldsymbol{R}^+$.*

(IV) *There exists a ring $\mathscr{A}$ of subsets of Z generating $\mathscr{Z}$ such that for every $A \in \mathscr{A}$ and $t \in \boldsymbol{R}^+$*

$$(\psi m)\big(A \times (t, \infty)\big) < \infty.$$

The above conditions, especially (III) and (IV), have a rather technical character. We shall see that in many cases investigated earlier they were fulfilled in a quite natural way. For instance, if X is a real separable Banach space and m is a Lévy measure, then as a function f we can take a so-called weight function (cf. Section 3).

PROPOSITION 1. *If a σ-finite measure m on $(X, \mathcal{M})$ satisfies conditions (I)–(IV), then there exist a probability measure γ on $(Z, \mathcal{Z})$ and a function $k\colon Z \times \mathbf{R}^+ \to \mathbf{R}^+$ such that $k(\cdot, t)$ is measurable, $k(x, \cdot)$ is non-increasing and left-continuous, $\lim_{t\to\infty} k(x, t) = 0$, and such that for $A \in \mathcal{M}_0$*

$$m(A) = \int_Z \int_{\mathbf{R}^+} 1_A(\psi^{-1}(x, t))\frac{k(x, t)}{t} dt\gamma(dx).$$

Conversely, each measure of the above-mentioned form satisfies the inequality in (II).

Proof. From (I) and (III) we infer that the measure

(1) $$\gamma_0(A) := c^{-1} \int_{\{x:\psi(x)\in A \times \mathbf{R}^+\}} f(x)m(dx), \qquad A \in \mathcal{Z},$$

is a probability measure on $(Z, \mathcal{Z})$ whenever

$$c = \int_{X_0} f dm.$$

Further, by (I) and (IV), the measures $v_r(\cdot) := (\psi m)(\cdot \times (e^r, \infty))$ are σ-finite and absolutely continuous with respect to γ_0 ($r \in \mathbf{R}$). Therefore, by the Radon-Nikodym theorem, there are non-negative measurable functions $h(\cdot, r)$ such that

(2) $$v_r(A) = \int_A h(x, r)\gamma_0(dx) \quad \text{for all } A \in \mathcal{Z}.$$

Moreover, if $r_1 \leqslant r_2$, then

(3) $$h(x, r_1) \geqslant h(x, r_2) \quad \text{for } \gamma_0\text{-almost every } x.$$

By (II) and (III), for arbitrary $r_1 \leqslant r_2$ and $h > 0$ we obtain

$$\begin{aligned} v_{r_1}(A) - v_{r_2}(A) &\geqslant m(\{x \in X_0\colon U_{e^{-h}} x \in \psi^{-1}(A \times (e^{r_1}, e^{r_2}])\}) \\ &= m(\{\psi^{-1}(z, s)\colon \psi U_{e^{-h}} \psi^{-1}(z, s) \in A \times (e^{r_1}, e^{r_2}]\}) \\ &= v_{r_1+h}(A) - v_{r_2+h}(A), \end{aligned}$$

i.e. for every $A \in \mathcal{Z}$ the function $r \to v_r(A)$ is convex, right-continuous, and non-increasing on $\mathbf{R}$. Consequently, by (2), for $r_1, r_2 \in \mathbf{R}$ and $0 < \alpha < 1$, we get

(4)
$$h(x, \alpha r_1 + (1-\alpha)r_2) \leqslant \alpha h(x, r_1) + (1-\alpha)h(x, r_2) \quad \text{for } \gamma_0\text{-almost every } x \in Z.$$

Therefore, there exists $A_1 \in \mathcal{Z}$ such that $\gamma_0(A_1) = 1$ and inequality (4) is fulfilled for all rationals r_1, r_2, α ($r_1, r_2 \in \mathbf{R}$, $0 < \alpha < 1$) and for all $x \in A_1$. Now, for $x \in A_1$ and $t \in \mathbf{R}$, we put

$$\tilde{h}(x, t) := \sup_{r \geqslant t} h(x, r),$$

where the supremum is taken over all rationals r. It is easy to see that $\tilde{h}$ is non-increasing and convex as a function of t, measurable with respect of x, and formula (2) holds also for $\tilde{h}$. Further,

$$\lim_{t \to \infty} \tilde{h}(x, t) = 0 \quad \text{for } \gamma_0\text{-almost every } x.$$

The structural characterization of the convex functions implies that there is a function $\tilde{k}(x, t)$ (non-increasing and left-continuous with respect of t and measurable with respect of x) such that

$$\tilde{h}(x, t) = \int_t^\infty \tilde{k}(x, s)\, ds \quad \text{for } \gamma_0\text{-almost every } x.$$

Consequently, by (2), we get

$$(\psi m)(A \times I) = \int_A \int_I \frac{\tilde{k}(x, \ln s)}{s}\, ds\, \gamma_0(dx).$$

Putting $k(x, t) := \tilde{k}(x, \ln t)$ for $x \in A_1$, $t \in \boldsymbol{R}^+$, and defining k outside A_1 as an arbitrary function satisfying the required conditions and $\gamma(A) := \gamma_0(A \cap A_1)$ for $A \in \mathscr{Z}$, we obtain the formula in Proposition 1. Thus the necessity is proved.

Since U is a semi-group and $k(x, \cdot)$ is a non-increasing function, for $0 < s < 1$ we get

$$(U_s m)(A) = \int_Z \int_{\boldsymbol{R}^+} 1_A\big(\psi^{-1}(x, r)\big) \frac{k(x, r/s)}{r}\, dr\gamma(dx) \leqslant m(A),$$

i.e. the measure m satisfies the inequality in (II), which proves the sufficiency.

Note that if a measure m satisfies condition (II), then $m_s := m - U_s m$ for $0 < s < 1$ are also positive measures. This simple remark suggests the following sequence of conditions defined inductively.

Namely, let condition (II) be denoted by (II_0), i.e.

(II_0) *for every* t $(0 < t < 1)$ *and* $A \in \mathscr{M}_0$, $(U_t m)(A) \leqslant m(A)$.

Next, for positive integers $j \geqslant 1$,

(II_j) *the measures* m *and* $m_s := m - U_s m$ $(0 < s < 1)$ *satisfy conditions* (II_0) *and* (II_{j-1}), *respectively.*

Finally, we introduce the condition

(II_∞) *the measure* m *satisfies* (II_0) *and the measures* m_s $(0 < s < 1)$ *satisfy* (II_j) *for* $j = 0, 1, 2, \ldots$

Now we are going to describe the measures m satisfying (II_j), so satisfying (II_0), in terms of the function k and the probability measure γ occurring in Proposition 1. To this end we need the notion of monotonicity of order j $(j = 0, 1, 2, \ldots, \infty)$. We say that a function f is *monotone of order zero* if it is non-increasing, non-negative, and left-continuous. For $j \geqslant 1$, a function f is *monotone of order* j if it is $j-1$ times differentiable, $(-1)^k f^{(k)} \geqslant 0$ for $k = 0, 1, 2, \ldots, j-1$, and $f^{(j-1)}$ has a left-hand derivative $D^- f^{(j-1)}$ such that $(-1)^j D^- f^{(j-1)}$ is monotone of order zero. Of course, functions *monotone of order* ∞ are completely monotone ones which are characterized by the Bernstein theorem.

PROPOSITION 2. *Assume that a* σ*-finite measure* m *on* $(X, \mathscr{M})$ *satisfies* (I), (III), *and* (IV). *Then* m *satisfies* (II_j), $j = 0, 1, 2, \ldots, \infty$, *if and only if* m *satisfies* (II_0) *and the function* $g(x, t) := k(x, e^t)$ $(x \in Z,\ t \in \mathbf{R})$ *is monotone of order* j *for* γ*-almost every* x, *where* k *and* γ *are parameters occurring in Proposition* 1.

Proof. Proposition 1 gives the validity of this statement for $j = 0$. Assume that it is true for $j-1$. If m satisfies (II_j), then for m and $m_s := m - U_s m$, $0 < s < 1$, condition (II_{j-1}) holds true. Hence $g(x, \cdot)$ and $g(x, t) - g(x, t - \log s)$ are monotone functions of order $j-1$ for γ-almost every x. Hence the function $(-1)^{j-1} \{g^{(j-1)}(x, t) - g^{(j-1)}(x, t - \log s)\}$ is non-increasing for γ-almost every x, and so $(-1)^{j-1} g^{(j-1)}(x, \cdot)$ is convex. Therefore, $(-1)^j D^- g^{(j-1)}(x, \cdot)$ exists and is non-increasing, left-continuous, and non-negative, which proves that $g(x, \cdot)$ is monotone of order j for γ-almost every x.

The sufficiency for $j = 0$ is trivial. Suppose that the statement is proved for $j-1$. If g is a γ-almost everywhere monotone function of order j, then for $0 < s < 1$ the function $h_s(x, t) := g(x, t) - g(x, t - \log s)$ is monotone of order $j-1$ for γ-almost every x. Indeed, the functions $(-1)^l g^{(l)}(x, t)$ are non-increasing for $l = 0, 1, 2, \ldots, j-1$. Thus $(-1)^l h_s^{(l)}(x, \cdot) \geqslant 0$ for $l = 0, 1, 2, \ldots, m-2$ and $(-1)^{j-1} D^- h_s^{(j-2)}(x, \cdot)$ is monotone of order zero. Hence, by induction, m_s satisfies (II_{j-1}) for $0 < s < 1$, i.e. m satisfies (II_j), which completes the proof of the sufficiency.

As a consequence of the definition of monotonicity of order ∞ and condition (II_∞) we obtain Proposition 2 for $j = \infty$ as well.

Applying an integral representation of the monotone function of order j (cf. [16] or [6]) we get the following characterization of measures satisfying condition (II_j) for $j = 0, 1, 2, \ldots, \infty$ (cf. [6], Theorems 7.1 and 7.2).

COROLLARY 1. *Let a* σ*-finite measure* m *on* $(X, \mathscr{M})$ *satisfy conditions* (I), (III), *and* (IV). *Then*

(a) m *satisfies* (II_j), $j = 0, 1, 2, \ldots$, *if and only if there exist Borel measures* F_z *on* $\mathbf{R}^+$, *finite on compact subsets*, $\int_e^\infty (\log v)^j F_z(dv) < \infty$, *the mapping* $z \to F_z$ $(z \in Z)$ *is measurable, and there exists a probability measure* γ *on* Z *such that*

$$m(A) = \int_Z \int_{\mathbf{R}^+} \int_0^v 1_A(\psi^{-1}(z, t)) \frac{1}{t} \left(\log \frac{v}{t}\right)^j dt F_z(dv) \gamma(dz);$$

(b) *m satisfies* (II_∞) *if and only if there exist Borel measures F_z on $\mathbf{R}^+$ ($z \in Z$) and a probability measure γ on $\mathscr{Z}$ such that the mapping $z \to F_z$ is measurable and*

$$m(A) = \int_Z \int_{\mathbf{R}^+} \int_{\mathbf{R}^+} 1_A(\psi^{-1}(z, t)) t^{-(s+1)} dt F_z(ds) \gamma(dz).$$

Remark 1. If for the product space $Z \times \mathbf{R}^+$ the Existence Theorem of Regular Conditional Distributions can be applied, then there are probability measures γ_s on Z ($s \in \mathbf{R}^+$) and a Borel measure F on $\mathbf{R}^+$ such that the mapping $s \to \gamma_s$ is measurable and the first two integrals in the above-mentioned formulas can be replaced (cf. [6], Section 7). For instance, this is true if Z is a Borel subset of some complete separable metric space (cf. [15], Chapter V, Theorem 8.1).

Considering classes of limit distributions analogous to the stable ones we obtain an equation instead of the inequality in (II). Therefore, we introduce the following condition:

(II') *There exists a Borel measurable function g: $\mathbf{R}^+ \to \mathbf{R}^+$ such that, for every $t \in \mathbf{R}^+$ and every $A \in \mathscr{M}_0 := \mathscr{M} \cap X_0$,*

$$(U_t m)(A) = g(t) \cdot m(A).$$

In particular cases one can find a general form of m satisfying (II') following Urbanik's approach, i.e. via the Choquet theorem (cf. [3] and [9]). On the other hand, the equation in (II') can be solved directly as is shown in the following proposition:

Proposition 3. *If a measure m on $(X, \mathscr{M})$ satisfies conditions* (I), (II'), (III), *and* (IV), *then $g(t) = t^p$, p is a positive constant, and there exists a measure γ on $(Z, \mathscr{Z})$ such that, for $A \in \mathscr{M}_0$,*

$$m(A) = \int_Z \int_{\mathbf{R}^+} 1_A(\psi^{-1}(z, t)) \frac{dt}{t^{p+1}} \gamma(dz).$$

Conversely, each measure of the above form satisfies (II').

Proof. Since U is a semi-group, by (III) we get

$$g(ts) m(\psi^{-1}(A \times I)) = U_t((U_s m)(\psi^{-1}(A \times I))) = g(t) g(s) m(\psi^{-1}(A \times I))$$

for arbitrary $t, s \in \mathbf{R}^+$, $A \in \mathscr{A}$, and a Borel subset I of $\mathbf{R}^+$. Hence, if m is a non-zero measure, we obtain $g(ts) = g(t) g(s)$, $t, s \in \mathbf{R}^+$. Thus, by (II') and (IV), $g(t) = t^p$ for some $p > 0$. Further, putting $f(r) := (\psi m)(A \times (r, \infty))$, $r \in \mathbf{R}^+$ ($A \in \mathscr{A}$ is fixed), we infer from (II') that

$$f(r/t) = t^p f(r) \quad \text{for all } t, r \in \mathbf{R}^+.$$

Moreover, putting $\gamma(A) := p(\psi m)(A \times (1, \infty))$ for $A \in \mathscr{A}$, we get $f(r)$

$= p^{-1} r^{-p} \gamma(A)$ and, therefore, for an arbitrary interval I in R^+ we obtain

$$(\psi m)(A \times I) = \int_A \int_I \frac{dt}{t^{p+1}} \gamma(dx).$$

This formula has an extension for whole σ-algebra $\mathscr{M}_0$, which completes the proof of the necessity of Proposition 3. One can easily check that the above converse part holds true with $g(t) = t^p$.

COROLLARY 2. *A measure m on* $(X, \mathscr{M})$ *satisfying conditions* (I), (II'), (III), *and* (IV) *fulfills also* (II$_j$) *for* $j = 0, 1, \ldots, \infty$.

3. Applications. In this section we show some applications of the results given in Section 2. We find characterizations for some subclasses of the class $\mathrm{ID}(E)$ of all infinitely divisible probability measures on a real separable Banach space E. For the necessary information we refer to [1]. In particular, each $\mu \in \mathrm{ID}(E)$ has a unique representation $[x_0, R, M]$, where $x_0 \in E$, R is a covariance operator of a Gaussian measure, and M is a *Lévy measure* of μ (it is sometimes called a *generalized Poisson exponent*). Moreover, Lévy measures vanish at zero, are σ-finite, finite outside every neighbourhood of zero, and there is a so-called *weight function* Φ (cf. [22], p. 299) such that

$$(5) \qquad \int_{E\setminus\{0\}} \Phi(x) M(dx) < \infty \quad \text{for every Lévy measure } M.$$

If E is a Hilbert space, as a function Φ we can take

$$\Phi(x) = \frac{\|x\|^2}{1+\|x\|^2},$$

and an arbitrary Borel measure M on E satisfying (5) is a Lévy measure of some $\mu \in \mathrm{ID}(E)$. For $t \in R^+$ and $\mu = [x, R, M]$, we denote by μ^{*t} the measure $[tx, tR, tM]$.

(a) *The classes* $L(U)$ *and* $S(U)$. Let U be a strongly continuous one-parameter group of bounded linear operators on E satisfying the condition

$$\lim_{t\to 0} U_t x = 0 \quad \text{for } x \in E.$$

The classes $L(U)$ and $S(U)$ were introduced in [6] as limit measures of sequences $U_{t_n}(\mu_1 * \ldots * \mu_2) * \delta_{x_n}$ with the infinitesimality assumption. Moreover, we know that

$$\mu = \begin{cases} [x_0, 0, M] \in L(U) & \text{iff } M \geqslant U_t M \text{ for every } t\ (0 < t < 1), \\ [x_0, 0, M] \in S(U) & \text{iff } U_t M = t^{\alpha} M \text{ for every } t > 0. \end{cases}$$

By Proposition 2.1 in [6] we see that (I)-(III) are fulfilled where X_0 and Z are some Borel subsets of $E\setminus\{0\}$ and $\psi^{-1}(z, t) := U_t z$. Further, as the function f we take Φ, and as the family $\mathscr{A}$ in (IV) we can take all Borel subsets of Z separated from zero, i.e. $A \in \mathscr{A}$ if $\inf\{\|a\|: a \in A\} > 0$ and A is a Borel subset of Z.

Therefore, according to Remark 1, Corollary 1 contains Theorems 7.1 and 7.2 from [6].

(b) *Operator-selfdecomposable and operator-stable measures.* In a particular case where U is a uniform continuous group of bounded linear operators, i.e. $U = \{t^Q: t \in \boldsymbol{R}^+\}$ for some bounded Q and

$$\lim_{t \to 0} t^Q = 0,$$

the classes $L(U)$ and $S(U)$ represent so-called *operator-selfdecomposable* and *operator-stable measures*, respectively. It is worth noting that these measures are limit distributions of partial sums of sequences of independent vectors normed by some affine transformations (cf. [22] and [8]). Moreover, $E \setminus \{0\}$ is Borel isomorphic to $S_Q \times \boldsymbol{R}^+$, where

$$S_Q := \{x: \|x\| = 1 \wedge \forall\ (t > 1)\ \|t^Q x\| > 1\} \quad \text{and} \quad \psi^{-1}(z, t) := t^Q z$$

(cf. [7], Proposition 2). Of course, (IV) is fulfilled for an arbitrary Borel subset A of S_Q. Finally, by Theorem 5.1 in [22] and Corollary 1 we obtain a new characterization of the characteristic functionals of operator-selfdecomposable measures (cf. [22], Theorem 5.3). For operator-stable measures the measure γ in Proposition 3 is concentrated on the set S_Q (cf. [4] and [8]). Taking the identity operator as Q, by Corollary 1 we obtain the description of n-times and completely-selfdecomposable measures (cf. [13]).

(c) *V-decomposable and V-stable measures.* Let H be a real separable Hilbert space and let $V := \{V_t: t \in \boldsymbol{R}^+\}$ denote a strongly continuous, unitary representation of the multiplicative group $\boldsymbol{R}^+$ in H. For the one-parameter group $U := \{U_t: t \in \boldsymbol{R}^+\}$, where $U_t := tV_t$, we define V-decomposability and V-stability as follows. We say that μ is *V-decomposable* if for every t $(0 < t < 1)$ there exists a probability measure μ_t such that $\mu = U_t \mu * \mu_t$, and μ is *V-stable* if for every $a, b \in \boldsymbol{R}^+$ there exist $c \in \boldsymbol{R}^+$ and $x \in H$ such that $U_a \mu * U_b \mu = U_c \mu * \delta_x$. Consequently, Lévy measures M of V-decomposable ones satisfy condition (II) and V-stable ones satisfy the equation in (II') (cf. [2]). Putting

$$X_0 = H \setminus \{0\}, \quad f(x) = \frac{\|x\|^2}{1 + \|x\|^2},$$

$$Z = \{x \in H: \|x\| = 1\}, \quad \psi^{-1}(z, t) := U_t z,$$

and taking as the family $\mathscr{A}$ in (IV) all Borel subsets of Z, we see that Proposition 1, Corollary 1, and Proposition 3 give characterizations of V-decomposable measures and its subclasses. In particular, we obtain

COROLLARY 3. *M is a Lévy measure of a V-decomposable one if and only if there exist a probability measure γ on $Z = \{x: \|x\| = 1\}$ and a function*

k: $Z \times \mathbf{R}^+ \to \mathbf{R}^+$, measurable in x, non-increasing and left-continuous in t, such that

$$M(A) = \int_Z \int_{\mathbf{R}^+} 1_A(U_t x) \frac{k(x, t)}{t} dt \gamma(dx).$$

Moreover, the constant

$$c := \int_{\mathbf{R}^+} \frac{t \cdot k(x, t)}{1+t^2} dt$$

does not depend on x.

COROLLARY 4. *M is a Lévy measure of a V-stable one if and only if there exist a constant p $(0 < p < 2)$ and a finite Borel measure γ on the unit sphere Z such that*

$$M(A) = \int_Z \int_{\mathbf{R}^+} 1_A(U_t x) \frac{dt}{t^{p+1}} \gamma(dx).$$

(d) *Selfdecomposable measures in generalized convolution algebras.* The theory of generalized convolutions was introduced by Urbanik [25] to study binary operations $\circ$ (on measures on the non-negative half-line) having appropriate analogues of the most important properties of the ordinary convolution $*$. By $(\mathscr{P}, \circ)$ we denote the convolution algebra, where $\mathscr{P}$ is the class of all probability measures on $[0, \infty)$. Let $U := \{U_a \colon a \geqslant 0\}$ be a semigroup of mappings on $[0, \infty)$ defined by the formula $U_a x := ax$. We say that $\mu \in \mathscr{P}$ is *selfdecomposable in* $(\mathscr{P}, \circ)$ if for every t $(0 < t < 1)$ there exists $\mu_t \in \mathscr{P}$ such that

$$\mu = U_t \mu \circ \mu_t. \tag{6}$$

The class of selfdecomposable measures is contained in the class of all infinitely divisible measures (μ is *infinitely divisible* if $\mu = \mu_n^{\circ n}$ for every positive integer n and $\mu_n \in \mathscr{P}$). Moreover, μ is infinitely divisible if and only if its characteristic functional Φ_μ is of the form

$$\Phi_\mu(t) = \exp\{-ct^\varkappa + \int_0^\infty (\Omega(tx)-1) m(dx)\}, \tag{7}$$

where c is a non-negative constant, $\varkappa$ is the characteristic exponent of the algebra $(\mathscr{P}, \circ)$, m is a σ-finite measure on $[0, \infty)$, finite outside every neighbourhood of zero, $m(\{0\}) = 0$, and

$$\int_0^1 \omega(x) m(dx) < \infty. \tag{8}$$

Since the representation (7) is unique (cf. [25] and [23]), we write $\mu = [c, m]$

if the characteristic function Φ_μ is of the form (7). Finally, recall that the kernel Ω is associated with a homomorphism of $(\mathscr{P}, \circ)$ into the real line,

$$\omega(x) := \begin{cases} 1-\omega(x) & \text{if } 0 < x \leqslant x_0, \\ 1-\omega(x_0) & \text{if } x > x_0, \end{cases}$$

and x_0 is a positive real number such that $\Omega(x) < 1$ whenever $0 < x \leqslant x_0$ (cf. [25]). From (6) we infer that $[c, m]$ is selfdecomposable in $(\mathscr{P}, \circ)$ if and only if $m \geqslant U_t m$ for every t $(0 < t < 1)$. Further, we say that μ is *n times selfdecomposable* $(n = 1, 2, \ldots)$ if μ is selfdecomposable and μ_t is $n-1$ times selfdecomposable, where 0 times selfdecomposability means selfdecomposability (cf. [14] and [23]). Since, by (8), conditions (I)-(IV) and (II_j), $j = 1, 2, \ldots$, are fulfilled, Corollary 2 gives the following characterizations:

COROLLARY 5. (i) *The class of the characteristic functions of n times selfdecomposable measures in* $(\mathscr{P}, \circ)$, $n = 0, 1, 2, \ldots$, *coincides with the class of all functions of the form*

$$\Phi_\mu(t) = \exp\left\{-ct^\varkappa + \int_0^\infty \int_0^s \frac{\Omega(tx)-1}{x} \left(\log\frac{s}{x}\right)^n dx F(ds)\right\},$$

where c is a non-negative constant, $\varkappa$ *is the characteristic exponent, and F is a Borel measure on* $\boldsymbol{R}^+$ *such that*

$$\int_e^\infty (\log x)^n F(dx) < \infty \quad \textit{and} \quad \int_0^1 \int_0^s \frac{\omega(x)}{x} \log\left(\frac{s}{x}\right)^n dx F(ds) < \infty.$$

(ii) *The class of the characteristic functions of completely selfdecomposable measures in* $(\mathscr{P}, \circ)$ *coincides with the class of all functions of the form*

$$\Phi_\mu(t) = \exp\left\{-ct^\varkappa + \int_0^\varkappa \int_0^\infty \frac{\Omega(tx)-1}{x^{1+s}} dx F(ds)\right\},$$

where c is a non-negative constant, $\varkappa$ *is the characteristic exponent of the algebra* $(\mathscr{P}, \circ)$, *and F is a Borel measure on the interval* $(0, \varkappa)$ *such that*

$$\int_0^\varkappa \int_0^\infty \frac{\omega(x)}{x^{1+s}} dx F(ds) < \infty.$$

(e) *S-selfdecomposable probability measures.* All the transformations U_t considered in subsections (a)-(d) have been linear on the underlying space X. Now we shall quote a semi-group of non-linear mappings on a real separable Banach space E. Namely, for $t \in \boldsymbol{R}^+$ we define a *shrinking operation* (for short: *s-operation*) $T_t\colon E \to E$ as follows:

$$T_t x := \begin{cases} 0 & \text{if } \|x\| \leqslant t, \\ (1-t/\|x\|)x & \text{if } \|x\| > t. \end{cases}$$

It is easy to check that the set $\{T_t\colon t \in \boldsymbol{R}^+\}$ is an additive semi-group with composition of non-linear transformations. Further, putting $U_t := T_{-\log t}$, $0 < t < 1$, we get a multiplicative semi-group. We say that $\mu = [x_0, R, M] \in \mathrm{ID}(E)$ is *s-selfdecomposable* if for every t $(0 < t < 1)$

$$(9) \qquad M \geqslant U_t M \quad \text{on Borel subsets of } E \setminus \{0\}$$

or, equivalently, $M \geqslant T_t M$ for every $t \in \boldsymbol{R}^+$. It is worth noting that in a Hilbert space the class of all s-selfdecomposable measures coincides with the class of limits of the sequences

$$U_{r_n}\mu_1 * U_{r_n}\mu_2 * \ldots * U_{r_n}\mu_n * \delta_{x_n},$$

where the triangular array $U_{r_n}\mu_j$ $(j = 1, 2, \ldots, n)$ is infinitesimal (cf. [5]). Further, we say that $\mu = [x_0, R, M]$ is *n times s-selfdecomposable* $(n \geqslant 1)$ if μ is s-selfdecomposable and $[x_0, R, M - U_t M]$ is $n-1$ times s-selfdecomposable for every t $(0 < t < 1)$. Finally, if μ is n times s-selfdecomposable for $n = 0, 1, 2, \ldots$, then it is called a *completely s-selfdecomposable measure.* (By 0 *times s-selfdecomposability* we mean s-selfdecomposability). Following the results of [5] we say that μ is an *s-stable measure* if either μ is Gaussian or $\mu = [x_0, 0, M]$ and there is a constant p $(0 < p < \infty)$ such that for every s $(0 < s < 1)$

$$(10) \qquad U_s M = s^p M \quad \text{on Borel subsets of } E \setminus \{0\}.$$

From (10) we infer that the measure M is finite on $E \setminus \{0\}$ because the Lévy measures are always finite on subsets separated from zero. In order to solve inequality (8) or equation (10) we can follow the arguments of the proof of Proposition 1. Namely, taking $X_0 = E \setminus \{0\}$, $f = \Phi$ (Φ is the weight function on E) and $\psi(x) := (x/\|x\|, \|x\|)$ we infer that conditions (I), (II), (III) (i), and (IV) are fulfilled with the unit sphere Z and the family of all Borel subsets of Z as $\mathscr{A}$. Moreover, assuming that, for every $z \in Z$, $(z, 0)$ represents a zero vector in E, instead of (III) (ii) we have the condition

$$U_t(\psi^{-1}(z, s)) = \psi^{-1}(z, \max(0, \log te^s))$$

since $T_t(\psi^{-1}(z, s)) = \psi^{-1}(z, \max(0, s-t))$ for $t, s \in \boldsymbol{R}^+$ and $z \in Z$. Hence the function $r \to v_r(A) := (\psi M)(A \times (r, \infty))$ is convex on $\boldsymbol{R}^+$ and we can repeat arguments from the proof of Proposition 1. Therefore, we obtain the following

PROPOSITION 4. *A measure* $\mu = [x_0, R, M]$ *on a Banach space* E *is s-selfdecomposable if and only if there exist a Borel probability measure* γ *on the unit sphere* Z *in* E *and a function* k: $Z \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ *such that* $k(\cdot, t)$ *is measurable,* $k(x, \cdot)$ *is non-increasing and left-continuous,* $\lim_{t\to\infty} k(x, t) = 0$, *and such that for a Borel subset* A *of* $E \setminus \{0\}$

$$M(A) = \int_Z \int_{\boldsymbol{R}^+} 1_A(tz)\, k(z, t)\, dt\, \gamma(dz).$$

In view of Corollary 1 and Remark 1 in Section 1 we get

PROPOSITION 5. (a) *A measure $\mu = [x_0, R, M]$ on a Banach space E is n times s-selfdecomposable $(n = 0, 1, 2, \ldots)$ if and only if there exist probability Borel measures γ_s on the unit sphere Z in E and Borel measure F on $\mathbf{R}^+$ such that the mapping $s \to \gamma_s$ is measurable and for a Borel subset A of $E \setminus \{0\}$*

$$M(A) = \int_{\mathbf{R}^+} \int_Z \int_0^v 1_A(tz)(v-t)^n \, dt \gamma_s(dv) F(ds).$$

(b) *A measure $\mu = [x_0, R, M]$ on a Banach space E is completely s-selfdecomposable if and only if there exist Borel probability measures γ_s on Z and a Borel measure F on $\mathbf{R}^+$ such that the mapping $s \to \gamma_s$, $s \in \mathbf{R}^+$, is measurable and for a Borel subset A of $E \setminus \{0\}$*

$$M(A) = \int_{\mathbf{R}^+} \int_Z \int_{\mathbf{R}^+} 1_A(tz) e^{-st} \, dt \gamma_s(dz) F(ds).$$

Using the characterization of the infinitely divisible probability measures on a Banach space in terms of the characteristic functionals (cf. [1]), we can get analogous characterizations of multiply s-selfdecomposable measures from Proposition 5.

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