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SEQUENTIAL ESTIMATION FOR THE SPECTRAL DENSITY PARAMETER OF A STATIONARY GAUSSIAN PROCESS

BY

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Abstract. In this paper we consider the problem of sequential estimation for the stationary zero-mean Gaussian process whose spectral density is of the form $[2\pi(\lambda^2+\vartheta^2)]^{-1}$, where $\vartheta > 0$ is an unknown parameter. We find the class of Markov stopping times determining optimal sequential estimation plans for a given function $g(\vartheta)$. A sequential plan is optimal if the lower bound in the information inequality is attained. Moreover, the form of efficient sequential estimators is derived and the class of efficiently estimable functions is investigated.

1. Preliminaries. Let $\xi(t) = \xi_\vartheta(t)$, $t \in T = [0, \infty)$, be a separable stationary zero-mean Gaussian process with the spectral density

$$\varphi_\vartheta(\lambda) = \frac{1}{2\pi(\lambda^2+\vartheta^2)}, \quad -\infty < \lambda < \infty, \tag{1}$$

where $\vartheta \in D = (0, \infty)$ is an unknown parameter. Such a process is a Markov one and has continuous sample functions with probability 1. The covariance function of the process $\xi_\vartheta(t)$, $t \in T$, is defined by

$$B_\vartheta(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \varphi_\vartheta(\lambda)\, d\lambda = (2\vartheta)^{-1} \exp(-\vartheta |t|)$$

and the variance of this process is equal to $B_\vartheta(0) = (2\vartheta)^{-1}$.

The processes $\{\xi_\vartheta(t),\ t \in T\}$, $\vartheta \in D$, can be determined in the following way. Assume that $W(t)$, $t \in T$, is a Wiener process on a probability space $(\Omega, \mathscr{F}, P)$ and X is a random variable on $(\Omega, \mathscr{F}, P)$, independent of $W(t)$, $t \in T$, and standard normally distributed. Let $\mathscr{F}_t = \sigma\{W(s),\ s \leqslant t;\ X\}$ be a family of σ-algebras. Then, for every $\vartheta \in D$, a unique solution $\xi(t) = \xi_\vartheta(t)$, $t \in T$, of the stochastic integral equation

(2) $$\xi(t) - \xi(0) = -\vartheta \int_0^t \xi(s)\,ds + W(t), \qquad \xi(0) = \frac{1}{\sqrt{2\vartheta}} X,$$

considered with respect to $W(t)$, $t \in T$, and $\mathscr{F}_t$, is the process with the above-mentioned properties.

In the sequel, if no ambiguity arises, we omit the index ϑ and write simply $\xi(t)$, $t \in T$.

2. Sufficient statistics. An equivalent model of the processes $\{\xi_\vartheta(t), t \in T\}$, $\vartheta \in D$, obtained by the canonical form will be useful in our considerations. Let C be the space of all continuous real-valued functions $x = x(t)$, $t \in T$, and let $\mathscr{C} = \sigma\{x = x(t), t \in T\}$ denote the minimal σ-algebra consisting of all cylinder sets of C. By μ_ϑ we denote the measure on $(C, \mathscr{C})$ corresponding to the process $\xi_\vartheta(t)$, $t \in T$:

$$\mu_\vartheta(B) = P(\xi_\vartheta(\cdot) \in B), \qquad B \in \mathscr{C}.$$

Let $\mu_{\vartheta,t}$ be the truncation of the measure μ_ϑ on

$$\mathscr{C}_t = \sigma\{x = x(s), s \leqslant t, s, t \in T\}.$$

Let us consider the sequential statistical space $(C, \mathscr{C}_t, \{\mu_{\vartheta,t}, \vartheta \in D\})$, $t \in T$, corresponding to the family of processes $\{\xi_\vartheta(t), t \in T\}$, $\vartheta \in D$. Let R be the real line and let $\mathscr{B}_R$ denote the σ-algebra of Borel subsets of R. A function $Z(t, x)\colon T \times C \to R^k$ such that for every $t \in T$ the mapping $Z(t, \cdot)$ is $(\mathscr{C}_t, \mathscr{B}_{R^k})$-measurable will be called a (k-dimensional) *statistic* on the space $(C, \mathscr{C}_t, \{\mu_{\vartheta,t}, \vartheta \in D\})$, $t \in T$.

LEMMA 1. (a) *The statistical space $(C, \mathscr{C}_t, \{\mu_{\vartheta,t}, \vartheta \in D\})$, $t \in T$, is dominated by a measure $\mu_{\vartheta_0,t}$ for some $\vartheta_0 \in D$.*

(b) *The densities $d\mu_{\vartheta,t}/d\mu_{\vartheta_0,t}$ are defined by*

(3) $$\frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}}(x) = \exp\left\{\frac{1}{2}[\log \vartheta - \log \vartheta_0 - (\vartheta - \vartheta_0) Z_1(t, x) - (\vartheta^2 - \vartheta_0^2) Z_2(t, x)]\right\}$$

$$= h(Z(t, x); \vartheta, \vartheta_0), \qquad x \in C,$$

where $Z(t, x) = (Z_1(t, x), Z_2(t, x))$, and

(4) $$Z_1(t, x) = x^2(0) + x^2(t) - t,$$

(5) $$Z_2(t, x) = \int_0^t x^2(s)\,ds.$$

(c) *The statistic* $Z(t, x) = (Z_1(t, x), Z_2(t, x))$ *with* $Z_1(t, x)$ *and* $Z_2(t, x)$ *defined by* (4) *and* (5), *respectively, is a (two-dimensional) sufficient statistic on the space* $(C, \mathscr{C}_t, \{\mu_{\vartheta,t}, \vartheta \in D\})$, $t \in T$.

Proof. Using the results on absolutely continuous substitution of measures [4] or the Skorohod theorems [6] we get (a) and the formula

$$\frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}}(\xi_{\vartheta_0}(\cdot)) = p_{\vartheta_0}(\xi_{\vartheta_0}(0); \vartheta) \exp\left[-\frac{(\vartheta-\vartheta_0)^2}{2}\int_0^t \xi_{\vartheta_0}^2(s)\,ds - (\vartheta-\vartheta_0)\int_0^t \xi_{\vartheta_0}(s)\,dW(s)\right], \tag{6}$$

where $p_{\vartheta_0}(\cdot; \vartheta)$ denotes the distribution density of values of the process $\xi_\vartheta(t)$, $t \in T$, at time $t = 0$ relative to this distribution for the process $\xi_{\vartheta_0}(t)$, $t \in T$. The function $p_{\vartheta_0}(\cdot; \vartheta)$ is defined as $p_{\vartheta_0}(\cdot; \vartheta) = p(\cdot; \vartheta)/p(\cdot; \vartheta_0)$, where $p(\cdot; \vartheta)$ is the distribution density of values of the process $\xi_\vartheta(t)$, $t \in T$, at time $t = 0$. We have

$$p(\xi_{\vartheta_0}(0); \vartheta) = \frac{1}{\sigma_\vartheta\sqrt{2\pi}} \exp\left[-\frac{\xi_{\vartheta_0}^2(0)}{2\sigma_\vartheta^2}\right],$$

where $\sigma_\vartheta^2 = (2\vartheta)^{-1}$ is the variance of the process $\xi_\vartheta(t)$, $t \in T$. Thus

$$p_{\vartheta_0}(\xi_{\vartheta_0}(0); \vartheta) = (\vartheta/\vartheta_0)^{1/2} \exp\left[-(\vartheta-\vartheta_0)\,\xi_{\vartheta_0}^2(0)\right]. \tag{7}$$

From Ito's formula for processes satisfying equation (2) we obtain

$$\int_0^t \xi_{\vartheta_0}(s)\,dW(s) = \tfrac{1}{2}\left[\xi_{\vartheta_0}^2(t) - \xi_{\vartheta_0}^2(0) - t\right] + \vartheta_0 \int_0^t \xi_{\vartheta_0}^2(s)\,ds. \tag{8}$$

Substituting (7) and (8) into (6) we get

$$\frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}}(\xi_{\vartheta_0}(\cdot)) = \exp\left\{\tfrac{1}{2}\left[\log\vartheta - \log\vartheta_0 - (\vartheta-\vartheta_0)\left(\xi_{\vartheta_0}^2(0) + \xi_{\vartheta_0}^2(t) - t\right) - (\vartheta^2-\vartheta_0^2)\int_0^t \xi_{\vartheta_0}^2(s)\,ds\right]\right\},$$

which is equivalent to (3).

(c) follows from the Fisher-Neyman theorem on factorization (see, e.g., [2], Chap. II, § 2).

3. Absolute continuity of the measures generated by a Markov stopping time and a sufficient statistic. Let $\tau = \tau(x)$ be a finite Markov time with respect to the family $\mathscr{C}_t$, $t \in T$, i.e., $\tau\colon C \to [0, \infty]$ so that $\{x\colon \tau(x) \leqslant t\} \in \mathscr{C}_t$ for every $t \in T$ and $\mu_\vartheta(\{x\colon \tau(x) < \infty\}) = 1$ for all $\vartheta \in D$. Let $U = T \times R^2$ and let $t = t(u)$ and $z = z(u) = (z_1(u), z_2(u))$ be the components of the point $u \in U$. The pair $\mathscr{X}(x) = (\tau(x), Z(\tau(x), x))$ of $\mathscr{C}_\tau$-measurable functions generates for every $\vartheta \in D$ the measure m_ϑ on $(U, \mathscr{B}_U)$ in the standard way: for every $A \in \mathscr{B}_U$,

$$m_\vartheta(A) = \mu_\vartheta(\mathscr{X}^{-1}(A)) = \mu_\vartheta\big((\tau(x), Z(\tau(x), x)) \in A\big).$$

LEMMA 2. *For every finite Markov time τ there exists a σ-finite measure m_τ on $(U, \mathscr{B}_U)$ independent of ϑ and such that for every $A \in \mathscr{B}_U$ and every $\vartheta \in D$*

(9) $$m_\vartheta(A) = \int_A \exp\{\tfrac{1}{2}[\log\vartheta - \vartheta z_1(u) - \vartheta^2 z_2(u)]\}\, m_\tau(du).$$

Proof. From the modification of the Sudakov lemma obtained in [5] for right-continuous functionals it follows that the measures m_ϑ, $\vartheta \in D$, are absolutely continuous with respect to m_{ϑ_0} and

$$\frac{dm_\vartheta}{dm_{\vartheta_0}}(u) = h(z(u); \vartheta, \vartheta_0),$$

i.e. (see formula (3))

$$\frac{dm_\vartheta}{dm_{\vartheta_0}}(u) = \exp\{\tfrac{1}{2}[\log\vartheta - \log\vartheta_0 - (\vartheta-\vartheta_0) z_1(u) - (\vartheta^2 - \vartheta_0^2) z_2(u)]\}.$$

Introducing the measure m_τ defined by

$$m_\tau(du) = \exp\{\tfrac{1}{2}[-\log\vartheta_0 + \vartheta_0 z_1(u) + \vartheta_0^2 z_2(u)]\}\, m_{\vartheta_0}(du)$$

we complete the proof.

4. Sequential plans. Let $g(\vartheta)$ be a real-valued function of the parameter $\vartheta \in D$. We observe the process $\xi(t)$, $t \in T$, up to time τ and want to estimate the function $g(\vartheta)$. A $(\mathscr{B}_U, \mathscr{B}_R)$-measurable function $f: U \to R$ will be called an *estimator for* $g(\vartheta)$.

Definition. By a *sequential estimation plan* for $g(\vartheta)$ we mean any pair $\delta = (\tau, f)$ consisting of a Markov time τ satisfying, for all $\vartheta \in D$, the condition

(10) $$P(0 < \tau(\xi_\vartheta) < \infty) = 1$$

and of an estimator f such that, for every $\vartheta \in D$,

(11) $$\mathrm{E}_\vartheta f^2(\mathscr{X}(\xi)) = \int_U f^2(u) \exp\{\tfrac{1}{2}[\log\vartheta - \vartheta z_1(u) - \vartheta^2 z_2(u)]\}\, m_\tau(du) < \infty$$

and

(12) $$\mathrm{E}_\vartheta f(\mathscr{X}(\xi)) = \int_U f(u) \exp\{\tfrac{1}{2}[\log\vartheta - \vartheta z_1(u) - \vartheta^2 z_2(u)]\}\, m_\tau(du) = g(\vartheta).$$

It follows from (10) that the observation of the process $\xi(t)$, $t \in T$, terminates in a finite time. Condition (12) means that f is an unbiased estimator for $g(\vartheta)$.

From (10) and Lemma 2 we have

$$\int_U \exp\{\tfrac{1}{2}[\log \vartheta - \vartheta z_1(u) - \vartheta^2 z_2(u)]\}\, m_\tau(du) = 1 \tag{13}$$

for every $\vartheta \in D$.

For simplicity, in the sequel we put $Z_1(\tau) = Z_1(\tau(\xi), \xi)$ and $Z_2(\tau) = Z_2(\tau(\xi), \xi)$.

Now, we formulate the following regularity conditions:

(i) $g(\vartheta)$ is differentiable and not identically constant on D;

(ii) $0 < \mathrm{E}_\vartheta[\frac{1}{\vartheta} - Z_1(\tau) - 2\vartheta Z_2(\tau)]^2 < \infty$ for every $\vartheta \in D$;

(iii) the differentiation and repeated differentiation of the integral with respect to ϑ in identities (12) and (13), respectively, is allowed;

(iv) $\mathrm{E}_\vartheta Z_2(\tau)$ is a differentiable function of the variable $\vartheta \in D$.

LEMMA 3. *If the regularity conditions* (i)-(iv) *are satisfied for a sequential plan* (τ, f), *then the following identities hold*:

$$2\vartheta^2 \mathrm{E}_\vartheta Z_2(\tau) = 1 - \vartheta \mathrm{E}_\vartheta Z_1(\tau), \tag{14}$$

$$\vartheta^2 \mathrm{E}_\vartheta [1/\vartheta - Z_1(\tau) - 2\vartheta Z_2(\tau)]^2 = 4\vartheta^2 \mathrm{E}_\vartheta Z_2(\tau) + 2, \tag{15}$$

$$\mathrm{E}_\vartheta \{f(\tau, Z(\tau))[1/\vartheta - Z_1(\tau) - 2\vartheta Z_2(\tau)]\} = 2g'(\vartheta), \tag{16}$$

$$\mathrm{E}_\vartheta \{Z_2(\tau)[1/\vartheta - Z_1(\tau) - 2\vartheta Z_2(\tau)]\} = 2\mathrm{E}'_\vartheta Z_2(\tau), \tag{17}$$

$$\vartheta^2 \mathrm{D}_\vartheta Z_1(\tau) = 4\vartheta^4 \mathrm{D}_\vartheta Z_2(\tau) + 4\vartheta^2 \mathrm{E}_\vartheta Z_2(\tau) + 8\vartheta^3 \mathrm{E}'_\vartheta Z_2(\tau) + 2 \tag{18}$$

($\mathrm{D}_\vartheta(\cdot)$ *denotes the variance evaluated at* ϑ).

A simple proof of Lemma 3 is omitted. Identity (18) is obtained from (14), (15), and (17).

Using (14)-(16) and the Schwarz inequality we obtain

THEOREM 1. *For every sequential plan* (τ, f) *satisfying conditions* (i)-(iii) *the inequality*

$$\mathrm{D}_\vartheta f(\tau, Z(\tau)) \geqslant \frac{2\vartheta^2 [g'(\vartheta)]^2}{1 + 2\vartheta^2 \mathrm{E}_\vartheta Z_2(\tau)} \tag{19}$$

holds for all $\vartheta \in D$. *The equality holds at a particular value of* ϑ *if and only if*

$$f(u) = c(\vartheta)[1/\vartheta - z_1(u) - 2\vartheta z_2(u)] + g(\vartheta) \quad m_\tau\text{-a.e.}, \quad \text{where } c(\vartheta) \neq 0. \tag{20}$$

Condition (i) implies that a sequential estimation plan (τ, f) for $g(\vartheta)$ cannot consist of the estimator $f(u) = \text{const}$ m_τ-a.e. Indeed, if $f(u) = \text{const}$ m_τ-a.e., then $\mathrm{E}_\vartheta f(\tau, Z(\tau)) = \text{const} = g(\vartheta)$ for all $\vartheta \in D$, which contradicts the assumption.

A sequential estimation plan (τ, f) for $g(\vartheta)$ is said to be *efficient at* (a fixed value) ϑ if (19) becomes equality at ϑ. The estimator f is then called *efficient at this value* ϑ and the function $g(\vartheta)$ is *efficiently estimable at the point* ϑ.

A sequential estimation plan (τ, f) for $g(\vartheta)$ is said to be *efficient* if it is efficient at each $\vartheta \in D$. The estimator f is then called *efficient* and the function $g(\vartheta)$ is *efficiently estimable.*

It follows from Theorem 1 that a sequential estimation plan (τ, f) for $g(\vartheta)$ is efficient at a point ϑ if and only if the estimator f is of the form (20).

THEOREM 2. *If (τ, f) is an efficient sequential estimation plan for $g(\vartheta)$, then there exist constants α_1, α_2 not both equal to zero and a constant α_3 such that*

$$\alpha_1 z_1(u)+\alpha_2 z_2(u)+\alpha_3 = 0 \quad m_\tau\text{-}a.e. \tag{21}$$

Proof. By assumption we can choose points ϑ_1 and ϑ_2 in D $(\vartheta_1 \neq \vartheta_2)$ and then we write equality (20) in the form

$$f(u) = c(\vartheta_1)[1/\vartheta_1 - z_1(u) - 2\vartheta_1 z_2(u)] + g(\vartheta_1) \quad m_\tau\text{-a.e.}$$

and

$$f(u) = c(\vartheta_2)[1/\vartheta_2 - z_1(u) - 2\vartheta_2 z_2(u)] + g(\vartheta_2) \quad m_\tau\text{-a.e.},$$

where $c(\vartheta_1)$ and $c(\vartheta_2)$ are both different from zero.

Subtracting one equality from the other we obtain

$$[c(\vartheta_2) - c(\vartheta_1)] z_1(u) + 2[\vartheta_2 c(\vartheta_2) - \vartheta_1 c(\vartheta_1)] z_2(u) + \\ + c(\vartheta_1)/\vartheta_1 - c(\vartheta_2)/\vartheta_2 + g(\vartheta_1) - g(\vartheta_2) = 0 \quad m_\tau\text{-a.e.},$$

which completes the proof.

THEOREM 3. *In a given sequential plan (τ, f) the function $g(\vartheta)$ is efficiently estimable at a point $\vartheta = \vartheta^0$ if and only if it is of the form*

$$g(\vartheta) = c(\vartheta^0)\{1/\vartheta^0 - 1/\vartheta - 2(\vartheta^0 - \vartheta) \mathrm{E}_\vartheta [Z_2(\tau)]\} + g(\vartheta^0). \tag{22}$$

Proof. By Theorem 1 the only efficient estimators at a point $\vartheta = \vartheta^0$ are those which take the form

$$f(\mathscr{Z}) = c(\vartheta^0)[1/\vartheta^0 - Z_1(\tau) - 2\vartheta^0 Z_2(\tau)] + g(\vartheta^0) \tag{23}$$

with probability 1, where $c(\vartheta^0) \neq 0$. Thus the function $g(\vartheta)$ is efficiently estimable at $\vartheta = \vartheta^0$ if and only if it is equal to the expected value of the estimator defined by (23). Therefore

$$g(\vartheta) = \mathrm{E}_\vartheta [f(\mathscr{Z})] = c(\vartheta^0)\{1/\vartheta^0 - \mathrm{E}_\vartheta [Z_1(\tau) - 2\vartheta^0 \mathrm{E}_\vartheta [Z_2(\tau)]\} + g(\vartheta^0).$$

Hence, making use of (14) we obtain (22), which completes the proof.

THEOREM 4. *If in a given sequential plan (τ, f) the function $g(\vartheta)$ is efficiently estimable, then it must be of the form*

$$g(\vartheta) = \frac{k_0 + k_1 \vartheta + k_2 \vartheta^2}{l_1 \vartheta + l_2 \vartheta^2}, \tag{24}$$

where $k_0 \neq 0$, and k_1, k_2, l_1, l_2 are arbitrary constants.

Proof. Suppose that ϑ_1 and ϑ_2 belong to D and $\vartheta_1 \neq \vartheta_2$. Since the function $g(\vartheta)$ is efficiently estimable at these points, it follows from Theorem 3 that the equalities

$$g(\vartheta) = c(\vartheta_1)\{1/\vartheta_1 - 1/\vartheta - 2(\vartheta_1 - \vartheta)\mathrm{E}_\vartheta[Z_2(\tau)]\} + g(\vartheta_1)$$

and

$$g(\vartheta) = c(\vartheta_2)\{1/\vartheta_2 - 1/\vartheta - 2(\vartheta_2 - \vartheta)\mathrm{E}_\vartheta[Z_2(\tau)]\} + g(\vartheta_2)$$

must hold. Eliminating $\mathrm{E}_\vartheta[Z_2(\tau)]$ from these equalities we get

$$g(\vartheta)\{\vartheta_1 c(\vartheta_1) - \vartheta_2 c(\vartheta_2) + [c(\vartheta_2) - c(\vartheta_1)]\vartheta\} = \vartheta^{-1} c(\vartheta_1) c(\vartheta_2)(\vartheta_2 - \vartheta_1) + \\ + \vartheta[c(\vartheta_1)c(\vartheta_2)(1/\vartheta_1 - 1/\vartheta_2) + c(\vartheta_2)g(\vartheta_1) - c(\vartheta_1)g(\vartheta_2)] + \\ + c(\vartheta_1)c(\vartheta_2)(\vartheta_1/\vartheta_2 - \vartheta_2/\vartheta_1) + \vartheta_1 c(\vartheta_1)g(\vartheta_2) - \vartheta_2 c(\vartheta_2) g(\vartheta_1).$$

Since neither $c(\vartheta_1)$ nor $c(\vartheta_2)$ can be equal to zero, the coefficients $l_1 = \vartheta_1 c(\vartheta_1) - \vartheta_2 c(\vartheta_2)$ and $l_2 = c(\vartheta_2) - c(\vartheta_1)$ cannot vanish simultaneously and $k_0 = c(\vartheta_1)c(\vartheta_2)(\vartheta_2 - \vartheta_1) \neq 0$. Thus the function standing by $g(\vartheta)$ in the above-given equality cannot vanish and, consequently, by a proper choice of coefficients we obtain formula (24).

It follows from Theorem 2 that one should seek the efficient sequential plans for a given function $g(\vartheta)$ from the class described in Theorem 4 among the plans determined by Markov stopping times for which (21) holds.

Let us consider the Markov times

$$\tau^{(1)}(x) = \inf\{t\colon Z_1(t, x) = a\}, \tag{25}$$

$$\tau^{(2)}(x) = \inf\{t\colon Z_2(t, x) = b\}, \quad 0 < b < \infty, \tag{26}$$

$$\tau^{(3)}(x) = \inf\{t\colon Z_2(t, x) = c_1 Z_1(t, x) + c_2\}, \tag{27}$$

where a, b, c_1, c_2, are boundaries given in advance and $Z_1(t, x)$, $Z_2(t, x)$ are defined by (4), (5), respectively. A sequential plan determined by $\tau^{(2)}$ will be called a *fixed-energy plan.*

Observe that from Ito's formula for processes satisfying the stochastic integral equation (2) it follows that the relation

$$Z_1(t, \xi) = 2\xi^2(0) - 2\vartheta Z_2(t, \xi) + 2I(t, \xi) \tag{28}$$

holds with probability 1, where $I(t, \xi)$ denotes the stochastic integral $\int_0^t \xi(s)\,dW(s)$.

LEMMA 4. *If* $-\infty < a < 0$, *then*

$$\mathrm{E}_\vartheta(\tau^{(1)})^n < \infty \tag{29}$$

for every $n = 1, 2, \ldots$ *and all* $\vartheta \in D$.

Proof. Observe that (see formula (4) and Fig. 1)

$$\tau^{(1)}(\xi) = \inf\{t:\ \xi^2(0)+\xi^2(t) = t+a\}$$

and

$$P(\tau^{(1)}(\xi) > t) \leqslant P(\xi^2(0)+\xi^2(t) > t+a) \leqslant P\left(\xi^2(0) > \frac{t+a}{2}\right)+P\left(\xi^2(t) > \frac{t+a}{2}\right).$$

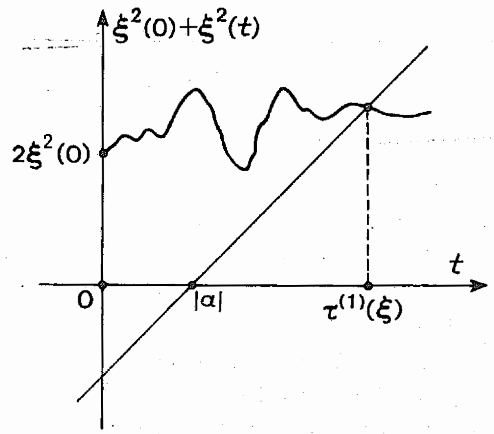

Fig. 1

Since the process $\xi(t)$, $t \in T$, is stationary, the terms on the right-hand side of the above inequality are equal. Thus

$$P(\tau^{(1)}(\xi) > t) \leqslant 2P\left(\xi^2(0) > \frac{t+a}{2}\right). \tag{30}$$

Taking into account the fact that the randon variable $\xi(0)$ is normally distributed with mean zero and variance $(2\vartheta)^{-1}$ and using the inequality

$$P(|X| > \lambda) \leqslant 2(\sqrt{2\pi}\,\lambda)^{-1}\exp(-\tfrac{1}{2}\lambda^2), \quad \lambda > 0,$$

for the standard normally distributed random variable X, from (30) we obtain

$$P(\tau^{(1)}(\xi) > t) \leqslant 2P(\sqrt{2\vartheta}\,|\xi(0)| > \sqrt{\vartheta(t+a)}) \leqslant 4[2\pi\vartheta(t+a)]^{-1/2}\exp[-\tfrac{1}{2}\vartheta(t+a)]$$

for $t > |a|$. Consequently, putting $k = 4(2\pi\vartheta)^{-1/2}\exp(-\frac{1}{2}\vartheta a)$ we have

$$P(\tau^{(1)}(\xi) > t) \leqslant k(t+a)^{-1/2}\exp(-\tfrac{1}{2}\vartheta t), \tag{31}$$

which implies

$$\int_{|a|}^{\infty} t^{n-1} P\left(\tau^{(1)}(\xi) > t\right) dt < \infty$$

for all $\vartheta \in D$. Thus the lemma is proved.

In particular, it follows from Lemma 4 that for $-\infty < a < 0$ condition (10) of the closedness of a sequential plan $(\tau^{(1)}, f)$ is satisfied. Henceforth, we shall suppose that $-\infty < a < 0$.

THEOREM 5. *A sequential plan* $\delta^{(1)} = (\tau^{(1)}, f^{(1)})$ *with*

$$f^{(1)} = \lambda_1 Z_2(\tau^{(1)}) + \lambda_2 \tag{32}$$

is efficient for

$$g(\vartheta) = \frac{\lambda_1(1-a\vartheta) + 2\lambda_2 \vartheta^2}{2\vartheta^2}, \tag{33}$$

where $\lambda_1 \neq 0$ *and* λ_2 *denote arbitrary constants. The variance of the estimator* $f^{(1)}$ *is equal to*

$$\mathrm{D}_\vartheta f^{(1)} = \frac{\lambda_1^2(2-a\vartheta)}{2\vartheta^4}. \tag{34}$$

Proof. First we show that $\mathrm{E}_\vartheta Z_2^2(\tau^{(1)}) < \infty$ for all $\vartheta \in D$.

Using the Schwarz inequality we get

$$\mathrm{E}_\vartheta Z_2(\tau^{(1)}) = \mathrm{E}_\vartheta \int_0^{\tau^{(1)}} \xi^2(s)\,ds = \mathrm{E}_\vartheta \int_0^{\infty} \chi_{\{s \leqslant \tau^{(1)}\}} \xi^2(s)\,ds$$

$$\leqslant l \int_0^{\infty} (\mathrm{E}_\vartheta \chi_{\{s \leqslant \tau^{(1)}\}})^{1/2} ds = l \int_0^{\infty} \left[P\left(\tau^{(1)}(\xi_\vartheta) > s\right)\right]^{1/2} ds,$$

where χ_A denote the indicator function of the set A and $l = \left(\mathrm{E}_\vartheta \xi^4(s)\right)^{1/2}$ is a positive and finite constant. Thus, by (31), we obtain $\mathrm{E}_\vartheta Z_2(\tau^{(1)}) < \infty$ for all $\vartheta \in D$. Then it follows from the properties of stochastic integrals with random upper limits (see, e.g., [3], Part I, § 4) that $\mathrm{E}_\vartheta I(\tau^{(1)}) = 0$ and

$$\mathrm{E}_\vartheta I^2(\tau^{(1)}) = \mathrm{E}_\vartheta Z_2(\tau^{(1)}). \tag{35}$$

For the sequential plan $\delta^{(1)}$ we have $Z_1(\tau^{(1)}) = a$ with probability 1, and formula (28) implies that for this plan the relation

$$2\vartheta Z_2(\tau^{(1)}) + a = 2\xi^2(0) + 2I(\tau^{(1)})$$

is valid with probability 1. Taking into account (35) we obtain

$$\mathrm{E}_\vartheta [2\vartheta Z_2(\tau^{(1)}) + a]^2 \leqslant 8\mathrm{E}_\vartheta \xi^4(0) + 8\mathrm{E}_\vartheta I^2(\tau^{(1)})$$

$$= 8\mathrm{E}_\vartheta \xi^4(0) + 8\mathrm{E}_\vartheta Z_2(\tau^{(1)}).$$

Since $E_\vartheta \xi^4(0) < \infty$ and $E_\vartheta Z_2(\tau^{(1)}) < \infty$ for all $\vartheta \in D$, we have $E_\vartheta Z_2^2(\tau^{(1)}) < \infty$ for all $\vartheta \in D$.

Taking into account the finiteness of $E_\vartheta Z_2^2(\tau^{(1)})$ it is easy to verify that regularity condition (iii) is satisfied for the plan $\delta^{(1)}$.

By (14) and (18), for the plan $\delta^{(1)}$ we have

$$E_\vartheta Z_2(\tau^{(1)}) = \frac{1-a\vartheta}{2\vartheta^2} \tag{36}$$

and

$$D_\vartheta Z_2(\tau^{(1)}) = \frac{2-a\vartheta}{2\vartheta^4}, \tag{37}$$

and formula (34) follows from (37).

Now we shall prove that an efficient sequential estimator $f(\tau^{(1)}, Z(\tau^{(1)}))$ in the plan $\delta^{(1)}$ is indeed of the form defined by (32) and the function (33) is the only efficiently estimable one in this plan.

Let $f(\tau^{(1)}, Z(\tau^{(1)}))$ be an efficient estimator in the plan $\delta^{(1)}$. Then it is efficient at a certain point $\vartheta_1 \in D$ and, by Theorem 1, takes the form

$$f(\tau^{(1)}, Z(\tau^{(1)})) = c(\vartheta_1)[1/\vartheta_1 - a - 2\vartheta_1 Z_2(\tau^{(1)})] + g(\vartheta_1)$$

with probability 1, where $c(\vartheta_1) \neq 0$. Hence, if $f(\tau^{(1)}, Z(\tau^{(1)}))$ is an efficient estimator, then there exist constants $\lambda_1 \neq 0$ and λ_2 such that

$$f(\tau^{(1)}, Z(\tau^{(1)})) = \lambda_1 Z_2(\tau^{(1)}) + \lambda_2 = f^{(1)}.$$

Moreover, only the function

$$g(\vartheta) = E_\vartheta f^{(1)} = \lambda_1 E_\vartheta Z_2(\tau^{(1)}) + \lambda_2 \tag{38}$$

is efficiently estimable in the plan $\delta^{(1)}$. By (36), from (38) we obtain (33), which completes the proof of the theorem.

In particular, it follows from Theorem 5 that

$$Z_2(\tau^{(1)}) = \int_0^{\tau^{(1)}} \xi^2(s)\,ds$$

is an efficient sequential estimator for $g(\vartheta) = (1-a\vartheta)/2\vartheta^2$.

Let us now consider the fixed-energy plan. From the ergodic theorem we obtain the following lemma:

LEMMA 5. *We have*

$$P(\tau^{(2)}(\xi_\vartheta) < \infty) = 1 \quad \textit{for all } \vartheta \in D. \tag{39}$$

THEOREM 6. *A sequential plan* $\delta^{(2)} = (\tau^{(2)}, f^{(2)})$ *with*

$$f^{(2)} = \lambda_1 Z_1(\tau^{(2)}) + \lambda_2 \tag{40}$$

is efficient for

$$g(\vartheta) = \frac{\lambda_1(1-2b\vartheta^2)+\lambda_2\vartheta}{\vartheta}, \tag{41}$$

where $\lambda_1 \neq 0$ *and* λ_2 *are arbitrary constants. The variance of the estimator* $f^{(2)}$ *is given by*

$$D_\vartheta f^{(2)} = \frac{2\lambda_1^2}{\vartheta^2}(1+2b\vartheta^2).$$

Proof. In the plan $\delta^{(2)}$ we have $Z_2(\tau^{(2)}) = b$ with probability 1, and relation (28) takes the form

$$Z_1(\tau^{(2)}) - 2\xi^2(0) = 2I(\tau^{(2)}) - 2b\vartheta. \tag{42}$$

Since

$$E_\vartheta Z_2(\tau^{(2)}) = E_\vartheta \int_0^{\tau^{(2)}} \xi^2(s)\,ds = b < \infty,$$

using the properties of stochastic integrals with random upper limits and relation (42), we infer in an analogous way as in Theorem 5 that $E_\vartheta Z_1^2(\tau^{(2)}) < \infty$ for all $\vartheta \in D$ and the regularity conditions for the sequential plan are satisfied.

By (14) and (18), for the plan $\delta^{(2)}$ we have

$$E_\vartheta Z_1(\tau^{(2)}) = \frac{1-2b\vartheta^2}{\vartheta} \tag{43}$$

and

$$D_\vartheta Z_1(\tau^{(2)}) = \frac{2}{\vartheta^2}(1+2b\vartheta^2).$$

Let $f(\tau^{(2)}, Z(\tau^{(2)}))$ be an efficient estimator in the plan $\delta^{(2)}$. Then, similarly as in Theorem 5 we infer from Theorem 1 that it is equal with probability 1 to the estimator defined by (40). By (43) we have

$$E_\vartheta f^{(2)} = \lambda_1 E_\vartheta Z_1(\tau^{(2)}) + \lambda_2 = \frac{\lambda_1(1-2b\vartheta^2)+\lambda_2\vartheta}{\vartheta} = g(\vartheta).$$

Thus the fixed-energy plan $\delta^{(2)}$ is efficient and $g(\vartheta)$ defined by (41) is the only efficiently estimable function in this plan.

By Theorem 6 we conclude that, e.g., $Z_1(\tau^{(2)}) = \xi^2(0) + \xi^2(\tau^{(2)}) - \tau^{(2)}$ is an efficient sequential estimator for $g(\vartheta) = (1-2b\vartheta^2)\vartheta^{-1}$.

LEMMA 6. *If* $c_1 > 0$ *and* $c_2 > 0$, *then*

$$P(\tau^{(3)}(\xi_\vartheta) < \infty) = 1 \quad \textit{for all } \vartheta \in D. \tag{44}$$

Proof. Let us observe that, by (28),

$$\tau^{(3)}(\xi_\vartheta) = \inf\{t\colon (1+2c_1\vartheta)\int_0^t \xi^2(s)\,ds - 2c_1\int_0^t \xi(s)\,dW(s) = 2c_1\xi^2(0)+c_2\}.$$

Put $\alpha = 1+2c_1\vartheta$ and $\beta = 2c_1$. It is easy to see that condition (44) is satisfied if, for all $\vartheta\in D$,

$$P\big(\lim_{t\to\infty}[\alpha\int_0^t \xi^2(s)\,ds - \beta\int_0^t \xi(s)\,dW(s)] = \infty\big) = 1.$$

Let

$$\lim_{T_0\to\infty}\ \sup_{0\leqslant t\leqslant T_0}[\alpha\int_0^t \xi^2(s)\,ds - \beta\int_0^t \xi(s)\,dW(s)] = \eta.$$

Then for every $K>0$ and all $\vartheta\in D$ we have

$$\begin{aligned}(45)\quad P(\eta\geqslant K) &\geqslant P\big(\sup_{0\leqslant t\leqslant T_0}[\alpha\int_0^t \xi^2(s)\,ds - \beta\int_0^t \xi(s)\,dW(s)]\geqslant K\big)\\ &\geqslant P\big(\alpha\int_0^{T_0}\xi^2(s)\,ds - \beta\sup_{0\leqslant t\leqslant T_0}|\int_0^t \xi(s)\,dW(s)|\geqslant K\big)\\ &\geqslant P\big(\alpha\int_0^{T_0}\xi^2(s)\,ds\geqslant 2K,\ \beta\sup_{0\leqslant t\leqslant T_0}|\int_0^t \xi(s)\,dW(s)|\leqslant K\big)\\ &\geqslant P\big(\alpha\int_0^{T_0}\xi^2(s)\,ds\geqslant 2K\big)+P\big(\beta\sup_{0\leqslant t\leqslant T_0}|\int_0^t \xi(s)\,dW(s)|\leqslant K\big)-1.\end{aligned}$$

Let $T_0 = K^{3/2}$. Then

$$P\big(\alpha\int_0^{T_0}\xi^2(s)\,ds\geqslant 2K\big) = P\Big(\frac{\alpha}{K^{3/2}}\int_0^{K^{3/2}}\xi^2(s)\,ds\geqslant\frac{2}{\sqrt{K}}\Big)$$

and by the ergodic theorem this probability tends to 1 as $K\to\infty$. Moreover, from the well-known inequality for stochastic integrals (see, e.g., [1], Theorem 5.1.1, or [3], Part I, § 3) we get

$$P\Big(\sup_{0\leqslant t\leqslant T_0}|\int_0^t \xi(s)\,dW(s)|\leqslant\frac{K}{\beta}\Big)\geqslant 1-\frac{\beta^2}{K^2}\int_0^{K^{3/2}} \mathrm{E}_\vartheta\,\xi^2(s)\,ds = 1-\frac{\beta\gamma}{\sqrt{K}},$$

where $\gamma = \mathrm{E}_\vartheta\,\xi^2(s) = (2\vartheta)^{-1}$. Thus the second probability on the right-hand side of (45) tends also to 1 as $K\to\infty$. We then have $P(\eta=\infty)=1$, for all $\vartheta\in D$, which proves the closedness of the plan.

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