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WEAK CONVERGENCE OF A NUMERICAL SCHEME FOR STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract. In this paper a numerical scheme approximating the solution to a stochastic differential equation is presented. On bounded subsets of time, this scheme has a finite state space, which allows us to decrease the round-off error when the algorithm is implemented. At the same time, the scheme introduced turns out locally consistent for any step size of time. Weak convergence of the scheme to the solution of the stochastic differential equation is shown.

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1. INTRODUCTION

This paper deals with numerical schemes for stochastic differential equations. A number of numerical methods for approximating solutions of such equations have been developed and, probably, the most important reference to this subject is the book by Kloeden and Platen [14]. However, authors such as Debrabant and Rössler [4], [5], Fard [7], Fleury [9], and Janković and Ilić [13], among others, have done important contributions in this address. The purpose of this paper is to present a new scheme of approximation, which is an extension of a scheme of approximation by Fierro and Torres [8] for ordinary differential equations. For a given stochastic differential equation (SDE), we propose a method consisting in approximating its solution by means of a sequence of Markov chains, where each of them, starting at a suitable initial condition and on any bounded time interval, could take values in a finite state space of rational numbers previously defined. One of the advantages of relaxing the requirement of strong convergence is that the Wiener process (WP) in the SDE could be approximated by random variables not

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defined in terms of the paths of this WP. The scheme we are introducing is a modification of the so-called Euler–Maruyama (EM) approximation method, however some important considerations have to be done when comparing our method with the EM scheme. In the latter, the Wiener process in the SDE is approximated by random variables depending on the same Wiener process, which produces the state space of the scheme is uncountable. This fact could be a serious problem because the digital computers are restricted to rational numbers when doing their calculations. Furthermore, digital computers are restricted to the use of a finite number of decimal places. Severe round-off errors could be presented whenever this consideration is not taken into account. Indeed, when approximation schemes are to be applied, their inherent round-off errors are an important subject which must be considered. Usually, while finer the discretization method, greater will be the rounding error. Some authors interested in decreasing the round-off error have contributed in this direction. For instance, this circumstance was considered by Bykov in [2] for systems of linear ordinary differential equations, by Srinivasu and Venkatesulu in [21] for nonstandard initial value problems, and by Wollman in [22] for the one-dimensional Vlasov–Poisson system, among others. We refer to Henrici in [10] for a complete discussion on methods for solving ordinary differential equation problems, error propagation and rate of convergence. Round-off errors for stochastic systems have been discussed, for instance, by Delanttre and Jacod [6], Rosenbaum [19] and Daumas et al. [3]. The scheme of approximation which is introduced in this paper aims to reduce the round-off error when numerical methods are used for approximating the unique solution of a stochastic differential equation. For this purpose, we introduce a method of approximation which considers in a number of cases, on bounded time intervals, a finite state space with states having few digits. However, to obtain this simplification we need to augment the randomness on the drift coefficient, which means that the states of the scheme are, in part, chosen according to a conditional probability depending on its drift coefficient.

The proposed scheme gives quite freedom to approximating the WP in the SDE, and moreover, it allows us to choose suitable state spaces for the scheme. However, since the convergence is weak, it is not possible in general to guarantee a good convergence rate, at least when the state space of the scheme is finite on bounded time intervals. For this reason, in Remark 3.1, we present an alternative scheme which converges strongly, with a convergence order equalling the standard EM scheme, although in this case, the state space is denumerable and stops to be finite on bounded time intervals.

Convergence in distribution of stochastic integrals is necessary to prove our main result. Hence, our methodology technically differs from that used for proving strong convergence of schemes towards the solution of an SDE. In this work, the condition (UT) (uniform tightness) for the convergence in distribution of stochastic integrals is used. This condition was introduced by Jakubowski et al. in [12] and by Słomiński in [20]. Theorems for proving convergence in distribution of stochastic integrals are found in [12], [15], [17], [20], among other references.

Other approximation schemes having a denumerable state space can be found in the books by Kloeden and Platen, Section 6.2 in [14], and by Kushner and Dupuis, Section 4.1 in [16]. These schemes are asymptotically locally consistent, which means the consistency has a place for small step sizes. As seen below, the scheme of approximation, we are introducing here, is unbiased and locally consistent for any step size and, at the same time, it has a finite state space on bounded subsets of time.

The plan of this paper is as follows. In Section 2, we define the approximation scheme, the main result is stated in Section 3, and its proof is deferred to Section 4.

2. THE APPROXIMATION SCHEME

In what follows, $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ stands for a stochastic basis satisfying the usual Dellacherie conditions, let $W = (W_1, \ldots, W_r)^\top$ be an r-dimensional Brownian motion defined on this stochastic basis, where $A^\top$ denotes the transpose of a matrix A, and consider the stochastic differential equation

$$\mathrm{d}X(t) = a\big(t, X(t)\big)\mathrm{d}t + b\big(t, X(t)\big)\mathrm{d}W(t), \quad X(0) = x_0, \tag{2.1}$$

where $x_0 \in \mathbb{R}^d$, $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$ satisfy sufficient conditions of regularity which ensure existence and uniqueness of the solution to (2.1). Here, $\mathbb{R}^{d\times r}$ stands for the space of all real $d \times r$-matrices. In the sequel, we denote by $\|x\|$ the Euclidean norm of a vector $x \in \mathbb{R}^d$ and by $|||A|||$ the Frobenius (or trace) norm of a matrix $A \in \mathbb{R}^{d\times r}$.

A Markov chain approximating the unique solution to (2.1) is defined by means of their probability transitions.

Let $N \in \mathbb{N} \setminus \{0\}$, and $H_N : \mathbb{R}^{d\times r} \to \mathbb{R}^{d\times r}$ be defined as

$$H_N(b) = \begin{pmatrix} [Nb_{11}]/N & \cdots & [Nb_{1r}]/N \\ \vdots & \ddots & \vdots \\ [Nb_{d1}]/N & \cdots & [Nb_{dr}]/N \end{pmatrix},$$

where $[x]$ stands for the integer part of $x \in \mathbb{R}$. By identifying $\mathbb{R}^d$ with $\mathbb{R}^{d\times 1}$, H_N is applied to any $a \in \mathbb{R}^d$. For each $n \in \mathbb{N} \setminus \{0\}$, let $0 = t_0^n < \ldots < t_k^n < \ldots$ be the partition of $\mathbb{R}_+$ defined recursively as $t_k^n = t_{k-1}^n + \Delta t_k^n$, where $\Delta t_k^n = 1/n$, $k \in \mathbb{N}$. In the sequel, for each $t \geqslant 0$, we define $c_n(t) = H_n(t)$, so that $c_n(t) = t_k^n$ if $t_k^n \leqslant t < t_{k+1}^n$.

On the space $(\Omega, \mathcal{F}, \mathbb{P})$ and for each $n \in \mathbb{N} \setminus \{0\}$, let $\{\zeta_k^n\}_{k\in\mathbb{N}}$ be a sequence of independent random vectors such that for each $k \geqslant 0$, $\zeta_k^n = \big(\zeta_k^n(1), \ldots, \zeta_k^n(r)\big)^\top$ with $\zeta_k^n(1), \ldots, \zeta_k^n(r)$ being independent random variables with mean zero and taking values in the set $\frac{1}{mn^q}S = \big\{\frac{x}{mn^q} : x \in S\big\}$, where $S \subset \mathbb{Z}$, $m \in \mathbb{N} \setminus \{0\}$ and $q \in \mathbb{N}$. Additionally, we assume

$$\sup_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{n} \sum_{k=0}^{n} \mathbb{E}(\|\zeta_k^n\|^2) < \infty, \tag{2.2}$$

and $\{W^n\}_{n\in\mathbb{N}\setminus\{0\}}$ converges in distribution to W, where $W^n(t) = \frac{1}{\sqrt{n}}\sum_{k=0}^{[nt]}\zeta_k^n$. For example, by the Donsker theorem, these conditions are fulfilled whenever $\zeta_k^n = \zeta_k = \big(\zeta_k(1),\dots,\zeta_k(r)\big)^\top$ and $\{\zeta_k\}_{k\in\mathbb{N}}$ is a sequence of independent and identically distributed random vectors such that $\mathbb{E}\big(\zeta_k(j)^2\big) = 1$ for all $k\in\mathbb{N}$ and $j = 1,\dots,r$.

Next, we define the sequence of random vectors $\{\xi_k^n\}_{k\in\mathbb{N}}$ and the Markov chain $X^n = \{X_k^n\}_{k\in\mathbb{N}}$, starting at $X_0^n = x_0$, recursively as follows. Let $\xi_0^n = 0 \in \mathbb{R}^d$ and suppose, for $k \geqslant 1$, X_{k-1}^n has been defined. Choose a random vector ξ_k^n such that

$$\mathbb{P}(\xi_k^n = e|\, X_{k-1}^n)(\omega) = \mu_{k-1}^1(\{e_1\},\omega)\dots\mu_{k-1}^d(\{e_d\},\omega), \tag{2.3}$$

where $e = (e_1,\dots,e_d)^\top \in \{0,1\}^d$, $\omega\in\Omega$, and $\mu_{k-1}^i(\cdot,\omega)$, $i = 1,\dots,d$, is the Bernoulli law with parameter

$$p_i^n(t_{k-1}^n)(\omega) = a_i\big(t_{k-1}^n, X_{k-1}^n(\omega)\big) - \big[Na_i\big(t_{k-1}^n, X_{k-1}^n(\omega)\big)\big]/N.$$

Next we define

$$X_k^n = X_{k-1}^n + \frac{1}{n}\big\{H_N\big(a(t_{k-1}^n, X_{k-1}^n)\big) + \xi_k^n + H_N\big(\sqrt{n}b(t_{k-1}^n, X_{k-1}^n)\big)\zeta_k^n\big\}. \tag{2.4}$$

The main aim of this paper is to approximate the solution to (2.1) by means of $X^n = \{X^n(t)\}_{t\geqslant 0}$, where

$$X^n(t) = \sum_{k=1}^{\infty} X_{k-1}^n \mathrm{I}_{[t_{k-1}^n, t_k^n)}(t),$$

and I_A stands for the indicator function of a set A. It is clear that X^n is an $\mathbb{F}^n$-adapted process.

Note that, for each $n\in\mathbb{N}$ and any $T > 0$, the Markov chain X^n, on $[0,T]$, has its states in the denumerable set $E_T = \big\{x_0 + \frac{z}{mNn^{q+1}} : z\in S^r\big\}$. We are particularly interested in the case when S is finite because so is E_T. For example, this is obtained, with $m = 1$ and $q = 0$, if $\mathbb{P}\big(\zeta_k^n(i) = 1\big) = \mathbb{P}\big(\zeta_k^n(i) = -1\big) = 1/2$ for all $i = 1,\dots,r$ and $n\in\mathbb{N}\setminus\{0\}$.

Due to the fact that H_N converges to the identity function as N goes to infinity, it is worth noticing that the approximation scheme we are proposing is similar to the Euler scheme for large N, since in this case, $p_i^n(t_{k-1}^n)$, $i = 1,\dots,d$, take small values, and hence, the random vectors ξ_k^n have their distributions close to δ_0, the Dirac measure at zero. On the other hand, by taking smaller values of N, the size of the state space decreases, however, in order to obtain convergence of X^n, the presence of the random vectors ξ_k^n turns out relevant. Hence, the smaller state space that we choose, the bigger relevancy of the random vectors ξ_k^n becomes.

In the sequel, given a local square integrable martingale Z (with respect to a given stochastic basis), we denote by $\langle Z\rangle$ the predictable increasing process of

Z, i.e. $\langle Z\rangle$ is the unique predictable increasing process having the property that $Z^2 - \langle Z\rangle$ is a martingale.

LEMMA 2.1. *Let $\mathbb{F} = \{\mathcal{F}_k\}_{k\in\mathbb{N}}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\eta_k\}_{k\in\mathbb{N}}$ be a sequence of d-dimensional $(\mathbb{F}, \mathbb{P})$-martingale differences such that $\mathbb{E}(\|\eta_k\|^2) < \infty$, $n \in \mathbb{N}$, and $Z^n = \{Z^n(t)\}_{t\geqslant 0}$ be the process defined as $Z^n(t) = \sum_{k=0}^{[nt]} \eta_k$. Then, Z^n is a d-dimensional $(\mathbb{F}^n, \mathbb{P})$-martingale, where $\mathbb{F}^n = \{\mathcal{F}_{[nt]}\}_{t\geqslant 0}$. Moreover, if $d = 1$, the predictable increasing process of Z^n is given by*

$$\langle Z^n\rangle(t) = \sum_{k=1}^{[nt]} \mathbb{E}(\eta_k^2|\mathcal{F}_{k-1}).$$

Proof. Clearly, Z^n is $\mathbb{F}^n$-adapted, and for each $t \geqslant 0$, $\mathbb{E}\big(\|Z^n(t)\|^2\big) < \infty$. Let $s, t \in \mathbb{R}_+$ and $s < t$. Hence

$$\mathbb{E}(\eta_k|\mathcal{F}_s^n) = \begin{cases} \eta_k & \text{if } k \leqslant [ns], \\ 0 & \text{if } k > [ns], \end{cases}$$

and, consequently,

$$\mathbb{E}\big(Z^n(t)|\mathcal{F}_s^n\big) = \sum_{k=1}^{[ns]} \eta_k = Z^n(s),$$

which proves that Z^n is a d-dimensional $(\mathbb{F}^n, \mathbb{P})$-martingale.

Next, assume $d = 1$ and let $Y = \{Y(t)\}_{t\geqslant 0}$ with $Y(t) = \sum_{k=1}^{[nt]} \mathbb{E}(\eta_k^2|\mathcal{F}_{k-1})$. Clearly, for $t > 0$, Y is $\mathcal{F}_{t-}$-measurable, increasing, and since the jumps of Y are deterministic, Y is predictable. Let

$$Q^n(t) = (Z^n)(t)^2 - \sum_{k=1}^{[nt]} \mathbb{E}(\eta_k^2|\mathcal{F}_{k-1}).$$

We need to prove that $\mathbb{E}\big(Q(t)|\mathcal{F}_t^n\big) = Q(s)$. We have

$$\begin{aligned}
&\mathbb{E}\big(Q^n(t) - Q^n(s)|\mathcal{F}_s^n\big) \\
&\qquad = \mathbb{E}\left(Z^n(t)^2 - Z^n(s)^2|\mathcal{F}_s^n\right) - \sum_{k=[ns]+1}^{[nt]} \mathbb{E}\big(\mathbb{E}(\eta_k^2|\mathcal{F}_{k-1})|\mathcal{F}_s^n\big) \\
&\qquad = \mathbb{E}\Big(\big(Z^n(t) - Z^n(s)\big)^2|\mathcal{F}_s^n\Big) - \sum_{k=[ns]+1}^{[nt]} \mathbb{E}(\eta_k^2|\mathcal{F}_s^n) \\
&\qquad = \mathbb{E}\Big(\big(\sum_{k=[ns]+1}^{[nt]} \eta_k\big)^2|\mathcal{F}_s^n\Big) - \sum_{k=[ns]+1}^{[nt]} \mathbb{E}(\eta_k^2|\mathcal{F}_s^n) = 0.
\end{aligned}$$

The last equality is a consequence of the fact that, for $[ns] < i < j$, $\mathbb{E}(\eta_j|\mathcal{F}_i) = 0$, and

$$\mathbb{E}(\eta_i\eta_j|\mathcal{F}_s^n) = \mathbb{E}\big(\eta_i\mathbb{E}(\eta_j|\mathcal{F}_i)|\mathcal{F}_s^n\big) = 0.$$

Therefore, $\mathbb{E}\big(Q^n(t)|\mathcal{F}_s^n\big) = Q^n(s)$, and the proof is complete. ∎

In the sequel, for each $n \in \mathbb{N}$, $\mathbb{F}^n = \{\mathcal{F}_t^n; t \geqslant 0\}$, where for each $t \geqslant 0$, $\mathcal{F}_t^n = \mathcal{F}_{[nt]}$ and $\mathcal{F}_k$ is the completion of $\sigma(\xi_0^n, \ldots, \xi_k^n, \zeta_0^n, \ldots, \zeta_k^n)$, the σ-field generated by $\xi_0^n, \ldots, \xi_k^n, \zeta_0^n, \ldots, \zeta_k^n$, by aggregation of sets from $\mathcal{F}$ having $\mathbb{P}$-measure zero.

For each $t \geqslant 0$ and $n \geqslant 1$, we define

$$L_i^n(t) = \frac{1}{n}\sum_{k=1}^{[nt]}\{\xi_k^n(i) - p_i^n(t_{k-1}^n)\}, \quad i = 1, \ldots, d,$$

where $\xi_k^n = \big(\xi_k^n(1), \ldots, \xi_k^n(d)\big)^\top$. Let $L^n(t) = \big(L_1^n(t), \ldots, L_d^n(t)\big)^\top$. Lemma 2.1 implies that $L^n = \{L^n(t)\}_{t\geqslant 0}$ is a d-dimensional $(\mathbb{F}^n, \mathbb{P})$-martingale and $W^n = \{W^n(t)\}_{t\geqslant 0}$ is an r-dimensional $(\mathbb{F}^n, \mathbb{P})$-martingale. Moreover,

$$\mathbb{E}\big(\{\xi_k^n(i) - p_i^n(t_{k-1}^n)\}^2|\mathcal{F}_{k-1}\big) = p_i^n(t_{k-1}^n)\big(1 - p_i^n(t_{k-1}^n)\big)$$

and from Lemma 2.1 we obtain

$$\langle L_i^n\rangle(t) = \frac{1}{n^2}\sum_{k=1}^{[nt]} p_i^n(t_{k-1}^n)\big(1 - p_i^n(t_{k-1}^n)\big). \tag{2.5}$$

Also, by Lemma 2.1,

$$\langle W_j^n\rangle(t) = \frac{1}{n}\sum_{k=1}^{[nt]}\mathbb{E}\big(\zeta_k^n(j)^2\big), \tag{2.6}$$

where $W^n = (W_1^n, \ldots, W_r^n)^\top$.

Let

$$M^n(t) = \int_0^t \left\{\frac{H_N\big(\sqrt{n}b\big(u-, X^n(u-)\big)\big)}{\sqrt{n}} - b\big(u-, X^n(u-)\big)\right\}\mathrm{d}W^n(u).$$

By putting $M^n = (M_1^n, \ldots, M_d^n)^\top$ and

$$b = \begin{pmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{d1} & \cdots & b_{dr}\end{pmatrix},$$

we observe that, for each $i = 1, \ldots, d$,

(2.7)

$$M_i^n(t) = \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}\sum_{j=1}^{r}\left\{\frac{H_N\big(\sqrt{n}b_{ij}(t_{k-1}^n, X_{k-1}^n)\big)}{\sqrt{n}} - b_{ij}(t_{k-1}^n, X_{k-1}^n)\right\}\zeta_k^n(j).$$

Since $M^n(t)$ is a stochastic integral of an $\mathbb{F}^n$-predictable process with respect to an $(\mathbb{F}^n, \mathbb{P})$-martingale, $M^n = \{M^n(t)\}_{t \geqslant 0}$ is a d-dimensional $(\mathbb{F}^n, \mathbb{P})$-martingale. Moreover, by (2.7) and Lemma 2.1, we have

(2.8)
$$\langle M_i^n \rangle(t) = \frac{1}{n} \sum_{k=1}^{[nt]} \sum_{j=1}^{r} \left\{ \frac{H_N\big(\sqrt{n} b_{ij}(t_{k-1}^n, X_{k-1}^n)\big)}{\sqrt{n}} - b_{ij}(t_{k-1}^n, X_{k-1}^n) \right\}^2 \mathbb{E}\big(\zeta_k^n(j)^2\big).$$

Note that, for each $t \geqslant 0$,

$$\begin{aligned} X^n(t) = x_0 &+ \int_0^t a\big(u-, X^n(u-)\big)\, \mathrm{d}c_n(u) \\ &+ \int_0^t b\big(u-, X^n(u-)\big)\, \mathrm{d}W^n(u) + R^n(t), \end{aligned} \tag{2.9}$$

where $R^n(t) = L^n(t) + M^n(t)$.

On the other hand,

$$\mathbb{E}(X_k^n - X_{k-1}^n | \mathcal{F}_{k-1}^n) = a(t_{k-1}^n, X_{k-1}^n)\Delta t_k^n, \quad k \in \mathbb{N} \setminus \{0\}.$$

Since the above equality holds for all $n \in \mathbb{N}$, and consequently for any step size, we say that the scheme of approximation, $X^n = \{X^n(t)\}_{t \geqslant 0}$, is unbiased and locally consistent.

3. MAIN RESULT

In this section we state the main results, and their proofs are deferred to the next section. We assume the vectorial function $a = (a_1, \ldots, a_d)^\top$ and the matricial function $b = (b_{ij}; 1 \leqslant i \leqslant d, 1 \leqslant j \leqslant r)$ are continuous and there exists a constant $C > 0$ such that the following two conditions hold for all $x, y \in \mathbb{R}^d$ and $s, t \in \mathbb{R}_+$:

(A1) $\quad \|a(t,x) - a(t,y)\|^2 + \|\!|b(t,x) - b(t,y)|\!\|^2 \leqslant C\|x - y\|^2.$

(A2) $\quad \|a(t,x)\|^2 + \|\!|b(t,x)|\!\|^2 \leqslant C(1 + \|x\|^2).$

It is well known that under conditions (A1) and (A2), equation (2.1) has a unique solution, which we assume, along with the Wiener process W, to be defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, these conditions enable us to prove that, as in the classical Euler–Maruyama scheme for SDEs (see Theorem 10.2.2 in [14]), we can choose a good sequence $\{\zeta_k^n\}_{n \in \mathbb{N} \setminus \{0\}}$ such that the order of strong convergence of $\{X^n\}_{n \in \mathbb{N} \setminus \{0\}}$ equals $1/2$ (see Remark 3.1 below). This fact means that, for each $T > 0$, there exists a constant $c(T) > 0$ such that

$$\mathbb{E}\big(\|X^n(T) - X(T)\|\big) \leqslant c(T)/n^{1/2},$$

where X is the unique solution to (2.1).

The main result is the following:

THEOREM 3.1. *Let X be the unique solution to* (2.1). *Then*, $\{X^n\}_{n\in\mathbb{N}\setminus\{0\}}$ *converges in distribution to* X.

REMARK 3.1. *By choosing a suitable sequence* $\{\zeta_k^n\}_{n\in\mathbb{N}\setminus\{0\}}$ *in the scheme*, *it is seen that each* X^n *has a finite number of states on a bounded interval* $[0,T]$. *But*, *because the convergence of the scheme is weak*, *it is not possible to guarantee a good order of convergence. However*, *by defining*

$$\zeta_k^n(j) = \frac{1}{n}\left(\frac{1}{2} + [n\eta_k^n(j)]\right),$$

where $\eta_k^n(j) = \sqrt{n}\big(W_j(t_k^n) - W_j(t_{k-1}^n)\big)$, $j = 1,\dots,r$, $\{\zeta_k^n\}_{n\in\mathbb{N}}$ *takes values in* $\frac{1}{n}\left(\frac{1}{2} + \mathbb{Z}^r\right)$, *clearly satisfies the assumptions*, *and the scheme* $\{X^n\}_{n\in\mathbb{N}\setminus\{0\}}$ *converges strongly with order of convergence* $1/2$, *which equals the order of strong convergence of the standard Euler–Maruyama scheme* (see Theorem 10.2.2 in [14]). *Although*, *in this case*, X^n *does not have a finite number of states on* $[0,T]$, *its state space is contained in a discrete subset of* $\mathbb{Q}$ *on* $[0,\infty)$.

The proof of what is mentioned in this remark is similar to the classical one of the Euler–Maruyama scheme and is given in the Appendix.

4. PROOF OF THE MAIN RESULT

Some notation are used in what follows. As usual, $\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)$ denotes the space of all functions $x:\mathbb{R}_+\to\mathbb{R}^d$, which are right-continuous and have left-hand limits. In the sequel, we consider this space provided with the usual metrizable Skorokhod topology. Given a subset E of $\mathbb{R}_+$, the space of continuous functions from E to $\mathbb{R}^d$ is denoted by $\mathrm{C}(E,\mathbb{R}^d)$ and is considered endowed with the uniform topology.

A sequence $\{Z^n\}_{n\in\mathbb{N}\setminus\{0\}}$ with trajectories in $\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)$ is said to be C-*tight* if it is tight and any subsequence $\{Z^{n_k}\}_{k\in\mathbb{N}}$ of $\{Z^n\}_{n\in\mathbb{N}\setminus\{0\}}$, converging in distribution to Z, satisfies $\mathbb{P}\big(Z\in\mathrm{C}(\mathbb{R}_+,\mathbb{R}^d)\big) = 1$.

In order to prove C-tightness of certain processes, we define, for any fix $T>0$, $\omega_T:\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)\times]0,\infty[\to\mathbb{R}$ as

$$\omega_T(x,\delta) = \sup_{s,t\in[0,T],|s-t|<\delta}\|x(s)-x(t)\|.$$

Given a process $Y = \{Y(t)\}_{t\geqslant 0}$ with trajectories in $\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)$, we write $\Delta Y(t) = Y(t) - Y(t-)$.

In order to prove the main result stated in Section 3, we need the following lemmas.

LEMMA 4.1. *For each $T > 0$ and $N \in \mathbb{N} \setminus \{0\}$, there exists a positive constant $C(T,N)$ such that*

$$\mathbb{E}\big(\sup_{0\leqslant t\leqslant T} \|L^n(t)\|^2\big) \leqslant \frac{C(T,N)}{n} \quad \textit{and} \quad \mathbb{E}\big(\sup_{0\leqslant t\leqslant T} \|M^n(t)\|^2\big) \leqslant \frac{C(T,N)}{n}.$$

Proof. Fix $T > 0$. From (2.5), for each $i = 1, \ldots, d$, and $j = 1, \ldots, r$, we obtain

$$\langle L_i^n \rangle(t) = \frac{1}{n}\int_0^t p_i^n\big(c_n(u)\big)\Big(1 - p_i^n\big(c_n(u)\big)\Big)\,\mathrm{d}c_n(u)$$

and, by Doob's inequality,

$$\begin{aligned}\mathbb{E}\big(\sup_{0\leqslant s\leqslant T} \|L^n(s)\|^2\big) &\leqslant \frac{4}{n}\sum_{i=1}^{d}\int_0^T \mathbb{E}\Big(p_i^n\big(c_n(u)\big)\Big(1 - p_i^n\big(c_n(u)\big)\Big)\Big)\mathrm{d}c_n(u)\\ &\leqslant \frac{c_n(T)d}{n} \leqslant \frac{Td}{n}.\end{aligned}$$

On the other hand, from (2.8) we have

$$\begin{aligned}&\langle M_i^n\rangle(t)\\ &= \frac{1}{n}\sum_{k=1}^{[nt]}\sum_{j=1}^{r}\left(\frac{\big[N\sqrt{n}b_{ij}\big(u-, X^n(u-)\big)\big]}{N\sqrt{n}} - b_{ij}\big(u-, X^n(u-)\big)\right)^2 \mathbb{E}\big(\zeta_k^n(j)^2\big)\\ &\leqslant \frac{1}{nN^2}\left(\frac{1}{n}\sum_{k=1}^{[nt]}\mathbb{E}(\|\zeta_k^n\|^2)\right).\end{aligned}$$

Thus, $\sum_{i=1}^d \langle M_i^n\rangle(t) \leqslant \frac{d}{nN^2}\big(\frac{1}{n}\sum_{k=1}^{[nt]}\mathbb{E}(\|\zeta_k^n\|^2)\big)$ and, by Doob's inequality,

$$\mathbb{E}\big(\sup_{0\leqslant s\leqslant T}\|M^n(s)\|^2\big) \leqslant 4\sum_{i=1}^{d}\mathbb{E}\big(\langle M_i^n\rangle(T)\big) \leqslant \frac{4dT}{nN^2}\left(\frac{1}{[nT]}\sum_{k=1}^{[nT]}\mathbb{E}(\|\xi_k^n\|^2)\right).$$

Hence, condition (2.2) implies that there exists $C(T,N) > 0$ such that

$$\mathbb{E}\big(\sup_{0\leqslant t\leqslant T} \|L^n(t)\|^2\big) \leqslant \frac{C(T,N)}{n} \quad \text{and} \quad \mathbb{E}\big(\sup_{0\leqslant t\leqslant T} \|M^n(t)\|^2\big) \leqslant \frac{C(T,N)}{n},$$

which completes the proof. ■

LEMMA 4.2. *For each $T > 0$, we have*

$$\sup_{n\in\mathbb{N}\setminus\{0\}} \mathbb{E}\big(\sup_{0\leqslant t\leqslant T}\|X^n(t)\|^2\big) < \infty.$$

P r o o f. By (2.9) and Jensen's inequality, we have

$$\sup_{0\leqslant s\leqslant t}\|X^n(s)\|^2 \leqslant 4\Big(\|x_0\|^2 + c_n(t)\int_0^t \sup_{0\leqslant u\leqslant s}\big\|a\big(u,X^n(u)\big)\big\|^2\,\mathrm{d}c_n(s) + \sup_{0\leqslant s\leqslant t}\Big\|\int_0^s b\big(u,X^n(u)\big)\,\mathrm{d}W^n(u)\Big\|^2 + \sup_{0\leqslant s\leqslant t}\|R^n(s)\|^2\Big).$$

By taking expectation and applying Doob's inequality and Fubini's theorem, we obtain

$$\mathbb{E}\big(\sup_{0\leqslant s\leqslant t}\|X^n(s)\|^2\big)\leqslant 4\Big[\|x_0\|^2+c_n(t)\int_0^t \mathbb{E}\Big(\sup_{0\leqslant u\leqslant s}\big\|a\big(u,X^n(u)\big)\big\|^2\Big)\,\mathrm{d}c_n(s) + 4\int_0^t \mathbb{E}\Big(\big|\!\big|\!\big|b\big(u,X^n(u)\big)\big|\!\big|\!\big|^2\Big)\,\mathrm{d}c_n(u) + \mathbb{E}\big(\sup_{0\leqslant s\leqslant t}\|R^n(s)\|^2\big)\Big].$$

By (A2), there exist two positive constants, C_T and D_T, such that

$$\mathbb{E}\big(\sup_{0\leqslant s\leqslant t}\|X^n(s)\|^2\big) \leqslant r_T^n + C_T + D_T\int_0^t \mathbb{E}\big(\sup_{0\leqslant s\leqslant u}\|X^n(u)\|^2\big)\,\mathrm{d}c_n(u), \tag{4.1}$$

where $r_T^n = 4\mathbb{E}\big(\sup_{0\leqslant s\leqslant T}\|R^n(s)\|^2\big)$.

Let $\varphi_n(t) = \mathbb{E}\big(\sup_{0\leqslant s\leqslant t}\|X^n(s)\|^2\big)$. By (4.1) and a slight modification of Gronwall's inequality, we have

$$\varphi_n(t) \leqslant (r_T^n + C_T)\exp\big(D_T c_n(t)\big).$$

From Lemma 4.1 we infer that $\{r_T^n\}_{n\in\mathbb{N}\setminus\{0\}}$ converges to zero, and therefore

$$\sup_{n\in\mathbb{N}\setminus\{0\}} \varphi_n(T) < \infty,$$

which concludes the proof of the lemma. ∎

LEMMA 4.3. *The sequence $\{X^n\}_{n\in\mathbb{N}\setminus\{0\}}$ is C-tight.*

P r o o f. Fix $T > 0$ and notice that $X^n = x_0 + A^n + B^n + R^n$, where, for each $t \in [0,T]$,

$$A^n(t) = \int_0^t a\big(u-,X^n(u-)\big)\,\mathrm{d}c_n(u) \quad \text{and} \quad B^n(t) = \int_0^t b\big(u,X^n(u-)\big)\,\mathrm{d}W^n(u).$$

By Lemmas 4.1 and 4.2, and Proposition 3.26 in [11], it suffices to prove that

(4.2) for all $T > 0$ and $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that
$$\mathbb{E}\big(\omega_T(A^n + B^n,\delta)\big) < \epsilon \quad \text{for all } n \geqslant n_0.$$

Condition (A2) implies that

$$\sup_{0\leqslant t\leqslant T}\|\Delta A^n(t)\| \leqslant \frac{C^{1/2}T}{n}\max_{0\leqslant k\leqslant [nT]}(1+\|X^n_{k-1}\|^2)^{1/2}$$

and

$$\omega_T(A^n,\delta) \leqslant \frac{\delta C^{1/2}T}{n}\max_{0\leqslant k\leqslant [nT]}(1+\|X^n_{k-1}\|^2)^{1/2}. \tag{4.3}$$

Notice that $B^n(t)=\sqrt{n}\int_0^t b(t^n_{[nu]-1},X^n_{[nu]-1})\zeta^n_{[nu]}\,\mathrm{d}c^n(u)$ and from (A2), for any $s,t\in[0,T]$ such that $0\leqslant t-s<\delta$, we obtain

$$\|B^n(t)-B^n(s)\| \leqslant \frac{C^{1/2}T}{\sqrt{n}}\max_{0\leqslant k\leqslant [nT]}(1+\|X^n_{k-1}\|^2)^{1/2}\max_{0\leqslant k\leqslant [nT]}\|\zeta^n_k\|(\delta+1/n)$$

and

$$\omega_T(B^n,\delta) \leqslant (\delta+1/n)C^{1/2}T\max_{0\leqslant k\leqslant [nT]}(1+\|X^n_{k-1}\|^2)^{1/2}\max_{0\leqslant k\leqslant [nT]}\frac{\|\zeta^n_k\|}{\sqrt{n}}. \tag{4.4}$$

Hence conditions (4.3) and (4.4) imply that

$$\begin{aligned}\mathbb{E}\big(\omega_T(A^n+B^n,\delta)\big) &\leqslant \mathbb{E}\big(\omega_T(A^n,\delta)\big)+\mathbb{E}\big(\omega_T(B^n,\delta)\big)\\ &\leqslant (\delta+1/n)C^{1/2}T\alpha(n)\beta(n),\end{aligned}$$

where

$$\alpha(n)=\Big[1+\mathbb{E}\big(\sup_{0\leqslant t\leqslant T}\|X^n(t)\|^2\big)\Big]^{1/2}$$

and

$$\beta(n)=1+\left[\frac{T}{[nT]}\sum_{k=0}^{[nT]}\mathbb{E}(\|\zeta^n_k\|^2)\right]^{1/2}.$$

Consequently, (4.2) is obtained from Lemma 4.2 and condition (2.2), which completes the proof. ■

For each $n\in\mathbb{N}$, let $\Lambda_n:\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)\times\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)$ be the function defined as

$$\Lambda_n(x,y)(t)=x(t)-x(0)-\int_0^t a\big(u-,x(u-)\big)\,\mathrm{d}c_n(u)-y(t).$$

Also, we define $\Lambda:\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)\times\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)$ as

$$\Lambda(x,y)(t)=x(t)-x(0)-\int_0^t a\big(u,x(u)\big)\,\mathrm{d}u-y(t).$$

LEMMA 4.4. *Let $\{Z^n\}_{n\in\mathbb{N}}$ be a sequence of processes having their paths in* $\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)\times\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)$, *and suppose this sequence converges in distribution to a process Z satisfying $\mathbb{P}\big(Z\in\mathrm{C}(\mathbb{R}_+,\mathbb{R}^d)\times\mathrm{C}(\mathbb{R}_+,\mathbb{R}^d)\big)=1$. Then, $\{\Lambda_n(Z^n)\}_{n\in\mathbb{N}}$ converges in distribution to $\Lambda(Z)$.*

Proof. Let E be the set of all $(x,y)\in\mathrm{C}(\mathbb{R}_+,\mathbb{R}^d)\times\mathrm{C}(\mathbb{R}_+,\mathbb{R}^d)$ such that $\{\Lambda(x_n,y_n)\}_{n\in\mathbb{N}}$ does not converge to $\Lambda(x,y)$ for some sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ in $\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)\times\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)$ converging to (x,y). Since

$$\mathbb{P}\big(Z\in\mathrm{C}(\mathbb{R}_+,\mathbb{R}^d)\times\mathrm{C}(\mathbb{R}_+,\mathbb{R}^d)\big)=1,$$

in order to prove that $\mathbb{P}(Z\in E)=0$, it suffices to verify that

$$E\cap[\mathrm{C}(\mathbb{R}_+,\mathbb{R}^d)\times\mathrm{C}(\mathbb{R}_+,\mathbb{R}^d)]=\emptyset.$$

Take $(x,y)\in\mathrm{C}(\mathbb{R}_+,\mathbb{R}^d)\times\mathrm{C}(\mathbb{R}_+,\mathbb{R}^d)$, and let $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ be any sequence in $\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)\times\mathrm{D}(\mathbb{R}_+,\mathbb{R}^d)$ converging to (x,y). Since (x,y) is continuous, this sequence converges to (x,y) uniformly on compact subsets of $\mathbb{R}_+$.

From (A1), for each $T>0$, we get

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\|\Lambda_n(x_n,y_n)(t)-\Lambda(x,y)(t)\|\\
&\leqslant 2\sup_{0\leqslant t\leqslant T}\|x_n(t)-x(t)\|+2\sup_{0\leqslant t\leqslant T}\|y_n(t)-y(t)\|\\
&\quad+\sup_{0\leqslant t\leqslant T}\big\|a\big(t,x_n(t)\big)\big\|\int_0^T \mathrm{d}\big(u-c_n(u)\big)+C^{1/2}\int_0^T\|x_n(u)-x(u)\|\,\mathrm{d}u,
\end{aligned}$$

and since, by (A2),

$$\sup_{0\leqslant t\leqslant T}\big\|a\big(t,x_n(t)\big)\big\|\int_0^T \mathrm{d}\big(u-c_n(u)\big)\leqslant\frac{C^{1/2}}{n}\big(1+\sup_{0\leqslant t\leqslant T}\|x_n(t)\|^2\big)^{1/2},$$

we obtain

$$\lim_{n\to\infty}\sup_{0\leqslant t\leqslant T}\|\Lambda_n(x_n)(t)-\Lambda(x)(t)\|=0.$$

Consequently, $(x,y)\notin E$ and $\mathbb{P}(Z\in E)=0$. Theorem 5.5 in [1] implies that $\{\Lambda_n(Z^n)\}_{n\in\mathbb{N}}$ converges in distribution to $\Lambda(Z)$, which completes the proof. ■

Proof of Theorem 3.1. Let $\{n_m\}_{m\in\mathbb{N}}$ be any increasing sequence of positive integers. The Donsker theorem implies that $\{W^{n_m}\}_{m\in\mathbb{N}}$ is C-tight, and, by Lemma 4.3, so is $\{X^{n_m}\}_{m\in\mathbb{N}}$. Thus, there exists a subsequence $\{n_{m_k}\}_{k\in\mathbb{N}}$ of $\{n_m\}_{m\in\mathbb{N}}$ such that $\{(X^{n_{m_k}},W^{n_{m_k}})\}_{k\in\mathbb{N}}$ converges in distribution to a pair of processes (X, W). Let $Y^n = \{Y^n(t)\}_{t\geqslant 0}$ be the process defined as $Y^n(t) = \int_0^t b\big(u, X^n(u)\big)\,\mathrm{d}W^n(u)$. From (2.9), for each $k \in \mathbb{N}$, we obtain

$$\Lambda_{n_{m_k}}(X^{n_{m_k}}, Y^{n_{m_k}}) = R^{n_{m_k}}. \tag{4.5}$$

Since $\mathbb{E}\left(\sup_{t\geqslant 0} |\Delta W^{n_{m_k}}(t)|\right) \leqslant 1/\sqrt{n_{m_k}}$, Theorem 2.6 and Proposition 3.2 in [12] (see also Theorem 1-8 in [17] and Theorem 1 in [20]) imply that the sequence $\{(X^{n_{m_k}}, Y^{n_{m_k}})\}_{k\in\mathbb{N}}$ converges in distribution to (X, Y), where $Y = \{Y(t)\}_{t\geqslant 0}$ is defined as $Y(t) = \int_0^t b\big(u, X(u)\big)\,\mathrm{d}W(u)$. Hence, (4.5) and Lemmas 4.1 and 4.4 imply that $\Lambda(X, Y) = 0$, which proves that $\{X^n\}_{n\in\mathbb{N}}$ converges in distribution to the unique solution X to (2.1). Therefore, the proof is complete. ■

5. APPENDIX

Proof of Remark 3.1. For $t \in [0, T]$, we set

$$Z^n(t) = \mathbb{E}\big(\sup_{0\leqslant u\leqslant t} \|X^n(u) - X(u)\|^2\big).$$

Theorem 3.1 implies $C_1 := \sup_{n\in\mathbb{N}\setminus\{0\}} \mathbb{E}\big(\sup_{0\leqslant t\leqslant T} \big\|a\big(t, X^n(t)\big)\big\|^2\big) < \infty$ and $C_2 := \sup_{n\in\mathbb{N}\setminus\{0\}} \mathbb{E}\big(\sup_{0\leqslant t\leqslant T} \big\|\big|b\big(t, X^n(t)\big)\big\|\big|^2\big) < \infty$. Hence,

$$\begin{aligned}\mathbb{E}\Big(\Big|\!\Big|\!\Big|\int_0^t b\big(u-, X^n(u-)\big)\,\mathrm{d}\big(W^n(u) - W(u)\big)\Big|\!\Big|\!\Big|\Big) &\leqslant \frac{C_2^{1/2}}{\sqrt{n}} \sum_{k=1}^{[nt]} \mathbb{E}(\|\zeta_k^n - \eta_k^n\|^2)^{1/2} \\ &\leqslant \frac{\sqrt{C_2 r}[nt]}{2n^{3/2}},\end{aligned}$$

and condition (2.9) implies that

$$\begin{aligned}X^n(t) - X(t) &= \int_0^t \Big(a\big(u-, X^n(u-)\big) - a\big(u-, X(u-)\big)\Big)\,\mathrm{d}u \\ &+ \int_0^t \Big(b\big(u-, X^n(u-)\big) - b\big(u-, X(u-)\big)\Big)\,\mathrm{d}W(u) + R^n(t) \\ &+ \int_0^t a\big(u-, X^n(u-)\big)\,\mathrm{d}\big(c_n(u) - u\big) \\ &+ \int_0^t b\big(u-, X^n(u-)\big)\,\mathrm{d}\big(W^n(u) - W(u)\big).\end{aligned}$$

Consequently, by conditions (A1) and (A2), the Doob inequality, Lemma 4.1 and a standard deduction, there exists a constant $c_T > 0$ such that

$$Z^n(t) \leqslant c_T\Big(\int_0^t Z^n(u)\,\mathrm{d}u + \frac{1}{n}\Big).$$

From the Gronwall inequality we obtain $Z^n(T) \leqslant c_T \exp(c_T T)/n$, and since

$$\mathbb{E}\big(\|X^n(T) - X(T)\|\big) \leqslant Z^n(T)^{1/2},$$

the scheme converges strongly with order of convergence $1/2$, and therefore, the proof is complete. ■

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