A LIMIT PROCESS FOR A SEQUENCE OF PARTIAL SUMS OF RESIDUALS OF A SIMPLE REGRESSION ON ORDER STATISTICS WITH MARKOV-MODULATED NOISE*

BY

ARTYOM KOVALEVSKII (Novosibirsk) AND EVGENY SHATALIN (Novosibirsk)

Abstract. We consider a simple regression model where a regressor is composed of order statistics, and a noise is Markov-modulated. We introduce an empirical bridge of regression residuals and prove its weak convergence to a centered Gaussian process. In the proof we use convergence properties of order statistics.

2010 AMS Mathematics Subject Classification: Primary: 60F17; Secondary: 62J05.

Key words and phrases: Simple regression model, order statistics, Markov-modulated noise, regression residuals, empirical bridge.

1. INTRODUCTION AND A MAIN RESULT

Brown et al. [4] proposed a test for change of regression at unknown time. Their approach is based on computation of recursive residuals. MacNeill [11] studied a linear regression against values of continuously differentiable functions. He obtained limit processes for sequences of partial sums of regression residuals. Later Bischoff [3] showed that the MacNeill theorem holds in a more general setting, namely for continuous regressor functions. Aue et al. [1] introduced a new test for polynomial regression functions which is analogous to the classical likelihood test. Stute [13] proposed a class of tests that are based on regression residuals. His general approach also allows for analysing models where regressors are order statistics.

We consider another model of a simple linear regression on order statistics where the noise is Markov-modulated. The need for this model comes from applications. Kovalevskii [10] analysed dependence of logarithm of a car price on a production year basing on a list of ads. The standard homoscedasticity test shows that there is significant dependence of variance on a date of submission of an ad.

Our model includes a case of heteroscedasticity by allowing of Markov-modulated noise. Variance can variate in wide limits in an initial sequence under this model but an asymptotic result holds true and does not depend on limiting distribution of Markov chain.

So we have Y_i (a logarithm of price in this example) that is assumed to depend linearly on production year X_i and noise ε_i^v with zero mean and non-constant variance. Deviations of variance are modelled as Markov-dependence. Then we reorder the data to correspond to ascending order of X_i. We need in a statistical criterium to verify the model.

To define the model, we introduce three mutually independent families of random variables:

1) $\{\varepsilon_i^v, i \geqslant 1, 1 \leqslant v \leqslant M\}$, a family of independent random variables where $\{\varepsilon_i^v, i \geqslant 1\}$ are identically distributed for each v, $\mathbf{E}\varepsilon_1^v = 0$, $\mathbf{Var}\varepsilon_1^v = \sigma_v^2 \geqslant 0$ and $\sum_{v=1}^{M} \sigma_v^2 > 0$;

2) $\{X_i\}_{i=1}^{\infty}$, a sequence of i.i.d. random variables with distribution function F and finite positive variance $\mathbf{Var}X_1$;

3) $\{V_i\}_{i=1}^{\infty}$, an irreducible aperiodic Markov chain on the finite state space $\{1, \ldots, M\}$ with stationary distribution $\{\pi_i\}_{i=1}^{M}$.

A regression model before ordering is of the form

$$Y_i = a + bX_i + \varepsilon_i^{V_i}, \quad i = 1, \ldots, n.$$

So we have a three-dimensional vector $(Y_i, X_i, \varepsilon_i^{V_i})$. Then we order it on the second component (X_i) and obtain the vector $(Y_{ni}, X_{ni}, \varepsilon_{ni}^{V_{ni}})$. Here $X_{ni} = X_{i:n}$ is the i-th order statistic of the first n random variables $X_1, \ldots, X_n$. In particular, $X_{n1} = \min_{1 \leqslant i \leqslant n} X_i$ and $X_{nn} = \max_{1 \leqslant i \leqslant n} X_i$. Values Y_{ni}, $\varepsilon_{ni}^{V_{ni}}$ are values of Y and ε^V corresponding to X_{ni} (that is, induced order statistics, concomitants).

We have the following regression model after ordering:

$$Y_{ni} = a + bX_{ni} + \varepsilon_{ni}^{V_{ni}}, \quad i = 1, \ldots, n.$$

For this model, we introduce an *empirical bridge* and show its weak convergence to a centered Gaussian process.

The novelty of our model lies in consideration of both ordered regressors and Markov-modulated noise.

Let

$$\widehat{b}_n = \frac{\overline{XY} - \overline{X}\,\overline{Y}}{\overline{X^2} - \overline{X}^2}, \quad \widehat{a}_n = \overline{Y} - \widehat{b}_n\,\overline{X}$$

be the classical Gauss–Markov estimators for a and b. Here $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_{ni} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_{ni} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, etc. Note that a sum over all i does not depend on order, therefore estimators coincide for models before and after ordering.

Define *fitted values* $\widehat{Y}_{ni}$, *regression residuals* $\widehat{\varepsilon}_{ni}$ and their *partial sums* $\widehat{\Delta}_{ni}$ by $\widehat{Y}_{ni} = \widehat{a}_n + \widehat{b}_n X_{ni}, \widehat{\varepsilon}_{ni} = Y_{ni} - \widehat{Y}_{ni}$ and $\widehat{\Delta}_{ni} = \widehat{\varepsilon}_{n1} + \ldots + \widehat{\varepsilon}_{ni}$ for $1 \leqslant i \leqslant n$, $\widehat{\Delta}_{n0} = 0$. Note that $\widehat{\Delta}_{nn} = 0$.

An *empirical bridge* is a random polygon $\widehat{Z}_n$ with nodes

$$\left(k/n, \widehat{\Delta}_{nk}/\sqrt{n\widehat{\sigma^2}} \right), \quad k = 0, \ldots, n,$$

where $\widehat{\sigma^2} = \overline{\widehat{\varepsilon^2}}$ is an estimator of variance $\sigma^2 = \sum_{v=1}^M \sigma_v^2 \pi_v$.

Let $GL_F(t) = \int_0^t F^{-1}(s)\, ds$ be the *theoretical general Lorenz curve* (see Gastwirth [6], Davydov and Zitikis [5]), where $F^{-1}(s) = \sup\{x : F(x) < s\}$ is the inverse of distribution function $F(x)$. Let $GL_F^0(t) = GL_F(t) - tGL_F(1)$ be its centered version. Let $GL_n(t) = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} X_{ni}$ be the *empirical Lorenz curve*. Goldie [7] showed that, as $n \to \infty$, the empirical Lorenz curve converges a.s. to the theoretical curve in the uniform metric, i.e.

$$\sup_{t \in \mathbf{R}} |GL_n(t) - GL_F(t)| \to 0 \text{ a.s.}$$

Now we formulate the main result of the paper.

THEOREM 1.1. *The empirical bridge $\widehat{Z}_n$ converges weakly, as $n \to \infty$, to the centered Gaussian process Z_F with covariance kernel $K_F(t, s)$ given by*

$$K_F(t, s) = \min\{t, s\} - ts - \frac{GL_F^0(t)GL_F^0(s)}{\mathbf{Var}X_1}, \quad t, s \in [0, 1].$$

Here weak convergence holds in the space $C(0, 1)$ of continuous functions on $[0, 1]$ endowed by the uniform metric.

2. PROOF OF THEOREM 1.1

Let $X_{ni}^0 = X_{ni} - \overline{X}$, $\varepsilon_{ni}^0 = \varepsilon_{ni}^{V_{ni}} - \overline{\varepsilon}$, where $\overline{\varepsilon} = \frac{1}{n} \sum_{i=1}^n \varepsilon_{ni}^{V_{ni}} = \frac{1}{n} \sum_{i=1}^n \varepsilon_i^{V_i}$, because the sum over all i does not depend on order.

The proof includes four steps. In the first step, we show that, in the formulae under consideration, the sum $\sum_{i=1}^n \varepsilon_{ni}^0 X_{ni}^0 / \sqrt{n}$ may be replaced by the sum $\sum_{i=1}^n (\varepsilon_{ni}^0 \mathbf{E} X_{ni}^0) / \sqrt{n}$. Secondly, we prove weak convergence of a normalized vector with coordinates $(\widehat{\Delta}_{nk_1}, \ldots, \widehat{\Delta}_{nk_m})$ to a normalized vector with coordinates $(\Delta_{nk_1}, \ldots, \Delta_{nk_m})$, where Δ_{nk_i} are defined below. Then we prove weak convergence of finite-dimensional distributions. The third step contains a proof of relative compactness of the family $\{\widehat{Z}_n(t), 0 \leqslant t \leqslant 1\}$. We complete with a proof of the convergence of sample variance $\widehat{\sigma^2}$ to variance σ^2.

In what follows, the notation $\xrightarrow{\mathbf{P}}$ means convergence in probability.

S t e p 1. Note that

$$\widehat{\Delta}_{nk} = \sum_{i=1}^{k} \left(\varepsilon_{ni}^0 - \frac{\overline{X^0 \varepsilon^0}}{(X^0)^2} X_{ni}^0 \right).$$

We show that

$$(2.1) \qquad \frac{1}{\sqrt{n}} \Big(\sum_{i=1}^{n} \varepsilon_{ni}^0 X_{ni}^0 - \sum_{i=1}^{n} \varepsilon_{ni}^0 \mathbf{E} X_{ni}^0 \Big) \xrightarrow{\mathbf{P}} 0.$$

Theorem 1 together with Theorem 2 in Hoeffding [8] imply $\frac{1}{n} \sum_{i=1}^{n} \mathbf{Var} X_{ni} \to 0$ as $n \to \infty$. Note that $\mathbf{Var}\overline{X} = \mathbf{Var} X_1/n$, and

$$\frac{1}{n} \sum_{i,j=1}^{n} \mathbf{cov}(X_{ni}, X_{nj}) = \frac{1}{n} \mathbf{Var} \sum_{i=1}^{n} X_{ni} = \mathbf{Var} X_1.$$

As $\sum_{i=1}^{n} (X_{ni}^0 - \mathbf{E} X_{ni}^0) = 0$, we have

$$\sum_{i=1}^{n} \varepsilon_{ni}^0 (X_{ni}^0 - \mathbf{E} X_{ni}^0) = \sum_{i=1}^{n} \varepsilon_{ni}^{V_{ni}} (X_{ni}^0 - \mathbf{E} X_{ni}^0).$$

Due to Chebyshev's inequality, we get

$$\mathbf{P}\left\{ \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_{ni}^{V_{ni}} (X_{ni}^0 - \mathbf{E} X_{ni}^0) \right| \geqslant \delta \right\} \leqslant \frac{\mathbf{Var} \sum\limits_{i=1}^{n} \varepsilon_{ni}^{V_{ni}} (X_{ni}^0 - \mathbf{E} X_{ni}^0)}{n \delta^2}.$$

As $\{\varepsilon_{ni}^{V_{ni}}\}$ are conditionally independent for fixed $\{V_{ni}\}$ and do not depend on $\{X_{ni}\}$, we have

$$\mathbf{Var} \sum_{i=1}^{n} \varepsilon_{ni}^{V_{ni}} (X_{ni}^0 - \mathbf{E} X_{ni}^0) = \sum_{i=1}^{n} \mathbf{Var}\varepsilon_{ni}^{V_{ni}} \mathbf{Var}(X_{ni}^0 - \mathbf{E} X_{ni}^0)$$

$$= \sum_{i=1}^{n} \mathbf{Var}\varepsilon_{ni}^{V_{ni}} \mathbf{Var} X_{ni}^0.$$

$\mathbf{Var}\varepsilon_{ni}^{V_{ni}}$ have an upper bound and

$$\sum_{i=1}^{n} \mathbf{Var} X_{ni}^0 = \sum_{i=1}^{n} \mathbf{Var} X_{ni} - 2 \sum_{i=1}^{n} \mathbf{cov}(X_{ni}, \overline{X}) + n \mathbf{Var}\overline{X}$$

$$= \sum_{i=1}^{n} \mathbf{Var} X_{ni} - \frac{2}{n} \sum_{i,j=1}^{n} \mathbf{cov}(X_{ni}, X_{nj}) + \mathbf{Var} X_1$$

$$= \sum_{i=1}^{n} \mathbf{Var} X_{ni} - \mathbf{Var} X_1 = o(n).$$

Thus

$$\frac{1}{n}\mathbf{Var}\sum_{i=1}^{n}\varepsilon_{ni}^{V_{ni}}(X_{ni}^0 - \mathbf{E}X_{ni}^0) \to 0,$$

so (2.1) is proved.

S t e p 2. Let $\lfloor t \rfloor$ be the integer part of t. For any fixed m and for $0 \leqslant s_1 < \ldots < s_m \leqslant 1$, $k_i = \lfloor ns_i \rfloor$, we establish the weak convergence, as $n \to \infty$, of the vector $\vec{\eta} = \frac{1}{\sigma\sqrt{n}}(\widehat{\Delta}_{nk_1}, \ldots, \widehat{\Delta}_{nk_m})$ to the vector $\vec{Z_F} = (Z_F(s_1), \ldots, Z_F(s_m))$.

By (2.1) and by the convergences

$$\overline{(X^0)^2} \to \mathbf{Var}X_1 \text{ a.s.} \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{k_i} X_{ni}^0 \to GL_F^0(s_i) \text{ a.s.}$$

(see Goldie [7]), it is enough to prove $\vec{\zeta} \Rightarrow \vec{Z_F}$, where

$$\vec{\zeta} = \frac{1}{\sigma\sqrt{n}}(\Delta_{nk_1}, \ldots, \Delta_{nk_m}),$$

$$\Delta_{nk_j} = \sum_{i=1}^{k_j}\varepsilon_{ni}^0 - \frac{GL_F^0(s_j)}{\mathbf{Var}X_1}\sum_{i=1}^{n}\varepsilon_{ni}^0\mathbf{E}X_{ni}^0 = \sum_{i=1}^{k_j}\varepsilon_{ni}^0 - \frac{GL_F^0(s_j)}{\mathbf{Var}X_1}\sum_{i=1}^{n}\varepsilon_{ni}^{V_{ni}}\mathbf{E}X_{ni}^0.$$

We prove the weak convergence $\vec{\zeta} \Rightarrow \vec{Z_F^0}$ using the characteristic function

$$\varphi_{\vec{\zeta}}(\vec{t}) = \mathbf{E}\prod_{j=1}^{m}\exp\left(\mathbf{i}\frac{t_j\Delta_{nk_j}}{\sigma\sqrt{n}}\right).$$

Notice that

$$\sum_{j=1}^{m}t_j\left(\sum_{i=1}^{k_j}(\varepsilon_{ni}^{V_{ni}} - \overline{\varepsilon}) - \frac{GL_F^0(s_j)}{\mathbf{Var}X_1}\sum_{i=1}^{n}\varepsilon_{ni}^{V_{ni}}\mathbf{E}X_{ni}^0\right)$$

$$= \sum_{i=1}^{n}\varepsilon_{ni}^{V_{ni}}\sum_{j=1}^{m}t_j\left(\mathbf{I}\{i \leqslant k_j\} - \frac{k_j}{n} - \frac{GL_F^0(s_j)}{\mathbf{Var}X_1}\mathbf{E}X_{ni}^0\right).$$

It is well known that the finiteness of $\mathbf{E}\psi_1$ implies the convergence $\psi_{n:n}/n \to 0$ a.s. and in mean for a sequence of i.i.d random variables $\psi_1, \ldots, \psi_n$, and, more generally, for a stationary ergodic sequence as a consequence of the subadditive ergodic theorem (see Kingman [9]).

Applying this fact and using Hölder's inequality we have $\mathbf{E}X_{ni}^0 = o(\sqrt{n})$ uniformly in $1 \leqslant i \leqslant n$.

Let

$$\beta_{ni} = \sum_{j=1}^{m}t_j\left(\mathbf{I}\{i \leqslant k_j\} - \frac{k_j}{n} - \frac{GL_F^0(s_j)}{\mathbf{Var}X_1}\mathbf{E}X_{ni}^0\right).$$

Then $\beta_{ni}/\sqrt{n} \to 0$, and

$$\sum_{i=1}^{n} \frac{\beta_{ni}^2}{n} \to C_F := \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} t_{j_1} t_{j_2} K_F(s_{j_1}, s_{j_2}).$$

As for any $1 \leqslant v \leqslant M$, $t \to 0$ it follows that

$$\mathbf{E} \exp(\mathbf{i} t \varepsilon_{ni}^v) = \exp\left(-\frac{1}{2} t^2 \mathbf{Var} \varepsilon_{ni}^v\right) (1 + o(1)),$$

and $\mathbf{Var} \varepsilon_{ni}^{V_{ni}} \to \sigma^2$ as $i, n \to \infty$, we have, by integration on Markov chain distribution (as in Step 1),

$$\varphi_{\vec{\zeta}}(\vec{t}\,) = \mathbf{E} \prod_{i=1}^{n} \exp\left(\mathbf{i} \frac{\varepsilon_{ni}^{V_{ni}} \beta_{ni}}{\sigma \sqrt{n}}\right)$$

$$= \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \frac{\beta_{ni}^2 \mathbf{Var} \varepsilon_{ni}^{V_{ni}}}{n \sigma^2}\right) (1 + o(1)) \to \exp(-C_F/2).$$

Consequently, we have $\varphi_{\vec{\zeta}}(\vec{t}\,) \to \exp(-C_F/2)$. Thus, the convergence of finite-dimensional distributions is proved.

S t e p 3. We show that the family of distributions $\{\widehat{Z}_n(t), 0 \leqslant t \leqslant 1\}$ is relatively compact.

Let $S_{nk} = \sum_{i=1}^{k} X_{ni}$, $k = 1, \ldots, n$, $S_{n0} = 0$.

By Prokhorov's theorem (see Section 1 §6 in Billingsley [2]) it suffices to show that the family of distributions of random processes $\{\widehat{\Delta}_{n,\lfloor nt \rfloor}/(\sigma \sqrt{n}), 0 \leqslant t \leqslant 1\}$, $n = 1, 2, \ldots$, is tight. Put $k = \lfloor nt \rfloor$ and let

$$\widehat{\Delta}_{nk}^0 = \sum_{i=1}^{k} \left(\varepsilon_{ni}^{V_{ni}} - \frac{\overline{X^0 \varepsilon^0}}{(X^0)^2} X_{ni}\right).$$

Then $\widehat{\Delta}_{nk} = \widehat{\Delta}_{nk}^0 - \frac{k}{n} \widehat{\Delta}_{nn}^0$.

As $\{\varepsilon_{ni}^v\}$ are i.i.d. for any v, the invariance principle implies the tightness of the family $\left\{\left(\sum_{i=1}^{\lfloor nt \rfloor} \varepsilon_{ni}^v\right)/(\sigma \sqrt{n}), 0 \leqslant t \leqslant 1\right\}$ for any $v \in \{1, \ldots, M\}$. Thus the family $\left\{\left(\sum_{i=1}^{\lfloor nt \rfloor} \varepsilon_{ni}^{V_{ni}}\right)/(\sigma \sqrt{n}), 0 \leqslant t \leqslant 1\right\}$ is tight. The invariance principle for this Markov-modulated sequence goes from Corollary 3.9 in McLeish [12].

So, it is enough to establish the tightness of

$$\left\{\frac{\overline{X^0 \varepsilon^0} \sqrt{n}}{\sigma \overline{(X^0)^2}} \frac{S_{n,\lfloor nt \rfloor}}{n}, 0 \leqslant t \leqslant 1\right\}.$$

In turn, by Theorem 8.3 in Billingsley [2], it suffices to prove that, for any $\varepsilon > 0$, $\alpha > 0$, there are $0 < \delta < 1$, $n_0 \in \mathbf{N}$ such that, for all $n > n_0$, $0 \leqslant t \leqslant 1$,

(2.2) $$\frac{1}{\delta} \mathbf{P} \left\{\sup_{t \leqslant s \leqslant t+\delta} \left| \frac{\overline{X^0 \varepsilon^0} \sqrt{n}}{\sigma \overline{(X^0)^2}} \frac{S_{n,\lfloor ns \rfloor} - S_{n,\lfloor nt \rfloor}}{n} \right| \geqslant \varepsilon \right\} \leqslant \alpha.$$

Notice that $(\overline{X^0\varepsilon^0}\sqrt{n})/\sigma\overline{(X^0)^2} \Rightarrow \zeta/\sqrt{\mathbf{Var}\,X_1}$, and (see Goldie [7])

$$\sup_{t\leqslant s\leqslant t+\delta} \left| \frac{S_{n,\lfloor ns\rfloor} - S_{n,\lfloor nt\rfloor}}{n} \right| \rightarrow \sup_{t\leqslant s\leqslant t+\delta} |GL_F(s) - GL_F(t)| \text{ a.s.}$$

Here ζ is a standard normal random variable and $GL_F(x)$ is the general Lorenz curve.

By the Cauchy–Bunyakovsky inequality,

$$\sup_{t\leqslant s\leqslant t+\delta} |GL_F(s) - GL_F(t)| \leqslant \sup_{t\leqslant s\leqslant t+\delta} \int_t^s |F^{-1}(x)|dx \leqslant \sqrt{\delta\mathbf{E}X_1^2}.$$

Thus, one may choose a positive δ that satisfies (2.2).

S t e p 4. It remains to prove $\widehat{\sigma^2} \xrightarrow{\mathbf{P}} \sigma^2$. Indeed, $\overline{\varepsilon} = 0$, $\overline{X\varepsilon} \xrightarrow{\mathbf{P}} 0$, $\overline{\varepsilon^2} \xrightarrow{\mathbf{P}} \sigma^2$, and

$$\overline{\widehat{\varepsilon^2}} = \frac{1}{n}\sum_{i=1}^n (Y_{ni} - \widehat{a} - \widehat{b}X_{ni})^2$$

$$= \frac{1}{n}\sum_{i=1}^n \left(\varepsilon_{ni}^{V_{ni}} - \frac{\overline{X\varepsilon}}{\overline{X^2} - (\overline{X})^2}(X_{ni} - \overline{X}) \right)^2 = \overline{\varepsilon^2} - \frac{(\overline{X\varepsilon})^2}{\overline{X^2} - (\overline{X})^2} \xrightarrow{\mathbf{P}} \sigma^2.$$

This completes the proof of Theorem 1.1.

REFERENCES

[1] A. A u e, L. H o r v a t h, M. H u s k o v a, and P. K o k o s z k a, *Testing for change in polynomial regression*, Bernoulli 14 (2008), pp. 637–660.

[2] P. B i l l i n g s l e y, *Convergence of Probability Measures*, Wiley, New York 1968.

[3] W. B i s c h o f f, *A functional central limit theorem for regression models*, Ann. Statist. 26 (1998), pp. 1398–1410.

[4] R. L. B r o w n, J. D u r b i n, and J. M. E v a n s, *Techniques for testing the constancy of regression relationships over time*, J. Roy. Statist. Soc. 37 (1975), pp. 149–192.

[5] Y. D a v y d o v and R. Z i t i k i s, *Convex rearrangements of random elements*, Fields Inst. Commun. 44 (2004), pp. 141–171.

[6] J. L. G a s t w i r t h, *A general definition of the Lorenz curve*, Econometrica 39 (1971), pp. 1037–1039.

[7] C. M. G o l d i e, *Convergence theorems for empirical Lorenz curves and their inverses*, Adv. in Appl. Probab. 9 (1977), pp. 765–791.

[8] W. H o e f f d i n g, *On the distribution of the expected values of the order statistics*, Ann. Math. Statist. 24 (1953), pp. 93–100.

[9] J. F. C. K i n g m a n, *The ergodic theory of subadditive stochastic processes*, J. Roy. Statist. Soc. Ser. B 30 (1968), pp. 499–510.

[10] A. Kovalevskii, *A regression model for prices of second-hand cars*, in: *Applied Methods of Statistical Analysis. Applications in Survival Analysis, Reliability and Quality Control*, B. Lemeshko, M. Nikulin, and N. Balakrishnan (Eds.), Novosibirsk 2013, pp. 124–128.

[11] I. B. MacNeill, *Limit processes for sequences of partial sums of regression residuals*, Ann. Probab. 6 (1978), pp. 695–698.

[12] D. L. McLeish, *Invariance principles for dependent variables*, Z. Wahrsch. Verw. Gebiete 32 (1975), pp. 165–178.

[13] W. Stute, *Nonparametric model checks for regression*, Ann. Statist. 25 (1997), pp. 613–641.

Artyom Kovalevskii
Novosibirsk State Technical University
Karl Marx str. 20
Novosibirsk, 630073, Russia
and
Novosibirsk State University
Pirogova str. 2
Novosibirsk, 630090, Russia
E-mail: pandorra@ngs.ru

Evgeny Shatalin
Sobolev Institute of Mathematics
Koptyuga str. 4
Novosibirsk, 630090, Russia
E-mail: sh_e_v_89@list.ru