PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 34, Fasc. 2 (2014), pp. 337–362

THE QUANTUM DECOMPOSITION ASSOCIATED WITH THE LÉVY WHITE NOISE PROCESSES WITHOUT MOMENTS

BY

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Abstract. The theory of one-mode type Interacting Fock Space (IFS) allows us to construct the quantum decomposition associated with stochastic processes on $\mathbb{R}$ with moments of any order. The problem to extend this result to processes without moments of any order is still open but the Araki–Woods–Parthasarathy–Schmidt characterization of Lévy processes in terms of boson Fock spaces, canonically associated with the Lévy–Khintchine functions of these processes, provides a quantum decomposition for them which is based on boson creations, annihilation and preservation operators rather than on their IFS counterparts. In order to compare the two quantum decompositions in their common domain of application (i.e., the Lévy processes with moments of all orders) the first step is to give a precise formulation of the quantum decomposition for these processes and the analytical conditions of its validity. We show that these conditions distinguish three different notions of quantum decomposition of a Lévy process on $\mathbb{R}$ according to the existence of second or only first moments, or no moments at all. For the last class a multiplicative renormalization procedure is needed.

2000 AMS Mathematics Subject Classification: Primary: 60J65; Secondary: 60J45, 60H40.

Key words and phrases: Quantum decomposition, Kolmogorov representation, Lévy white noise processes, Araki–Woods–Parthasarathy–Schmidt characterization.

1. INTRODUCTION

The discovery that any classical random variable with all moments is canonically associated with a non-commutative structure and, in particular, admits a quantum decomposition [2], extended to processes in [3], naturally raises the question of comparing this quantum decomposition with another quantum decomposition, valid only for infinitely divisible random variables but, in this domain, not limited to random variables with all moments: this is the quantum decomposition related to the Araki–Woods–Parthasarathy–Schmidt (AWPS) algebraic approach to the theory of infinitely divisible distributions on a general class of topological groups (see

[23], [10], [7]). This approach extends to non-commutative groups Kolmogorov papers [15], [16] on helices in Hilbert space (*helix* is the name used by Kolmogorov for what is now called a *one-cocycle* for a given unitary representation).

Since the intersection of the domains of application of the two theories consists exactly of the infinitely divisible random variables with moments of any order, which are many and very important, the problem to compare them naturally arises. In fact, the two descriptions are both canonically associated with the same object (say, to fix ideas, an independent stationary increment process on $\mathbb{R}$ with all moments) and both describe the same object, i.e., the L^2-space associated with it, therefore one expects that there is a canonical isomorphism between them, where the term *canonical* means that the operators representing in the two spaces the random variables of the process are mapped into one-another by this isomorphism.

This problem is strictly related to another problem which has been studied by several authors in the past years (see [12]–[14], [6], [18], [17], [8], [9]), i.e., the extension of Hida white noise theory [11] to more general Lévy processes. The connection consists in the fact that, according to the Araki–Woods–Parthasarathy–Schmidt theory, any infinitely divisible process on $\mathbb{R}$ (in fact, on a much more general class of groups, but in the present paper we discuss only the case of the real line) can be realized in a boson Fock space constructed in terms of the Lévy–Khintchine function of the process.

In order to realize the program formulated above our first step has been to obtain a form of the AWPS representation of a Lévy process on the real line explicit enough to deduce a quantum decomposition for any infinitely divisible real-valued random variable and for the associated white noise: the case of a single random variable is discussed in the paper [5]; the associated white noise is constructed in the present paper. Our construction is different from the construction used by the authors who have applied the AWPS results to classical Lévy processes (e.g., [21], [26]) and, being applicable to any Lévy process, has the advantage of dispensing from some analytical assumptions like the existence of the Laplace transform [26] or of finite second moments [19], [20], which limits the class of processes to which the theory is applicable. Our investigation points out a natural subdivision of the Lévy processes on $\mathbb{R}$ into three classes according to the existence of second or first moments or no moments at all. For the last class the existence of the quantum decomposition requires a multiplicative renormalization procedure whose effect can be intuitively described as "subtracting an infinite constant to a classical real-valued random variable working with quantities which are finite at every step". The remark that a classical real-valued infinitely divisible random variable (respectively, process) with no moments is equivalent "up to subtraction of an infinite constant" (see Section 6 below for a precise formulation of this statement) to a classical random variable with moments of any order was an unexpected result for the authors of this paper.

2. GENERAL FRAMEWORK

2.1. White noise measures. Recall (see [25]) that any infinitely divisible probability measure μ on $\mathbb{R}$ is canonically associated with a triple (α, σ, β) such that:

- α is a real constant,
- β is a positive finite measure on $\mathbb{R}$ with

$$\sigma^2 = \beta(\{0\}),$$

- denoting by $\widehat{\mu}$ the Fourier transform of μ, and by Ψ the Lévy–Khintchine function given by

$$\Psi(x) = i\alpha x - \frac{\sigma^2}{2}x^2 + \int\limits_{\mathbb{R}\setminus\{0\}} \left(e^{ixt} - 1 - \frac{ixt}{1+t^2}\right)\frac{1+t^2}{t^2} d\beta(t), \qquad x \in \mathbb{R}, \tag{2.1}$$

we obtain the measure on $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ in the form

$$d\nu(t) = \frac{1+t^2}{t^2} d\beta(t) \tag{2.2}$$

which is called the *Lévy measure* of μ, and we denote by X_1 the real-valued random variable with distribution μ.

Let $\mathcal{C}^\infty([-a, a])$ $(a > 0)$ be the space of all infinitely differentiable functions on $\mathbb{R}$ having compact supports in the interval $[-a, a]$. Then $\mathcal{C}^\infty([-a, a])$ is a nuclear space with a family $\{|\cdot|_{a,p}\}$ of countable Hilbert norms defined by

$$|f|_{a,p}^2 := \sum_{m=0}^{p} \int\limits_{-a}^{a} |f^{(m)}(t)|^2 dt,$$

where $f^{(m)}$ denotes the m-th derivative. Let $\mathcal{C}_c^\infty(\mathbb{R})$ be the union of the spaces $\mathcal{C}^\infty([-a, a])$ endowed with the inductive limit topology. Defining $H = L^2(\mathbb{R}, dt)$ we obtain the real nuclear triple

$$\mathcal{C}_c^\infty(\mathbb{R}) \subset H \subset \mathcal{C}_c^\infty(\mathbb{R})',$$

where $\mathcal{C}_c^\infty(\mathbb{R})'$ is the dual of $\mathcal{T}$ (see below) with the weak topology.

Under the assumption that β has finite absolute second moments, Lee and Shih [19], [20] have shown that there exists a unique probability measure, which can be identified with the white noise measure Λ_μ on the space $\mathcal{C}_c^\infty(\mathbb{R})'$, of tempered distributions with the Borel σ-algebra induced by the topology described above with characteristic functional $\mathcal{C}_{\alpha,\beta}$ on $\mathcal{C}_c^\infty(\mathbb{R})$ given by

$$\mathcal{C}_{\alpha,\beta}(\xi) := \int\limits_{\mathcal{C}_c^\infty(\mathbb{R})'} e^{i\langle w,\xi\rangle} \Lambda_{L}(dw) = \exp\Big\{\int\limits_{-\infty}^{+\infty} \Psi\big(\xi(t)\big)dt\Big\}, \qquad \xi \in \mathcal{T}, \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ is the duality $\langle\mathcal{T}', \mathcal{T}\rangle$.

Since we want to drop the restriction on second moments, we will use a different test function space based on the fact that every infinitely divisible probability measure μ on $\mathbb{R}$ defines an independent, stationary, real-valued increment process (X_t) (simply called *white noise*) on $\mathbb{R}$, that is unique up to stochastic equivalence. The probability space of this white noise will be denoted by $(\Omega_\mu, \mathcal{F}_\mu, \Lambda_\mu)$ and the probability measure Λ_μ will be called the *white noise measure associated with* μ. We will also use the shorthand notation $\mathbb{L}^2(\Lambda_\mu)$ for the space $\mathbb{L}^2_{\mathbb{C}}(\Omega_\mu, \mathcal{F}_\mu, \Lambda_\mu)$.

Finally, we fix the test function space

$$\mathcal{T} := \{\text{finite range, compact support step functions } \mathbb{R} \to \mathbb{R}\}$$

and the random variables

$$X_\xi := \sum_{j\in F} \xi_j (X_{t_{j+1}} - X_{t_j}) := \int_{\mathbb{R}} \xi_t dX_t, \qquad \xi := \sum_{j\in F} \xi_j \chi_{(t_j,t_{j+1}]} \in \mathcal{T},$$

where $\big((t_j, t_{j+1}]\big)_{j\in F}$ is a partition of the support of ξ and for any set $I \subset \mathbb{R}$

$$\chi_I(x) = \begin{cases} 0 & \text{if } x \notin I, \\ 1 & \text{if } x \in I. \end{cases}$$

Using the fact that for $\omega \in \Omega_\mu$ the map

$$T_\omega : f \in \mathcal{T} \to X_f(\omega) \in \mathbb{R}$$

is linear, we identify the measurable space $(\Omega_\mu, \mathcal{F}_\mu)$ with $(\mathcal{T}', \mathcal{F}'_\mu)$, where $\mathcal{F}_\mu$ is the pullback of $\mathcal{F}_\mu$ through the map $T : \Omega_\mu \to \mathcal{T}'$, i.e.,

$$\mathcal{F}' := \{A \subset \mathcal{T}', T^{-1}(A) \in \mathcal{F}_\mu\}.$$

With this identification we use the same symbol Λ_μ for the measure induced by Λ_μ on $(\mathcal{T}', \mathcal{F}')$ and the notation

$$\langle x, \xi\rangle := X_\xi(x), \qquad x \in \mathcal{T}',$$

where $\langle \mathcal{T}', \mathcal{T}\rangle$ is the natural duality between $\mathcal{T}'$ and $\mathcal{T}$.

2.2. The Araki–Woods–Parthasarathy–Schmidt approach. According to Kolmogorov representation theorem a $\mathbb{C}$-valued kernel k on a set $\mathcal{X}$ is positive definite if and only if there exists a Hilbert space $\mathcal{H}$ and a map

$$e. : x \in \mathcal{X} \mapsto e_x \in \mathcal{H}$$

such that $k(x, y) = \langle e_x, e_y\rangle_{\mathcal{H}}$ and $\{e_x;\ x \in \mathcal{X}\}$ is total in $\mathcal{H}$. The pair $(\mathcal{H}, e.)$ is unique up to unitary isomorphism and is called the *Kolmogorov pair* associated with the positive definite kernel k.

If $\mathcal{H}$ is a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, the exponential kernel

$$Exp\langle\cdot,\cdot\rangle_{\mathcal{H}} : (x,y) \in \mathcal{H}\times\mathcal{H} \to \mathbb{C} \tag{2.4}$$

is also a positive definite kernel on $\mathcal{H}$. The Kolmogorov pair associated with the positive definite kernel $\exp\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$ is denoted by

$$\big(\Gamma(\mathcal{H}), Exp(\,\cdot\,)\big)$$

and called the *exponential space* (or *boson Fock space*) over $\mathcal{H}$. The total set

$$Exp(\mathcal{H}) := \left\{ Exp(f) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} f^{\otimes n} \in \Gamma(\mathcal{H});\ f \in \mathcal{H} \right\}$$

is called the *set of exponential vectors* of $\Gamma(\mathcal{H})$.

The *characterizing property* of the exponential vectors is

$$\langle Exp(f), Exp(g)\rangle = \langle e_f, e_g\rangle = e^{\langle f,g\rangle_{\mathcal{H}}} \quad \text{for all } f,g \in \mathcal{H}. \tag{2.5}$$

A kernel k is called *infinitely divisible* if the map $t \in \mathbb{N} \mapsto k(x,y)^t$ admits an extension $t \in \mathbb{R} \mapsto k(x,y)$ such that for each $t \in \mathbb{R}_+$ the kernel $k^t(x,y) = \big(k(x,y)\big)^t$ is positive definite. Clearly, any exponential kernel (2.5) is infinitely divisible. The converse statement is the starting point of the Araki–Woods–Parthasarathy–Schmidt theory.

THEOREM 2.1 (Araki–Woods–Parthasarathy–Schmidt). *For a kernel k on a set $\mathcal{X}$ the following statements are equivalent*:

(i) *k is infinitely divisible positive definite.*

(ii) *There exists a kernel q_0 such that k has the form*

$$k(f,g) = e^{q_0(f,g)}, \quad f,g \in \mathcal{X},$$

and q_0 is conditionally positive definite, i.e., for any $f_0 \in \mathcal{X}$ the kernel q on $\mathcal{X}$, defined by

$$q(f,g) = q_0(f,g) - q_0(f,f_0) - q_0(f_0,g), \tag{2.6}$$

is positive definite.

(iii) *In the above notation, the map*

$$\kappa : f \in \mathcal{X} \mapsto \kappa_f := -q_0(f_0,f) \in \mathbb{C} \tag{2.7}$$

is such that, denoting by $(\mathcal{H},e)$ and $(\mathcal{K},u)$ the Kolmogorov pairs associated with k and q, respectively, then the map

$$U : e^{-\kappa_f} Exp(u_f) \in \Gamma(\mathcal{K}) \mapsto e_f \in \mathcal{H} \tag{2.8}$$

extends to a unitary isomorphism between $\mathcal{H}$ and the Fock space $\Gamma(\mathcal{K})$ over $\mathcal{K}$.

3. THE KOLMOGOROV ISOMORPHISM ASSOCIATED WITH THE WHITE NOISE MEASURE

For a given white noise measure Λ_μ we define the kernel q_0 on $\mathcal{T}$ by

$$q_0(\xi,\eta) = \int_{-\infty}^{+\infty} \Psi\big(\eta(t)-\xi(t)\big)dt =: \langle\Psi(\eta-\xi)\rangle \tag{3.1}$$

which is conditionally positive definite, and hence the kernel

$$k(\xi,\eta) := e^{\langle\Psi(\eta-\xi)\rangle}$$

is infinitely divisible and positive definite. Moreover, in this case, we can concretely realize the Araki–Woods–Parthasarathy–Schmidt isomorphism in Theorem 2.1 as follows.

THEOREM 3.1. *For each $\xi \in \mathcal{T}$ and for any $b \in \mathbb{C}$ such that $|b|^2 = \sigma$, let us define the following*:

(i) *the trigonometric exponential $e_\xi \in L^2(\Lambda_\mu)$,*

$$e_\xi(x) := e^{i\langle x,\xi\rangle}, \quad x \in \mathcal{T}'; \tag{3.2}$$

(ii) *the function v_ξ,*

$$v_\xi(y,t) := e^{iy\xi(t)} - 1, \quad y \in \mathbb{R}^*, \tag{3.3}$$

and the space $\mathcal{K}_0$,

$$\mathcal{K}_0 := \textit{closed linear span of } \{v_\xi;\ \xi \in \mathcal{T}\} \subseteq L^2(\nu \otimes dt); \tag{3.4}$$

(iii) *the vector u_ξ in $H \oplus \mathcal{K}_0$,*

$$u_\xi = b\xi \oplus v_\xi \in H \oplus \mathcal{K}_0, \tag{3.5}$$

where v_ξ is defined by (3.3).

Then, *in the notation* (3.1), *the unique linear operator U_L such that*, *for all $\xi \in \mathcal{T}$,*

$$U_L : e^{\langle\Psi(\xi)\rangle} Exp(u_\xi) \in \Gamma(H \oplus \mathcal{K}_0) \to U_L\big(e^{\langle\Psi(\xi)\rangle} Exp(u_\xi)\big) := e_\xi \in L^2(\Lambda_\mu) \tag{3.6}$$

is a unitary isomorphism from the Fock space $\Gamma(H \oplus \mathcal{K}_0)$, over $H \oplus \mathcal{K}_0$, to $L^2(\Lambda_\mu)$.

P r o o f. Clearly, $v_\xi \in L^2(\nu \otimes dt)$, which proves the inclusion (3.4). In the notation of Theorem 2.1 we choose:

$$\mathcal{X} = \mathcal{T}, \quad k(\xi,\eta) = e^{\langle\Psi(\eta-\xi)\rangle}, \quad q_0(\xi,\eta) = \langle\Psi(\eta-\xi)\rangle.$$

Then, using the expression (2.1) for the Lévy–Khintchine function and with the above choices, we see that the kernel q defined by (2.6) is given by

$$(3.7) \qquad \begin{aligned} q(\xi,\eta) &= q_0(\xi,\eta) - q_0(\xi,0) - q_0(0,\eta) \\ &= \int_{-\infty}^{+\infty} \big\{\Psi\big(\eta(t)-\xi(t)\big) - \Psi\big(-\xi(t)\big) - \Psi\big(\eta(t)\big)\big\}dt \\ &= \int_{-\infty}^{+\infty} \sigma^2\xi(t)\eta(t)dt + \int_{-\infty}^{+\infty}\int_{|y|>0} \overline{v_\xi(y,t)}v_\eta(y,t)\nu(dy)dt, \end{aligned}$$

where v_ξ is defined by

$$v_\xi : (y,t) \mapsto e^{iy\xi(t)} - 1.$$

The right-hand side of (3.7) suggests a natural choice for a Kolmogorov representation of the kernel q. On the other hand, there exists a subset $\mathcal{T}_0 \subseteq \mathcal{T}$ such that $\{v_\xi;\ \xi \in \mathcal{T}_0\}$ is a linearly independent set and $\mathcal{K}_0$ is the closed linear span of $\{v_\xi;\ \xi \in \mathcal{T}_0\}$. Then one can see that the first term in the last line of (3.7) is a scalar product on H and the second, due to the linear independence of the v_ξ's, $\xi \in \mathcal{T}_0$, extends to a scalar product on the space $\mathcal{K}_0$ defined by (3.4). Then (3.7) gives the scalar product on

$$(3.8) \qquad \mathcal{K} := H \oplus \mathcal{K}_0,$$

which takes the form

$$(3.9) \qquad \langle\cdot,\cdot\rangle_{\mathcal{K}} := \langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_H + \langle\cdot,\cdot\rangle_{L^2(\nu\otimes dt)}.$$

From the definition of $\mathcal{K}_0$ it is clear that the range of the map $\xi \mapsto u_\xi$ in (3.5) is total in $H \oplus \mathcal{K}_0$. Therefore, the pair $(\mathcal{K}, u_\xi)$ defined respectively by (3.8) and (3.5) is a Kolmogorov representation of the kernel q. Passing to the exponential space $\Gamma(\mathcal{K})$ of $\mathcal{K}$, we see that the exponential kernel of the scalar product (3.9) is of the form

$$\langle Exp(u_\xi), Exp(u_\eta)\rangle = e^{\langle u_\xi, u_\eta\rangle} = e^{q(\xi,\eta)}.$$

On the other hand, we have

$$e^{\langle\Psi(\eta-\xi)\rangle} = \int_{\mathcal{T}'} e^{-i\langle x,\xi\rangle}e^{i\langle x,\eta\rangle}\Lambda_\mu(dx) = \widehat{\Lambda}_L(\eta-\xi) = \langle e_\xi, e_\eta\rangle_{L^2(\Lambda_\mu)},$$

and the family $\{e_\xi;\ \xi \in \mathcal{T}\}$ is total in $L^2(\Lambda_\mu)$. It follows that, if we define the linear map

$$U_L : \Gamma(\mathcal{K}) = \Gamma\big(H \oplus \mathcal{K}_0\big) \to L^2(\Lambda_\mu)$$

by linear extension of

$$U_L\big(e^{\langle\Psi(\xi)\rangle}Exp(u_\xi)\big) = e_\xi, \quad \xi \in \mathcal{T},$$

then we get

$$\begin{aligned}&\langle e_\xi, e_\eta\rangle_{L^2(\Lambda_\mu)} = \widehat{\Lambda}_L(\eta-\xi) = e^{\langle\Psi(\eta-\xi)\rangle} = \exp\{q_0(\xi,\eta)\}\\ &= \exp\{q(\xi,\eta)+q_0(\xi,0)+q_0(0,\eta)\} = \exp\{\langle u_\xi, u_\eta\rangle + \overline{\langle\Psi(\xi)\rangle} + \langle\Psi(\eta)\rangle\}\\ &= \langle e^{\langle\Psi(\xi)\rangle} Exp(u_\xi), e^{\langle\Psi(\eta)\rangle} Exp(u_\eta)\rangle_{\Gamma(\mathcal{K})}.\end{aligned}$$

Therefore, the pair

$$(\mathcal{H}, e) := \left(L^2(\Lambda_\mu), \{e_\xi;\ \xi\in\mathcal{T}\}\right)$$

is a Kolmogorov representation of the kernel $Exp\langle\cdot,\cdot\rangle_{\mathcal{K}}$. ■

4. GENERALIZED FIELD OPERATORS

Recall that the *differential second quantization* of a self-adjoint operator T on the Hilbert space $\mathcal{H}$, denoted by $\Lambda(T)$, is defined via the Stone theorem by

$$\Gamma(e^{itT}) =: e^{it\Lambda(T)}, \quad t\in\mathbb{R}, \tag{4.1}$$

where for a unitary operator $X, \Gamma(X)$ is the second quantization of X, characterized by the condition

$$\Gamma(X)Exp(x) := Exp(Xx).$$

Recall that the creation (respectively, annihilation) operator $A^+(u)$ (respectively, $A^-(u)$) acts on the domain of exponential vectors as follows:

(4.2)

$$A^+(u)Exp(x) := \left.\frac{d}{ds}\right|_{s=0} Exp(x+su), \quad A^-(u)Exp(x) := \langle u, x\rangle Exp(x).$$

It follows that if $x\in\mathrm{dom}(T)$, then

$$\Lambda(T)Exp(x) = -i\left.\frac{d}{ds}\right|_{s=0} Exp(e^{isT}x) = A^+(Tx)Exp(x). \tag{4.3}$$

DEFINITION 4.1. Let q_ξ be the multiplication operator by the random variable $\langle\cdot,\xi\rangle$ in $L^2(\Lambda_\mu)$:

$$(q_\xi F)(x) := \langle x,\xi\rangle F(x), \quad F\in L^2(\Lambda_L),\ x\in\mathcal{T}'.$$

Define the operator Q_ξ on $\Gamma(H\oplus\mathcal{K}_0)$ by

$$Q_\xi := U_L^* q_\xi U_L,$$

where U_L is the isomorphism defined by (3.6). Since Λ_μ is a probability measure on $\mathcal{T}'$, q_ξ is self-adjoint (see [24], Chapter VIII.3, Proposition 1) and

$$e^{itQ_\xi} = U_L^* e^{itq_\xi} U_L, \quad t\in\mathbb{R}.$$

Q_ξ is called the *generalized field operator*.

LEMMA 4.1. *The one-parameter unitary group*

$$t \mapsto \exp\{itQ_\xi\}$$

acts on the total set $\{\exp\{u_\eta\}; \eta \in \mathcal{T}\}$ *as follows*:

$$\exp\{itQ_\xi\}Exp(u_\eta) = \exp\{\langle \Psi(\eta + t\xi) - \Psi(\eta)\rangle\}Exp(u_{\eta+t\xi}). \tag{4.4}$$

P r o o f. Using the isomorphism U_L defined by (3.6) we get

$$\begin{aligned}
&\exp\{itQ_\xi\}Exp(u_\eta) = U_L^{-1}\exp\{itq_\xi\}U_L Exp(u_\eta)\\
&= U_L^{-1}\exp\{itq_\xi\}(\exp\{\kappa_\eta\}e_\eta) = \exp\{\kappa_\eta\}U_L^{-1}(\exp\{itq_\xi\}\exp\{i\langle\cdot,\eta\rangle\})\\
&= \exp\{\kappa_\eta\}U_L^{-1}(\exp\{i\langle\cdot,\eta + t\xi\rangle\}) = \exp\{\kappa_\eta\}U_L^{-1}(e_{\eta+t\xi})\\
&= \exp\{\kappa_\eta\}\exp\{-\kappa_{\eta+t\xi}\}U_L^{-1}(\exp\{\kappa_{\eta+t\xi}\}e_{\eta+t\xi})\\
&= \exp\{\kappa_\eta\}\exp\{-\kappa_{\eta+t\xi}\}Exp(u_{\eta+t\xi})\\
&= \exp\{\langle\Psi(\eta+t\xi) - \Psi(\eta)\rangle\}Exp(u_{\eta+t\xi}). \quad \blacksquare
\end{aligned}$$

LEMMA 4.2. *For all* $\xi \in \mathcal{T}$, *the following statements are equivalent*:

(i) *The second moment of* Λ_μ *is finite.*

(ii) *The vacuum vector is in the domain* $\mathcal{D}(Q_\xi)$ *of* Q_ξ.

(iii) *The total set* $\{Exp(u_\eta);\ \eta \in \mathcal{T}\}$ *is in the domain of* Q_ξ.

P r o o f. The domain $\mathcal{D}(q_\xi)$ of the multiplication operator by the random variable $\langle\cdot,\xi\rangle$ is defined by

$$\mathcal{D}(q_\xi) := \{g \in L^2(\Lambda_\mu);\ \langle\cdot,\xi\rangle g \in L^2(\Lambda_\mu)\}.$$

Therefore, given $\eta \in \mathcal{T}$, $Exp(u_\eta) \in \mathcal{D}(Q_\xi)$ if and only if

$$\begin{aligned}
&+\infty > \big\|Q_\xi\big(Exp(u_\eta)\big)\big\|^2\\
&= \big\|U_L^* q_\xi U_L\big(Exp(u_\eta)\big)\big\|^2 = \big\|q_\xi\big(\exp\{-\langle\Psi(\eta)\rangle\}e_\eta\big)\big\|^2\\
&= \exp\big\{-2\Re\big(\langle\Psi(\eta)\rangle\big)\big\}\langle e_\eta, q_\xi^2 e_\eta\rangle = \exp\big\{-2\Re\big(\langle\Psi(\eta)\rangle\big)\big\}\int_{\mathcal{T}'}\langle x,\xi\rangle^2\Lambda(dx)\\
&= \exp\big\{-2\Re\big(\langle\Psi(\eta)\rangle\big)\big\}\langle\Phi, Q_\xi^2\Phi\rangle_{\Gamma(\mathcal{K})},
\end{aligned}$$

where Φ is the vacuum vector. Thus, the assertion holds immediately. $\blacksquare$

Let $\widetilde{q}_\xi = q \otimes \mathrm{Mult}(\xi)$ be the operator acting on the Hilbert space $L^2(\nu \otimes dt)$ as follows:

$$\widetilde{q}_\xi F(y,t) := y\xi(t)F(y,t). \tag{4.5}$$

Using (3.3) one can see that

$$\widetilde{q}_\xi(v_\eta + 1)(y,t) = y\xi(t)\exp\{iy\eta(t)\}.$$

REMARK 4.1. *The equivalent conditions of Lemma* 4.2 *coincide with the assumption in Lee and Shih paper* [20] *and it is known that they allow us to differentiate twice with respect to* r, *at* $r = 0$, *the characteristic function* $\mathcal{C}_{\alpha,\beta}(r\xi)$ *given by* (2.3). *The result is*

$$\mathbb{E}(\langle\cdot,\xi\rangle^2) = \Big(\mathbb{E}(X_1)\int_{\mathbb{R}}\xi(t)dt\Big)^2 + \|\xi\|_0^2\int_{\mathbb{R}^*}y^2\nu(dy), \qquad \xi\in\mathcal{T},$$

where $\|\cdot\|_0$ *is the norm of* H. *This gives a simple proof of the known fact* (see [25]) *that* Λ_μ *has finite second moments if and only if* ν *has this property.*

PROPOSITION 4.1. *If the second moment of* Λ_μ *is finite*, *the generalized field operator* Q_ξ *acts on the total set* $\{Exp(u_\eta);\ \eta\in\mathcal{T}\}$ *as follows*:

$$Q_\xi\big(Exp(u_\eta)\big) = \big(A^+(\Omega_{\xi,\eta}) + A^-(\Omega_{\xi,\eta}) + \lambda(\xi,\eta)\big)Exp(u_\eta),$$

where

$$\Omega_{\xi,\eta} = -ib\xi\oplus\widetilde{q}_\xi(v_\eta+1) \quad \textit{and} \quad \lambda(\xi,\eta) = \mathbb{E}(X_1)\langle\xi\rangle - 2\Re\langle\Omega_{\xi,\eta},u_\eta\rangle.$$

Proof. By Remark 4.1 one can see that Ψ is twice differentiable. Thus we can take the derivative at $t = 0$ of equation (4.4) obtaining

$$iQ_\xi Exp(u_\eta) = \langle\xi\Psi'(\eta)\rangle Exp(u_\eta) + \frac{d}{dt}\bigg|_{t=0} Exp(u_{\eta+t\xi}). \tag{4.6}$$

But with the notation $g_{\xi,\eta}(t) = u_{\eta+t,\xi} - u_\eta$ we have

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} Exp(u_{\eta+t\xi}) &= \lim_{t\to0}\sum_{n=0}^{+\infty}\frac{1}{\sqrt{n!}}\frac{(u_{\eta+t\xi})^{\otimes n}-(u_\eta)^{\otimes n}}{t}\\ &= \sum_{n=1}^{+\infty}\frac{1}{\sqrt{n!}}\lim_{t\to0}\frac{(u_{\eta+t\xi})^{\otimes n}-(u_\eta)^{\otimes n}}{t}\\ &= \sum_{n=1}^{+\infty}\frac{1}{\sqrt{n!}}\lim_{t\to0}\frac{\big(g_{\xi,\eta}(t)+u_\eta\big)^{\otimes n}-(u_\eta)^{\otimes n}}{t}\\ &= \sum_{n=1}^{+\infty}\frac{1}{\sqrt{n!}}\lim_{t\to0}\sum_{k=1}^{n}\binom{n}{k}\frac{g_{\xi,\eta}(t)}{t}\widehat{\otimes}\big(g_{\xi,\eta}(t)\big)^{\widehat{\otimes}(k-1)}\widehat{\otimes}(u_\eta)^{\widehat{\otimes}(n-k)}\\ &= \sum_{n=1}^{+\infty}\frac{\sqrt{n}}{\sqrt{(n-1)!}}g'_{\xi,\eta}(0)\widehat{\otimes}(u_\eta)^{\widehat{\otimes}(n-1)}.\end{aligned}$$

Note that

$$\begin{aligned}g'_{\xi,\eta}(0) &= \lim_{t\to0}\frac{u_{\eta+t\xi}-u_\eta}{t} = \lim_{t\to0}\frac{b(\eta+t\xi)-b\eta}{t}\oplus\frac{v_{\eta+t\xi}-v_\eta}{t}\\ &= b\xi\oplus i\widetilde{q}_\xi(v_\eta+1) = i\Omega_{\xi,\eta}.\end{aligned}$$

Then

$$\frac{d}{dt}\bigg|_{t=0} Exp(u_{\eta+t\xi}) = \sum_{n=1}^{+\infty} \frac{\sqrt{n}}{\sqrt{(n-1)!}} i\Omega_{\xi,\eta}\widehat{\otimes}(u_\eta)^{\otimes(n-1)} = iA^+(\Omega_{\xi,\eta})Exp(u_\eta).$$

On the other hand, we have

$$\langle u_\eta, \Omega_{\xi,\eta}\rangle = -i\sigma^2\langle \eta, \xi\rangle - \int_{\mathbb{R}^*\times\mathbb{R}} y\xi(t)\big(\exp\{iy\eta(t)\} - 1\big)\nu(dy)dt. \tag{4.7}$$

Moreover, using (2.3) we have

$$\Psi'(x) = i\mathbb{E}(X_1) - \sigma^2 x + i\int_{\mathbb{R}^*} y(\exp\{ixy\} - 1)\nu(dy). \tag{4.8}$$

Hence (4.7) and (4.8) yield

$$\begin{aligned}\langle \xi\Psi'(\eta)\rangle &= i\mathbb{E}(X_1)\langle\xi\rangle - \sigma^2\langle \xi, \eta\rangle + i\int_{\mathbb{R}^*\times\mathbb{R}} y\xi(t)\big(\exp\{iy\eta(t)\} - 1\big)\nu(dy)dt \\ &= i\mathbb{E}(X_1)\langle\xi\rangle - i\langle u_\eta, \Omega_{\xi,\eta}\rangle.\end{aligned}$$

This gives

$$\langle \xi\Psi'(\varepsilon\eta)\rangle = i\big(\mathbb{E}(X_1)\langle\xi\rangle - 2\Re(\langle\Omega_{\xi,\eta}, u_\eta\rangle)\big) + i\langle\Omega_{\xi,\eta}, u_\eta\rangle =: i\lambda(\xi,\eta) + i\langle\Omega_{\xi,\eta}, u_\eta\rangle,$$

and we get

$$\begin{aligned}Q_\xi Exp(u_\eta) &= \lambda(\xi,\eta)Exp(u_\eta) + \langle\Omega_{\xi,\eta}, u_\eta\rangle Exp(u_\eta) + A^+(\Omega_{\xi,\eta})Exp(u_\eta) \\ &= A^+(\Omega_{\xi,\eta})Exp(u_\eta) + A^-(\Omega_{\xi,\eta})Exp(u_\eta) + \lambda(\xi,\eta)Exp(u_\eta). \quad \blacksquare\end{aligned}$$

THEOREM 4.1. *Assume that the second moment of Λ_μ is finite. Then under the identification*

$$\Gamma(H \oplus \mathcal{K}_0) \equiv \Gamma(H) \otimes \Gamma(\mathcal{K}_0),$$

$$Exp(g \oplus f) \equiv Exp(g) \otimes Exp(f), \tag{4.9}$$

the generalized field operator Q_ξ has the form

$$Q_\xi = Q_{G,\xi} \otimes 1 + 1 \otimes Q_{CP,\xi}, \tag{4.10}$$

where

$$Q_{G,\xi} = A^+(-ib\xi) + A^-(-ib\xi),$$

$$Q_{CP,\xi} = A_\nu^+(\widetilde{q}_\xi \cdot 1) + A_\nu^-(\widetilde{q}_\xi \cdot 1) + \Lambda_\nu(\widetilde{q}_\xi) + \mathbb{E}(X_1)\langle\xi\rangle\mathbf{1}, \tag{4.11}$$

and $A_\nu^+, A_\nu^-, \Lambda_\nu$ are the creation, annihilation, and preservation operators in the Fock representation of $L^2(\nu \otimes dt)$ and $\widetilde{q}_\xi$ is defined by (4.5).

P r o o f. Using Proposition 4.1 and the identification (4.9), we have

$$
\begin{aligned}
&Q_\xi Exp(u_\eta)\\
&= \frac{d}{ds}\Big|_{s=0} Exp(u_\eta + s\Omega_{\xi,\eta}) + \langle -ib\xi \oplus \widetilde{q}_\xi(v_\eta+1), b\eta \oplus v_\eta\rangle Exp(b\eta\oplus v_\eta)\\
&\quad + \lambda(\xi,\eta)Exp(b\eta\oplus v_\eta)\\
&= \frac{d}{ds}\Big|_{s=0} Exp\Big(\big(b\eta + s(-ib\xi)\big)\oplus\big(v_\eta + s\widetilde{q}_\xi(v_\eta+1)\big)\Big)\\
&\quad + \big(\langle -ib\xi, b\eta\rangle + \langle \widetilde{q}_\xi(v_\eta+1), v_\eta\rangle\big)Exp(b\eta\oplus v_\eta)\\
&\quad + \lambda(\xi,\eta)Exp(b\eta\oplus v_\eta)\\
&= \frac{d}{ds}\Big|_{s=0}\Big(Exp\big(b\eta + s(-ib\xi)\big)\otimes Exp\big(v_\eta + s\widetilde{q}_\xi(v_\eta+1)\big)\Big)\\
&\quad + \big(\langle -ib\xi, b\eta\rangle + \langle \widetilde{q}_\xi(v_\eta+1), v_\eta\rangle\big)Exp(b\eta\oplus v_\eta)\\
&\quad + \lambda(\xi,\eta)Exp(b\eta\oplus v_\eta)\\
&= \left(\frac{d}{ds}\Big|_{s=0} Exp\big(b\eta + s(-ib\xi)\big)\right)\otimes Exp(v_\eta)\\
&\quad + Exp(b\eta)\otimes\left(\frac{d}{ds}\Big|_{s=0} Exp\big(v_\eta + s\widetilde{q}_\xi(v_\eta+1)\big)\right)\\
&\quad + \langle -ib\xi, b\eta\rangle Exp(b\xi)\otimes Exp(v_\eta) + \langle \widetilde{q}_\xi(v_\eta+1), v_\eta\rangle Exp(b\eta)\otimes Exp(v_\eta)\\
&\quad + \lambda(\xi,\eta)Exp(b\eta)\otimes Exp(v_\eta).
\end{aligned}
$$

Hence, by (4.2), we have

$$
\begin{aligned}
&(4.12)\qquad Q_\xi Exp(u_\eta)\\
&= \big(A^+(-ib\xi)Exp(b\eta)\big)\otimes Exp(v_\eta) + Exp(b\eta)\otimes\Big(A^+_\nu\big(\widetilde{q}_\xi(v_\eta+1)\big)Exp(v_\eta)\Big)\\
&\quad + \big(A^-(-ib\xi)Exp(b\eta)\big)\otimes Exp(v_\xi) + Exp(b\eta)\otimes\Big(A^-_\nu\big(\widetilde{q}_\xi(v_\eta+1)\big)Exp(v_\eta)\Big)\\
&\quad + Exp(b\eta)\otimes\big(\lambda(\xi,\eta)Exp(v_\eta)\big)\\
&= \big[\big(A^+(-ib\xi) + A^-(-ib\xi)\big)Exp(b\eta)\big]\otimes Exp(v_\eta)\\
&\quad + Exp(b\eta)\otimes\Big[\Big(A^+_\nu\big(\widetilde{q}_\xi(v_\eta+1)\big) + A^-_\nu\big(\widetilde{q}_\xi(v_\eta+1)\big) + \lambda(\xi,\eta)\Big)Exp(v_\eta)\Big].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\lambda(\xi,\eta) &= \mathbb{E}(X_1)\langle\xi\rangle - 2\Re(\langle\Omega_{\xi,\eta}, u_\eta\rangle)\\
&= \mathbb{E}(X_1)\langle\xi\rangle - 2\Re\big(\langle -ib\xi\oplus\widetilde{q}_\xi(v_\eta+1), b\eta\oplus v_\eta\rangle\big)\\
&= \mathbb{E}(X_1)\langle\xi\rangle - 2\Re\big(\langle\widetilde{q}_\xi(v_\eta+1), v_\eta\rangle\big)\\
&= \mathbb{E}(X_1)\langle\xi\rangle - \big(\langle\widetilde{q}_\xi(v_\eta+1), v_\eta\rangle + \langle v_\eta, \widetilde{q}_\xi(v_\eta+1)\rangle\big).
\end{aligned}
$$

Therefore, using the fact that

$$\langle v_\eta, \widetilde{q}_\xi(v_\eta + 1)\rangle = -\langle \widetilde{q}_\xi \cdot 1, v_\eta\rangle$$

and equation (4.3), we obtain

$$\begin{aligned}
&\Big(A_\nu^+\big(\widetilde{q}_\xi(v_\eta+1)\big) + A_\nu^-\big(\widetilde{q}_\xi(v_\eta+1)\big) + \lambda(\xi,\eta)\Big) Exp(v_\eta)\\
&\quad = A_\nu^+(\widetilde{q}_\xi \cdot 1 + \widetilde{q}_\xi v_\eta) Exp(v_\eta) + \langle \widetilde{q}_\xi(v_\eta+1), v_\eta\rangle Exp(v_\eta)\\
&\quad\quad - \big(\langle \widetilde{q}_\xi(v_\eta+1), v_\eta\rangle + \langle v_\eta, \widetilde{q}_\xi(v_\eta+1)\rangle + \mathbb{E}(X_1)\langle\xi\rangle\big) Exp(v_\eta)\\
&\quad = A_\nu^+(\widetilde{q}_\xi \cdot 1) Exp(v_\eta) + A_\nu^+(\widetilde{q}_\xi v_\eta) Exp(v_\eta)\\
&\quad\quad + \big(\mathbb{E}(X_1)\langle\xi\rangle - \langle v_\eta, \widetilde{q}_\xi(v_\eta+1)\rangle\big) Exp(v_\eta)\\
&\quad = A_\nu^+(\widetilde{q}_\xi \cdot 1) Exp(v_\eta) + \Lambda_\nu(\widetilde{q}_\xi) Exp(v_\eta)\\
&\quad\quad + \langle \widetilde{q}_\xi \cdot 1, v_\eta\rangle Exp(v_\eta) + \mathbb{E}(X_1)\langle\xi\rangle Exp(v_\eta)\\
&\quad = A_\nu^+(\widetilde{q}_\xi \cdot 1) Exp(v_\eta) + \Lambda_\nu(\widetilde{q}_\xi) Exp(v_\eta)\\
&\quad\quad + A_\nu^-(\widetilde{q}_\xi \cdot 1) Exp(v_\eta) + \mathbb{E}(X_1)\langle\xi\rangle Exp(v_\eta).
\end{aligned}$$

Finally, the previous equation and (4.12) yield

$$\begin{aligned}
Q_\xi &= \big(A^+(-ib\xi) + A^-(-ib\xi)\big) \otimes 1\\
&\quad + 1 \otimes \big(A_\nu^+(\widetilde{q}_\xi \cdot 1) + A_\nu^-(\widetilde{q}_\xi \cdot 1) + \Lambda_\nu(\widetilde{q}_\xi) + \mathbb{E}(X_1)\langle\xi\rangle\mathbf{1}\big)\\
&= Q_{G,\xi} \otimes 1 + 1 \otimes Q_{CP,\xi}. \quad \blacksquare
\end{aligned}$$

REMARK 4.2. *The identities* (4.10) *and* (4.11) *define the quantum decomposition of the generalized field associated with the Lévy white noise processes. The technique applied in the proof of the previous theorem uses heavily the existence of the second order moment of* ν. *It is therefore natural to ask what is the quantum decomposition associated with the Lévy white noise processes in the following cases*:

(i) *only the first moment of* ν *is finite*,

(ii) ν *has no finite moments.*

5. THE WEAK QUANTUM DECOMPOSITION

In the present section we assume that only the first order moment of ν is finite, meaning by this that the vector $\widetilde{q}_\xi \cdot 1 \in L^1(\nu \otimes dt)$ but $\widetilde{q}_\xi \cdot 1 \notin L^2(\nu \otimes dt)$. This implies that, for any $F \in L^\infty(\nu \otimes dt)$, $\widetilde{q}_\xi \cdot F \in L^1(\nu \otimes dt)$. Under this assumption we prove that the quantum decomposition associated with the Lévy white noise processes can be given a meaning in a weak sense (see Definition 5.3 below).

Without loss of generality we neglect the Gaussian part, which can be dealt with independently and for which the problem does not subsist, and we denote by the same symbol Ψ the characteristic exponent associated with the compound Poisson process.

Since, by assumption, $\widetilde{q}_\xi \cdot 1 \notin L^2(\nu \otimes dt)$, the first problem is to define objects like

$$A_\nu^+(\widetilde{q}_\xi \cdot 1), \quad A_\nu^-(\widetilde{q}_\xi \cdot 1).$$

To this goal recall that (see [22]) the Weyl operator (in the normally ordered form)

$$\Gamma(u, T, v, z) = \exp\{A^+(u)\}\Gamma(T)\exp\{A^-(v)\}\exp\{z\} \tag{5.1}$$

is well defined on the domain of the exponential vectors and maps the scalar multiples of these vectors into themselves because

$$\Gamma(u, T, v, z)Exp(f) = \exp\{z + \langle v, f\rangle\}Exp(Tf + u), \quad f \in L^2(\nu \otimes dt). \tag{5.2}$$

Moreover, for $u_1, u_2, v_1, v_2 \in L^2(\nu \otimes dt)$, $z_1, z_2 \in \mathbb{C}$, and for any two unitary operators T_1, T_2 on $L^2(\nu \otimes dt)$, we have

$$\begin{aligned} &\Gamma(u_1, T_1, v_1, z_1)\Gamma(u_2, T_2, v_2, z_2) \\ &\qquad = \Gamma(u_1 + Tu_2, T_1T_2, v_2 + T_2^* v_1, z_1 + z_2 + \langle v_1, u_2\rangle). \end{aligned} \tag{5.3}$$

The following result can be deduced from the existing literature, but we include a direct proof for completeness.

THEOREM 5.1. *The operator-valued function*

$$\begin{aligned} t \mapsto W_\xi(t) &:= \Gamma\big(v_{t\xi}, \exp\{it\widetilde{q}_\xi\}, v_{-t\xi}, \langle\Psi(t\xi)\rangle\big) \\ &= \exp\{A^+(v_{t\xi})\}\Gamma(\exp\{it\widetilde{q}_\xi\})\exp\{A^-(v_{-t\xi})\}\exp\{\langle\Psi(t\xi)\rangle\} \end{aligned} \tag{5.4}$$

is a strongly continuous one-parameter unitary group with generator $Q_{CP,\xi}$.

Proof. Step 1. In the notation already used above, it is known that (v_ξ) is a one-cocycle for the group $(\exp\{i\widetilde{q}_\xi\})_{\xi \in \mathcal{C}_c^\infty(\mathbb{R})}$, i.e.,

$$v_{\xi+\eta} = \exp\{i\widetilde{q}_\xi\}v_\eta + v_\xi, \quad v_{-\xi} = -\exp\{-i\widetilde{q}_\xi\}v_\xi,$$

and that the two-coboundary associated with Ψ has the form

$$\begin{aligned} \big\langle\Psi\big((t+s)\xi\big) - \Psi(s\xi) - \Psi(t\xi)\big\rangle &= \big\langle\Psi\big(t\xi - (-s)\xi\big) - \overline{\Psi(-s\xi)} - \Psi(t\xi)\big\rangle \\ &= q(-s\xi, t\xi) = \langle v_{-s\xi}, v_{t\xi}\rangle. \end{aligned}$$

Therefore, for $f, g \in \mathcal{K}_0$, we have

$$
\begin{aligned}
&\langle W_\xi(t)Exp(f), W_\xi(t)Exp(g)\rangle \\
&= \Big\langle \exp\{\langle\Psi(t\xi)\rangle + \langle v_{-t\xi}, f\rangle\}Exp(\exp\{it\widetilde{q}_\xi\}f + v_{t\xi}), \\
&\qquad\qquad\qquad\qquad \exp\{\langle\Psi(t\xi)\rangle + \langle v_{-t\xi}, g\rangle\}Exp(\exp\{it\widetilde{q}_\xi\}g + v_{t\xi})\Big\rangle \\
&= \exp\{\overline{\langle\Psi(t\xi)\rangle} + \langle\Psi(t\xi)\rangle + \overline{\langle v_{-t\xi}, f\rangle} \\
&\quad + \langle v_{-t\xi}, g\rangle\}\langle Exp(\exp\{it\widetilde{q}_\xi\}f + v_{t\xi}), Exp(\exp\{it\widetilde{q}_\xi\}g + v_{t\xi})\rangle \\
&= \exp\big\{\overline{\langle\Psi(t\xi)\rangle} + \langle\Psi(t\xi)\rangle + \overline{\langle v_{-t\xi}, f\rangle} + \langle v_{-t\xi}, g\rangle + \langle\exp\{it\widetilde{q}_\xi\}f + v_{t\xi}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \exp\{it\widetilde{q}_\xi\}g + v_{t\xi}\rangle\big\} \\
&= \exp\{\langle v_{t\xi}, v_{t\xi}\rangle + \overline{\langle\Psi(t\xi)\rangle} + \langle\Psi(t\xi)\rangle + \overline{\langle -\exp\{-it\widetilde{q}_\xi\}v_{t\xi}, f\rangle} \\
&\quad + \langle -\exp\{-it\widetilde{q}_\xi\}v_{t\xi}, g\rangle + \langle f, \exp\{-it\widetilde{q}_\xi\}v_{t\xi}\rangle + \langle v_{t\xi}, \exp\{it\widetilde{q}_\xi\}g\rangle + \langle f, g\rangle\} \\
&= \exp\{q(t\xi, t\xi) + \overline{\langle\Psi(t\xi)\rangle} + \langle\Psi(t\xi)\rangle + \langle f, g\rangle\} \\
&= \exp\{\langle f, g\rangle\} \\
&= \langle Exp(f), Exp(g)\rangle.
\end{aligned}
$$

Thus $W_\xi(t)$ is unitary for all $t \in \mathbb{R}$.

S t e p 2. Now we prove the group property and the strong continuity.

It is easily seen that

$$W_\xi(0) = \Gamma\big(v_0, \mathbf{1}, v_0, \langle\Psi(0)\rangle\big) = \Gamma(0, \mathbf{1}, 0, 0) = \mathbf{1},$$

and by equation (5.3) we obtain

$$
\begin{aligned}
W_\xi(t)W_\xi(s) &= \Gamma\big(v_{t\xi}, \exp\{it\widetilde{q}_\xi\}, v_{-t\xi}, \langle\Psi(t\xi)\rangle\big) \\
&\quad \times \Gamma\big(v_{s\xi}, \exp\{is\widetilde{q}_\xi\}, v_{-s\xi}, \langle\Psi(s\xi)\rangle\big) \\
&= \Gamma\big(v_{t\xi} + \exp\{it\widetilde{q}_\xi\}v_{s\xi}, \exp\{it\widetilde{q}_\xi\}\exp\{is\widetilde{q}_\xi\}, \\
&\qquad v_{-s\xi} + (\exp\{is\widetilde{q}_\xi\})^* v_{-t\xi}, \langle\Psi(t\xi) + \Psi(s\xi)\rangle + \langle v_{-t\xi}, v_{s\xi}\rangle\big).
\end{aligned}
$$

From the identities

$$v_{t\xi} + \exp\{it\widetilde{q}_\xi\}v_{s\xi} = v_{(t+s)\xi},$$

$$v_{-s\xi} + (\exp\{is\widetilde{q}_\xi\})^* v_{-t\xi} = v_{-s\xi} + \exp\{-is\widetilde{q}_\xi\}v_{-t\xi} = v_{-s\xi-t\xi},$$

$$
\begin{aligned}
\langle\Psi(t\xi) + \Psi(s\xi)\rangle + \langle v_{-t\xi}, v_{s\xi}\rangle &= \langle\Psi(t\xi) + \Psi(s\xi)\rangle + q(-t\xi, s\xi) \\
&= \big\langle\Psi(t\xi) + \Psi(s\xi) + \Psi\big(s\xi - (-t)\xi\big)\big\rangle - \overline{\langle\Psi(-t\xi)\rangle} - \langle\Psi(s\xi)\rangle \\
&= \big\langle\Psi\big((t+s)\xi\big)\big\rangle
\end{aligned}
$$

we deduce that

$$W_\xi(t)W_\xi(s)=\Gamma\Big(v_{(t+s)\xi},\exp\{i(t+s)\widetilde{q}_\xi\},v_{-(t+s)\xi},\big\langle\Psi\big((t+s)\xi\big)\big\rangle\Big)=W_\xi(t+s).$$

For the strong continuity, it is sufficient to prove that

$$\lim_{t\to 0}\|W_\xi(t)Exp(f)-Exp(f)\|=0 \quad \text{for all } f\in L^2(\nu\otimes ds).$$

We have

$$\begin{aligned}
(5.5)\quad & \|W_\xi(t)Exp(f)-Exp(f)\|^2\\
&=\|W_\xi(t)Exp(f)\|^2+\|Exp(f)\|^2-2\Re\langle Exp(f),W_\xi(t)Exp(f)\rangle\\
&=2\|Exp(f)\|^2-2\Re\langle Exp(f),W_\xi(t)Exp(f)\rangle\\
&=2\exp\{\|f\|^2\}\\
&\quad -2\Re\big(\big\langle Exp(f),\exp\{\langle\Psi(t\xi)\rangle+\langle v_{-t\xi},f\rangle\}Exp(\exp\{it\widetilde{q}_\xi\}f+v_{t\xi})\big\rangle\big)\\
&=2\exp\{\|f\|^2\}-2\Re\big(\exp\big\{\langle\Psi(t\xi)\rangle+\langle v_{-t\xi},f\rangle+\langle f,\exp\{it\widetilde{q}_\xi\}f\rangle+\langle f,v_{t\xi}\rangle\big\}\big).
\end{aligned}$$

By the dominated convergence theorem, for all $f\in\mathcal{K}_0$ we have

$$(5.6)\quad \langle f,\exp\{it\widetilde{q}_\xi\}f\rangle=\int_{\mathbb{R}^*\times\mathbb{R}}|f(y,s)|^2\exp\{ity\xi(s)\}\nu(dy)ds\to\|f\|^2 \quad \text{as } t\to 0.$$

On the other hand, for $|t|\leqslant 1$, we obtain

$$\big|f(y,s)\big(\exp\{ity\xi(s)\}-1\big)\big|\leqslant 2|y\xi(s)f(y,s)|=\varphi_\xi(y,s)$$

and $\varphi_\xi\in L^1(\nu\otimes dt)$ because

$$\begin{aligned}
\int_{\mathbb{R}^*\times\mathbb{R}}\varphi_\xi(y,s)\nu(dy)ds&=2\int_{\mathbb{R}^*\times\mathbb{R}}|y\xi(s)f(y,s)|\nu(dy)ds\\
&\leqslant 2\Big(\int_{\mathbb{R}^*\times\mathbb{R}}y^2\xi^2(s)\nu(dy)ds\Big)^{1/2}\Big(\int_{\mathbb{R}^*\times\mathbb{R}}|f(y,s)|^2\nu(dy)ds\Big)^{1/2}<\infty.
\end{aligned}$$

Again by the dominated convergence theorem we conclude that

$$(5.7)\quad \lim_{t\to 0}\langle f,v_{t\xi}\rangle=\int_{\mathbb{R}^*\times\mathbb{R}}\lim_{t\to 0}\overline{f(y,s)}\big(\exp\{ity\xi(s)\}-1\big)\nu(dy)ds=0.$$

Taking the limit $t\to 0$ in equation (5.5) and using (5.6) and (5.7) we obtain

$$\lim_{t\to 0}\|W_\xi(t)Exp(f)-Exp(f)\|^2=2\big(\exp\{\|f\|^2\}-\Re(\exp\{\|f\|^2\})\big)=0.$$

Finally, from (4.4) we can see that, for all $s, t \in \mathbb{R}$,

$$
\begin{aligned}
W_\xi(t)Exp(v_\eta) &= \exp\{\langle\Psi(t\xi)\rangle + \langle v_{-t\xi}, v_\eta\rangle\}Exp(\exp\{it\widetilde{q}_\xi\}v_\eta + v_{t\xi}) \\
&= \exp\{\langle\Psi(t\xi)\rangle + q(-t\xi, \eta)\}Exp(v_{\eta+t\xi}) \\
&= \exp\{\langle\Psi(\eta + t\xi) - \Psi(\eta)\rangle\}Exp(v_{\eta+t\xi}) \\
&= \exp\{itQ_{CP,\xi}\}Exp(v_\eta). \quad \blacksquare
\end{aligned}
$$

Now, we extend the definition of creation, annihilation, and preservation operators to include the case when the images of some vectors in the Fock space are not vectors of the same space but elements in its algebraic dual of a dense subspace. In this sense we speak of *distribution-valued operators*.

DEFINITION 5.1. A *distribution-valued operator* T on a Hilbert space $\mathcal{H}$ with dense domain $\mathcal{D}$ is a linear map from $\mathcal{D} \subseteq \mathcal{H}$ to its algebraic dual $\mathcal{D}'$. Moreover, the natural embedding

$$\xi \in \mathcal{H} \mapsto \langle\xi, \cdot\rangle \in \mathcal{H}' \subset \mathcal{D}'$$

allows us to adopt the language of standard triplets

$$\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}'$$

and to interpret elements of $\mathcal{D}'$ as vector-valued distributions on $\mathcal{D}$.

Let $\mathcal{D}_\nu$ be a total set in $\mathcal{K}_0$ with the following properties:

(C1) $\mathcal{D}_\nu$ is invariant under complex conjugate.

(C2) For all $g \in \mathcal{D}_\nu$ and $\xi \in \mathcal{T}$, the distributions

$$\widetilde{q}_\xi \cdot 1 : f \mapsto \langle\widetilde{q}_\xi \cdot 1, f\rangle \quad \text{and} \quad \widetilde{q}_\xi \cdot g : f \mapsto \langle\widetilde{q}_\xi \cdot g, f\rangle = \langle\widetilde{q}_\xi \cdot 1, \bar{g}f\rangle$$

are well defined on Lin-span$(\mathcal{D}_\nu)$.

DEFINITION 5.2. For $f \in \mathcal{D}_\nu$, define the operators $A_\nu^-(\widetilde{q}_\xi \cdot 1)$ and $A_\nu^-(\widetilde{q}_\xi f)$ on the invariant domain $Exp(\mathcal{D}_\nu) \subseteq \Gamma(\mathcal{K}_0)$ (the linear subspace of $\Gamma(\mathcal{K}_0)$ generated by $\{Exp(g);\ g \in \mathcal{D}_\nu\}$) by linear extension of

$$A_\nu^-(\widetilde{q}_\xi \cdot 1)Exp(g) := \langle\widetilde{q}_\xi \cdot 1, g\rangle Exp(g), \quad g \in \mathcal{D}_\nu, \tag{5.8}$$

$$A_\nu^-(\widetilde{q}_\xi f)Exp(g) := \langle\widetilde{q}_\xi \cdot 1, \bar{f}g\rangle Exp(g), \quad g \in \mathcal{D}_\nu. \tag{5.9}$$

The distribution-valued operators $A_\nu^+(\widetilde{q}_\xi \cdot 1)$ and $\Lambda_\nu(\widetilde{q}_\xi)$ on the domain $Exp(\mathcal{D}_\nu) \subset \Gamma(\mathcal{K}_0)$ are defined, for each $f, g \in \mathcal{D}_\nu$, as follows:

$$\langle A_\nu^+(\widetilde{q}_\xi \cdot 1)Exp(f), Exp(g)\rangle := \langle Exp(f), A_\nu^-(\widetilde{q}_\xi \cdot 1)Exp(g)\rangle, \tag{5.10}$$

$$\langle\Lambda_\nu(\widetilde{q}_\xi)Exp(f), Exp(g)\rangle := \langle Exp(f), A_\nu^-(\widetilde{q}_\xi f)Exp(g)\rangle. \tag{5.11}$$

REMARK 5.1. *One easily proves that the definition* (5.11) *of* $\Lambda_\nu(\widetilde{q}_\xi)$ *is compatible with the usual one, as the differential second quantization of* $\widetilde{q}_\xi$, *in the sense that the two definitions coincide on the set of exponential vectors with test functions in the domain of* $\widetilde{q}_\xi$.

DEFINITION 5.3. Let $U(t) = \exp\{itA\}$ be a strongly continuous one-parameter unitary group on a Hilbert space $\mathcal{H}$ with generator A. Define the *weak domain* wk-dom(A) of A as the maximal subspace $\mathcal{D}$ of $\mathcal{H}$ such that for all $\varphi, \phi \in \mathcal{D}$ the limit

$$\lim_{t\to 0}\left\langle \frac{U(t)-1}{t}\varphi, \phi\right\rangle$$

exists.

REMARK 5.2. *Clearly, the weak domain of* A *contains the domain of* A. *In particular,* wk-dom(A) *is a dense subspace of* $\mathcal{H}$ *and* A *can be defined as a distribution-valued operator on its weak domain by the formula*

$$A\psi := i\lim_{t\to 0}\left\langle \frac{U(t)-1}{t}\psi,\ \cdot\ \right\rangle.$$

LEMMA 5.1. *Let* $\mathcal{D}_\nu := \{v_\xi;\ \xi \in \mathcal{T}\} \subset \mathcal{K}_0$. *Then* $\mathcal{D}_\nu$ *is a total set in* $\mathcal{K}_0$ *satisfying conditions* (C1) *and* (C2). *Moreover, for all* $f \in \mathcal{D}_\nu$, *the function*

$$F : s \mapsto \langle v_{s\xi}, f\rangle$$

is derivable at $s = 0$ *and*

$$F'(0) = -i\langle \widetilde{q}_\xi \cdot 1, f\rangle := -i\int\limits_{\mathbb{R}^*\times\mathbb{R}} y\xi(t)f(y,t)\nu(dy)dt.$$

Proof. The totality of $f \in \mathcal{D}_\nu$ in $\mathcal{K}_0$ follows immediately from the definitions of these sets.

From the relation $\bar{v_\xi} = v_{-\xi}$ we deduce that $\mathcal{D}_\nu$ is invariant under complex conjugate. Then condition (C1) holds.

Since the first order moment of ν exists, the map

$$(y,t) \mapsto y\xi(t)v_\xi(y,t)$$

belongs to $L^1(\nu\otimes dt)$. This implies that

$$\langle \widetilde{q}_\xi \cdot 1, v_\xi\rangle := \int\limits_{\mathbb{R}^*\times\mathbb{R}} y\xi(t)v_\xi(y,t)\nu(dy)dt$$

exists for all $\xi \in \mathcal{C}_c^\infty(\mathbb{R})$. Then, by a linear extension, the distribution

$$\text{Lin-span}(\mathcal{D}_\nu) \ni f \mapsto \langle \widetilde{q}_\xi \cdot 1, f\rangle := \int\limits_{\mathbb{R}^*\times\mathbb{R}} y\xi(t)f(y,t)\nu(dy)dt$$

is well defined. Moreover, from the relation

$$v_\xi v_\eta = v_{\xi+\eta} - v_\xi - v_\eta, \quad \xi, \eta \in \mathcal{T},$$

we deduce that Lin-span$(\mathcal{D}_\nu)$ is invariant under multiplication. Then, for all $g \in \mathcal{D}_\nu$, the distribution

$$\text{Lin-span}(\mathcal{D}_\nu) \ni f \mapsto \langle \widetilde{q}_\xi \cdot 1, f\bar{g}\rangle := \int\limits_{\mathbb{R}^*\times\mathbb{R}} y\xi(t) f(y,t)\overline{g(y,t)}\nu(dy)dt$$

is also well defined, which proves that condition (C2) is satisfied.

For any $\xi, \eta \in \mathcal{T}$, the function

$$F(s) := \langle v_{s\xi}, v_\eta\rangle = \int\limits_{\mathbb{R}^*\times\mathbb{R}} \overline{v_{s\xi}(y,t)} v_\eta(y,t)\nu(dy)dt$$

is well defined because $v_{s\xi}, v_\eta \in L^2(\nu \otimes dt)$. On the other hand,

$$\begin{aligned}\left|\frac{\partial}{\partial s}\big(\overline{v_{s\xi}(y,t)} v_\eta(y,t)\big)\right| &= |-iy\xi(t)\exp\{-isy\xi(t)\} v_\eta(y,t)| \\ &= |y\xi(t) v_\eta(y,t)| =: \varphi(y,t)\end{aligned}$$

with $\varphi \in L^1(\nu \otimes dt)$. Therefore, F is derivable at any $s \in \mathbb{R}$ and

$$F'(s) = -i \int\limits_{\mathbb{R}^*\times\mathbb{R}} y\xi(t)\exp\{-isy\xi(t)\} v_\eta(y,t)\nu(dy)dt;$$

in particular, $F'(0) = -i\langle \widetilde{q}_\xi \cdot 1, v_\eta\rangle$. ■

In the remaining of this section, we take $\mathcal{D}_\nu := \{v_\xi;\ \xi \in \mathcal{T}\}$.

THEOREM 5.2. *The exponential vectors $Exp(\mathcal{D}_\nu)$ are in the weak domain* wk-dom$(Q_{CP,\xi})$ *of $Q_{CP,\xi}$. Moreover, on the domain $Exp(\mathcal{D}_\nu)$, the operator $Q_{CP,\xi}$ coincides with the distribution-valued operator*

$$A_\nu^+(\widetilde{q}_\xi \cdot 1) + A_\nu^-(\widetilde{q}_\xi \cdot 1) + \Lambda_\nu(\widetilde{q}_\xi) + \mathbb{E}(X_1)\langle\xi\rangle\mathbf{1}. \tag{5.12}$$

P r o o f. By Theorem 5.1 we know that $Q_{CP,\xi}$ is the generator of $\big(W_\xi(t)\big)_{t\in\mathbb{R}}$ and that

$$\begin{aligned}&\langle W_\xi(t) Exp(v_\zeta), Exp(v_\eta)\rangle \\ &\quad= \big\langle \exp\{\langle\Psi(t\xi)\rangle + \langle v_{-t\xi}, v_\zeta\rangle\} Exp(v_{t\xi} + \exp\{it\widetilde{q}_\xi\} v_\zeta), Exp(v_\eta)\big\rangle \\ &\quad= \exp\{\langle\Psi(-t\xi)\rangle + \langle v_\zeta, v_{-t\xi}\rangle + \langle v_{t\xi}, v_\eta\rangle + \langle \exp\{it\widetilde{q}_\xi\} v_\zeta, v_\eta\rangle\} \\ &\quad= \exp\{h(t)\},\end{aligned}$$

where

$$\begin{aligned}h(t) &:= \langle \Psi(-t\xi)\rangle + \langle v_\zeta, v_{-t\xi}\rangle + \langle v_{t\xi}, v_\eta\rangle + \langle \exp\{it\widetilde{q}_\xi\}v_\zeta, v_\eta\rangle \\ &= \langle \Psi(-t\xi)\rangle + \langle v_{t\xi}, v_{-\zeta}\rangle + \langle v_{t\xi}, v_\eta\rangle + \langle v_{t\xi}, v_{-\zeta}v_\eta\rangle + \langle v_\zeta, v_\eta\rangle \\ &= \langle \Psi(-t\xi)\rangle + \langle v_{t\xi}, v_{-\zeta} + v_\eta + v_{-\zeta}v_\eta\rangle + \langle v_\zeta, v_\eta\rangle \\ &= \langle \Psi(-t\xi)\rangle + \langle v_{t\xi}, v_{\eta-\zeta}\rangle + \langle v_\zeta, v_\eta\rangle.\end{aligned}$$

Lemma 5.1 proves that the function h is derivable at $t = 0$ with

$$\begin{aligned}h'(0) &= -\langle \xi\Psi'(0)\rangle - i\langle \widetilde{q}_\xi \cdot 1, v_{\eta-\zeta}\rangle \\ &= -i\langle \widetilde{q}_\xi \cdot 1, v_{\eta-\zeta}\rangle - i\mathbb{E}(X_1)\langle\xi\rangle \\ &= -i\big(\overline{\langle \widetilde{q}_\xi \cdot 1, v_\zeta\rangle} + \langle \widetilde{q}_\xi \cdot 1, v_\eta\rangle + \langle \widetilde{q}_\xi \cdot 1, \overline{v_\zeta}v_\eta\rangle + \mathbb{E}(X_1)\langle\xi\rangle\big).\end{aligned}$$

Then

$$\begin{aligned}\lim_{t\to 0}\left\langle \frac{W_\xi(t)-1}{t}Exp(v_\zeta), Exp(v_\eta)\right\rangle &= \frac{d}{dt}\Big|_{t=0}\langle W_\xi(t)Exp(v_\zeta), Exp(v_\eta)\rangle \\ &= h'(0)\exp\{h(0)\}.\end{aligned}$$

Therefore, for each $v_\zeta \in \mathcal{D}_\nu$, we have $Exp(v_\zeta) \in \text{wk-dom}(Q_{CP,\xi})$, i.e., $Exp(\mathcal{D}_\nu) \subset \text{wk-dom}(Q_{CP,\xi})$ and we know that $Exp(\mathcal{D}_\nu)$ is in the domain of the operator-valued distributions $A^-_\nu(\widetilde{q}_\xi \cdot 1)$, $A^+_\nu(\widetilde{q}_\xi \cdot 1)$, $\Lambda_\nu(\widetilde{q}_\xi)$. Consequently,

$$\begin{aligned}&\langle Q_\xi Exp(v_\zeta), Exp(v_\eta)\rangle = i\lim_{t\to 0}\left\langle \frac{W_\xi(t)-1}{t}Exp(v_\zeta), Exp(v_\eta)\right\rangle \\ &= ih'(0)\exp\{h(0)\} \\ &= \big(\overline{\langle \widetilde{q}_\xi \cdot 1, v_\zeta\rangle} + \langle \widetilde{q}_\xi \cdot 1, v_\eta\rangle + \langle \widetilde{q}_\xi \cdot 1, \overline{v_\zeta}v_\eta\rangle + \mathbb{E}(X_1)\langle\xi\rangle\big)\exp\{\langle v_\zeta, v_\eta\rangle\} \\ &= \big\langle \langle \widetilde{q}_\xi \cdot 1, v_\zeta\rangle Exp(v_\zeta), Exp(v_\eta)\big\rangle + \big\langle Exp(v_\zeta), \langle \widetilde{q}_\xi \cdot 1, v_\eta\rangle Exp(v_\eta)\big\rangle \\ &\quad + \big\langle Exp(v_\zeta), \langle \widetilde{q}_\xi \cdot 1, \bar{v}_\zeta v_\eta\rangle Exp(v_\eta)\big\rangle + \big\langle \mathbb{E}(X_1)\langle\xi\rangle Exp(v_\zeta), Exp(v_\eta)\big\rangle \\ &= \langle A^-_\nu(\widetilde{q}_\xi \cdot 1)Exp(v_\zeta), Exp(v_\eta)\rangle + \langle Exp(v_\zeta), A^-_\nu(\widetilde{q}_\xi \cdot 1)Exp(v_\eta)\rangle \\ &\quad + \langle Exp(v_\zeta), A^-_\nu(\widetilde{q}_\xi v_\zeta)Exp(v_\eta)\rangle + \big\langle \mathbb{E}(X_1)\langle\xi\rangle Exp(v_\zeta), Exp(v_\eta)\big\rangle \\ &= \langle A^-_\nu(\widetilde{q}_\xi \cdot 1)Exp(v_\zeta), Exp(v_\eta)\rangle + \langle A^+_\nu(\widetilde{q}_\xi \cdot 1)Exp(v_\zeta), Exp(v_\eta)\rangle \\ &\quad + \langle \Lambda_\nu(\widetilde{q}_\xi)Exp(v_\zeta), Exp(v_\eta)\rangle + \big\langle \mathbb{E}(X_1)\langle\xi\rangle Exp(v_\zeta), Exp(v_\eta)\big\rangle \\ &= \big\langle \big(A^-_\nu(\widetilde{q}_\xi \cdot 1) + A^+_\nu(\widetilde{q}_\xi \cdot 1) + \Lambda_\nu(\widetilde{q}_\xi) + \mathbb{E}(X_1)\langle\xi\rangle\mathbf{1}\big)Exp(v_\zeta), Exp(v_\eta)\big\rangle. \quad \blacksquare\end{aligned}$$

6. THE RENORMALIZED QUANTUM DECOMPOSITION

In this section we assume that ν has no moment at all.

Comparing the expressions (5.12) and (4.11) one sees that even if they should be understood in different ways, they look formally the same and that the existence of a finite first order moment is a necessary condition for both expressions to make sense. Therefore, for random variables not satisfying this condition, one must look for a notion of quantum decomposition different from the one given by expressions of the form (4.11).

On the other hand, the above-mentioned fact that all the results of Section 5 do not require the existence of the first order moment suggests that all the problems related with the extension of expression (4.11) to random variables without moments is concentrated on the scalar term in the sense that, after subtracting this term, which is infinite in the case of random variables without moments, one obtains the distribution-valued operator

$$A_\nu^+(\widetilde{q}_\xi \cdot 1) + A_\nu^-(\widetilde{q}_\xi \cdot 1) + \Lambda_\nu(\widetilde{q}_\xi), \tag{6.1}$$

which is meaningful because of the arguments discussed in the previous section.

In physics the procedure of subtracting infinite constants to some expressions, in order to transform them into meaningful and physically measurable ones, is well known and called *additive renormalization*.

In the present case a mere additive renormalization would not be sufficient because it would leave open the question of the connection between the resulting expression (6.1) after additive renormalization and the original random variable without moments. In other words, we want the renormalized quantum decomposition (6.1) to be canonically associated with the random variable $Q_{CP,\xi}$ or, equivalently, with the one-parameter group $\exp\{itQ_{CP,\xi}\}$ generated by it.

In the following we prove that such a canonical connection can be established by using a multiplicative renormalization procedure. In mathematical terms this means the transition from a representation of the additive group $\mathbb{R}$ to a projective representation of the same group.

The idea of the construction of this projective representation is naturally suggested by the proof of Theorem 5.2. In fact, from this proof one can see that the emergence of the first moment in the quantum decomposition is due to the derivative of the scalar term in the normally ordered form (5.4) of the one-parameter unitary group $\exp\{itQ_{CP,\xi}\}$, i.e., $\exp\{\langle\Psi(t\xi)\rangle\}$. Therefore, the emergence of the "infinite constant" $\mathbb{E}(X_1)$ in the formal expression (5.12) is a manifestation of the fact that if the first moment of the random variable X_1 is infinite, then the function Ψ is not differentiable.

In order to *remove this constant* from $\exp\{itQ_{CP,\xi}\}$ notice that if $(W_t)_{t\in\mathbb{R}}$ is a unitary representation of $\mathbb{R}$ and $t \in \mathbb{R} \mapsto \mu_t \in \mathbb{C}$ is any measurable function, then the one-parameter family

$$V_t := \exp\{-\mu_t\}W_t$$

is a projective, in general non-unitary, representation of $\mathbb{R}$ with multiplier (or two-cocycle, which in this case is in fact a two-coboundary):

$$\hat{\sigma}(s,t) := \exp\{\mu_{t+s} - \mu_s - \mu_t\}.$$

In other terms,

$$V_s V_t = \hat{\sigma}(s,t) V_{s+t}, \qquad s,t \in \mathbb{R}.$$

Now we apply this remark to the case when $\mu_t = \langle \Psi(t\xi) \rangle$ and W_t is given by (5.5).

LEMMA 6.1. *Let p be the orthogonal projection from $L^2(\nu \otimes dt)$ onto $\mathcal{K}_0$ and let $C_{c,0}^{\infty}(\mathbb{R}^* \times \mathbb{R})$ be the dense subspace of $L^2(\nu \otimes dt)$ consisting of the infinitely differentiable functions with compact support not containing zero. Let*

$$\mathcal{D}_\nu := p\big(C_{c,0}^{\infty}(\mathbb{R}^* \times \mathbb{R})\big) = \{p(\varphi);\ \varphi \in C_{c,0}^{\infty}(\mathbb{R}^* \times \mathbb{R})\} \subset \mathcal{K}_0$$

be the image under p of $C_{c,0}^{\infty}(\mathbb{R}^* \times \mathbb{R})$. Then $\mathcal{D}_\nu$ is a dense subspace of $\mathcal{K}_0$ satisfying conditions (C1) *and* (C2).

Proof. Step 1. Let $f \in \mathcal{K}_0 \subset L^2(\nu \otimes dt)$. By the density of $C_{c,0}^{\infty}(\mathbb{R}^* \times \mathbb{R})$ in $L^2(\nu \otimes dt)$, there exists $\varphi_n \in C_{c,0}^{\infty}(\mathbb{R}^* \times \mathbb{R})$ converging to f as $n \to +\infty$. Then we have

$$\|\varphi_n - f\|^2 = \|\varphi_n - p(\varphi_n)\|^2 + \|p(\varphi_n) - f\|^2$$

and we get

$$\|p(\varphi_n) - f\| \leqslant \|\varphi_n - f\| \to 0 \quad \text{ as } n \to +\infty.$$

Hence f is a limit of a sequence of $\mathcal{D}_\nu$, which proves the density.

For $\varphi \in C_{c,0}^{\infty}(\mathbb{R}^* \times \mathbb{R})$, let $f = p(\varphi) \in \mathcal{D}_\nu$. While $C_{c,0}^{\infty}(\mathbb{R}^* \times \mathbb{R})$ is invariant under complex conjugate, then to see this property for $\mathcal{D}_\nu$, it is sufficient to prove that $\bar{f} = p(\bar{\varphi})$. In fact, we have $f - \varphi \in \mathcal{K}_0^{\perp}$. Then

$$\langle \bar{f} - \bar{\varphi}, v_\xi \rangle = \langle v_{-\xi}, f - \varphi \rangle = 0 \quad \text{ for all } \xi \in \mathcal{T}.$$

This gives $\bar{f} - \bar{\varphi} \in \mathcal{K}_0^{\perp}$, and using the fact that $\bar{f} \in \mathcal{K}_0$, we deduce that $\bar{f} = p(\bar{\varphi})$, which proves the condition (C1).

Step 2. For $f \in \mathcal{D}_\nu$, let us consider the function $\Xi_f(t) = \langle v_{t\xi}, f \rangle$ and let $\varphi \in C_{c,0}^{\infty}(\mathbb{R}^* \times \mathbb{R})$ be such that $f = p(\varphi)$. Then $f - \varphi \in \mathcal{K}_0^{\perp}$, which gives

$$\Xi_f(t) = \langle v_{t\xi}, f \rangle = \langle v_{t\xi}, \varphi \rangle,$$

and one can see that Ξ_f is derivable at $t = 0$ and

$$\Xi_f'(0) = -i \int\limits_{\mathbb{R}^* \times \mathbb{R}} y\xi(s)\varphi(y,s)\nu(dy)ds = -i\langle \widetilde{q}_\xi \cdot 1, \varphi \rangle.$$

It is obvious that $\langle \widetilde{q}_\xi \cdot 1, \varphi \rangle$ does not depend on the choice of φ but only on f. Then the distribution

$$f \mapsto \langle \widetilde{q}_\xi \cdot 1, f \rangle := \langle \widetilde{q}_\xi \cdot 1, \varphi \rangle = i\Xi_f'(0) \tag{6.2}$$

is well defined. On the other hand, for all $g \in \mathcal{K}_0, v_\xi g \in \mathcal{K}_0$. In fact, g is a limit in $L^2(\nu \otimes ds)$ of a sequence $(g_n)_n \subset$ Lin-span$\{v_\xi;\ \xi \in \mathcal{T}\}$. Then it follows that $v_\xi g_n \in$ Lin-span$\{v_\xi;\ \xi \in \mathcal{T}\}$. Moreover,

$$\begin{aligned}\|v_\xi g_n - v_\xi g\|^2 &= \int\limits_{\mathbb{R}^*\times\mathbb{R}} \left|v_\xi(y,s)\big(g_n(y,s) - g(y,s)\big)\right|^2 \nu(dy)ds \\ &\leqslant 4\|g_n - g\|^2 \to 0 \quad \text{ as } n \to +\infty.\end{aligned}$$

Hence $v_\xi g$ is a limit of some sequence of Lin-span$\{v_\xi;\ \xi \in \mathcal{T}\}$, so it belongs to $\mathcal{K}_0$.

Moreover, for $\varphi \in C^\infty_{c,0}(\mathbb{R}^* \times \mathbb{R})$, let $f = p(\varphi) \in \mathcal{D}_\nu$. Then, by the definition of p, $(f - \varphi) \perp v_\xi g$, which gives

$$\langle v_\xi, f\bar{g}\rangle - \langle v_\xi, \varphi\bar{g}\rangle = \langle v_\xi, f\bar{g} - \varphi\bar{g}\rangle = \langle v_\xi g, f - \varphi\rangle = 0.$$

Let us consider the function

$$\Theta_{g,f}(t) := \langle v_{t\xi}, f\bar{g}\rangle = \langle v_{t\xi}, \varphi\bar{g}\rangle = \int\limits_{\mathbb{R}^*\times\mathbb{R}} v_{-t\xi}(y,s)\overline{g(y,s)}\varphi(y,s)\nu(dy)ds.$$

One can see that $\Theta_{g,f}$ is derivable at $t = 0$ and

$$\Theta'_{g,f}(0) = -i \int\limits_{\mathbb{R}^*\times\mathbb{R}} y\xi(s)\overline{g(y,s)}\varphi(y,s)\nu(dy)ds.$$

Define

$$\langle \widetilde{q}_\xi g, f\rangle := \int\limits_{\mathbb{R}^*\times\mathbb{R}} y\xi(s)\overline{g(y,s)}\varphi(y,s)\nu(dy)ds = i\Theta'_{g,f}(0).$$

Clearly, $\langle \widetilde{q}_\xi g, f\rangle$ does not depend on the choice of φ. Then the distribution

$$f \mapsto \langle \widetilde{q}_\xi g, f\rangle = \langle \widetilde{q}_\xi \cdot 1, f\bar{g}\rangle$$

is well defined on $\mathcal{D}_\nu$, which gives the condition (C2). ■

THEOREM 6.1. *Let $\big(W_\xi(t)\big)_{t\in\mathbb{R}}$ be the one-parameter unitary group defined by* (5.5) *and define*

$$V_\xi(t) := \exp\{-\langle\Psi(t\xi)\rangle\}W_\xi(t) = \exp\{A^+(v_{t\xi})\}\exp\{\Lambda(it\widetilde{q}_\xi)\}\exp\{A^-(v_{-t\xi})\}.$$

Then $\{V_\xi(t)\,;\, t \in \mathbb{R}\}$ is a strongly continuous projective representation of $\mathbb{R}$ with multiplier

$$\hat{\sigma}_\xi(s,t) := \exp\left\{\left\langle \Psi\big((s+t)\xi\big) - \Psi(s\xi) - \Psi(t\xi)\right\rangle\right\}.$$

Its generator Q_Ψ contains $Exp(\mathcal{D}_\nu)$ in its weak domain and, on $Exp(\mathcal{D}_\nu)$, coincides with

$$A^+_\nu(\widetilde{q}_\xi \cdot 1) + A^-_\nu(\widetilde{q}_\xi \cdot 1) + \Lambda_\nu(\widetilde{q}_\xi).$$

P r o o f. Since $W_\xi(t)$ is a strongly continuous one-parameter unitary group and Ψ is continuous, the strong continuity of $V_\xi(t)$ is clear because, for any $\varphi \in \mathcal{K}_0$ and $t_0 \in \mathbb{R}$, defining $\varphi_0 := V_\xi(t_0)\varphi$, we get, as $s \to 0$,

$$\|V_\xi(t_0+s)\varphi - V_\xi(t_0)\varphi\| = \big\| \exp\big\{ -\big\langle \Psi\big((t_0+s)\xi\big)\big\rangle\big\} W_\xi\big((t_0+s)\big)\varphi - \varphi_0 \big\|$$
$$= \big\| \exp\big\{ -\big\langle \Psi\big((t_0+s)\xi\big) + \Psi(t_0\xi)\big\rangle\big\} W_\xi(s)\varphi_0 - \varphi_0 \big\|$$
$$\leqslant \Big\| W_\xi(s)\Big(\exp\big\{ -\big\langle \Psi\big((s+t_0)\xi\big) + \Psi(t_0\xi)\big\rangle\big\}\varphi_0 - \varphi_0\Big)\Big\| + \|W_\xi(s)\varphi_0 - \varphi_0\|$$
$$= \big\| \exp\big\{\big\langle \Psi(t_0\xi) - \Psi\big((t_0+s)\xi\big)\big\rangle\big\}\varphi_0 - \varphi_0\big\| + \|W_\xi(s)\varphi_0 - \varphi_0\| \to 0.$$

Let $f, g \in \mathcal{D}_\nu$. We have

$$\Big\langle \frac{V_\xi(t)-1}{t} Exp(f), Exp(g)\Big\rangle = \frac{1}{t}\big(\langle V_\xi(t) Exp(f), Exp(g)\rangle - \exp\{\langle f, g\rangle\}\big)$$
$$= \frac{1}{t}\big(\big\langle \exp\{\overline{\langle v_{-t\xi}, f\rangle}\} Exp(\exp\{it\widetilde{q}_\xi\}f + v_{t\xi}), Exp(g)\big\rangle - \exp\{\langle f, g\rangle\}\big)$$
$$= \frac{1}{t}\big(\exp\big\{\overline{\langle v_{-t\xi}, f\rangle} + \langle \exp\{it\widetilde{q}_\xi\}f + v_{t\xi}, g\rangle\big\} - \exp\{\langle f, g\rangle\}\big)$$
$$= \frac{1}{t}\big(\exp\{\gamma(t)\} - \exp\{\gamma(0)\}\big),$$

where, in the notation of the proof of Lemma 6.1,

$$\gamma(t) = \overline{\langle v_{-t\xi}, f\rangle} + \langle \exp\{it\widetilde{q}_\xi\}f + v_{t\xi}, g\rangle$$
$$= \overline{\langle v_{-t\xi}, f\rangle} + \langle v_{t\xi} f, g\rangle + \langle f, g\rangle + \langle v_{t\xi}, g\rangle = \overline{\Xi_f(-t)} + \Theta_{f,g}(t) + \Xi_g(t) + \langle f, g\rangle.$$

But it is clear from the above calculations that γ is derivable at $t = 0$ and

$$\gamma'(0) = -\overline{\Xi'_f(0)} + \Xi'_g(0) + \Theta'_{f,g}(0) = -i(\overline{\langle \widetilde{q}_\xi \cdot 1, f\rangle} + \langle \widetilde{q}_\xi \cdot 1, g\rangle + \langle \widetilde{q}_\xi \cdot 1, \overline{f} g\rangle).$$

Then the limit

$$\lim_{t\to 0} \Big\langle \frac{V_\xi(t)-1}{t} Exp(f), Exp(g)\Big\rangle$$

exists. Hence $Exp(f) \in$ wk-dom(Q_Ψ) and

$$\langle Q_\Psi Exp(f), Exp(g)\rangle$$
$$= i\Big\langle \lim_{t\to 0} \frac{V_\xi(t)-1}{t} Exp(f), Exp(g)\Big\rangle = i\gamma'(0)\exp\{\gamma(0)\}$$
$$= (\overline{\langle \widetilde{q}_\xi \cdot 1, f\rangle} + \langle \widetilde{q}_\xi \cdot 1, g\rangle + \langle \widetilde{q}_\xi \cdot 1, \overline{f} g\rangle)\exp\{\langle f, g\rangle\}$$
$$= \big\langle \langle \widetilde{q}_\xi \cdot 1, f\rangle Exp(f), Exp(g)\big\rangle + \big\langle Exp(f), \langle \widetilde{q}_\xi \cdot 1, g\rangle Exp(g)\big\rangle$$
$$+ \big\langle Exp(f), \langle \widetilde{q}_\xi \cdot 1, \overline{f} g\rangle Exp(g)\big\rangle$$
$$= \langle A_\nu^-(\widetilde{q}_\xi \cdot 1) Exp(f), Exp(g)\rangle + \langle Exp(f), A_\nu^-(\widetilde{q}_\xi \cdot 1) Exp(g)\rangle$$
$$+ \langle Exp(f), A_\nu^-(\widetilde{q}_\xi f) Exp(g)\rangle$$
$$= \langle A_\nu^-(\widetilde{q}_\xi \cdot 1) Exp(f), Exp(g)\rangle + \langle A_\nu^+(\widetilde{q}_\xi \cdot 1) Exp(f), Exp(g)\rangle$$
$$+ \langle \Lambda_\nu(\widetilde{q}_\xi) Exp(f), Exp(g)\rangle$$
$$= \big\langle \big(A_\nu^-(\widetilde{q}_\xi \cdot 1) + A_\nu^+(\widetilde{q}_\xi \cdot 1) + \Lambda_\nu(\widetilde{q}_\xi)\big) Exp(f), Exp(g)\big\rangle.$$

This gives the statement. ■

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Received on 14.2.2014;
revised version on 24.9.2014