PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 34, Fasc. 1 (2014), pp. 127–145

# A WICK FUNCTIONAL LIMIT THEOREM

BY

PETER PARCZEWSKI (MANNHEIM)

*Abstract.* We prove that weak convergence of multivariate discrete Wiener integrals towards the continuous counterparts carries over to the application of discrete and continuous Wick calculus. This is done by the representation of arbitrary Wick products of Wiener integrals in terms of generalized Hermite polynomials and a discrete analog of the Hermite recursion. The result is a multivariate non-central limit theorem in the form of a Wick functional limit theorem. As an application we give approximations of multivariate processes based on fractional Brownian motions for arbitrary Hurst parameters $H \in (0,1)$.

**2000 AMS Mathematics Subject Classification:** Primary: 60F05; Secondary: 60G15, 60H07.

**Key words and phrases:** Wick calculus, discrete Wick calculus, weak convergence, central limit theorem.

## 1. INTRODUCTION

The Wick calculus on Wiener integrals $I(f)$ is mainly based on the Hermite expansion

$$\exp^{\diamond}\big(I(f)\big) = \sum_{k=0}^{\infty} \frac{h^k\big(I(f)\big)}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} I(f)^{\diamond k}, \tag{1.1}$$

where

$$\exp^{\diamond}\big(I(f)\big) := \exp\left(I(f) - \frac{1}{2}\int_0^1 f^2(s)ds\right) \tag{1.2}$$

is the Wick exponential,

$$h^k(x) = h^k_{\sigma^2}(x) := (-\sigma^2)^k \exp(x^2/2\sigma^2) \frac{d^k}{dx^k} \exp(-x^2/2\sigma^2) \tag{1.3}$$

is the Hermite polynomial of degree $k$ with parameter $\sigma^2$, and $\diamond$ denotes the Wick product. We refer to the standard monographs [12], [10], and [13] for these objects from stochastic analysis and white noise theory. A brief introduction to Wick calculus is included in the next section. Here we assume that $(\Omega, \mathcal{F}, P)$ is a probability space which carries a Brownian motion $(B_t)_{t\in[0,1]}$ on the interval $[0,1]$ and $\mathcal{F}$ is the $\sigma$-field generated by the Brownian motion. Then $L^2(\Omega, \mathcal{F}, P)$ becomes a Gaussian Hilbert space.

As a discrete counterpart we consider, for every $n \in \mathbb{N}$, a binary random walk

$$B_t^n := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^n.$$

Here, $(\xi_1^n, \ldots, \xi_n^n)$ is an $n$-tuple of independent symmetric Bernoulli random variables with $P_n(\xi_i^n = \pm 1) = 1/2$ on a probability space $(\Omega_n, \mathcal{F}_n, P_n)$ and $\mathcal{F}_n$ is the $\sigma$-field generated by the binary trials $(\xi_1^n, \ldots, \xi_n^n)$. Defining the discrete Wiener integral via

$$I^n(f^n) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i^n \xi_i^n,$$

we have the following variant of the central limit theorem:

THEOREM 1.1. *For all $m$-tuples of Wiener integrals $\big(I(f_1), \ldots, I(f_m)\big)$ and discrete Wiener integrals $\big(I^n(f_1^n), \ldots, I^n(f_m^n)\big)$, respectively, the following three assertions are equivalent as $n$ tends to infinity*:

(a) $\big(I^n(f_1^n), \ldots, I^n(f_m^n)\big) \xrightarrow{d} \big(I(f_1), \ldots, I(f_m)\big)$.

(b) *For all $l_1, \ldots, l_k \in \{1, \ldots, m\}$*

$$\mathbb{E}[I^n(f_{l_1}^n) \ldots I^n(f_{l_k}^n)] \to \mathbb{E}[I(f_{l_1}) \ldots I(f_{l_k})].$$

(c) $\lim\limits_{n\to\infty} \max\limits_{l\leqslant m} \max\limits_{i\leqslant n} \frac{1}{\sqrt{n}} |f_{l,i}^n| = 0$ *and, for all $k, l$,*

$$\frac{1}{n} \sum_{i=1}^{n} (f_{k,i}^n)(f_{l,i}^n) \to \int_0^1 f_k(u) f_l(u) du.$$

For a proof we refer to Theorem 1.1 in [3].

Here we prove that the weak convergence in Theorem 1.1 carries over to applications of continuous and discrete Wick calculus. This is a multivariate extension of results in [2] and [3] which leads to a multivariate Wick functional limit theorem. We refer to [4] for an alternative approach to such strong convergence results by Skorokhod embedding and the approximation of the S-transforms by the discrete counterparts.

The paper is organized as follows:

In Section 2 we introduce the continuous Wick product and give analytic formulas for Wick products of Wiener integrals by generalized Hermite polynomials which have been introduced in [6].

The discrete Wick calculus is introduced in Section 3. After some $L^2$ formulas and estimates on Wick products and Wiener integrals, we give a discrete analog of the Hermite recursion for arbitrary Wick products of Wiener integrals. Then, based on the weak convergence in Theorem 1.1, we conclude the weak convergence of arbitrary multivariate Wick products in Theorem 3.1.

This convergence of arbitrary Wick products is extended in Section 4 to the main result, the weak convergence of arbitrary Wick products of Wick analytic functionals on the multivariate Wiener integrals in Theorem 1.1. Finally, we present an application on fractional Brownian motion based on a fractional Donsker theorem from [15].

The proofs are subsumed in the Appendix.

## 2. WICK CALCULUS AND GENERALIZED HERMITE POLYNOMIALS

The Wick exponential is closely related to the S-transform which plays an important role in the white noise distribution theory. For every $X \in L^2(\Omega, \mathcal{F}, P)$ and $h \in L^2([0,1])$, the *S-transform* of $X$ at $h$ is defined as

$$(SX)(h) := \mathbb{E}\big[X \exp^{\diamond}\big(I(h)\big)\big].$$

Thanks to the injectivity ([12], Theorem 16.11), the S-transform can be applied to characterize random variables. In particular, it leads to an elegant introduction of the Wick product:

PROPOSITION 2.1. *Define the* Wick product *by*

$$\exp^{\diamond}\big(I(f)\big) \diamond \exp^{\diamond}\big(I(g)\big) = \exp^{\diamond}\big(I(f+g)\big), \tag{2.1}$$

*where $I(f)$ and $I(g)$ are two possibly correlated Wiener integrals. This is equivalent to*

$$\Big(S \exp^{\diamond}\big(I(f)\big) \diamond \exp^{\diamond}\big(I(g)\big)\Big)(h) = \Big(S \exp^{\diamond}\big(I(f)\big)\Big)(h)\Big(S \exp^{\diamond}\big(I(g)\big)\Big)(h)$$

*and we set*

$$\begin{aligned}\mathcal{D} := \big\{&(X,Y) \in L^2(\Omega) \times L^2(\Omega) : \\ &\exists Z_{X,Y} \in L^2(\Omega)\ \forall h \in L^2(\mathbb{R})\ \ (SZ_{X,Y})(h) = (SX)(h)(SY)(h)\big\},\end{aligned}$$

$$\diamond : \mathcal{D} \to L^2(\Omega), \quad (X,Y) \mapsto Z_{X,Y}.$$

*Then $\mathcal{D}$ is a dense subset of $L^2(\Omega) \times L^2(\Omega)$, the Wick product $\diamond$ is well-defined (i.e., $Z_{X,Y}$ is uniquely determined), and the following properties hold true:*

(a) *For every $f, g \in L^2([0,1])$, $\big(\exp^{\diamond}\big(I(f)\big), \exp^{\diamond}\big(I(g)\big)\big) \in \mathcal{D}$ and* (2.1) *are valid.*

(b) *The Wick product is bilinear.*

(c) *The Wick product is closed, i.e., if* $(X_k, Y_k)_{k\in\mathbb{N}} \subset \mathcal{D}$, $(X_k, Y_k) \to (X, Y)$ *in* $L^2(\Omega)\times L^2(\Omega)$, *and* $X_k \diamond Y_k \to Z$ *in* $L^2(\Omega)$, *then* $(X, Y) \in \mathcal{D}$ *and* $X \diamond Y = Z$.

For more details on Wick calculus we refer to [12] and [10]. Here we notice the following generalization of a formula in [11]:

PROPOSITION 2.2. *Suppose* $(X_1, \ldots, X_k)$ *is jointly normal distributed and* $(Y_1, \ldots, Y_k)$ *is an independent copy of* $(X_1, \ldots, X_k)$. *Then*

$$X_1 \diamond \ldots \diamond X_k = \mathbb{E}[(X_1 + iY_1) \ldots (X_k + iY_k)|X_1, \ldots, X_k].$$

P r o o f. The assertion follows by

$$\begin{aligned}\exp^{\diamond}\big(\sum_{l=1}^{k} t_l X_l\big) &= \exp\Big(\sum_{l=1}^{k} t_l X_l - \frac{1}{2}\mathbb{E}\big[\big(\sum_{l=1}^{k} t_l Y_l\big)^2\big]\Big)\\ &= \mathbb{E}\big[\exp\big(\sum_{l=1}^{k} t_l (X_l + iY_l)\big)\big|X_1, \ldots, X_k\big],\end{aligned}$$

the Taylor expansion and identifying the coefficient terms for $t_1 \ldots t_k$. ∎

By the definition via the S-transform or thanks to Proposition 2.2 we obtain

$$\begin{aligned} I(f) \diamond I(g) &= I(f)I(g) - \langle f, g\rangle,\\ I(f) \diamond I(g) \diamond I(h) &= I(f)I(g)I(h) - \langle f, g\rangle I(h) - \langle f, h\rangle I(g) - \langle g, h\rangle I(f), \end{aligned} \tag{2.2}$$

where $\langle f, g\rangle = \int_0^1 f(s)g(s)ds = \mathbb{E}[I(f)I(g)]$. In the following we generalize this representation of the Wick product of Wiener integrals and illustrate the connection to the Hermite polynomials (1.3). For these reasons we consider a generalization of the Hermite polynomials to different variables as introduced in [6]. We define for the symmetric constants $\sigma_{i,j} = \sigma_{j,i} \in \mathbb{R}^+$, $i, j \in \mathbb{N}$, the following polynomials:

$$h^1(x_1) := x_1, \quad h^2_{\sigma_{1,2}}(x_1, x_2) := x_1x_2 - \sigma_{1,2},$$

and recursively

$$\begin{aligned} h^{n+1}_{\sigma}(x_1, \ldots, x_{n+1}) &:= x_{n+1}h^n_{\sigma}(x_1, \ldots, x_n)\\ &\quad - \sum_{l=1}^{n} \sigma_{l,n+1} h^{n-1}_{\sigma}(x_1, \ldots, \widehat{x_l}, \ldots, x_n), \end{aligned} \tag{2.3}$$

where $\sigma$ denotes the appropriate set of constants $\sigma$ for all pairs of the variables $x_1, \ldots, x_n$ and $\widehat{x}$ denotes the absence of the variable $x$. Thus we have, in particular,

$$h^3_{\sigma}(x_1, x_2, x_3) = x_1x_2x_3 - (x_1\sigma_{2,3} + x_2\sigma_{1,3} + x_3\sigma_{1,2}) \tag{2.4}$$

with $\sigma = \{\sigma_{1,2}, \sigma_{1,3}, \sigma_{2,3}\}$. For completeness we define $h^0 := 1$. For constant $\sigma_{i,j} = \sigma^2$ and $x_i = x$ for all $i, j$, we obtain the ordinary Hermite polynomials with parameter $\sigma^2$. This is a reformulation of the products of Hermite polynomials in [10]. These polynomials are included in multivariate Appell polynomials in [7]. In the following proposition we collect some properties of these polynomials related to a Wick exponential representation as in (1.1).

PROPOSITION 2.3. *The generalized Hermite polynomials have the following properties*:

(i) *For all $n \in \mathbb{N}$,*

$$h_\sigma^n(x_1, \ldots, x_n) = \frac{\partial^n}{\partial t_1 \ldots \partial t_n} \exp^\diamond \big( \sum_{i=1}^n t_i x_i \big)\big|_{t_1 = \ldots = t_n = 0}.$$

(ii) *The generalized Hermite polynomials are symmetric if the constants $\sigma_{i,j}$ are interchanged in according terms. This means that if we interchange $x_i \leftrightarrow x_j$, then the interchange $\sigma_{i,m} \leftrightarrow \sigma_{j,m}$ for all $m$ holds true.*

(iii) *The derivative recursion formula*

$$\frac{\partial}{\partial x_l} h_\sigma^n(x_1, \ldots, x_n) = h_\sigma^{n-1}(x_1, \ldots, \widehat{x}_l, \ldots, x_n) \tag{2.5}$$

*for all $n \geqslant 1$ and $l = 1, \ldots, n$ holds true.*

The proof goes by induction and is omitted. We refer to the computations in [7] and [1] for further details. Thus we conclude the following representation which is implied by the diagram formulae for Gaussian random variables (cf. [7], Theorem 4, or [12], Theorems 3.4 and 3.9).

LEMMA 2.1. (i) *For all $k \in \mathbb{N}$ and $f_1, \ldots, f_k \in L^2([0,1])$,*

$$I(f_1) \diamond \ldots \diamond I(f_k) = h^k_{\{\langle f_i, f_j \rangle, 1 \leqslant i < j \leqslant k\}} \big( I(f_1), \ldots, I(f_k) \big), \tag{2.6}$$

*where $I(f_i)$ are standard Wiener integrals.*

(ii) *For all $n, m \in \mathbb{N}$,*

$$\mathbb{E}\big[\big(I(f_1) \diamond \ldots \diamond I(f_n)\big)\big(I(g_1) \diamond \ldots \diamond I(g_m)\big)\big] = \delta_{n,m} \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^n \mathbb{E}[I(f_i) I(g_{\sigma(i)})],$$

*where $\mathcal{S}_n$ denotes the group of permutations on $\{1, \ldots, n\}$.*

## 3. DISCRETE WICK CALCULUS AND A DISCRETE HERMITE RECURSION

Concerning the discrete setting, a *discrete Wick exponential* of the discrete Wiener integral $I^n(f^n)$ is defined as

$$\exp^{\diamond_n} \big( I^n(f^n) \big) := \prod_{i=1}^n \left( 1 + \frac{1}{\sqrt{n}} f_i^n \xi_i^n \right). \tag{3.1}$$

Obviously, $L^2(\Omega_n, \mathcal{F}_n, P_n)$ is a $2^n$-dimensional Hilbert space. A canonical orthonormal basis of $L^2(\Omega_n, \mathcal{F}_n, P_n)$ consists of the set

$$\Big\{\Xi_A^n := \prod_{i\in A} \xi_i^n,\ A \subset \{1,\ldots,n\}\Big\}.$$

Every $X^n \in L^2(\Omega_n, \mathcal{F}_n, P_n)$ has a unique expansion in terms of this basis, which is called the *Walsh decomposition*,

$$X^n = \sum_{A\subset\{1,\ldots,n\}} X_A^n \Xi_A^n,$$

where $X_A^n \in \mathbb{R}$. It follows immediately that expectation and $L^2$-inner product can be computed in terms of the Walsh decomposition by the equalities $\mathbb{E}[X^n] = X_\emptyset^n$ and $\mathbb{E}\left[X^n Y^n\right] = \sum_{A\subset\{1,\ldots,n\}} X_A^n Y_A^n$. The discrete Wick exponential corresponds to the terminal value of a discrete exponential martingale and was studied in [8] in the context of discrete stochastic analysis.

Analogously to (2.1), for two, possibly correlated, discrete Wiener integrals $I^n(f^n)$, $I^n(g^n)$ a discrete Wick product of the corresponding discrete Wick exponentials is defined via the property

$$\exp^{\diamond_n}\big(I^n(f^n)\big) \diamond_n \exp^{\diamond_n}\big(I^n(g^n)\big) = \exp^{\diamond_n}\big(I^n(f^n+g^n)\big). \tag{3.2}$$

For the proof of the following result we refer to [3], Lemma 1.1:

LEMMA 3.1. *The discrete Wick product in* (3.2) *is well-defined and equivalent to the following characterization in terms of the canonical basis* $\big\{\Xi_A^n, A \subset \{1,\ldots,n\}\big\}$ *as introduced in* [9]: *For every* $A, B \subset \{1,\ldots,n\}$,

$$\Xi_A^n \diamond_n \Xi_B^n := \Xi_{A\cup B}^n \mathbf{1}_{A\cap B=\emptyset}.$$

*Moreover*, $\big(L^2(\Omega_n, \mathcal{F}_n, P_n), +, \diamond_n\big)$ *is a commutative ring.*

EXAMPLE 3.1. Suppose $f^n, g^n \in \mathbb{R}^n$. Then

$$I^n(f^n)^{\diamond_n N} = N! n^{-N/2} \sum_{\substack{A\subset\{1,\ldots,n\}\\ |A|=N}} f_A^n \Xi_A^n.$$

Hence, for different discrete Wick powers ($N, M \in \mathbb{N}$), we obtain

$$\begin{aligned} \mathbb{E}[I^n(f^n)^{\diamond_n N} I^n(g^n)^{\diamond_n M}] &= \delta_{N,M}(N!)^2 n^{-N} \sum_{\substack{A\subset\{1,\ldots,n\}\\ |A|=N}} f_A^n g_A^n \\ &= \mathbf{1}_{\{N,M\leqslant n\}} \delta_{N,M} N! \mathbb{E}[I^n(f^n) I^n(g^n)]^N, \end{aligned} \tag{3.3}$$

which is the discrete counterpart of the continuous formula. Analogously to (1.1), we obtain the Wick power series representation

$$\exp^{\diamond_n}\big(I^n(f^n)\big) = \sum_{A\subset\{1,\dots,n\}} n^{-|A|/2} f_A^n \Xi_A^n = \sum_{k=0}^{n} \frac{1}{k!} I^n(f^n)^{\diamond_n k}. \tag{3.4}$$

In contrast to the continuous case, we have

$$\begin{aligned} &I^n(f_1^n)\diamond_n I^n(f_2^n)\diamond_n I^n(f_3^n) \\ &= h^3_{\{\mathbb{E}[I^n(f_i^n)I^n(f_j^n)],1\leqslant i<j\leqslant 3\}}\big(I^n(f_1^n),I^n(f_2^n),I^n(f_3^n)\big) + 2\sum_{i=1}^{n} n^{-3/2} f_{1,i}^n f_{2,i}^n f_{3,i}^n \xi_i^n. \end{aligned}$$

Before stating the discrete analog of the Hermite recursion formula, we relate the $L^2$-norms of discrete Wick products to the continuous counterparts. The proofs of the following statements are included in the Appendix.

PROPOSITION 3.1. (i) *Suppose* $k_1,\dots,k_m\in\mathbb{N}$, $\sum_{i=1}^m k_i\leqslant n$. *Then*

$$\mathbb{E}\Big[\big(I^n(f_1^n)^{\diamond_n k_1}\diamond_n\dots\diamond_n I^n(f_m^n)^{\diamond_n k_m}\big)^2\Big] \leqslant m!\prod_{i=1}^{m}(k_i)!\,\mathbb{E}\Big[\big(I^n(f_i^n)\big)^2\Big]^{k_i}.$$

(ii) *Suppose* $\big(I^n(f_1^n),\dots,I^n(f_N^n),I^n(g_1^n),\dots,I^n(g_N^n)\big)$ *converges weakly to the continuous counterpart. Then*

$$\begin{aligned} \lim_{n\to\infty}\mathbb{E}\Big[\Big(\big(I^n(f_1^n)\diamond_n\dots\diamond_n I^n(f_N^n)\big)-\big(I^n(g_1^n)\diamond_n\dots\diamond_n I^n(g_N^n)\big)\Big)^2\Big] \\ = \mathbb{E}\Big[\Big(\big(I(f_1)\diamond\dots\diamond I(f_N)\big)-\big(I(g_1)\diamond\dots\diamond I(g_N)\big)\Big)^2\Big]. \end{aligned}$$

REMARK 3.1. (i) *For all* $\big(I^n(f^n),I^n(g^n)\big)\xrightarrow{d}\big(I(f),I(g)\big)$, $N,M\in\mathbb{N}$,

$$\mathbb{E}[I^n(f^n)^{\diamond_n N}I^n(g^n)^{\diamond_n M}]\to\mathbb{E}[I(f)^{\diamond N}I(g)^{\diamond M}].$$

(ii) *From Proposition* 3.1 (ii) *we infer that the assertion in Proposition* 3.1 (i) *holds true for continuous Wiener integrals as well.*

Now we can proceed with the derivation of the discrete analog of the Hermite recursion.

LEMMA 3.2 (Discrete Hermite recursion). *For all* $k\in\mathbb{N}$, $f_1^n,f_2^n,\dots,f_k^n\in\mathbb{R}^n$, *we have*

$$\begin{aligned} &I^n(f_1^n)\diamond_n\dots\diamond_n I^n(f_k^n) \\ &\quad= \big(I^n(f_1^n)\diamond_n\dots\diamond_n I^n(f_{k-1}^n)\big)\,I^n(f_k^n) \\ &\quad\quad-\sum_{l=1}^{k-1}\mathbb{E}[I^n(f_l^n)I^n(f_k^n)]\big(I^n(f_1^n)\diamond_n\dots\diamond_n\widehat{I^n(f_l^n)}\diamond_n\dots\diamond_n I^n(f_{k-1}^n)\big) \\ &\quad\quad+R^{k,n}\big(I^n(f_1^n),\dots,I^n(f_k^n)\big), \end{aligned} \tag{3.5}$$

*and*

$$\text{(3.6)}\quad \mathbb{E}\Big[\Big(R^{k,n}\big(I^n(f_1^n),\dots,I^n(f_k^n)\big)\Big)^2\Big] \leqslant (k-1)!(k-1)^3 \max_{l\leqslant k}\sup_{i\leqslant n}\frac{|f_{l,i}^n|^4}{n^2}\Big(\max_{l\leqslant k}\mathbb{E}\big[\big(I^n(f_l^n)\big)^2\big]\Big)^{k-1}.$$

REMARK 3.2. *This generalizes the discrete Hermite recursion formula for Wick powers in Proposition* 1.2 *of* [3] *to arbitrary Wick products.*

THEOREM 3.1. *Suppose that* $\big(I^n(f_1^n),\dots,I^n(f_k^n)\big)\xrightarrow{d}\big(I(f_1),\dots,I(f_k)\big)$. *We define, for all* $N\geqslant 1$, $\mathbf{I}_N^n$ *as the random vector of all discrete Wick products of the components in the vector* $\big(I^n(f_1^n),\dots,I^n(f_k^n)\big)$ *up to the order* $N$ *such that it contains for every* $A\subset\{1,\dots,N\}$ *and* $m: A\to\{1,\dots,k\}$ *the component*

$$(\Diamond_n)_{i\in A} I^n(f_{m(i)}^n).$$

*Analogously, for the random vector of Wiener integrals* $\big(I(f_1),\dots,I(f_k)\big)$, *we define* $\mathbf{I}_N$ *as the continuous counterpart of* $\mathbf{I}_N^n$ *in terms of continuous Wick products.*

*Then, for all* $N\geqslant 1$,

$$\mathbf{I}_N^n\xrightarrow{d}\mathbf{I}_N \quad \text{as } n\to\infty.$$

## 4. A WICK FUNCTIONAL LIMIT THEOREM

By Theorem 3.1 we are able to extend the weak convergence of the Wiener integrals in Theorem 1.1 to square integrable Wick analytic functionals.

THEOREM 4.1. *Assume that* $\big(I^n(f_1^n),\dots,I^n(f_m^n)\big)\xrightarrow{d}\big(I(f_1),\dots,I(f_m)\big)$. *Additionally suppose that the coefficients in the Wick analytic functionals*

$$F_l^{\diamond}(x)=\sum_{k=0}^{\infty}\frac{a_k^l}{k!}x^{\diamond k},\qquad F_l^{\diamond_n}(x)=\sum_{k=0}^{n}\frac{a_{n,k}^l}{k!}x^{\diamond_n k},\qquad l\in\{1,\dots,m\},$$

*satisfy the following conditions*:

(1) $\lim_{n\to\infty}a_{n,k}^l=a_k^l$ *exists for all* $k\in\mathbb{N}$ *and* $l=1,\dots,m$.

(2) *There exists a* $C\in\mathbb{R}_+$ *with* $|a_{n,k}^l|\leqslant C^k$ *for all* $n,k\in\mathbb{N}$, $l=1,\dots,m$.

*We define* $\mathbf{F}_m^n$ *as the random vector of all discrete Wick products of different components in the vector* $\big(F_1^{\diamond_n}\big(I^n(f_1^n)\big),\dots,F_m^{\diamond_n}\big(I^n(f_m^n)\big)\big)$ *up to the order* $m$ *such that it contains for every* $A\subset\{1,\dots,m\}$, $A\neq\emptyset$, *the component*

$$(\Diamond_n)_{i\in A}F_i^{\diamond_n}\big(I^n(f_i^n)\big).$$

*Analogously, we define the continuous counterpart* $\mathbf{F}_m$ *as the random vector of all Wick products of different components in the vector* $\big(F_1^{\diamond}\big(I(f_1)\big),\dots,F_m^{\diamond}\big(I(f_m)\big)\big)$. *Then it holds true that*

$$\mathbf{F}_m^n\xrightarrow{d}\mathbf{F}_m \quad \text{as } n\to\infty.$$

REMARK 4.1. (i) *For a simple case of the assertion* (cf. the proof) *a weaker assumption on the coefficients is*

(1) $\lim_{n\to\infty} a^l_{n,k} = a^l_k$ *exists for all* $k \in \mathbb{N}$ *and* $l = 1, \ldots, m$.

(2*) *For all* $l = 1, \ldots, m$,

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sup_{n\in\mathbb{N}} \left\{ (a^l_{n,k})^2 \left( \frac{1}{n} \sum_{i=1}^{n} (f^n_{l,i})^2 \right)^k \right\} < \infty.$$

*For technical reasons in the following convergence results we deal with the simpler assumption* (2*)*. By the boundedness of* $\frac{1}{n}\sum_{i=1}^n (f^n_{l,i})^2$ *for all* $l = 1, \ldots, m$, *the assumption* (2) *in Theorem* 4.1 *implies* (2*)*.*

(ii) *By Theorem* 1.1, *condition* (2) *implies that*

$$\sum_{k=0}^{\infty} \frac{(a^l_k)^2}{k!} \Big( \int_0^1 f_l^2(s) ds \Big)^k < \infty, \quad l = 1, \ldots, m,$$

*which is equivalent to the existence of* $F_l^{\diamond}\big(I(f_l)\big)$ *in the space* $L^2(\Omega, \mathcal{F}, P)$ *for all* $l = 1, \ldots, m$. *For further criteria for the existence in* $L^2(\Omega, \mathcal{F}, P)$ *we refer to* [11].

(iii) *The existence of the component* $\Diamond_{i\in A} F_i^{\diamond}\big(I(f_i)\big)$ *in* $L^2(\Omega, \mathcal{F}, P)$ *for some* $A \subset \{1, \ldots, m\}$ *follows by condition* (2) *as well.*

(iv) *The assumption on the convergence of the possibly correlated Wiener integrals in Theorem* 4.1 *is based on the weak convergence of the underlying processes*

$$B^n \xrightarrow{d} B.$$

*This inspires us to speak about a Wick functional limit theorem.*

EXAMPLE 4.1. The Molchan–Golosov representation of the fractional Brownian motion is given by

$$B^H_t = \int_0^t z_H(t,s) dB_s, \quad t \in [0,1],$$

for some deterministic kernel $z_H(t,s)$ (see [14], Chapter 5). In [15] we prove the following Donsker type theorem extending a result in [16]:

THEOREM 4.2. *For every* $H \in (0,1)$, *we define the discrete Volterra integrands as the pointwise approximation of the Molchan–Golosov kernel*:

$$b^{H,n}(l,i) := n \int_{(i-1)/n}^{i/n} z_H\,(l/n, s)\, ds \mathbf{1}_{\{i\leqslant l\}}.$$

*Then the sequence of processes* (*of discrete Wiener integrals*)

$$(B^{H,n}_t)_{t\in[0,1]} := I^n\big(b^{H,n}(\lfloor nt \rfloor, \cdot)\big)_{t\in[0,1]}$$

*converges weakly to the fractional Brownian motion* $(B^H_t)_{t\in[0,1]}$ *in the Skorokhod space* $\mathbb{D}([0,1], \mathbb{R})$.

Thus we obtain the following non-central limit theorems. Let $H_1, \ldots, H_k \in (0,1)$. By the embedding methods in [4], we obtain, for all $t_1, \ldots, t_k \in [0,1]$,

$$(B_{t_1}^{H_1,n}, \ldots, B_{t_k}^{H_k,n}) \xrightarrow{d} (B_{t_1}^{H_1}, \ldots, B_{t_k}^{H_k}).$$

Then, from Theorem 4.1 we infer that the random vector of all Wick products of fractional geometric Brownian motions

$$\big(\exp^{\diamond}(B_{t_1}^{H_1}), \ldots, \exp^{\diamond}(B_{t_1}^{H_1} + B_{t_2}^{H_2}), \ldots, \exp^{\diamond}(B_{t_1}^{H_1} + \ldots + B_{t_k}^{H_k})\big)$$

is approximated weakly by the discrete counterpart. We notice that the different fractional Brownian motions are correlated by the same underlying Brownian motion. The tightness of such processes will be considered in an accompanying article.

## 5. APPENDIX

Proof of Proposition 3.1. (i) We observe that

$$n^{-N} \sum_{\sigma \in \mathcal{S}_N} \sum_{i_1,\ldots,i_N=1}^{n} \prod_{j=1}^{N} f_{j,i_j}^n g_{j,i_{\sigma(j)}}^n = \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^{N} \mathbb{E}[I^n(f_j^n) I^n(g_{\sigma(j)}^n)]. \tag{5.1}$$

Here, the objects of interest have the form

$$I^n(f_1^n) \diamond_n \ldots \diamond_n I^n(f_N^n) = n^{-N/2} \sum_{\substack{i_1,\ldots,i_N=1 \\ \text{pairwise different}}}^{n} f_{1,i_1}^n \cdots f_{N,i_N}^n \Xi_{\{i_1,\ldots,i_N\}}^n.$$

Then, by $\Xi_{\{i_1,\ldots,i_N\}}^n \Xi_{\{i_{\sigma(1)},\ldots,i_{\sigma(N)}\}}^n = 1$ for all $\sigma \in \mathcal{S}_N$ and interchanging sums, we obtain

$$\begin{aligned}
&\mathbb{E}\big[\big(I^n(f_1^n) \diamond_n \ldots \diamond_n I^n(f_N^n)\big)\big(I^n(g_1^n) \diamond_n \ldots \diamond_n I^n(g_N^n)\big)\big] \\
&= n^{-N} \sum_{\sigma \in \mathcal{S}_N} \sum_{\substack{i_1,\ldots,i_N=1 \\ \text{pairwise different}}}^{n} f_{1,i_1}^n \cdots f_{N,i_N}^n g_{1,i_{\sigma(1)}}^n \cdots g_{N,i_{\sigma(N)}}^n \\
&= n^{-N} \sum_{\sigma \in \mathcal{S}_N} \Big( \sum_{i_1,\ldots,i_N=1}^{n} \prod_{j=1}^{N} f_{j,i_j}^n g_{j,i_{\sigma(j)}}^n - \sum_{\substack{i_1,\ldots,i_N=1 \\ \exists k,l: i_k = i_l}}^{n} \prod_{j=1}^{N} f_{j,i_j}^n g_{j,i_{\sigma(j)}}^n \Big) \\
&= \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^{N} \mathbb{E}[I^n(f_j^n) I^n(g_{\sigma(j)}^n)] - n^{-N} \sum_{\substack{i_1,\ldots,i_N=1 \\ \exists k,l: i_k = i_l}}^{n} \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^{N} f_{j,i_j}^n g_{j,i_{\sigma(j)}}^n.
\end{aligned} \tag{5.2}$$

The remaining sum on the right-hand side in (5.2) allows for the following reformulation:

$$\begin{aligned}
&n^{-N} \sum_{\substack{i_1,\ldots,i_N=1 \\ \exists k,l: i_k = i_l}}^{n} \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^{N} f_{j,i_j}^n g_{j,i_{\sigma(j)}}^n \\
&= N! n^{-N} \sum_{\substack{i_1,\ldots,i_N=1 \\ \exists k,l: i_k = i_l}}^{n} \bigg( \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^{N} f_{j,i_{\sigma(j)}}^n \bigg) \bigg( \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^{N} g_{j,i_{\sigma(j)}}^n \bigg).
\end{aligned} \tag{5.3}$$

Therefore, we conclude that

$$
\begin{aligned}
(5.4)\quad & \frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^{N}\mathbb{E}[I^n(f_j^n)I^n(f_{\sigma(j)}^n)] + \frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^{N}\mathbb{E}[I^n(g_j^n)I^n(g_{\sigma(j)}^n)] \\
& - 2\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^{N}\mathbb{E}[I^n(f_j^n)I^n(g_{\sigma(j)}^n)] \\
& - \frac{1}{N!}\mathbb{E}\Big[\Big(\big(I^n(f_1^n)\diamond_n\ldots\diamond_n I^n(f_N^n)\big) - \big(I^n(g_1^n)\diamond_n\ldots\diamond_n I^n(g_N^n)\big)\Big)^2\Big] \\
& = n^{-N}\sum_{\substack{i_1,\ldots,i_N=1\\ \exists k,l:i_k=i_l}}^{n}\Bigg(\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^{N} f_{j,i_{\sigma(j)}}^n - \frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^{N} g_{j,i_{\sigma(j)}}^n\Bigg)^2 \geqslant 0.
\end{aligned}
$$

By (5.4) for $g_i^n = 0$, $i = 1,\ldots,N$, and the Cauchy–Schwarz inequality, we obviously have

$$
\begin{aligned}
(5.5)\quad & \mathbb{E}\big[\big(I^n(f_1^n)\diamond_n\ldots\diamond_n I^n(f_N^n)\big)^2\big] \\
& \qquad \leqslant \sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^{N}\mathbb{E}\big[\big(I^n(f_j^n)\big)^2\big]^{1/2}\mathbb{E}\big[\big(I^n(f_{\sigma(j)}^n)\big)^2\big]^{1/2}.
\end{aligned}
$$

Then, via (5.5) and the associativity of the discrete Wick product, we conclude the assertion.

(ii) By (5.4) and Lemma 2.1 (ii), we obtain

$$
\begin{aligned}
(5.6)\quad & \mathbb{E}\Big[\Big(\big(I^n(f_1^n)\diamond_n\ldots\diamond_n I^n(f_N^n)\big) - \big(I^n(g_1^n)\diamond_n\ldots\diamond_n I^n(g_N^n)\big)\Big)^2\Big] \\
& - \mathbb{E}\Big[\Big(\big(I(f_1)\diamond\ldots\diamond I(f_N)\big) - \big(I(g_1)\diamond\ldots\diamond I(g_N)\big)\Big)^2\Big] \\
& = \sum_{\sigma\in\mathcal{S}_N}\big(\prod_{j=1}^{N}\mathbb{E}[I^n(f_j^n)I^n(f_{\sigma(j)}^n)] - \prod_{j=1}^{N}\mathbb{E}[I(f_j)I(f_{\sigma(j)})] \\
& + \prod_{j=1}^{N}\mathbb{E}[I^n(g_j^n)I^n(g_{\sigma(j)}^n)] - \prod_{j=1}^{N}\mathbb{E}[I(g_j)I(g_{\sigma(j)})] \\
& - 2\prod_{j=1}^{N}\mathbb{E}[I^n(f_j^n)I^n(g_{\sigma(j)}^n)] + 2\prod_{j=1}^{N}\mathbb{E}[I(f_j)I(g_{\sigma(j)})]\big) \\
& - N!n^{-N}\sum_{\substack{i_1,\ldots,i_N=1\\ \exists k,l:i_k=i_l}}^{n}\Bigg(\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^{N} f_{j,i_{\sigma(j)}}^n - \frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^{N} g_{j,i_{\sigma(j)}}^n\Bigg)^2.
\end{aligned}
$$

Now, for simplicity, we write

$$
\sum(f,\sigma) := n^{-N}\sum_{\substack{i_1,\ldots,i_N=1\\ \exists k,l:i_k=i_l}}^{n}\prod_{j=1}^{N}(f_{i_{\sigma(j)}}^n)^2.
$$

Interchanging sums and using the Cauchy–Schwarz inequality we obtain

$$n^{-N} \sum_{\substack{i_1,\dots,i_N=1\\ \exists k,l: i_k=i_l}}^{n} \Big( \frac{1}{N!} \sum_{\sigma\in\mathcal{S}_N} \prod_{j=1}^{N} f^n_{j,i_{\sigma(j)}} \Big)^2 \leqslant \frac{1}{(N!)^2} \sum_{\sigma,\sigma'\in\mathcal{S}_N} \sum (f_j,\sigma)^{1/2} \sum (f_j,\sigma')^{1/2}.$$

Here we notice

$$\begin{aligned}\sum (f,\sigma) &\leqslant \binom{N}{2} \sup_{i\leqslant n} \frac{|f^n_i|^2}{n} n^{-(N-1)} \sum_{i_1,\dots,i_{N-1}=1}^{n} \big( \prod_{j=1}^{N-1} (f^n_{i_j})^2 \big) \\ &= \binom{N}{2} \sup_{i\leqslant n} \frac{|f^n_i|^2}{n} \mathbb{E}\big[ \big( I^n(f^n) \big)^2 \big]^{N-1}.\end{aligned}$$

Thus we have

$$\begin{aligned}(5.7)\quad & n^{-N} \sum_{\substack{i_1,\dots,i_N=1\\ \exists k,l: i_k=i_l}}^{n} \Big( \frac{1}{N!} \sum_{\sigma\in\mathcal{S}_N} \prod_{j=1}^{N} f^n_{j,i_{\sigma(j)}} \Big)^2 \\ &\qquad \leqslant \binom{N}{2} \max_{l\leqslant N} \sup_{i\leqslant n} \frac{|f^n_{l,i}|^2}{n} \Big( \max_{l\leqslant N} \mathbb{E}\big[ \big( I^n(f^n_l) \big)^2 \big] \Big)^{N-1},\end{aligned}$$

and the analogous upper bound for $g^n_1,\dots,g^n_N$, respectively. Hence, via (5.6), (5.7) and applying the convergences in Theorem 1.1 (c), we obtain the assertion (iii). ■

Proof of Lemma 3.2. For $k \leqslant 2$ we have $R^{k,n} = 0$ and the formula (3.5) is clear by Example 3.1 and $I^n(f^n)^{\diamond_n 0} = 1$. Moreover, we observe that the first non-zero remaining term appears for $k = 3$ as

$$\begin{aligned}& R^{3,n}\big( I^n(f^n_1), I^n(f^n_2), I^n(f^n_3) \big) \\ &= \sum_{l=1}^{2} n^{-1/2} \sum_{i_{\{1,2\}\setminus\{l\}}=1}^{n} f^n_{\{1,2\}\setminus\{l\}, i_{\{1,2\}\setminus\{l\}}} \xi^n_{\{1,2\}\setminus\{l\}} \big( n^{-1} \sum_{i_l \in \{i_{\{1,2\}\setminus\{l\}}\}} f^n_{l,i_l} f^n_{3,i_l} \big) \\ &= 2n^{-3/2} \sum_{i=1}^{n} f^n_{1,i} f^n_{2,i} f^n_{3,i} \xi^n_i.\end{aligned}$$

This proves the assertion for $k = 3$. For $k \geqslant 3$ we obtain

$$\begin{aligned}(5.8)\quad & \big( I^n(f^n_1) \diamond_n \dots \diamond_n I^n(f^n_{k-1}) \big) \diamond_n I^n(f^n_k) \\ &= n^{-(k-1)/2} \sum_{\substack{i_1,\dots,i_{k-1}=1\\ \text{pairwise different}}}^{n} f^n_{1,i_1} \cdots f^n_{k-1,i_{k-1}} \Xi^n_{\{i_1,\dots,i_{k-1}\}} \diamond_n \big( n^{-1/2} \sum_{i_k=1}^{n} f^n_{k,i_k} \xi^n_{i_k} \big) \\ &= \big( I^n(f^n_1) \diamond_n \dots \diamond_n I^n(f^n_{k-1}) \big) I^n(f^n_k) \\ &\quad - n^{-k/2} \sum_{\substack{i_1,\dots,i_{k-1}=1\\ \text{pairwise different}}}^{n} \sum_{i_k \in \{i_1,\dots,i_{k-1}\}} f^n_{1,i_1} \cdots f^n_{k-1,i_{k-1}} f^n_{k,i_k} \Xi^n_{\{i_1,\dots,i_{k-1}\}} \xi^n_{i_k}.\end{aligned}$$

Here we define the remaining term as

$$
\begin{aligned}
&n^{-k/2} \sum_{\substack{i_1,\dots,i_{k-1}=1\\ \text{pairwise different}}}^{n} \sum_{i_k\in\{i_1,\dots,i_{k-1}\}} f^n_{1,i_1}\cdots f^n_{k-1,i_{k-1}} f^n_{k,i_k}\,\Xi^n_{\{i_1,\dots,i_{k-1}\}}\xi^n_{i_k}\\
&= \sum_{l=1}^{k-1} \big(n^{-1}\sum_{i_l=1}^{n} f^n_{l,i_l} f^n_{k,i_l}\big)\\
&\quad\times \big(n^{-(k/2-1)} \sum_{\substack{i_1,\dots,\widehat{i_l},\dots,i_{k-1}=1\\ \text{pairwise different}}}^{n} f^n_{1,i_1}\cdots\widehat{f^n_{l,i_l}}\cdots f^n_{k-1,i_{k-1}}\,\Xi^n_{\{i_1,\dots,\widehat{i_l},\dots,i_{k-1}\}}\big)\\
&\quad-\sum_{l=1}^{k-1} n^{-(k/2-1)} \sum_{\substack{i_1,\dots,\widehat{i_l},\dots,i_{k-1}=1\\ \text{pairwise different}}}^{n} f^n_{1,i_1}\cdots\widehat{f^n_{l,i_l}}\cdots f^n_{k-1,i_{k-1}}\,\Xi^n_{\{i_1,\dots,\widehat{i_l},\dots,i_{k-1}\}}\\
&\quad\times\big(n^{-1}\sum_{i_l\in\{i_1,\dots,\widehat{i_l},\dots,i_{k-1}\}} f^n_{l,i_l} f^n_{k,i_l}\big)\\
&= \sum_{l=1}^{k-1}\mathbb{E}\left[I^n(f^n_l)I^n(f^n_k)\right]\big(I^n(f^n_1)\diamond_n\dots\diamond_n \widehat{I^n(f^n_l)}\diamond_n\dots\diamond_n I^n(f^n_{k-1})\big)\\
&\quad- R^{k,n}\big(I^n(f^n_1),\dots,I^n(f^n_k)\big).
\end{aligned}
$$

For the estimate (3.6), due to the orthogonality of $\Xi^n_A$, $A\subset\{1,\dots,n\}$, we observe that

$$
\begin{aligned}
0&\leqslant \mathbb{E}\Big[\Big(R^{k,n}\big(I^n(f^n_1),\dots,I^n(f^n_k)\big)\Big)^2\Big]\\
&=\sum_{l,l'=1}^{k-1} n^{-(k-2)}\sum_{\sigma\in\mathcal{S}_{k-2}} \sum_{\substack{i_1,\dots,\widehat{i_l},\dots,i_{k-1}=1\\ \text{pairwise different}\\ (i'_1,\dots,\widehat{i'_l},\dots,i'_{k-1})=\sigma(i_1,\dots,\widehat{i_l},\dots,i_{k-1})}}^{n} \big(\prod_{\substack{j=1\\ j\neq l}}^{n} f^n_{j,i_j}\big)\big(\prod_{\substack{j=1\\ j\neq l'}}^{n} f^n_{j,i'_j}\big)\\
&\quad\times\big(n^{-1}\sum_{i_l\in\{i_1,\dots,\widehat{i_l},\dots,i_{k-1}\}} f^n_{l,i_l}f^n_{k,i_l}\big)\big(n^{-1}\sum_{i'_{l'}\in\{i'_1,\dots,\widehat{i'_l},\dots,i'_{k-1}\}} f^n_{l',i_{l'}}f^n_{k,i_{l'}}\big),
\end{aligned}
$$

where $\mathcal{S}_{k-2}$ is the group of permutations on $k-2$ elements. Clearly, we have

$$
\big(n^{-1}\sum_{i_l\in\{i_1,\dots,\widehat{i_l},\dots,i_{k-1}\}} f^n_{l,i_l}f^n_{k,i_l}\big)\leqslant (k-2)\max_{l\leqslant k}\sup_{i\leqslant n}\frac{|f^n_{l,i}|^2}{n}.
$$

Moreover, by the Cauchy–Schwarz inequality, for all $l,l'\in\{1,\dots,k-1\}$ and every fixed permutation $\sigma \in \mathcal{S}_{k-2}$, we get

$$
\begin{aligned}
n^{-(k-2)} & \sum_{\substack{i_1,\dots,\widehat{i_l},\dots,i_{k-1}=1 \\ \text{pairwise different} \\ (i'_1,\dots,\widehat{i'_l},\dots,i'_{k-1})=\sigma(i_1,\dots,\widehat{i_l},\dots,i_{k-1})}}^{n} \Big(\prod_{\substack{j=1 \\ j\neq l}}^{n} f^n_{j,i_j}\Big)\Big(\prod_{\substack{j=1 \\ j\neq l'}}^{n} f^n_{j,i'_j}\Big) \\
& \leqslant \prod_{j=1,j\neq l}^{k-1} \Big(n^{-1}\sum_{i=1}^{n} (f^n_{j,i})^2\Big)^{1/2} \prod_{j=1,j\neq l'}^{k-1} \Big(n^{-1}\sum_{i=1}^{n} (f^n_{j,i})^2\Big)^{1/2} \\
& \leqslant \Big(\max_{l\leqslant k} \mathbb{E}\big[\big(I^n(f^n_l)\big)^2\big]\Big)^{k-1}.
\end{aligned}
$$

Hence, by $|\mathcal{S}_{k-2}| = (k-2)!$, we conclude the asserted estimate (3.6). ∎

P r o o f o f T h e o r e m 3.1. The proof goes by induction on $N$. The weak convergence of $\mathbf{I}^n_N$ to $\mathbf{I}_N$ in the case $N=1$ equals the assumption on the weak convergence of the Wiener integrals. Suppose the convergence is already proved for $N \geqslant 1$. Making use of the assumption and Theorem 1.1, we see that the convergence of the deterministic terms $\lim_{n\to\infty} \mathbb{E}[I^n(f^n_i)I^n(f^n_j)] = \mathbb{E}[I(f_i)I(f_j)]$ holds true for all $i,j \in \{1,\dots,k\}$. Hence, from the Cramér–Wold device, we infer that the extension of $\mathbf{I}^n_N$ by these deterministic components to the random vector $\mathbf{I}^n_{N,E}$ defined as

$$
\Big(\underbrace{1, I^n(f^n_1), \dots}_{\text{all components of } \mathbf{I}^n_N}, \underbrace{\mathbb{E}[I^n(f^n_1)I^n(f^n_1)], \dots, \mathbb{E}[I^n(f^n_{k-1})I^n(f^n_k)]}_{\mathbb{E}[I^n(f^n_i)I^n(f^n_j)] \text{ for all } i,j\in\{1,\dots,k\}}\Big)
$$

converges weakly to the continuous counterpart $\mathbf{I}_{N,E}$. Moreover, we build from $\mathbf{I}^n_{N+1}$ the random vector $\mathbf{I}^n_{N+1,H}$ of the same dimension in the following two steps:

1. Take a copy of $\mathbf{I}_{N,E}$.
2. Every discrete Wick product of $N+1$ Wiener integrals,

$$
I^n(f^n_{i_1}) \diamond_n \dots \diamond_n I^n(f^n_{i_{N+1}}), \qquad i_1,\dots,i_{N+1} \in \{1,\dots,k\},
$$

is replaced by

$$
\begin{aligned}
& I^n(f^n_{i_{N+1}})\big(I^n(f^n_{i_1}) \diamond_n \dots \diamond_n I^n(f^n_{i_N})\big) \\
& - \sum_{j=1}^{N} \mathbb{E}[I^n(f^n_{i_j})I^n(f^n_{i_{N+1}})]\big(I^n(f^n_{i_1}) \diamond_n \dots \diamond_n \widehat{I^n(f^n_{i_j})} \diamond_n \dots \diamond_n I^n(f^n_{i_N})\big)
\end{aligned}
$$

on the right-hand side in the Hermite recursion formula (3.5). Hence, $\mathbf{I}^n_{N+1,H}$ is a continuous function of $\mathbf{I}^n_{N,E}$. Moreover, by (3.5), all components of the random vector

$$
\mathbf{I}^n_{N+1,R} := \mathbf{I}^n_{N+1} - \mathbf{I}^n_{N+1,H}
$$

are equal to zero or equal to some

$$R^{N+1,n}\big(I^n(f^n_{i_1}),\dots,I^n(f^n_{i_{N+1}})\big),\qquad i_1,\dots,i_{N+1}\in\{1,\dots,k\},$$

in (3.5). By Lemma 1.1 and the estimate (3.6) in Lemma 3.2, we obtain

$$\lim_{n\to\infty}\mathbb{E}[\|\mathbf{I}^n_{N+1,R}\|^2_{\mathbb{R}^{d(N+1)}}]=0, \tag{5.9}$$

where $\|\cdot\|_{\mathbb{R}^K}$ denotes the Euclidean norm on $\mathbb{R}^K$ and $d(N+1)$ is the dimension of $\mathbf{I}^n_{N+1}$. Since $\mathbf{I}^n_{N+1,H}$ is a continuous function of $\mathbf{I}^n_{N,E}$, by the induction hypothesis, the Hermite recursion formula (2.3), and the continuous mapping theorem we get

$$\mathbf{I}^n_{N+1,H}\xrightarrow{d}\mathbf{I}_{N+1}. \tag{5.10}$$

Hence, by (5.9), (5.10), and Slutsky's theorem we obtain the asserted convergence for $N+1$. ■

P r o o f o f T h e o r e m 4.1. Firstly we prove the simple case, i.e.,

$$\Big(F_1^{\diamond_n}\big(I^n(f_1^n)\big),\dots,F_m^{\diamond_n}\big(I^n(f_m^n)\big)\Big)\xrightarrow{d}\Big(F_1^{\diamond}\big(I(f_1)\big),\dots,F_m^{\diamond}\big(I(f_m)\big)\Big) \tag{5.11}$$

as $n$ tends to infinity. Here, we assume the weaker conditions on the coefficients, that is, (1) and (2*) in Remark 4.1 (i). By [5], Theorem 4.2, it is sufficient to show that the following three conditions are satisfied:

(5.12) $\forall N\in\mathbb{N}$,

$$\Big(F_1^{N,\diamond_n}\big(I^n(f_1^n)\big),\dots,F_m^{N,\diamond_n}\big(I^n(f_m^n)\big)\Big)\xrightarrow{d}\Big(F_1^{N,\diamond}\big(I(f_1)\big),\dots,F_m^{N,\diamond}\big(I(f_m)\big)\Big),$$

$$\forall l\leqslant m,\quad \lim_{N\to\infty}\limsup_{n\to\infty}\mathbb{E}\Big[\Big(F_l^{\diamond_n}\big(I^n(f_l^n)\big)-F_l^{N,\diamond_n}\big(I^n(f_l^n)\big)\Big)^2\Big]=0, \tag{5.13}$$

$$\forall l\leqslant m,\quad \lim_{N\to\infty}\mathbb{E}\Big[\Big(F_l^{\diamond}\big(I(f_l)\big)-F_l^{N,\diamond}\big(I(f_l)\big)\Big)^2\Big]=0. \tag{5.14}$$

Here $F^{N,\diamond}$ and $F^{N,\diamond_n}$ denote the partial sums of the Wick analytic functionals. Condition (5.12) is a consequence of the generalized continuous mapping theorem ([5], Theorem 5.5) and Theorem 3.1. Condition (5.14) follows directly by Remark 4.1 (ii) and due to the orthogonality of different Wick powers of Wiener integrals in (3.3) of Example 3.1. Finally, thanks to the orthogonality of the different discrete Wick powers of discrete Wiener integrals, we obtain, for the inner limit superior in condition (5.13),

$$\begin{aligned}\limsup_{n\to\infty}&\mathbb{E}\Big[\Big(F_l^{\diamond_n}\big(I^n(f_l^n)\big)-F_l^{N,\diamond_n}\big(I^n(f_l^n)\big)\Big)^2\Big]\\&=\limsup_{n\to\infty}\sum_{k=N+1}^{n}\frac{(a_{n,k}^l)^2}{(k!)^2}\mathbb{E}\big[\big|\big(I^n(f_l^n)\big)^{\diamond_n k}\big|^2\big]\\&=\sum_{k=N+1}^{\infty}\frac{(a_k^l)^2}{(k!)^2}\mathbb{E}\big[\big|\big(I(f_l)\big)^{\diamond k}\big|^2\big]=\mathbb{E}\Big[\Big(F_l^{\diamond}\big(I(f_l)\big)-F_l^{N,\diamond}\big(I(f_l)\big)\Big)^2\Big],\end{aligned}$$

where we are making use of the convergence in Remark 3.1 (i), assumption (2*) and, hence, by Fatou's lemma for sums for introducing the limit superior under the series. Thus, (5.13) follows from (5.14) and we conclude the simple case (5.11).

The proof of the convergence of the assertion in Theorem 4.1 proceeds similarly. Thus, by Theorem 4.2 in [5], it is sufficient to show that the following three conditions are satisfied:

(5.15) $\forall N\in\mathbb{N}$,

$$\begin{aligned}&\Big(F_1^{N,\diamond_n}\big(I^n(f_1^n)\big),\ldots,F_1^{N,\diamond_n}\big(I^n(f_1^n)\big)\diamond_n\ldots\diamond_n F_m^{N,\diamond_n}\big(I^n(f_m^n)\big)\Big)\\&\qquad\xrightarrow{d}\Big(F_1^{N,\diamond}\big(I(f_1)\big),\ldots,F_1^{N,\diamond}\big(I(f_1)\big)\diamond\ldots\diamond F_m^{N,\diamond}\big(I(f_m)\big)\Big),\end{aligned}$$

(5.16) $\forall A\subset\{1,\ldots,m\}, A\neq\emptyset$,

$$\lim_{N\to\infty}\limsup_{n\to\infty}\mathbb{E}\Big[\Big((\diamondsuit_n)_{i\in A}F_i^{\diamond_n}\big(I^n(f_i^n)\big)-(\diamondsuit_n)_{i\in A}F_i^{N,\diamond_n}\big(I^n(f_i^n)\big)\Big)^2\Big]=0,$$

(5.17) $\forall A\subset\{1,\ldots,m\}, A\neq\emptyset$,

$$\lim_{N\to\infty}\mathbb{E}\Big[\Big(\diamondsuit_{i\in A}F_i^{\diamond}\big(I(f_i)\big)-\diamondsuit_{i\in A}F_i^{N,\diamond}\big(I(f_i)\big)\Big)^2\Big]=0.$$

Condition (5.15) is a consequence of the generalized continuous mapping theorem ([5], Theorem 5.5) and Theorem 3.1.

Suppose $A\subset\{1,\ldots,m\}$. Then we have

(5.18) $\diamondsuit_{i\in A}F_i^{\diamond}\big(I(f_i)\big)-\diamondsuit_{i\in A}F_i^{N,\diamond}\big(I(f_i)\big)$

$$\begin{aligned}&=\sum_{k_l=0,l\in A}^{\infty}\Big(\prod_{l\in A}\frac{a_{k_l}^l}{(k_l)!}\Big)\Big(\diamondsuit_{l\in A}\big(I(f_l)\big)^{\diamond k_l}\Big)\\&\quad-\sum_{k_l=0,l\in A}^{N}\Big(\prod_{l\in A}\frac{a_{k_l}^l}{(k_l)!}\Big)\Big(\diamondsuit_{l\in A}\big(I(f_l)\big)^{\diamond k_l}\Big)\\&=\sum_{\substack{k_l=0,l\in A\\ \max_{l\in A}k_l>N}}^{\infty}\Big(\prod_{l\in A}\frac{a_{k_l}^l}{(k_l)!}\Big)\Big(\diamondsuit_{l\in A}\big(I(f_l)\big)^{\diamond k_l}\Big).\end{aligned}$$

Thus, by the Cauchy–Schwarz inequality and Remark 3.1 (ii), we obtain

$$
\begin{aligned}
(5.19)\quad & \mathbb{E}\Big[\Big(\Diamond_{i\in A}F_i^{\Diamond}\big(I(f_i)\big)-\Diamond_{i\in A}F_i^{N,\Diamond}\big(I(f_i)\big)\Big)^2\Big] \\
&= \sum_{\substack{k_l=0,l\in A\\ k_l'=0,l\in A\\ \max\limits_{l\in A}k_l,\max\limits_{l\in A}k_l'>N}}^{\infty} \Big(\prod_{l\in A}\frac{a^l_{k_l}}{(k_l)!}\frac{a^l_{k_l'}}{(k_l')!}\Big)\mathbb{E}\Big[\Big(\Diamond_{l\in A}\big(I(f_l)\big)^{\Diamond k_l}\Big)\Big(\Diamond_{l\in A}\big(I(f_l)\big)^{\Diamond k_l'}\Big)\Big] \\
&\leqslant \sum_{\substack{k_l=0,l\in A\\ k_l'=0,l\in A\\ \max\limits_{l\in A}k_l,\max\limits_{l\in A}k_l'>N}}^{\infty} |A|!\prod_{l\in A}\frac{|a^l_{k_l}|}{\sqrt{(k_l)!}}\frac{|a^l_{k_l'}|}{\sqrt{(k_l')!}}\mathbb{E}\big[\big(I(f_l)\big)^2\big]^{k_l/2}\mathbb{E}\big[\big(I(f_l)\big)^2\big]^{k_l'/2} \\
&= |A|!\Big(\sum_{\substack{k_l=0,l\in A\\ \max\limits_{l\in A}k_l>N}}^{\infty}\prod_{l\in A}\frac{|a^l_{k_l}|}{\sqrt{(k_l)!}}\mathbb{E}\big[\big(I(f_l)\big)^2\big]^{k_l/2}\Big)^2.
\end{aligned}
$$

By the assumptions on the coefficients and by the condition

$$\max_{l=1,\dots,m}\mathbb{E}[I(f_l)^2]\leqslant L<\infty,$$

we have

$$
\begin{aligned}
(5.20)\quad \sum_{\substack{k_l=0,l\in A\\ \max\limits_{l\in A}k_l>N}}^{\infty}\prod_{l\in A}\frac{|a^l_{k_l}|}{\sqrt{(k_l)!}}\mathbb{E}\big[\big(I(f_l)\big)^2\big]^{k_l/2} &\leqslant \sum_{\substack{k_l=0,l\in A\\ \max\limits_{l\in A}k_l>N}}^{\infty}\prod_{l\in A}\frac{C^{k_l}}{\sqrt{(k_l)!}}L^{k_l/2} \\
&\leqslant \sum_{q\in A}\Big(\sum_{k_q=N+1}^{\infty}\frac{C^{k_q}}{\sqrt{k_q!}}L^{k_q/2}\Big)\Big(\prod_{l\in A\setminus\{q\}}\Big(\sum_{k_l=0}^{\infty}\frac{C^{k_l}}{\sqrt{k_l!}}L^{k_l/2}\Big)\Big) \\
&= |A|\Big(\sum_{k=N+1}^{\infty}\frac{C^k}{\sqrt{k!}}L^{k/2}\Big)\Big(\sum_{k=0}^{\infty}\frac{C^k}{\sqrt{k!}}L^{k/2}\Big)^{|A|-1}.
\end{aligned}
$$

Hence, via

$$\sum_{k=0}^{\infty}\frac{C^k}{\sqrt{k!}}L^{k/2}<\infty,$$

we conclude that

$$\sum_{\substack{k_l=0,l\in A\\ \max\limits_{l\in A}k_l>N}}^{\infty}\prod_{l\in A}\frac{a^l_{k_l}}{\sqrt{(k_l)!}}\mathbb{E}\big[\big(I(f_l)\big)^2\big]^{k_l/2}\to 0 \quad \text{as } N\to\infty.$$

Thus, due to (5.19), the condition (5.17) is satisfied.

Analogously to (5.18) and (5.19), we obtain for the discrete counterpart

$$\mathbb{E}\Big[\Big((\Diamond_n)_{i\in A}F_i^{\Diamond_n}\big(I^n(f_i^n)\big)-(\Diamond_n)_{i\in A}F_i^{N,\Diamond_n}\big(I^n(f_i^n)\big)\Big)^2\Big]$$
$$\leqslant |A|!\Bigg(\sum_{\substack{k_l=0,l\in A\\ \max\limits_{l\in A}k_l>N}}^{n}\prod_{l\in A}\frac{a_{n,k_l}^l}{\sqrt{(k_l)!}}\mathbb{E}\big[\big(I^n(f_l^n)\big)^2\big]^{k_l/2}\Bigg)^2.$$

Hence, by (5.20) and the assumptions on the coefficients, we conclude condition (5.16). ■

**Acknowledgments.** The author thanks Christian Bender for helpful discussions and comments.

**REFERENCES**


[1] F. Avram and M. S. Taqqu, *Noncentral limit theorems and Appell polynomials*, Ann. Probab. 15 (2) (1987), pp. 767–775.

[2] C. Bender and P. Parczewski, *Approximating a geometric fractional Brownian motion and related processes via discrete Wick calculus*, Bernoulli 16 (2) (2010), pp. 389–417.

[3] C. Bender and P. Parczewski, *On the connection between discrete and continuous Wick calculus with an application to the fractional Black–Scholes model*, in: *Stochastic Processes, Filtering, Control and Their Applications*, S. Cohen et al. (Eds.), World Scientific, 2012, pp. 3–40.

[4] C. Bender and P. Parczewski, *On convergence of S-transforms and a Wiener chaos limit theorem*, in preparation (2014).

[5] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York 1968.

[6] R. Fox and M. S. Taqqu, *Multiple stochastic integrals with dependent integrators*, J. Multivariate Anal. 21 (1) (1987), pp. 105–127.

[7] L. Giraitis and D. Surgailis, *Multivariate Appell polynomials and the central limit theorem*, in: *Dependence in Probability and Statistics (Oberwolfach, 1985)*, E. Eberlein and M. S. Taqqu (Eds.), Progr. Probab. Statist., Vol. 11, Birkhäuser Boston, Boston, MA, 1986, pp. 21–71.

[8] H. Gzyl, *An exposé on discrete Wiener chaos expansions*, Bol. Asoc. Mat. Venez. 13 (1) (2006), pp. 3–27.

[9] H. Holden, T. Lindstrøm, B. Øksendal, and J. Ubøe, *Discrete Wick calculus and stochastic functional equations*, Potential Anal. 1 (3) (1992), pp. 291–306.

[10] H. Holden, B. Øksendal, J. Ubøe, and T. Zhang, *Stochastic Partial Differential Equations: A Modeling, White Noise Functional Approach*, second edition, Springer, New York 2010.

[11] Y. Hu and J. A. Yan, *Wick calculus for nonlinear Gaussian functionals*, Acta Math. Appl. Sinica (English Ser.) 25 (3) (2009), pp. 399–414.

[12] S. Janson, *Gaussian Hilbert Spaces*, Cambridge University Press, Cambridge 1997.

[13] H.-H. Kuo, *White Noise Distribution Theory,* Probability and Stochastics Series, CRC Press, Boca Raton, FL, 1996.

[14] D. Nualart, *The Malliavin Calculus and Related Topics*, second edition, Probab. Appl., Springer, New York 2006.

[15] P. Parczewski, *A fractional Donsker theorem*, Stoch. Anal. Appl. 32 (2014), pp. 328–347.
[16] T. Sottinen, *Fractional Brownian motion, random walks and binary market models*, Finance Stoch. 5 (2001), pp. 343–355.

Peter Parczewski
Institute of Mathematics
University of Mannheim
A5, 6
PO Box D-68131 Mannheim, Germany
*E-mail*: parczewski@math.uni-mannheim.de

*Received on 7.8.2013;*
*revised version on 1.12.2013*