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ON PATHWISE STOCHASTIC INTEGRATION WITH RESPECT TO SEMIMARTINGALES

BY

RAFAŁ M. ŁOCHOWSKI* (WARSZAWA AND AL KHOBAR)

Abstract. For any real-valued stochastic process X with *càdlàg* paths we define non-empty family of processes which have locally finite total variation, have jumps of the same order as the process X and uniformly approximate its paths on compacts. The application of the defined class is the definition of stochastic integral with semimartingale integrand and integrator as a limit of pathwise Lebesgue–Stieltjes integrals. This construction leads to the stochastic integral with some correction term (different from the Stratonovich integral). Using properties of a functional called truncated variation we compare the obtained result with classical results of Wong–Zakai and Bichteler on pathwise stochastic integration.

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1. INTRODUCTION

Let $X = (X_t)_{t \geqslant 0}$ be a real-valued stochastic process with *càdlàg* paths and let $T \geqslant 0$. The total variation of the process X on the interval $[0, T]$ is defined by the following formula:

$$TV(X, T) := \sup_n \sup_{0 \leqslant t_0 < t_1 < \ldots < t_n \leqslant T} \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|.$$

Unfortunately, many of the most important families of stochastic processes are characterized by a "wild" behavior, demonstrated by their infinite total variation. This fact caused arguably the need to develop the general theory of stochastic integration. The main idea allowing the authors to overcome the problematic infinite

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total variation and define stochastic integral with respect to a semimartingale utilizes the fact that the quadratic variation of the semimartingale is still finite. A similar idea may be applied when p-variation of the integrator is finite for some $p \in (1, 2)$. This approach utilizes the Love–Young inequality and may be used, e.g., to define stochastic integral with respect to fractional Brownian motion (cf. [11]), Lévy processes (cf. [16]) or Dirichlet processes of class $\mathcal{D}^p$ (cf. [5], [4]). Other developments led to the rough paths theory developed by T. Lyons and his co-workers (cf. [7]); some other generalization introduces Φ-variation and may be found in the recent book by Dudley and Norvaiša [6], Chapter 3. The approach used in this article is somewhat different. It is similar to the old approach of Wong and Zakai [19] and is based on the simple observation that in the neighborhood (in sup norm) of every *càdlàg* function defined on compact interval $[0, T]$ one finds easily another function with finite total variation but contrary to the Wong–Zakai approach we use an adapted sequence of approximations. Thus, for every $c > 0$, the process X may be decomposed as the sum

$$X = X^c + (X - X^c),$$

where X^c is a "nice", adapted process with finite total variation and the difference $X - X^c$ is a process with small amplitude (no greater than $K_T c$) but possibly "wild" behavior with infinite total variation. More precisely, let F be some fixed, right continuous filtration such that X is adapted to F. Now, for every $c > 0$ we introduce a (non-empty, as will be shown in the sequel) family $\mathcal{X}^c$ of processes with *càdlàg* paths, satisfying the following conditions. If $X^c \in \mathcal{X}^c$ then

(1) the process X^c has a locally finite total variation;

(2) X^c has *càdlàg* paths;

(3) for every $T \geqslant 0$ there exists a $K_T < +\infty$ such that, for every $t \in [0, T]$, $|X_t - X_t^c| \leqslant K_T c$;

(4) for every $T \geqslant 0$ there exists an $L_T < +\infty$ such that, for every $t \in [0, T]$, $|\Delta X_t^c| \leqslant L_T |\Delta X_t|$;

(5) the process X^c is adapted to the filtration $F = (\mathcal{F}_t)_{t \geqslant 0}$.

We will prove that if processes X and Y are *càdlàg* semimartingales on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, F)$, with a probability measure $\mathbb{P}$, such that usual hypotheses hold (cf. [18], Section 1.1), then the sequence of pathwise Lebesgue–Stieltjes integrals

$$\int_0^T Y_- \mathrm{d}X^c, \quad c > 0, \tag{1.1}$$

with $X^c \in \mathcal{X}^c$, tends uniformly in probability $\mathbb{P}$ on compacts to $\int_0^T Y_- \mathrm{d}X + [X^{cont}, Y^{cont}]_T$ as $c \downarrow 0$; $\int_0^T Y_- \mathrm{d}X$ denotes here the (semimartingale) stochastic integral and X^{cont} and Y^{cont} denote continuous martingale parts of X and Y respectively. Moreover, for any square summable sequence $\big(c(n)\big)_{n \geqslant 1}$ we get a.s., uniform on compacts, convergence of the sequence $\int_0^T Y_- \mathrm{d}X^{c(n)}, n = 1, 2, \ldots$ (cf. Theorem 3.2).

It should be stressed here that for each $c > 0$ and each pair of *càdlàg* paths $\big(X(\omega), Y(\omega)\big)$, $\omega \in \Omega$, the value of $\int_0^T Y_-(\omega)\mathrm{d}X^c(\omega)$ (and thus the limit, if it exists) is independent of the probability measure $\mathbb{P}$. Thus we obtain a result in the spirit of Wong and Zakai [19], Bichteler (see [2], Theorem 7.14, or see [10]) or the recent result of Nutz [17], where there are considered operations almost surely leading to the stochastic integral, independent of probability measures and filtrations. The old approach of Wong and Zakai is very straightforward, since it just replaces stochastic integral with Lebesgue–Stieltjes integral. However, it deals with very limited family of possible integrands and integrators (diffusions driven by a Brownian motion), $x_t = \int_0^t g(s)\mathrm{d}s + \int_0^t f(s)\mathrm{d}B_s$, and using an appropriate continuous, finite variation approximation of x, x^n, one gets, a.s. in the limit, the Stratonovich integral

$$\lim_{n\to\infty} \int_0^T \psi(x_t^n, t)\mathrm{d}x_t^n = \int_0^T \psi(x_t, t)\mathrm{d}x_t + \frac{1}{2}\int_0^T f^2(t)\frac{\partial\psi}{\partial x}(x_t, t)\mathrm{d}t.$$

(Modification of this approach is possible, cf. [8], but it requires introducing a probability measure on the Skorokhod space and a rather strong UT – uniform tightness – condition.)

Bichteler's remarkable approach makes it possible to integrate any adapted *càdlàg* process Y with semimartingale integrator X, and is based on the approximation

$$\text{(1.2)} \qquad \lim_{n\to\infty} \sup_{0\leqslant t\leqslant T} \Big| \sum_{i=1}^{\infty} Y_{\tau_{i-1}^n \wedge t}(X_{\tau_i^n \wedge t} - X_{\tau_{i-1}^n \wedge t}) - \int_0^t Y_- \mathrm{d}X \Big| = 0 \text{ a.s.},$$

where $\tau^n = (\tau_i^n)$, $i = 0, 1, 2, \ldots$, is the following sequence of stopping times: $\tau_0^n = 0$ and, for $i = 1, 2, \ldots$,

$$\tau_i^n = \inf\{t > \tau_{i-1}^n : |Y_t - Y_{\tau_{i-1}^n}| \geqslant 2^{-n}\}.$$

REMARK 1.1. *Following the proof of Theorem* 2 *in* [10] *it is easy to see that Bichteler's construction works for any sequence* $\tau^n = (\tau_i^n)$, $i = 0, 1, 2, \ldots$, *of stopping times such that* $\tau_0^n = 0$ *and*

$$\tau_i^n = \inf\{t > \tau_{i-1}^n : |Y_t - Y_{\tau_{i-1}^n}| \geqslant c(n)\}$$

for $i = 1, 2, \ldots$, *given* $c(n) > 0$, $\sum_{n=1}^{\infty} c^2(n) < +\infty$.

The new result of Nutz goes even further, since it does not assume the *càdlàg* property of the integrand, but to prove his result one needs the existence of Mokobodzki's medial limits (cf. [15]), which cannot be proved under standard Zermelo–Fraenkel set theory with the axiom of choice.

The results of this paper seem to indicate that Bichteler's approach is the most flexible (under standard Zermelo–Fraenkel set theory with the axiom of choice) since we will prove that even in the case when the integrator is a standard Brownian motion, our construction (1.1) cannot be extended to an arbitrarily adapted, continuous integrand Y, bounded by a constant. Moreover, a construction similar to the Wong–Zakai one, but more general, i.e.,

$$\int_0^T Z_-^c \mathrm{d}X^c, \qquad c > 0,$$

cannot be extended to an arbitrary continuous semimartingale integrand Z and semimartingale integrator X. Although we obtain rather negative results, we think that the examples as well as the techniques by which they were obtained are the most interesting and may be applied in other situations.

The construction of the appropriate Y and Z, adapted to the natural filtration of B and leading to divergent series of integrals $\int_0^T Y \mathrm{d}B^{\gamma(n)}$, $\int_0^T Z^{\delta(n)} \mathrm{d}\tilde{B}^{\delta(n)}$, where $B^{\gamma(n)}$, $Z^{\delta(n)}$, $\tilde{B}^{\delta(n)}$ satisfy conditions (1)–(5) for some semimartingales Z, $\tilde{B}$, with $\gamma(n), \delta(n) \downarrow 0$ as $n \uparrow +\infty$, will utilize the recent findings of Bednorz, Łochowski and Miłoś on truncated variation (see [1] and [14]) and its relation with the double Skorokhod map on $[-c, c]$ (cf. [3]).

Let us shortly comment on the organization of the paper. In the next section we prove, for any $c > 0$, the existence of a non-empty family of processes $\mathcal{X}^c$. In Section 3 we deal with the limit of pathwise, Lebesgue–Stieltjes integrals $\int_0^T Y_- \mathrm{d}X^c$ as $c \downarrow 0$. Section 4 is devoted to the construction of counterexamples. The last section – Appendix – summarizes the necessary facts on the relation between the truncated variation and the double Skorokhod map on $[-c, c]$.

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2. EXISTENCE OF THE APPROXIMATING SEQUENCE

In this section we will prove that for every $c > 0$ the family of processes $\mathcal{X}^c$ satisfying the conditions (1)–(5) of Section 1 is non-empty. For given $c > 0$ we will simply construct a process X^c satisfying all these conditions. We start with a few definitions.

For fixed $c > 0$ we define two stopping times

$$T_u^{2c} X = \inf\{s \geqslant 0 : \sup_{t \in [0,s]} X_t - X_0 > c\},$$

$$T_d^{2c} X = \inf\{s \geqslant 0 : X_0 - \inf_{t \in [0,s]} X_t > c\}.$$

Assume that $T_d^{2c}X \geqslant T_u^{2c}X$, i.e. the first upward movement of the process X from X_0 of size c appears before the first downward movement of the same size c or both times are infinite (there is no upward or downward jump of size c). Note that in the case $T_d^{2c}X < T_u^{2c}X$ we may simply consider the process $-X$. Now we define sequences $(T_{d,k}^{2c})_{k=1}^{\infty}$, $(T_{u,k}^{2c})_{k=1}^{\infty}$ in the following way: $T_{u,0}^{2c} = T_u^{2c}X$, and for $k = 0, 1, 2, \ldots$

$$T_{d,k}^{2c} = \begin{cases} \inf\{s \geqslant T_{u,k}^{2c} : \sup_{t\in[T_{u,k}^{2c},s]} X_t - X_s > 2c\} & \text{if } T_{u,k}^{2c} < +\infty, \\ +\infty & \text{otherwise,} \end{cases}$$

$$T_{u,k+1}^{2c} = \begin{cases} \inf\{s \geqslant T_{d,k}^{2c} : X_s - \inf_{t\in[T_{d,k}^{2c},s]} X_t > 2c\} & \text{if } T_{d,k}^{2c} < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

REMARK 2.1. *Let us note that for any $s > 0$ there exists a $K < \infty$ such that $T_{u,K}^{2c} > s$ or $T_{d,K}^{2c} > s$. Otherwise, we would obtain two infinite sequences $(s_k)_{k=1}^{\infty}$ and $(S_k)_{k=1}^{\infty}$ such that $0 \leqslant s(1) < S(1) < s(2) < S(2) < \ldots \leqslant s$ and $X_{S(k)} - X_{s(k)} \geqslant c$. But this is a contradiction since X is a* càdlàg *process and for any sequence such that $0 \leqslant s(1) < S(1) < s(2) < S(2) < \ldots \leqslant s$ sequences $(X_{S(k)})_{k=1}^{\infty}$ and $(X_{s(k)})_{k=1}^{\infty}$ have a common limit.*

Now, for the given process X we define the process X^c by the formulas

$$X_s^c = \begin{cases} X_0 & \text{if } s \in [0, T_{u,0}^{2c}), \\ \sup_{t\in[T_{u,k}^{2c},s]} X_t - c & \text{if } s \in [T_{u,k}^{2c}, T_{d,k}^{2c}),\ k = 0, 1, 2, \ldots, \\ \inf_{t\in[T_{d,k}^{2c},s]} X_t + c & \text{if } s \in [T_{d,k}^{2c}, T_{u,k+1}^{2c}),\ k = 0, 1, 2, \ldots \end{cases} \tag{2.1}$$

REMARK 2.2. *Note that, due to Remark* 2.1, *s belongs to one of the intervals $[0, T_{u,0}^{2c})$, $[T_{u,k}^{2c}, T_{d,k}^{2c})$, or $[T_{d,k}^{2c}, T_{u,k+1}^{c})$ for some $k = 0, 1, 2, \ldots$ and the process X_s^c is defined for every $s \geqslant 0$.*

Now we are ready to prove that X^c satisfies conditions (1)–(5).

P r o o f. (1) The process X^c has a finite total variation on compact intervals, since it is monotonic on intervals of the form $[T_{u,k}^{2c}, T_{d,k}^{2c})$, $[T_{d,k}^{2c}, T_{u,k+1}^{c})$ which sum up to the whole half-line $[0, +\infty)$.

(2) From formula (2.1) it follows that X^c is also *càdlàg*.

(3) In order to prove condition (3) we consider three possibilities.

- $s \in [0, T_{u,0}^{2c})$. In this case, since $0 \leqslant s < T_u^{2c}X \leqslant T_d^{2c}X$, by the definition of $T_u^{2c}X$ and $T_d^{2c}X$, we have

$$X_s - X_s^c = X_s - X_0 \in [-c, c].$$

- $s \in [T_{u,k}^{2c}, T_{d,k}^{2c})$ for some $k = 0, 1, 2, \ldots$ In this case, by the definition of

$T^{2c}_{d,k}$, $\sup_{t\in[T^{2c}_{u,k},s]} X_t - X_s$ belongs to the interval $[0,2c]$. Hence

$$X_s - X^c_s = X_s - \sup_{t\in[T^{2c}_{u,k},s]} X_t + c \in [-c,c].$$

• $s \in [T^{2c}_{d,k}, T^{2c}_{u,k+1})$ for some $k = 0,1,2,\ldots$ In this case $X_s - \inf_{t\in[T^{2c}_{d,k},s]} X_t$ belongs to the interval $[0,2c]$, hence

$$X_s - X^c_s = X_s - \inf_{t\in[T^{2c}_{d,k},s]} X_t - c \in [-c,c].$$

(4) We will prove a stronger fact than (4), namely that for every $s > 0$

$$|\Delta X^c_s| \leqslant |\Delta X_s|. \tag{2.2}$$

Indeed, from formula (2.1) it follows that for any $s \notin \{T^{2c}_{u,k}, T^{2c}_{d,k}\}$ the condition (2.2) holds. Hence let us assume that $s \in \{T^{2c}_{u,k}, T^{2c}_{d,k}\}$. We consider several possibilities. If $s = T^{2c}_{u,0}$ then, by the definition of $T^{2c}_{u,0}$,

$$X^c_s - X^c_{s-} = X_s - c - X_0 \geqslant 0 \quad \text{and} \quad X^c_s - X^c_{s-} = X_s - X_0 - c \leqslant X_s - X_{s-}.$$

If $s = T^{2c}_{u,k}$, $k = 1,2,\ldots$, then, by the definition of $T^{2c}_{u,k}$,

$$X^c_s - X^c_{s-} = X_s - c - \Big(\inf_{t\in[T^{2c}_{d,k-1},s]} X_t + c\Big) = X_s - \inf_{t\in[T^{2c}_{d,k-1},s]} X_t - 2c \geqslant 0$$

and, on the other hand,

$$X^c_s - X^c_{s-} = X_s - \inf_{t\in[T^{2c}_{d,k-1},s]} X_t - 2c \leqslant X_s - X_{s-}.$$

Similar arguments may be applied for $s = T^{2c}_{d,k}$, $k = 0,1,\ldots$

(5) The process X^c is adapted to the filtration F since it is adapted to any right-continuous filtration containing the natural filtration of the process X. ■

REMARK 2.3. *It is possible to define the process X^c in many different ways. For example, defining*

$$X^c = X_0 + UTV^c(X,\cdot) - DTV^c(X,\cdot) \tag{2.3}$$

we obtain a process satisfying all the conditions (1)–(5) *and having* (*on the intervals of the form* $[0,T]$, $T > 0$) *the smallest possible total variation among all processes, increments of which differ from the increments of the process X by no* *more than c.* $UTV^c(X,\cdot)$ *and* $DTV^c(X,\cdot)$ *denote here upward and downward truncated variation processes defined as*

$$UTV^c(X,t) := \sup_n \sup_{0\leqslant t_1<t_2<\ldots<t_n\leqslant t} \sum_{i=1}^n \max\{X_{t_i} - X_{t_{i-1}} - c, 0\},$$

$$DTV^c(X,t) := \sup_n \sup_{0\leqslant t_1<t_2<\ldots<t_n\leqslant t} \sum_{i=1}^n \max\{X_{t_{i-1}} - X_{t_i} - c, 0\}.$$

Moreover, for any $T > 0$ *we have*

$$\begin{aligned} TV^c(X,T) &:= \sup_n \sup_{0\leqslant t_1<t_2<\ldots<t_n\leqslant T} \sum_{i=1}^n \max\{|X_{t_i} - X_{t_{i-1}}| - c, 0\} \\ &= UTV^c(X,T) + DTV^c(X,T) = TV(X^c,T). \end{aligned}$$

We will call TV^c truncated variation. *For more on truncated variation, upward truncated variation, and downward truncated variation see* [13], [12], *or* [14].

The fact that the process X^c *defined by formula* (2.3) *satisfies conditions* (1) *and* (3) *with* $K_T \equiv 1$ *follows from Lemma* 3.10 *and Theorem* 4.1 *in* [13]. *By Lemma* 3.10 *in* [13] *we have*

$$|UTV^c(X,t) - DTV^c(X,t) - (X_t - X_0)| \leqslant c.$$

The fact that the process X^c *defined by formula* (2.3) *satisfies conditions* (2) *and* (4) *with* $L_T \equiv 1$ *follows from Remark* 3.7 *in* [13].

Some other construction may be done with the Skorokhod map on $[-\alpha^c, \beta^c]$ (cf. [3]), *where* $\alpha^c, \beta^c : [0,+\infty) \to (0,+\infty)$ *are (possibly time-dependent) continuous boundaries such that* $\sup_{0\leqslant t\leqslant T} \alpha^c(t) \leqslant K_T c$, $\sup_{0\leqslant t\leqslant T} \beta^c(t) \leqslant K_T c$, *and* $\inf_{t\geqslant 0} \big(\beta^c(t) + \alpha^c(t)\big) > 0$. *The Skorokhod map on* $[-\alpha^c, \beta^c]$ *allows one to construct a* càdlàg *process* $-X^c$ *with locally finite variation and such that*

$$X + (-X^c) \in [-\alpha^c, \beta^c]. \tag{2.4}$$

The process X^c *obtained via the Skorokhod map on* $[-\alpha^c, \beta^c]$ *applied to* $X - x_0$, *where* x_0 *is some arbitrary (possibly random) number, starts from*

$$X_0^c = X_0 - \max\big\{ -\alpha^c(0), \min\{X_0 - x_0, \beta^c(0)\}\big\} \tag{2.5}$$

and has minimal total variation on intervals $[0,T]$, $T > 0$, *among all processes satisfying* (2.4) *and starting from the point defined by* (2.5). *This and the fact that* X^c *satisfies condition* (4) *with* $L_T \equiv 1$ *follow from the observation that* X^c *is a "lazy" process, which does not change its value as long as* X^c *stays within the interval* $[X - \beta^c, X + \alpha^c]$.

In fact, the construction (2.1) *of* X^c *is based on a Skorokhod map on the interval* $[-c, c]$ *applied to* $X - X_0$. *In the Appendix we will prove this as well as other interesting properties of this map.*

Truncated variation TV^c is also related to the Skorokhod map. It is the total variation of the process X^c obtained via the Skorokhod map on $[-c/2, c/2]$ applied to $X - x_0$, with appropriate adjusted point x_0 such that the total variation of X^c is minimal possible.

3. PATHWISE LEBESGUE–STIELTJES INTEGRATION WITH RESPECT TO THE APPROXIMATING PROCESS

We consider now a measurable space $(\Omega, \mathcal{F})$ equipped with a right-continuous filtration F and two processes X and Y with *càdlàg* paths, adapted to F. For $T > 0$ and for a sequence of processes $(X^c)_{c>0}$ with $X^c \in \mathcal{X}^c$ let us consider the sequence

$$\int_0^T Y_- \mathrm{d}X^c. \tag{3.1}$$

The integral in (3.1) is understood in the pathwise, Lebesgue–Stieltjes sense (recall that, for any $c > 0$, X^c has bounded variation). We have

THEOREM 3.1. *Assume that $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ such that X and Y are semimartingales with respect to this measure and filtration F, which is complete under $\mathbb{P}$. Then*

$$\int_0^T Y_- \mathrm{d}X^c \overset{ucp\mathbb{P}}{\to} \int_0^T Y_- \mathrm{d}X + [X^{cont}, Y^{cont}]_T \quad \text{as } c \downarrow 0,$$

where $\overset{ucp\mathbb{P}}{\to}$ denotes uniform convergence on compacts in probability $\mathbb{P}$, and $[X^{cont}, Y^{cont}]_T$ denotes quadratic covariation of continuous parts X^{cont} and Y^{cont} of X and Y, respectively.

Proof. Fixing $c > 0$ and using integration by parts (cf. [9], formula (1), p. 519) we get

$$Y_T X_T^c - Y_0 X_0^c = \int_0^T Y_{t-} \mathrm{d}X_t^c + \int_0^T X_{t-}^c \mathrm{d}Y_t + [Y, X^c]_T$$

(the above equality and subsequent equalities in the proof hold $\mathbb{P}$ a.s.). By the uniform convergence, $X_t^c \to X_t$ as $c \downarrow 0$ (note that the bound $|X^c| \leqslant |X| + K_T c$ and a.s. pointwise convergence $X_t^c \to X_t$ as $c \downarrow 0$ are sufficient), we get

$$\int_0^T X_{t-}^c \mathrm{d}Y_t \overset{ucp\mathbb{P}}{\to} \int_0^T X_{t-} \mathrm{d}Y_t.$$

Since X^c has locally finite variation, we have (cf. [9], Theorem 26.6 (viii))

$$[Y, X^c]_T = \sum_{0<s\leqslant T} \Delta Y_s \Delta X_s^c.$$

We calculate the (pathwise) limit

$$\lim_{c\downarrow 0}[Y,X^c]_T = \lim_{c\downarrow 0}\sum_{0<s\leqslant T}\Delta Y_s \Delta X^c_s = \sum_{0<s\leqslant T}\Delta Y_s\Delta X_s$$

(notice that, for any $0\leqslant s\leqslant T$, $|\Delta X^c_s|\leqslant L_T|\Delta X_s|$, thus the above sum is convergent by dominated convergence) and finally obtain

(3.2)
$$\begin{aligned}\int_0^T Y_{t-}\mathrm{d}X^c_t &= \left\{Y_T X^c_T - Y_0X^c_0 - \int_0^T X^c_{t-}\mathrm{d}Y_t - [Y,X^c]_T\right\}\\ &\overset{ucp\mathbb{P}}{\to} Y_TX_T - Y_0X_0 - \int_0^T X_{t-}\mathrm{d}Y_t - \sum_{0<s\leqslant T}\Delta Y_s\Delta X_s \quad \text{as } c\downarrow 0.\end{aligned}$$

On the other hand, again by the integration by parts, we obtain

$$\int_0^T X_{t-}\mathrm{d}Y_t = Y_TX_T - Y_0X_0 - \int_0^T Y_{t-}\mathrm{d}X_t - [Y,X]_T. \tag{3.3}$$

Finally, comparing (3.2) and (3.3), and using [9], Corollary 26.15, we obtain

$$\begin{aligned}\int_0^T Y_{t-}\mathrm{d}X^c_t &\overset{ucp\mathbb{P}}{\to} \int_0^T Y_{t-}\mathrm{d}X_t + [Y,X]_T - \sum_{0<s\leqslant T}\Delta Y_s\Delta X_s \quad \text{as } c\downarrow 0\\ &= \int_0^T Y_{t-}\mathrm{d}X_t + [X^{cont},Y^{cont}]_T. \quad \blacksquare\end{aligned}$$

Note that to prove Theorem 3.1 we did not need the pathwise uniform convergence of the processes X^c to the process X; we might simply use local boundedness and a.s. pointwise convergence $X^c_t \to X_t$ as $c\downarrow 0$. Using the pathwise uniform convergence of the sequence $(X^c)_{c>0}$ we are able to prove a bit stronger result. We have

THEOREM 3.2. *Assume that $\mathbb{P}$ is a probability measure on $(\Omega,\mathcal{F})$ such that X and Y are semimartingales with respect to this measure and filtration F, which is complete under $\mathbb{P}$. Then for any $T>0$ and any sequence $\big(c(n)\big)_{n\geqslant 1}$ such that $c(n)>0$, $\sum_{n=1}^{\infty}c(n)^2<+\infty$ we have*

$$\lim_{n\to+\infty}\sup_{0\leqslant t\leqslant T}\Big|\int_0^t Y_-\mathrm{d}X^{c(n)} - \int_0^t Y_-\mathrm{d}X - [X^{cont},Y^{cont}]_t\Big| = 0 \ \mathbb{P}\ \textit{a.s.}$$

P r o o f. Let us fix $c > 0$. Using integration by parts and the inequality $|X^c - X| \leqslant K_T c$, we estimate

$$\begin{aligned}
&\Big|\int_0^t Y_- \mathrm{d}X^c - \int_0^t Y_- \mathrm{d}X - [X^{cont}, Y^{cont}]_t\Big| \\
&= \Big|Y_t(X_t^c - X_t) - Y_0(X_0^c - X_0) - \sum_{0<s\leqslant t} \Delta Y_s \Delta(X_s^c - X_s) - \int_0^t (X_-^c - X)\mathrm{d}Y\Big| \\
&\leqslant K_T c(|Y_0| + |Y_t|) + \Big|\sum_{0<s\leqslant t} \Delta Y_s \Delta(X_s^c - X_s)\Big| + \Big|\int_0^t (X_-^c - X)\mathrm{d}Y\Big|.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T} \Big|\int_0^t Y_- \mathrm{d}X^c - \int_0^t Y_- \mathrm{d}X - [X^{cont}, Y^{cont}]_t\Big| \\
&\qquad \leqslant K_T c(|Y_0| + \sup_{0\leqslant t\leqslant T} |Y_t|) + \sup_{0\leqslant t\leqslant T} \Big|\sum_{0<s\leqslant t} \Delta Y_s \Delta(X_s^c - X_s)\Big| \\
&\qquad + \sup_{0\leqslant t\leqslant T} \Big|\int_0^t (X_-^c - X)\mathrm{d}Y\Big|.
\end{aligned}$$

Since Y has *càdlàg* paths, it is locally bounded, and hence

$$K_T c(|Y_0| + \sup_{0\leqslant t\leqslant T} |Y_t|) \to 0 \ \mathbb{P} \text{ a.s.} \quad \text{as } c \downarrow 0.$$

Since for every $t \in [0, T]$ the inequality $|X_t^c - X_t| \leqslant K_T c$ holds true (condition (3)), for $s \in [0, t]$ we have $|\Delta(X_s^c - X_s)| \leqslant 2K_T c$. Similarly, by condition (4),

$$|\Delta(X_s^c - X_s)| \leqslant |\Delta X_s^c| + |\Delta X_s| \leqslant (L_T + 1)|\Delta X_s|.$$

Thus we obtain

$$\begin{aligned}
|\Delta(X_s^c - X_s)| &\leqslant \min\{2K_T c, (L_T + 1)|\Delta X_s|\} \\
&\leqslant (2K_T + L_T + 1) \min\{c, |\Delta X_s|\},
\end{aligned}$$

and using this, we estimate

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T} \Big|\sum_{0<s\leqslant t} \Delta Y_s (\Delta X_s^c - \Delta X_s)\Big| \\
&\leqslant \sup_{0\leqslant t\leqslant T} \sqrt{\sum_{0<s\leqslant t} |\Delta Y_s|^2} \sqrt{\sum_{0<s\leqslant t} |\Delta(X_s^c - X_s)|^2} \\
&= \sqrt{\sum_{0<s\leqslant T} |\Delta Y_s|^2} \sqrt{\sum_{0<s\leqslant T} |\Delta(X_s^c - X_s)|^2} \\
&\leqslant \sqrt{[\Delta Y]_T}(2K_T + L_T + 1) \sqrt{\sum_{0<s\leqslant T} \min\{c^2, |\Delta X_s|^2\}} \to 0 \ \mathbb{P} \text{ a.s.} \quad \text{as } c \downarrow 0.
\end{aligned}$$

In order to estimate

$$I^c(T) := \sup_{0\leqslant t\leqslant T} \Big|\int_0^t (X^c_- - X_-)\mathrm{d}Y\Big|$$

let us decompose the semimartingale Y into a local martingale M with bounded jumps (hence a local L^2-martingale) and a process A with locally finite variation (this is possible due to [9], Lemma 26.5, but the decomposition may depend on the measure $\mathbb{P}$), $Y = M + A$. Let $\big(\tau(k)\big)_{k\geqslant 1}$ be a sequence of stopping times increasing to $+\infty$ such that $(M_{t\wedge\tau(k)})_{t\geqslant 0}$ is a square-integrable martingale. We will use the elementary estimate $(a+b)^2 \leqslant 2a^2 + 2b^2$, the Burkholder inequality, and localization. On the set $\Omega_N = \{\omega\in\Omega : TV(A,T) \leqslant N\}$ we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau(k)} \Big|\int_0^t (X^c_- - X_-)\mathrm{d}Y\Big|^2;\Omega_N\Big]\\
&\leqslant 2\mathbb{E}\sup_{0\leqslant t\leqslant T\wedge\tau(k)} \Big|\int_0^t (X^c_- - X_-)\mathrm{d}M\Big|^2 + 2\Big[\mathbb{E}\Big|\int_0^T |X^c_- - X_-|\mathrm{d}A\Big|^2;\Omega_N\Big]\\
&\leqslant 2(4K_T^2c^2\mathbb{E}[M,M]_{T\wedge\tau(k)} + K_T^2c^2N^2) \leqslant 8(\mathbb{E}[M,M]_{T\wedge\tau(k)} + N^2)K_T^2c^2.
\end{aligned}$$

Let now $\big(c(n)\big)_{n\geqslant 1}$ be a sequence such that $c(n) > 0$, $\sum_{n=1}^{\infty} c(n)^2 < +\infty$. We have

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{n=1}^{\infty}\sup_{0\leqslant t\leqslant T\wedge\tau(k)} \Big|\int_0^t (X^{c(n)}_- - X_-)\mathrm{d}Y\Big|^2;\Omega_N\Big]\\
&\qquad = \sum_{n=1}^{\infty}\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T\wedge\tau(k)} \Big|\int_0^t (X^{c(n)}_- - X_-)\mathrm{d}Y\Big|^2;\Omega_N\Big]\\
&\qquad \leqslant 8(\mathbb{E}[M,M]_{T\wedge\tau(k)} + N^2)K_T^2\sum_{n=1}^{\infty} c(n)^2 < +\infty.
\end{aligned}$$

Hence, the sequence $I^{c(n)}\big(T\wedge\tau(k)\big), n = 1,2,\ldots,$ converges to zero on the set Ω_N. Since $\Omega = \bigcup_{N\geqslant 1}\Omega_N$, we infer that $I^{c(n)}\big(T\wedge\tau(k)\big)$ converges $\mathbb{P}$ a.s. to zero. Finally, since $\tau(k)\to+\infty$ a.s., it follows that $I^{c(n)}(T)$ converges $\mathbb{P}$ a.s. to zero. ∎

4. COUNTEREXAMPLES

In this section, using further properties of the sequence X^c defined in Section 2, which we will prove in the Appendix, we will show that even for the integrator $X = B$ being a standard Brownian motion Theorem 3.1 cannot be extended to the case when Y is not a semimartingale. To prove this we start with a few definitions. First, we define a sequence $\beta(n)$, $n = 1, 2, \ldots$, in the following way: $\beta(1) = 1$ and for $n = 2, 3, \ldots$

$$\beta(n) = n^2 \beta(n-1)^6.$$

Now we define $\alpha(n) := \beta(n)^{1/2}$, $\gamma(n) := \beta(n)^{-1}$, and

$$Y := \sum_{n=2}^{\infty} \alpha(n)(B - B^{\gamma(n)}),$$

where B is a standard Brownian motion and, for any $c > 0$, B^c is defined as in Section 2 (by formulas (2.1) or symmetric). Notice that Y is well defined, since

$$|\alpha(n)(B - B^{\gamma(n)})| \leqslant \alpha(n)\gamma(n) = \gamma(n)^{1/2}$$

and, for $n = 2, 3 \ldots,$

$$\begin{aligned} \gamma(n)^{1/2} &= \beta(n)^{-1/2} = n^{-1}\beta(n-1)^{-3} \\ &\leqslant 2^{-1}\beta(n-1)^{-1/2} = 2^{-1}\gamma(n-1)^{1/2}. \end{aligned}$$

Hence the series

$$\sum_{n=2}^{\infty} \alpha(n)(B - B^{\gamma(n)})$$

is uniformly convergent to a bounded, continuous process, adapted to the natural filtration of B. We will use the facts proved in the Appendix as well as [14], Theorem 1, stating that for any continuous semimartingale X

$$\lim_{c \downarrow 0} c \cdot TV^c(X, 1) = \langle X \rangle_1$$

(where $TV^c(X, T)$ was defined in Remark 2.3), from which it follows that

$$\lim_{c \downarrow 0} c \cdot TV^c(B, 1) = 1. \tag{4.1}$$

We will also use the Gaussian concentration of $TV^c(B, T)$ (see [1], Remark 6), from which it follows that, for $c \in (0, 1)$ and $k = 1, 2, \ldots,$

$$\mathbb{E}TV^c(B, 1)^k \leqslant C_k c^{-k}, \tag{4.2}$$

where C_k is a constant depending on k only.

We have

FACT 4.1. *The sequence of integrals*

$$\int_0^1 Y_- \mathrm{d}B^{\gamma(n)}$$

diverges.

P r o o f. Let us fix $n = 2, 3, 4, \ldots$ and split $\int_0^1 Y_- \mathrm{d}B^{\gamma(n)}$ into two summands, $\int_0^1 Y_- \mathrm{d}B^{\gamma(n)} = \mathrm{I} + \mathrm{II}$, where

$$\mathrm{I} = \sum_{m=2}^{n-1} \alpha(m) \int_0^1 (B - B^{\gamma(m)}) \mathrm{d}B^{\gamma(n)}$$

and

$$\mathrm{II} = \int_0^1 \Big\{ \alpha(n)(B - B^{\gamma(n)}) + \sum_{m=n+1}^{\infty} \alpha(m)(B - B^{\gamma(m)}) \Big\} \mathrm{d}B^{\gamma(n)}.$$

First, we consider the second summand, II. Let us notice that, for $m \geqslant 3$, $\gamma(m)^{1/2} \leqslant 3^{-1}\gamma(m-1)^{1/2}$, which implies

$$\begin{aligned} \Big| \sum_{m=n+1}^{\infty} \alpha(m)(B - B^{\gamma(m)}) \Big| &\leqslant \sum_{m=n+1}^{\infty} \alpha(m)\gamma(m) = \sum_{m=n+1}^{\infty} \gamma(m)^{1/2} \\ &\leqslant \gamma(n)^{1/2} \sum_{l=1}^{\infty} 3^{-l} = \frac{1}{2}\gamma(n)^{1/2}. \end{aligned}$$

Hence

$$\begin{aligned} \Big| \int_0^1 \sum_{m=n+1}^{\infty} \alpha(m)(B - B^{\gamma(m)}) \mathrm{d}B^{\gamma(n)} \Big| &\leqslant \frac{1}{2}\gamma(n)^{1/2} \int_0^1 |\mathrm{d}B^{\gamma(n)}| \\ &= \frac{1}{2}\gamma(n)^{1/2} \cdot TV(B^{\gamma(n)}, 1). \end{aligned}$$

By the equality (5.2) (see the Appendix),

$$\alpha(n) \int_0^1 (B - B^{\gamma(n)}) \mathrm{d}B^{\gamma(n)} = \gamma(n)^{1/2} TV(B^{\gamma(n)}, 1),$$

and by the last two estimates we get

$$\mathrm{II} \geqslant \frac{1}{2}\gamma(n)^{1/2} TV(B^{\gamma(n)}, 1) \geqslant \frac{1}{2}\gamma(n)^{1/2} TV^{2\gamma(n)}(B, 1), \tag{4.3}$$

where the last estimate follows from $TV(B^{\gamma(n)}, 1) \geqslant TV^{2\gamma(n)}(B, 1)$ (see (5.1) in the Appendix).

Now let us consider the first summand, I. For $m = 2, \ldots, n-1$, using integration by parts we calculate

$$\begin{aligned} \int_0^1 (B - B^{\gamma(m)}) \mathrm{d}B^{\gamma(n)} &= \int_0^1 B \mathrm{d}B^{\gamma(n)} - \int_0^1 B^{\gamma(m)} \mathrm{d}B^{\gamma(n)} \\ &= (B_1 - B_1^{\gamma(m)}) B_1^{\gamma(n)} + \int_0^1 B^{\gamma(n)} \mathrm{d}B^{\gamma(m)} - \int_0^1 B^{\gamma(n)} \mathrm{d}B. \end{aligned}$$

By this, the inequality $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, and the Itô isometry we estimate

$$(4.4)\quad \mathbb{E}\Big(\int_0^1 (B - B^{\gamma(m)})\mathrm{d}B^{\gamma(n)}\Big)^2$$

$$\leqslant 3\gamma(m)^2\mathbb{E}(B_1^{\gamma(n)})^2 + 3\mathbb{E}\{\sup_{0\leqslant s\leqslant 1}(B_s^{\gamma(n)})^2 TV(B^{\gamma(m)},1)^2\}$$

$$+ 3\int_0^1 \mathbb{E}(B_s^{\gamma(n)})^2\mathrm{d}s.$$

Further, from $a^2b^2 \leqslant \frac{1}{2}a^4 + \frac{1}{2}b^4$, and then $|B_s^{\gamma(n)}| \leqslant |B_s| + \gamma(n)$, $TV(B^{\gamma(m)},1) \leqslant TV^{2\gamma(m)}(B,1) + 2\gamma(m)$ (this follows from the estimate (5.1)), and $(a+b)^4 \leqslant 8(a^4+b^4)$, we get

$$\mathbb{E}\{\sup_{0\leqslant s\leqslant 1}(B_s^{\gamma(n)})^2 TV(B^{\gamma(m)},1)^2\}$$

$$\leqslant \frac{1}{2}\mathbb{E}\sup_{0\leqslant s\leqslant 1}(B_s^{\gamma(n)})^4 + \frac{1}{2}\mathbb{E}TV(B^{\gamma(m)},1)^4$$

$$\leqslant \frac{1}{2}8\mathbb{E}\sup_{0\leqslant s\leqslant 1}\big(B_s^4 + \gamma(n)^4\big) + \frac{1}{2}8\mathbb{E}\big(TV^{2\gamma(m)}(B,1)^4 + 2^4\gamma(n)^4\big)$$

$$\leqslant 4\mathbb{E}\sup_{0\leqslant s\leqslant 1}B_s^4 + 4\mathbb{E}\sup_{0\leqslant s\leqslant 1}TV^{2\gamma(m)}(B,1)^4 + 1.$$

Similarly, by $|B_s^{\gamma(n)}| \leqslant |B_s| + \gamma(n)$ and $(a+b)^2 \leqslant 2(a^2+b^2)$ we calculate

$$\mathbb{E}(B_1^{\gamma(n)})^2 \leqslant 2\mathbb{E}\big(B_1^2 + \gamma(n)^2\big) \leqslant 3$$

and

$$\int_0^1 \mathbb{E}(B_s^{\gamma(n)})^2\mathrm{d}s \leqslant 3.$$

Hence, by (4.4) and the last three estimates, we obtain

$$(4.5)\quad \mathbb{E}\Big(\sum_{m=2}^{n-1}\alpha(m)\int_0^1 (B - B^{\gamma(m)})\mathrm{d}B^{\gamma(n)}\Big)^2$$

$$\leqslant n\sum_{m=2}^{n-1}\alpha(m)^2\mathbb{E}\Big(\int_0^1 (B - B^{\gamma(m)})\mathrm{d}B^{\gamma(n)}\Big)^2$$

$$\leqslant n\sum_{m=2}^{n-1}\alpha(m)^2 3\big(3\gamma(m)^2 + 4\mathbb{E}\sup_{0\leqslant s\leqslant 1}B_s^4 + 4\mathbb{E}TV^{2\gamma(m)}(B,1)^4 + 4\big)$$

$$\leqslant n^2\alpha(n-1)^2 3\big(7 + 4\mathbb{E}\sup_{0\leqslant s\leqslant 1}B_s^4 + 4\mathbb{E}TV^{2\gamma(n-1)}(B,1)^4\big).$$

By the Gaussian concentration properties of $\sup_{0\leqslant s\leqslant 1} B_s$ and $TV^{2\gamma(n-1)}(B,1)$ (the estimate (4.2)), there exist universal constants $\tilde{C}$, C such that

$$\mathbb{E}TV^{2\gamma(n-1)}(B,1)^4 \leqslant \tilde{C}\gamma(n-1)^{-4}$$

and

$$3\big(7+4\mathbb{E}\sup_{0\leqslant s\leqslant 1} B_s^4 + 4\mathbb{E}TV^{2\gamma(n-1)}(B,1)^4\big) \leqslant C\gamma(n-1)^{-4} = C\beta(n-1)^4. \tag{4.6}$$

By (4.5) and (4.6), we have

$$\mathbb{E}\Big(\sum_{m=2}^{n-1}\alpha(m)\int_0^1(B-B^{\gamma(m)})\mathrm{d}B^{\gamma(n)}\Big)^2 \leqslant n^2\alpha(n-1)^2C\beta(n-1)^4 = Cn^2\beta(n-1)^5. \tag{4.7}$$

Now, by (4.7) and the Chebyshev inequality we get

$$\mathbb{P}\big(|\mathrm{I}| \geqslant \sqrt{3C}n\beta(n-1)^{5/2}\big) \leqslant \frac{1}{3}.$$

Thus, for the set $A_n := \{|\mathrm{I}| \leqslant \sqrt{3C}n\beta(n-1)^{5/2}\}$ we have $\mathbb{P}(A_n) \geqslant 2/3$, and by (4.3) on A_n we obtain

$$\begin{aligned}\int_0^1 Y_-\mathrm{d}B^{\gamma(n)} = \mathrm{I}+\mathrm{II} &\geqslant \frac{1}{2}\gamma(n)^{1/2}TV(B^{\gamma(n)},1) - \sqrt{2C}n\beta(n-1)^{5/2}\\ &\geqslant \frac{1}{2}\gamma(n)^{-1/2}\gamma(n)TV^{2\gamma(n)}(B,1) - \sqrt{2C}n\beta(n-1)^{5/2}\\ &= \frac{1}{2}\beta(n)^{1/2}\gamma(n)TV^{2\gamma(n)}(B,1) - \sqrt{2C}n\beta(n-1)^{5/2}.\end{aligned}$$

Let us choose N such that for any $n \geqslant N$

$$\mathbb{P}\left(\gamma(n)TV^{2\gamma(n)}(B,1) \geqslant \frac{1}{4}\right) \geqslant \frac{2}{3}$$

(this is possible by (4.1)). By the definition of $\beta(n)$, on the set $A_n \cap D_n$, where

$$D_n := \left\{\gamma(n)TV^{2\gamma(n)}(B,1) \geqslant \frac{1}{4}\right\},$$

we get

$$\frac{1}{2}\beta(n)^{1/2}\gamma(n)TV^{2\gamma(n)}(B,1) - \sqrt{3C}n\beta(n-1)^{5/2} \geqslant \frac{1}{8}n\beta(n-1)^3 - \sqrt{3C}n\beta(n-1)^{5/2}.$$

Since

$$\frac{1}{8}n\beta(n-1)^3 - \sqrt{3C}n\beta(n-1)^{5/2} \to +\infty$$

as $n \to +\infty$ and

$$\mathbb{P}(A_n \cap D_n) \geqslant \frac{1}{3},$$

we infer that the sequence of integrals $\int_0^1 Y_- \mathrm{d}B^{\gamma(n)}$ is divergent. ■

REMARK 4.1. *From Theorem* 3.1 *and just proved Fact* 4.1 *it follows that the bounded*, *continuous process*

$$Y = \sum_{n=2}^{\infty} \alpha(n)(B - B^{\gamma(n)}),$$

adapted to the natural filtration of B, *is not a semimartingale.*

The construction of sequences $Z^{\delta(n)}$, $\tilde{B}^{\delta(n)}$, $n = 1, 2, \ldots$, such that the sequence of integrals $\int_0^1 Z^{\delta(n)} \mathrm{d}\tilde{B}^{\delta(n)}$, $n = 1, 2, \ldots$, is divergent as $n \uparrow +\infty$ and $Z^{\delta(n)}$, $\tilde{B}^{\delta(n)}$ satisfy conditions (1)–(5) for some semimartingales $Z, \tilde{B}$ is much easier. We set $\delta(n) = 1/n$, $Z^{\delta(n)} = 2B^{1/n^2} + n(B^{1/(2n^2)} - B^{1/n^2})$, $\tilde{B}^{\delta(n)} = B^{1/n^2}$. We easily check that $Z^{\delta(n)}$ satisfies (1)–(5) for $Z = 2B$ and trivially $\tilde{B}^{\delta(n)}$ satisfies (1)–(5) for $\tilde{B} = B$. Since for any $c > 0$, on the set $B^c = B - c$, $\mathrm{d}B^c \geqslant 0$, and on the set $B^c = B + c$, $\mathrm{d}B^c \leqslant 0$ (see Lemma 5.2 in the Appendix), and $-c/2 \leqslant B - B^{c/2} \leqslant c/2$, it follows that $B^{c/2} - B^c \geqslant c/2$ on the set $\mathrm{d}B^c > 0$ and $B^{c/2} - B^c \leqslant -c/2$ on the set $\mathrm{d}B^c < 0$. Thus

$$\begin{aligned}
\int_0^1 Z^{\delta(n)} \mathrm{d}\tilde{B}^{\delta(n)} - \int_0^1 2B^{1/n^2} \mathrm{d}B^{1/n^2} &= \int_0^1 n(B^{1/(2n^2)} - B^{1/n^2}) \mathrm{d}B^{1/n^2} \\
&\geqslant n\frac{1}{2n^2} \int_0^1 |\mathrm{d}B^{1/n^2}| = \frac{n}{2} n^{-2} TV(B^{1/n^2}, 1) \\
&\geqslant \frac{n}{2} n^{-2} TV^{1/n^2}(B, 1).
\end{aligned}$$

Now, by the usual Lebesgue–Stieltjes integration, $\int_0^1 2B^{1/n^2} \mathrm{d}B^{1/n^2} = (B^{1/n^2})^2$, and by the just obtained estimate and (4.1) we see that

$$\int_0^1 Z^{\delta(n)} \mathrm{d}\tilde{B}^{\delta(n)} \to +\infty.$$

5. APPENDIX

In this Appendix we will prove estimates used in Section 4, concerning the process X^c constructed in Section 2. Before proceeding, let us recall the definitions of truncated variation, upward truncated variation, and downward truncated variation from Remark 2.3. Let us notice that for $c = 0$ it follows simply that TV^0 is the (finite or infinite) total variation and $UTV = UTV^0$ and $DTV = DTV^0$ are positive and negative parts of the total variation. Moreover, we have the Hahn–Jordan decomposition, $TV = UTV + DTV$.

LEMMA 5.1. *For the total variation of the process X^c, constructed in Section* 2, *one has the following estimates*:

$$TV^{2c}(X,T) \leqslant TV(X^c,T) \leqslant TV^{2c}(X,T) + 2c. \tag{5.1}$$

P r o o f. The lower bound in (5.1) follows directly from the estimate

$$|X_t^c - X_s^c| \geqslant \max\{|X_t - X_s| - 2c, 0\},$$

valid for any $0 \leqslant s < t \leqslant T$, which is implied directly by the inequalities $|X_s^c - X_s| \leqslant c$, $|X_t^c - X_t| \leqslant c$, and the triangle inequality.

To prove the opposite inequality, let us assume that $T_d^{2c}X \geqslant T_u^{2c}X$ and put $M_k^{2c} = \sup_{t\in[T_{u,k}^{2c}, T_{d,k}^{2c})} X_t$, $m_k^{2c} = \inf_{t\in[T_{d,k}^{2c}, T_{u,k+1}^{2c})} X_t$, $k = 0, 1, \ldots$, and consider three possibilities.

- $T \in [0, T_{u,0}^{2c})$. In this case $TV(X^c,T) = UTV(X^c,T) = DTV(X^c,T) = 0$.
- $T \in [T_{u,0}^{2c}, T_{d,0}^{2c})$. In this case

$$UTV(X^c,T) = \sup_{t\in[T_{u,0}^{2c},T]} X_t - c - X_0, \qquad DTV(X^c,T) = 0,$$

and

$$TV(X^c,T) = UTV(X^c,T) + DTV(X^c,T).$$

Now, by the definition of TV^{2c} it is not difficult to see that

$$TV^{2c}(X,T) \geqslant \max\{\sup_{t\in[T_{u,0}^{2c},T]} X_t - X_0 - 3c, 0\} \geqslant TV(X^c,T) - 2c.$$

- $T \in [T_{u,k}^{2c}, T_{d,k}^{2c})$ for some $k = 1, 2, \ldots$ In this case, using monotonicity of X^c on the intervals $[T_{u,k}^{2c}, T_{d,k}^{2c}]$ and $[T_{d,k}^{2c}, T_{u,k+1}^{2c}]$, $k = 0, 1, \ldots$, and formula (2.1) we calculate

$$\begin{aligned} UTV(X^c,T) = (M_0^{2c} - c - X_0) &+ \sum_{i=1}^{k-1} (M_i^{2c} - m_{i-1}^{2c} - 2c) \\ &+ \sup_{t\in[T_{u,k}^{2c},T]} X_t - m_{k-1}^{2c} - 2c, \end{aligned}$$

$$DTV(X^c,T) = \sum_{i=0}^{k-1} (M_i^{2c} - m_i^{2c} - 2c),$$

and

$$TV(X^c, T) = UTV(X^c, T) + DTV(X^c, T).$$

Now it is not difficult to see that

$$UTV^{2c}(X, T) \geqslant \max\{M_0^{2c} - X_0 - 3c, 0\} + \sum_{i=1}^{k-1} (M_i^{2c} - m_{i-1}^{2c} - 2c)$$
$$+ \sup_{t \in [T_{u,k}^{2c}, T]} X_t - m_{k-1}^{2c} - 2c \geqslant UTV(X^c, T) - 2c,$$

$$DTV^{2c}(X, T) \geqslant \sum_{i=0}^{k-1} (M_i^{2c} - m_i^{2c} - 2c) = DTV(X^c, T),$$

and

$$TV^{2c}(X, T) = UTV^{2c}(X, T) + DTV^{2c}(X, T) \geqslant TV(X^c, T) - 2c.$$

• $s \in [T_{d,k}, T_{u,k+1})$ for some $k = 0, 1, 2, \ldots$ The proof follows similarly to the previous case. ■

Now we will prove that the construction of X^c in Section 2 is based on a Skorokhod map on the interval $[-c, c]$. Let us recall the definition of the Skorokhod problem on the interval $[-c, c]$. Let $D[0, +\infty)$ denote the set of real-valued *càdlàg* functions, and $BV^+[0, +\infty)$, $BV[0, +\infty)$ denote subspaces of $D[0, +\infty)$ consisting of nondecreasing functions and functions of bounded variation, respectively. We have

DEFINITION 5.1. A pair of functions $(\phi, \eta) \in D[0, +\infty) \times BV[0, +\infty)$ is said to be a *solution of the Skorokhod problem on* $[-c, c]$ *for* ψ if the following conditions are satisfied:

(i) for every $t \geqslant 0$, $\phi^c(t) = \psi(t) + \eta^c(t) \in [-c, c]$;

(ii) $\eta(0-) = 0$ and $\eta = \eta_l - \eta_u$ for some $\eta_l, \eta_u \in BV^+[0, +\infty)$ such that the corresponding measures $d\eta_l$, $d\eta_u$ are carried by $\{t \geqslant 0 : \phi(t) = -c\}$ and $\{t \geqslant 0 : \phi(t) = c\}$, respectively.

It is possible to prove that for every $c > 0$ there exists a unique solution to the Skorokhod problem on $[-c, c]$ (cf. [3], Theorem 2.6, Proposition 2.3, and Corollary 2.4) and we will write $\phi^c = \Gamma^c(\psi)$ to denote the associated map, called the *Skorokhod map on* $[-c, c]$. Now we will prove

LEMMA 5.2. *The process X^c constructed in Section 2 and the Skorokhod map on $[-c, c]$ are related via the equality*

$$X^c = X - \Gamma^c(X - X_0),$$

and the mutually singular measures $\mathrm{d}UTV(X^c,\cdot)$ *and* $\mathrm{d}DTV(X^c,\cdot)$ *are carried by* $\{t>0: X_t - X_t^c = c\}$ *and* $\{t>0: X_t - X_t^c = -c\}$, *respectively. Thus, on these sets we have*

$$\mathrm{d}UTV(X^c,\cdot) = \mathrm{d}X^c \quad \textit{and} \quad \mathrm{d}DTV(X^c,\cdot) = -\mathrm{d}X^c,$$

respectively.

P r o o f. Let us put $V = X - X^c$. We have $V \in [-c,c]$, i.e. condition (i) in Definition 5.1 holds. Then to complete the proof it is enough to show that $X_0^c = X_0 - \Gamma^c(X - X_0)_0$ and for the finite variation process $-X^c$ the corresponding measures $\mathrm{d}UTV(-X^c,\cdot) = \mathrm{d}DTV(X^c,\cdot)$ and $\mathrm{d}DTV(-X^c,\cdot) = \mathrm{d}UTV(X^c,\cdot)$ are carried on $(0,+\infty)$ by $\{t>0: V_t = -c\}$ and $\{t>0: V_t = c\}$, respectively.

Let us observe that the condition $\eta(0-) = 0$ together with the remaining part of condition (ii) sets the value of $\Gamma^c(\psi)(0)$,

$$\Gamma^c(\psi)(0) = \max\big\{-c, \min\{\psi(0), c\}\big\}.$$

Hence we get the equality $X_0^c = X_0 = X_0 - \Gamma^c(X - X_0)_0$. Moreover, we have $\mathrm{d}UTV(-X^c,t) = 0$ and $\mathrm{d}DTV(-X^c,t) = 0$ for t from the interval $(0, T_{u,0}^{2c})$ (we assume again that $T_u^{2c}X \leqslant T_d^{2c}X$).

Now notice that by formula (2.1)

$$\mathrm{d}(-X_s^c) = \mathrm{d}DTV(X^c,s) = -\mathrm{d}\inf_{T_{d,k}^{2c}\leqslant t\leqslant s} X_t$$

and

$$\mathrm{d}(-X_s^c) = -\mathrm{d}UTV(X^c,s) = -\mathrm{d}\sup_{T_{u,k}^{2c}\leqslant t\leqslant s} X_t$$

on the intervals $(T_{d,k}^{2c}, T_{u,k+1}^{2c})$ and $(T_{u,k}^{2c}, T_{d,k}^{2c})$, $k = 0,1,2,\ldots$, respectively. Let us now notice that the only points of increase of the measure $\mathrm{d}UTV(X^c,\cdot)$ from the intervals $(T_{u,k}^{2c}, T_{d,k}^{2c})$, $k = 0,1,2,\ldots$, are the points where the process X attains new suprema on these intervals. But at every such point s we have

$$X_s^c = \sup_{t\in[T_{u,k}^{2c},s]} X_t - c = X_s - c,$$

and hence $V_s = X_s - X_s^c = c$. A similar assertion holds for $\mathrm{d}DTV(X^c,\cdot)$.

Next, notice that at the point $s = T_{u,0}$ one has $X_s^c = X_s - c \geqslant X_0 = X_{s-}$, and since for $T_{u,k+1}^{2c} < +\infty$, $k = 0,1,\ldots$, one has

$$T_{u,k+1}^{2c} = \inf\{s \geqslant T_{d,k}^{2c} : X_s - \inf_{t\in[T_{d,k}^{2c},s]} X_t > 2c\},$$

we obtain for $s = T^{2c}_{u,k+1} < +\infty, k = 0, 1, \ldots, \inf_{t\in[T^{2c}_{d,k},s]} X_t = \inf_{t\in[T^{2c}_{d,k},s)} X_t$ and

$$\begin{aligned} X^c_s = X_s - c &\geqslant \inf_{t\in[T_{d,k},s]} X_t + c \\ &= \inf_{t\in[T^{2c}_{d,k},s)} X_t + c = X^c_{s-}. \end{aligned}$$

Consequently, at the points $s = T^{2c}_{u,k}, k = 0, 1, \ldots,$ we have $\mathrm{d}DTV(X^c, s) = 0$, $\mathrm{d}UTV(X^c, s) \geqslant 0$ and $V_s = c$.

In a similar way one proves that the measure $\mathrm{d}DTV(X^c, \cdot)$ is carried by $\{t > 0 : V_t = -c\}$.

The last assertion follows from the fact that UTV and DTV are positive and negative parts of $\mathrm{d}X^c$. ■

The direct consequence of Lemma 5.2 is the equality

$$\int_0^T (X - X^c)\mathrm{d}X^c = c \cdot \int_0^T |\mathrm{d}X^c| = c \cdot TV(X^c, T), \tag{5.2}$$

which holds for any $c, T > 0$.

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Rafał M. Łochowski
Warsaw School of Economics
ul. Madalińskiego 6/8
02-513 Warszawa, Poland
and
Prince Mohammad Bin Fahd University
P.O. Box 1664
Al Khobar 31952, Saudi Arabia
E-mail: rlocho314@gmail.com

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