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LAW OF LARGE NUMBERS FOR MONOTONE CONVOLUTION

BY

JIUN-CHAU WANG* (SASKATOON) AND ENZO WENDLER** (SASKATOON)

Abstract. Using the martingale convergence theorem, we prove a law of large numbers for monotone convolutions $\mu_1 \triangleright \mu_2 \triangleright \ldots \triangleright \mu_n$, where μ_j's are probability laws on $\mathbb{R}$ with finite variances but not required to be identical.

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1. INTRODUCTION AND THE MAIN RESULT

The *monotone convolution* $\triangleright$ is an associative binary operation on $\mathcal{M}$, the set of all Borel probability measures on the real line $\mathbb{R}$. It was introduced by Muraki in [5], based on his notion of monotonic independence for operators acting on a certain type of Fock space. Later, a universal construction for the monotone convolution of measures was found in [2], which does not depend on the underlying Hilbert space. Thus, the monotone convolution $\mu \triangleright \nu$ for two measures $\mu, \nu \in \mathcal{M}$ is defined as the distribution of $X + Y$, where the (non-commutative) random variables X and Y are monotonically independent and having distributions μ and ν, respectively. Together with classical, free, and Boolean convolutions, the monotone convolution is one of the four natural convolution operations on the set $\mathcal{M}$ (see [7] and [6]).

The research of limit theorems for monotone convolution has been active in recent years. Notably, an equivalence between monotone and Boolean limit theorems has been proved in [1], making it possible to apply the classical Gnedenko type convergence criterion to the weak convergence for sums of monotonically independent and identically distributed random variables. In spite of these successful results, the literature lacks a treatment of limit theorems for non-identically

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distributed variables. The goal of the current paper is to supply one such limit theorem in the context of law of large numbers for variables with finite variances. The convergence condition we discovered here coincides with the one for the classical law of large numbers.

To explain our result in detail, we first recall some definitions. Following [3], a sequence of measures $\{\nu_n\}_{n=1}^{\infty}$ in $\mathcal{M}$ is said to be *stable* if one can find constants $a_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \nu_n(\{t \in \mathbb{R} : |t - a_n| \geqslant \varepsilon\}) = 0$$

for every $\varepsilon > 0$. Thus, the weak law of Khintchine states that if $\{X_n\}_{n=1}^{\infty}$ is an i.i.d. sequence of real random variables drawn from a law $\mu \in \mathcal{M}$ with finite expectation $m(\mu)$, then the sequence of distributions

$$D_{1/n}(\underbrace{\mu * \mu * \ldots * \mu}_{n \text{ times}})$$

is stable, with the asymptotic constants $a_n = m(\mu)$ for all $n \geqslant 1$. Here the notation $*$ means the classical convolution of measures. For $b > 0$, the measure $D_b\mu$ is the *dilation* of μ defined by $D_b\mu(A) = \mu(b^{-1}A)$ for Borel measurable $A \subset \mathbb{R}$.

Our main result is the following

THEOREM 1.1. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of probability laws with finite variances* $\mathrm{var}(\mu_n)$. *If the series*

$$\sum_{k=1}^{\infty} \frac{\mathrm{var}(\mu_k)}{b_k^2} < \infty \tag{1.1}$$

for some sequence $\{b_n\}_{n=1}^{\infty}$ with $0 < b_1 < b_2 < \ldots \to \infty$, then the sequence of measures

$$D_{1/b_n}(\mu_1 \triangleright \mu_2 \triangleright \ldots \triangleright \mu_n)$$

is stable.

As it will be seen from the proof of Theorem 1.1, a formula for the asymptotic constants a_n here is given by

$$a_n = \frac{1}{b_n} \sum_{k=1}^{n} m(\mu_k).$$

In particular, when $\mu_1 = \mu_2 = \ldots = \mu_n = \mu$ and $b_n = n$, Theorem 1.1 shows that laws of the normalized sums

$$\frac{Y_1 + Y_2 + \ldots + Y_n}{n}$$

converge weakly to the point mass at $m(\mu)$ as $n \to \infty$, where $\{Y_n\}_{n=1}^{\infty}$ is a monotonically independent sequence of random variables having the same distribution μ.

This is the weak law of large numbers obtained in [8]. Finally, we remark that the condition (1.1) also implies that the classical convolutions

$$D_{1/b_n}(\mu_1 * \mu_2 * \ldots * \mu_n)$$

are stable (see [3]).

We end this section with some comments on the method of our proof. The difficulty in proving limit theorems for monotone convolution comes from the fact that the computation of $\triangleright$ requires the composition of certain integral transforms. Precisely, recall that the *Cauchy transform* of a measure $\mu \in \mathcal{M}$ is defined as

$$G_\mu(z) = \int\limits_{-\infty}^{\infty} \frac{1}{z-t}\,\mu(dt), \qquad \Im z > 0,$$

and hence the map $F_\mu(z) = 1/G_\mu(z)$ is an analytic self-map of the complex upper half-plane $\mathbb{C}^+ = \{z = x + iy : \; y > 0\}$. For any $\mu, \nu \in \mathcal{M}$, it was shown in [2] that

$$F_{\mu \triangleright \nu}(z) = F_\mu \circ F_\nu(z), \qquad z \in \mathbb{C}^+.$$

Hence, proving Theorem 1.1 amounts to understanding the dynamics of the backward compositions

$$F_{\mu_1} \circ F_{\mu_2} \circ \ldots \circ F_{\mu_n}$$

of analytic functions. In general, a dynamical system of this sort is quite complicated to analyze using complex analysis. To go around this difficulty, we utilize the Markov chain approach in [4] and the L^2-martingale convergence theorem to treat the composition sequence of these F-transforms. We shall now begin to present these details.

2. PROOF OF THE MAIN RESULT

Let $\{\mu_n\}_{n=1}^{\infty}$ be the given sequence of probability measures with finite variances, and let $\mathcal{B}$ denote the Borel σ-field on $\mathbb{R}$. To each $n \geqslant 2$, we introduce the function

$$p_n(x, B) = \delta_x \triangleright \mu_n(B), \qquad x \in \mathbb{R}, B \in \mathcal{B},$$

where δ_x stands for the point mass at x. Then each $p_n(x, dy)$ is a transition probability function on $\mathbb{R} \times \mathcal{B}$. Indeed, denote by H the set of bounded and $\mathcal{B}$-measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that the function Tf is also $\mathcal{B}$-measurable, where

$$Tf(x) = \int\limits_{-\infty}^{\infty} f(y)\,p_n(x, dy), \qquad x \in \mathbb{R}.$$

First, since the map $x \mapsto \delta_x \triangleright \mu_n$ is weakly continuous, the set H contains all continuous and bounded real-valued functions on $\mathbb{R}$. Secondly, by the monotone convergence theorem, if $\{f_n\}_{n=1}^{\infty}$ is a monotonically increasing sequence of non-negative functions in H which converges pointwisely to a bounded function f, then the limit function f is also in H.

Now, consider the set $\mathcal{P} = \{(a,b) : -\infty \leqslant a < b \leqslant \infty\} \cup \{\phi\}$ and the set $\mathcal{L} = \{B \in \mathcal{B} : I_B \in H\}$. The set $\mathcal{P}$ is clearly a π-system that generates the field $\mathcal{B}$, and both ϕ and $\mathbb{R}$ belong to the set $\mathcal{L}$. Observe that for any finite interval (a,b) there exist continuous functions $0 \leqslant f_n \leqslant 1$ such that $f_n \nearrow I_{(a,b)}$. This implies that the indicator $I_{(a,b)}$ is in H because H is closed under bounded monotone convergence. Thus, the set $\mathcal{L}$ contains $\mathcal{P}$. Moreover, it is easy to see that $\mathcal{L}$ is a λ-system, and therefore we have $\mathcal{L} = \mathcal{B}$ by Dynkin's π-λ theorem. It follows that for each fixed $B \in \mathcal{B}$ the map $x \mapsto p_n(x,B)$ is $\mathcal{B}$-measurable, justifying that p_n is a transition probability.

Next, we consider the real-valued Markov chain $\{X_n\}_{n=1}^{\infty}$ generated by the transition probabilities $\{p_n\}_{n=2}^{\infty}$ and the initial distribution μ_1. The existence of such a Markov chain is guaranteed by the Kolmogorov extension theorem, and the finite-dimensional distributions of $\{X_n\}_{n=1}^{\infty}$ are determined by

$$\begin{aligned}&\Pr(X_j \in B_j;\, 1 \leqslant j \leqslant n)\\&\qquad = \int\limits_{x_1 \in B_1} \mu_1(dx_1) \int\limits_{x_2 \in B_2} p_2(x_1, dx_2) \ldots \int\limits_{x_n \in B_n} p_n(x_{n-1}, dx_n).\end{aligned}$$

Notice that one has

$$\begin{aligned}G_{\mu \triangleright \nu}(z) = G_\mu\big(F_\nu(z)\big) &= \int\limits_{x \in \mathbb{R}} \frac{1}{F_\nu(z) - x}\, \mu(dx) = \int\limits_{x \in \mathbb{R}} G_{\delta_x \triangleright \nu}(z)\, \mu(dx)\\&= \int\limits_{x \in \mathbb{R}} \int\limits_{t \in \mathbb{R}} \frac{1}{z-t}\, \delta_x \triangleright \nu(dt)\, \mu(dx)\\&= \int\limits_{t \in \mathbb{R}} \frac{1}{z-t} \int\limits_{x \in \mathbb{R}} \mu(dx)\, \delta_x \triangleright \nu(dt)\end{aligned}$$

for any $\mu, \nu \in \mathcal{M}$. (The $\mathcal{B}$-measurability of the function $\delta_x \triangleright \nu$ in x follows from the first two paragraphs of this section.) Since the Cauchy transform $G_{\mu \triangleright \nu}$ determines the measure $\mu \triangleright \nu$ uniquely, we deduce that

$$\int\limits_{-\infty}^{\infty} \mu(dx)\, \delta_x \triangleright \nu(dt) = \mu \triangleright \nu(dt).$$

In particular, if $B_1 = B_2 = \ldots = B_{n-1} = \mathbb{R}$, an easy induction argument shows that

$$\begin{aligned}\Pr(X_n \in B_n) &= \int\limits_{-\infty}^{\infty} \mu_1 \triangleright \mu_2 \triangleright \ldots \triangleright \mu_{n-1}(dx_{n-1})\, p_n(x_{n-1}, B_n)\\&= \int\limits_{-\infty}^{\infty} \mu_1 \triangleright \mu_2 \triangleright \ldots \triangleright \mu_{n-1}(dx_{n-1})\, \delta_{x_{n-1}} \triangleright \mu_n(B_n)\\&= \mu_1 \triangleright \mu_2 \triangleright \ldots \triangleright \mu_n(B_n),\end{aligned}$$

and therefore the distribution of X_n is precisely the monotone convolution

$$\mu_1 \triangleright \mu_2 \triangleright \ldots \triangleright \mu_n.$$

We now compute the first two conditional moments of the Markov chain $\{X_n\}_{n=1}^{\infty}$. The notation $m_2(\mu_n)$ means the second moment of the measure μ_n.

LEMMA 2.1. *For $n \geqslant 2$, we have*

$$E[X_n|\, X_{n-1}] \stackrel{a.s.}{=} X_{n-1} + m(\mu_n)$$

and

$$E[X_n^2|\, X_{n-1}] \stackrel{a.s.}{=} X_{n-1}^2 + 2m(\mu_n)X_{n-1} + m_2(\mu_n).$$

Proof. We write the function F_{μ_n} in the Nevanlinna form:

$$F_{\mu_n}(z) = z - m(\mu_n) + \int_{-\infty}^{\infty} \frac{1}{t-z}\, \sigma_n(dt),$$

where σ_n is a finite Borel measure on $\mathbb{R}$ with $\sigma_n(\mathbb{R}) = \text{var}(\mu_n)$. Because

$$F_{\delta_x \triangleright \mu_n}(z) = F_{\mu_n}(z) - x,$$

the uniqueness of the Nevanlinna representation implies that

$$m(\delta_x \triangleright \mu_n) = m(\mu_n) + x$$

and $\text{var}(\delta_x \triangleright \mu_n) = \text{var}(\mu_n)$. In other words, we have

$$\int_{-\infty}^{\infty} y\, p_n(x, dy) = m(\mu_n) + x$$

and

$$\int_{-\infty}^{\infty} y^2\, p_n(x, dy) = x^2 + 2m(\mu_n)x + m_2(\mu_n).$$

Hence, the desired result follows from the fact that the function $p_n(X_{n-1}, dy)$ serves as a regular conditional distribution for X_n given the σ-subfield $\sigma(X_{n-1})$, and the proof is complete. ■

We are now ready to prove the main result.

Proof of Theorem 1.1. For $n \geqslant 1$, define

$$Y_n = X_n - \sum_{k=1}^{n} m(\mu_k).$$

Then Lemma 2.1 and the Markov property of $\{X_n\}_{n=1}^{\infty}$ imply that $\{Y_n\}_{n=1}^{\infty}$ is an L^2-martingale. Consider the martingale differences

$$Z_n = Y_n - Y_{n-1}, \qquad n \geqslant 2,$$

and set $Z_1 = Y_1$. By Lemma 2.1 again, we have $E[Z_n | X_{n-1}] = 0$ and the second moment

$$E[Z_n^2] = \text{var}(\mu_n).$$

Our proof now follows a classical line. Recall that $\{b_n\}_{n=1}^{\infty}$ is a positive sequence increasing to infinity for which the condition (1.1) holds. For $n \geqslant 1$, let

$$S_n = \sum_{k=1}^{n} \frac{Z_k}{b_k}.$$

Note that $\{S_n\}_{n=1}^{\infty}$ also forms another L^2-martingale. Moreover, we have the second moment

$$\begin{aligned} E[S_n^2] &= E\big[E[(b_n^{-1}Z_n + S_{n-1})^2 | X_1, X_2, \ldots, X_{n-1}]\big] \\ &= b_n^{-2}E[Z_n^2] + E[S_{n-1}^2] + 2b_n^{-1}E\big[S_{n-1}E[Z_n | X_{n-1}]\big] \\ &= b_n^{-2}\text{var}(\mu_n) + E[S_{n-1}^2]. \end{aligned}$$

Proceeding inductively, we get

$$E[S_n^2] = \sum_{k=1}^{n} \frac{\text{var}(\mu_k)}{b_k^2},$$

which is bounded uniformly in n by the condition (1.1). Therefore, by the L^2-martingale convergence theorem, $\{S_n\}_{n=1}^{\infty}$ converges almost surely, and Kronecker's lemma further implies that

$$\frac{1}{b_n} \sum_{k=1}^{n} Z_k = \frac{1}{b_n}\big[X_n - \sum_{k=1}^{n} m(\mu_k)\big] \to 0$$

on the set of points in the sample space where $\{S_n\}_{n=1}^{\infty}$ converges. Thus, putting

$$a_n = \frac{1}{b_n} \sum_{k=1}^{n} m(\mu_k),$$

we see that $b_n^{-1}X_n - a_n$ converges in probability to zero as $n \to \infty$. Then the proof is completed, because $b_n^{-1}X_n$ has distribution $D_{1/b_n}(\mu_1 \triangleright \mu_2 \triangleright \ldots \triangleright \mu_n)$. ■

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University of Saskatchewan
106 Wiggins Road
Saskatoon, SK S7N 5E6
E-mail: jcwang@math.usask.ca

University of Saskatchewan
106 Wiggins Road
Saskatoon, SK S7N 5E6
E-mail: epw943@mail.usask.ca

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