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ASYMPTOTIC BEHAVIOUR OF LINEAR RANK STATISTICS FOR THE TWO-SAMPLE PROBLEM

BY

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Abstract. Applying the strong approximation technique we present a unified approach to asymptotic results for multivariate linear rank statistics for the two-sample problem. We reprove asymptotic normality of these statistics under the null hypothesis and under local alternatives convergent at a moderate rate to the null hypothesis. We also provide a moderate deviation theorem for these statistics under the null hypothesis. Proofs are short and use natural argumentation.

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1. INTRODUCTION

The asymptotic normality of linear rank statistics was studied by many authors, e.g., Wald and Wolfowitz [17], Chernoff and Savage [1], Hájek [7], [8], Govindarajulu et al. [5] and others. In their monograph, Hájek and Šidak [9] have presented their approach in a systematic and elegant form. The main tool they applied was the so-called method of Hájek's projection allowing for very general results. Pyke and Shorack [15] and Shorack and Wellner [16] generalized these results. Govindarajulu et al. [5] proposed a parallel set of sufficient conditions and obtained slightly stronger results. A detailed exposition of this approach can be found in Govindarajulu [4]. On the other hand, a Cramér type large deviation theorem for linear rank statistics was proved by Kallenberg [13]. His proof was based on elementary, but technically involved lemma of Hušková [11]. Basing on it, Ducharme and Ledwina [3] obtained some optimality results for data driven tests for the two-sample problem.

In the present note we propose a unified approach to the above-mentioned results by an application of the strong approximation technique, and the Komlós–Major–Tusnády inequality [14], in particular. Our method allows for much shorter and quite natural proofs. In some cases we are able to weaken assumptions. For example, we consider sample sizes of arbitrary orders.

Some motivation for our study was to simplify proofs of Theorems 3.1 and 3.2 in Ducharme and Ledwina [3]. Our approach was already used for the one-sample problem to study asymptotic behaviour as well as to prove asymptotic optimality of data driven tests for symmetry (see Inglot et al. [12]).

In Section 2 we state the problem and give some preliminary results and notation. Section 3 concerns the case of the null hypothesis while Section 4 deals with convergent alternatives. The main results are Theorems 3.3, 3.4, 3.5 and 4.3. In Section 5 we provide a proof of Theorem 3.5 and auxiliary results. It turns out that this proof requires a use of the independence of Brownian bridges defined in Section 2 while in other proofs the condition is not necessary. In a short Appendix we present, for the reader's convenience, an exponential inequality for the modulus of continuity of the Brownian bridge which is applied several times in this paper.

2. NOTATION AND STATING THE PROBLEM

Let $m = m_N$, $n = n_N$ be fixed sequences of natural numbers such that $N = m + n$ and let $F_N,\ G_N,\ H_N$ be sequences of continuous distribution functions on the real line. Let $X_1, \ldots, X_m$ be a sample from the distribution F_N, and $Y_1, \ldots, Y_n$ be a sample from the distribution G_N, independent of $X_1, \ldots, X_m$. Denote by

$$Z = (Z_1, \ldots, Z_N) = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$$

the pooled sample and by $V = (V_1, \ldots, V_N)$ the transformed sample into the interval $[0, 1]$, i.e. $V_i = H_N(Z_i)$, $i = 1, \ldots, N$. Further, denote by $\mathcal{H}_N(t)$ the empirical distribution function of V, by $\mathcal{F}_m(t)$ the empirical distribution function of $\big(H_N(X_1), \ldots, H_N(X_m)\big)$, and by $\varepsilon_m(t) = \sqrt{m}[\mathcal{F}_m(t) - F_N H_N^{-1}(t)]$ the empirical process of the first part of the transformed sample. Similarly we define $\mathcal{G}_n(t)$ and $\eta_n(t) = \sqrt{n}[\mathcal{G}_n(t) - G_N H_N^{-1}(t)]$ for the second part of the transformed sample. Then $\varepsilon_m(t) \stackrel{\mathcal{D}}{=} e_m\big(F_N H_N^{-1}(t)\big)$, where e_m is the uniform empirical process, and similarly $\eta_n(t) \stackrel{\mathcal{D}}{=} e_n\big(G_N H_N^{-1}(t)\big)$.

Let $\varphi_1, \varphi_2, \ldots$ be a sequence of linearly independent absolutely continuous functions on $[0, 1]$ satisfying $\int_0^1 \varphi_j(t)dt = 0,\ j \geqslant 1$. Let $\Phi(t) = \big(\varphi_1(t), \varphi_2(t), \ldots\big)^T$ stand for a vector of such functions.

The aim of the present paper is to study an asymptotic behaviour of linear rank statistics for the two-sample problem having the form

$$\begin{aligned} T_N &= \sqrt{\frac{mn}{N}}\Big(\int_0^1 \Phi\big(\mathcal{H}_N(t)\big)d\mathcal{F}_m(t) - \int_0^1 \Phi\big(\mathcal{H}_N(t)\big)d\mathcal{G}_n(t)\Big) \\ &= \sum_{i=1}^N c_{iN}\Phi\Big(\frac{R_i}{N}\Big), \end{aligned} \tag{2.1}$$

where $c_{iN} = \sqrt{n/(mN)}$ for $i \leqslant m$ and $c_{iN} = -\sqrt{m/(nN)}$ for $i > m$, while $R_1, \ldots, R_N$ are ranks of the variables in the pooled sample Z. Usually, one considers a smoothed version of T_N of the form

$$T_N^* = \sum_{i=1}^{N} c_{iN} \Phi\left(\frac{R_i - 1/2}{N}\right) \tag{2.2}$$

in which a correction for continuity is inserted. It is known that if Φ is sufficiently smooth, then $T_N^* - T_N$ is negligible both under the null hypothesis and convergent alternatives. Therefore, we focus our considerations on T_N.

Introduce two auxiliary processes

$$\zeta_N = \sqrt{\frac{n}{N}}\varepsilon_m - \sqrt{\frac{m}{N}}\eta_n, \quad \xi_N = \sqrt{\frac{m}{N}}\varepsilon_m + \sqrt{\frac{n}{N}}\eta_n. \tag{2.3}$$

Observe that

$$T_N = \int_0^1 \Phi\big(\mathcal{H}_N(t)\big) d\zeta_N(t) + \mathcal{S}_N, \tag{2.4}$$

where

$$\mathcal{S}_N = \sqrt{\frac{mn}{N}} \int_0^1 \Phi\big(\mathcal{H}_N(t)\big) d[F_N H_N^{-1}(t) - G_N H_N^{-1}(t)]$$

and

$$\frac{1}{\sqrt{N}}\xi_N(t) = \mathcal{H}_N(t) - \frac{1}{N}[mF_N H_N^{-1}(t) + nG_N H_N^{-1}(t)]. \tag{2.5}$$

The above suggests an introduction of an auxiliary statistic

$$T_N^0 = \int_0^1 \Phi(t) d\zeta_N(t) = -\int_0^1 \zeta_N(t)\Phi'(t)dt, \tag{2.6}$$

which corresponds to the first term on the right-hand side of (2.4).

A crucial task in a study of the asymptotics for T_N is to show that T_N and T_N^0 are close together both for $F_N = G_N$ and for convergent alternatives. To this end the following simple lemma will be useful.

LEMMA 2.1. *Under the above notation, if $\Phi(t)$ is a vector of absolutely continuous functions on $[0, 1]$ and $\int_0^1 \Phi(t)dt = 0$, then*

$$\Delta_N = \int_0^1 \big[\Phi\big(\mathcal{H}_N(t)\big) - \Phi(t)\big] d\zeta_N(t) = \sum_{i=1}^{N} \int_{(i-1)/N}^{i/N} \Phi'(u)[\zeta_N(u) - \zeta_N(V_{(i)})]du, \tag{2.7}$$

where $V_{(i)}$ are order statistics of the transformed sample V.

P r o o f. From the absolute continuity of Φ we have for every $j \geqslant 1$

$$\begin{aligned}\Delta_{Nj} &= \int_0^1 \big[\varphi_j\big(\mathcal{H}_N(t)\big) - \varphi_j(t)\big] d\zeta_N(t) = \int_0^1 \int_t^{\mathcal{H}_N(t)} \varphi_j'(u)\, du\, d\zeta_N(t) \\ &= \iint_D \operatorname{sgn}\big(\mathcal{H}_N(t) - t\big)\varphi_j'(u)\, du\, d\zeta_N(t) \\ &= \sum_{i=1}^N \iint_{D_i} \operatorname{sgn}\big(\mathcal{H}_N(t) - t\big)\varphi_j'(u)\, du\, d\zeta_N(t),\end{aligned}$$

where D is the set shown on Fig. 1, and $D_i,\ i = 1, \ldots, N$, its corresponding parts (cf. Fig. 1).

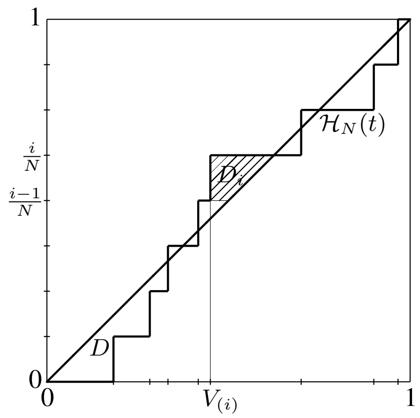

FIGURE 1

By the Fubini theorem we get

$$\Delta_{Nj} = \sum_{i=1}^N \int_{(i-1)/N}^{i/N} \varphi_j'(u) \int_{V_{(i)}}^u d\zeta_N(t)\, du = \sum_{i=1}^N \int_{(i-1)/N}^{i/N} \varphi_j'(u)[\zeta_N(u) - \zeta_N(V_{(i)})]\, du$$

which completes the proof. ■

REMARK 2.1. The above proof shows that the statement of Lemma 2.1 remains true if the process ζ_N in (2.7) is replaced by any deterministic (or random) function of bounded variation.

Up to now, the distribution function H_N was chosen arbitrarily. Observe that this choice does not influence values of the statistic T_N in (2.1) provided $\operatorname{supp} H_N \supset \operatorname{supp} F_N \cup \operatorname{supp} G_N$, where for a distribution function F, $\operatorname{supp} F$ denotes the support of the corresponding probability measure. However, the formula (2.5) suggests to take (and we shall do it throughout the paper) H_N of the form

$$H_N = \frac{m}{N} F_N + \frac{n}{N} G_N. \tag{2.8}$$

Then (2.5) simplifies to

$$\frac{1}{\sqrt{N}}\xi_N(t) = \mathcal{H}_N(t) - t \tag{2.9}$$

and $F_N H_N^{-1}$ and $G_N H_N^{-1}$ are absolutely continuous distribution functions on $[0,1]$. In the next two sections we shall investigate two important cases:

(A) The null distribution case, i.e. $F_N = G_N$ for every N. Then $H_N = F_N$.

(B) The alternative distribution case, i.e. $F_N \neq G_N$. Then we can write $F_N H_N^{-1}(t) - G_N H_N^{-1}(t) = \rho_N A_N(t)$, $t \in [0,1]$. This and (2.8) imply that the derivative $a_N(t)$ of $A_N(t)$ is a bounded function on $[0,1]$ for every N. So, we calibrate A_N by taking a_N such that

$$\int_0^1 a_N^2(t)dt = 1 \tag{2.10}$$

for every N and fitting $\rho_N > 0$, appropriately. Moreover, we assume that ρ_N satisfies

$$\rho_N \log^2 N \to 0, \qquad \frac{N\rho_N^2}{\log^2 N} \to \infty, \tag{2.11}$$

i.e. we consider alternatives convergent to the null hypothesis with a moderate rate.

Usually, it is assumed that sizes of both samples are proportional, i.e.

$$0 < \liminf_N \frac{m}{N} \leqslant \limsup_N \frac{m}{N} < 1. \tag{2.12}$$

However, we are going to consider a general case, i.e. arbitrary sequences $m, n \geqslant 1$ such that $m + n = N$.

The main tool in our proofs below is the method of strong approximations. So, consider two probability spaces and two sequences of Brownian bridges B'_m, B''_n defined on them and two sequences of uniform empirical processes e'_m, e''_n based on independent uniform random samples $U'_1, \ldots, U'_m$ and $U''_1, \ldots, U''_n$, respectively, such that the Komlós–Major–Tusnády inequality [14] (KMT inequality, for short) applies for them. Then $\varepsilon'_m = e'_m(F_N H_N^{-1})$ and $\eta''_n = e''_n(G_N H_N^{-1})$ are versions of ε_m and η_n. Let $(\Omega, \mathcal{B}, \mathbf{P})$ be the product of the probability spaces defined above. Now, $Z^{(2)} = \big(F_N^{-1}(U'_1), \ldots, F_N^{-1}(U'_m), G_N^{-1}(U''_1), \ldots, G_N^{-1}(U''_n)\big)$ defined on Ω is a version of Z, and $V^{(2)} = H_N(Z^{(2)})$ a version of V. The empirical distribution function of $V^{(2)}$ will be denoted by $\mathcal{H}_N^{(2)}$. Next, let us write

$$\zeta_N^{(1)} = \sqrt{\frac{n}{N}}\varepsilon'_m - \sqrt{\frac{m}{N}}\eta''_n, \quad \xi_N^{(2)} = \sqrt{\frac{m}{N}}\varepsilon'_m + \sqrt{\frac{n}{N}}\eta''_n \tag{2.13}$$

and

$$B_N^{(1)} = \sqrt{\frac{n}{N}}B'_m - \sqrt{\frac{m}{N}}B''_n, \quad B_N^{(2)} = \sqrt{\frac{m}{N}}B'_m + \sqrt{\frac{n}{N}}B''_n. \tag{2.14}$$

It follows that $B_N^{(1)}$, $B_N^{(2)}$ are independent Brownian bridges for every N, and $\zeta_N^{(1)}$, $\xi_N^{(2)}$ are versions of ζ_N and ξ_N for every N. Finally, let us put

$$T_N^{0(1)} = \int_0^1 \Phi(t)d\zeta_N^{(1)}(t) \quad \text{and} \quad T_N^{(1)} = \int_0^1 \Phi\big(\mathcal{H}_N^{(2)}(t)\big)d\zeta_N^{(1)}(t) + \mathcal{S}_N^{(2)}$$

for respective versions of T_N^0 and T_N, where in $\mathcal{S}_N^{(2)}$ we have inserted $\mathcal{H}_N^{(2)}$ in place of $\mathcal{H}_N$.

In the sequel we shall use letters $C,\ c,\ c_1,\ c_2$, etc. to denote positive constants possibly different in each case. For sequences $a_N,\ b_N$ of positive numbers we shall write $a_N \sim b_N$ if $cb_N \leqslant a_N \leqslant Cb_N$ for some positive constants c, C. Also, we shall write $|y|_r = \sqrt{y_1^2 + \ldots + y_r^2},\ r \geqslant 1$, for the r-th Euclidean norm of a vector $y = (y_1, y_2, \ldots)$. We restrict our attention to a finite number $d(N)$ of components of T_N including both $d(N) \equiv d \geqslant 1$ for all N and $d(N) \to \infty$ as $N \to \infty$.

3. THE NULL HYPOTHESIS

Throughout this section we shall consider the case (A), i.e. $F_N = G_N$. The following lemma is an immedite consequence of the KMT inequality.

LEMMA 3.1. *For any sequence x_N of positive numbers such that $mnx_N^2/(N\log^2 N) \to \infty$ as $N \to \infty$, we have for sufficiently large N*

$$\mathbf{P}\big(\sup_t |\zeta_N^{(1)}(t) - B_N^{(1)}(t)| \geqslant x_N\big) \leqslant 2L\exp\Big\{-lx_N\sqrt{\frac{mn}{2N}}\Big\} \tag{3.1}$$

and

$$\mathbf{P}\big(\sup_t |\xi_N^{(2)}(t) - B_N^{(2)}(t)| \geqslant x_N\big) \leqslant 2L\exp\Big\{-lx_N\sqrt{\frac{mn}{2N}}\Big\}, \tag{3.2}$$

where L, l are universal constants appearing in the KMT inequality.

Proof. Since $F_N = G_N$, we have $\varepsilon'_m = e'_m$ and $\eta''_n = e''_n$. So, applying the KMT inequality, the relation $\log N > \log m$ and the assumption of the lemma we get for sufficiently large N

$$\mathbf{P}\big(\sup_t |\varepsilon'_m(t) - B'_m(t)| \geqslant x_N\sqrt{n/N}\big) \leqslant L\exp\{-lx_N\sqrt{mn/2N}\}$$

and, similarly,

$$\mathbf{P}\big(\sup_t |\eta''_n(t) - B''_n(t)| \geqslant x_N\sqrt{m/N}\big) \leqslant L\exp\{-lx_N\sqrt{mn/2N}\}.$$

By the definition of $\zeta_N^{(1)}$ and $B_N^{(1)}$ and by the triangle inequality we obtain (3.1). Using additionally the relation $2\sqrt{mn} \leqslant N$ we get (3.2) similarly. This completes the proof. ■

Let $B(t)$ be a Brownian bridge on $[0,1]$. Then

$$\gamma = -\int_0^1 B(t)\Phi'(t)dt \tag{3.3}$$

is a Gaussian vector with mean zero and covariance matrix $\Gamma = \int_0^1 \Phi(t)\Phi^T(t)dt$.

From (2.4) and Lemma 3.1 we easily get an estimation of a distance between $T_N^{0(1)}$ and $\gamma_N^{(1)} = -\int_0^1 B_N^{(1)}(t)\Phi'(t)dt$.

THEOREM 3.1. *For any sequence x_N of positive numbers such that $mnx_N^2/(N\log^2 N) \to \infty$ as $N \to \infty$, we have for sufficiently large N*

$$\mathbf{P}\Big(|T_N^{0(1)} - \gamma_N^{(1)}|_{d(N)} \geqslant \psi\big(d(N)\big)x_N\Big) \leqslant C\exp\Big\{-cx_N\sqrt{\frac{mn}{N}}\Big\}, \tag{3.4}$$

where

$$\psi^2(r) = \sum_{j=1}^{r}\Big(\int_0^1 |\varphi_j'(t)|dt\Big)^2, \qquad r \geqslant 1. \tag{3.5}$$

Proof. From (2.6), (3.1) and (3.3) it follows that

$$\begin{aligned}
&\mathbf{P}\Big(|T_N^{0(1)} - \gamma_N^{(1)}|_{d(N)} \geqslant \psi\big(d(N)\big)x_N\Big) \\
&\qquad = \mathbf{P}\Big(\Big|\int_0^1 \big(\zeta_N^{(1)}(t) - B_N^{(1)}(t)\big)\Phi'(t)dt\Big|_{d(N)} \geqslant \psi\big(d(N)\big)x_N\Big) \\
&\qquad \leqslant \mathbf{P}\big(\sup_t |\zeta_N^{(1)}(t) - B_N^{(1)}(t)| \geqslant x_N\big) \leqslant C\exp\{-cx_N\sqrt{mn/N}\},
\end{aligned}$$

which completes the proof. ■

Observe that $\psi(r)$ is finite for every $r \geqslant 1$ due to the absolute continuity of $\Phi(t)$. In particular, for the Legendre polynomials $\psi(r) \sim r^{3/2}\log r$ while for the cosine system $\psi(r) \sim r^{3/2}$. The next theorem is crucial for the rest of this section.

THEOREM 3.2. *For every sequence x_N of positive numbers such that $x_N \to 0$ and*

$$\frac{Nx_N^2}{\log N} \to \infty, \qquad \frac{mnx_N^3}{\log^2 N} \to \infty \tag{3.6}$$

as $N \to \infty$, we have for sufficiently large N

$$\begin{aligned}
P\Big(|T_N - T_N^0|_{d(N)} \geqslant \psi\big(d(N)\big)\sqrt{Nx_N^3}\Big)& \\
\leqslant C\exp\{-cNx_N^2\} + C\exp\Big\{-c\sqrt{mnx_N^3}\Big\}&,
\end{aligned} \tag{3.7}$$

where $\psi(r)$ is given by (3.5).

Proof. Consider an event

$$E_N = \{\sup_t |\xi_N^{(2)}(t)| \geqslant x_N\sqrt{N}\}$$

on the space $(\Omega, \mathcal{B}, \mathbf{P})$ (defined in Section 2). Then from (3.2), (3.6) and the well-known inequality

$$\mathbf{P}\big(\sup_t |B_N^{(2)}(t)| \geqslant x\big) \leqslant 2\exp\{-2x^2\}, \tag{3.8}$$

which holds true for every $x > 0$, we get for sufficiently large N

$$\begin{aligned}\mathbf{P}(E_N) &\leqslant \mathbf{P}\bigg(\sup_t |\xi_N^{(2)}(t) - B_N^{(2)}(t)| \geqslant \frac{1}{4}x_N\sqrt{N}\bigg)\\ &\quad + \mathbf{P}\bigg(\sup_t |B_N^{(2)}(t)| \geqslant \frac{3}{4}x_N\sqrt{N}\bigg)\\ &\leqslant 2L\exp\{-lx_N\sqrt{mn/32}\} + C\exp\bigg\{-\frac{9}{8}Nx_N^2\bigg\}.\end{aligned} \tag{3.9}$$

From (2.9) it follows that for every $i = 1, 2, \ldots, N$ and $u \in [(i-1)/N, i/N]$ on the set E_N^c

$$\begin{aligned}|u - V_{(i)}^{(2)}| = \left|\frac{i}{N} - V_{(i)}^{(2)} + u - \frac{i}{N}\right| &\leqslant \frac{1}{\sqrt{N}}|\xi_N^{(2)}(V_{(i)}^{(2)})| + \frac{1}{N}\\ &\leqslant x_N + \frac{1}{N} \leqslant 2x_N,\end{aligned}$$

which means that on the set E_N^c we have

$$\begin{aligned}&\max_{1\leqslant i\leqslant N}\ \sup_{u\in[(i-1)/N, i/N]} |\zeta_N^{(1)}(u) - \zeta_N^{(1)}(V_{(i)}^{(2)})|\\ &\leqslant 2\sup_t |\zeta_N^{(1)}(t) - B_N^{(1)}(t)| + \sup_{0\leqslant t\leqslant 1-2x_N}\ \sup_{0\leqslant u\leqslant 2x_N} |B_N^{(1)}(t+u) - B_N^{(1)}(t)|.\end{aligned}$$

Hence and from (2.4), (2.5), (2.7) and (3.5) we get

$$\begin{aligned}&\mathbf{P}\Big(|T_N^{(1)} - T_N^{0(1)}|_{d(N)} \geqslant \psi\big(d(N)\big)\sqrt{Nx_N^3}\Big)\\ &\leqslant \mathbf{P}\Big(\max_{1\leqslant i\leqslant N}\ \sup_{u\in[(i-1)/N, i/N]} |\zeta_N^{(1)}(u) - \zeta_N^{(1)}(V_{(i)}^{(2)})| \geqslant \sqrt{Nx_N^3}\Big)\\ &\leqslant \mathbf{P}(E_N) + \mathbf{P}\Big(\sup_t |\zeta_N^{(1)}(t) - B_N^{(1)}(t)| \geqslant \sqrt{Nx_N^3}/3\Big)\\ &\quad + \mathbf{P}\Big(\sup_{0\leqslant t\leqslant 1-2x_N}\ \sup_{0\leqslant u\leqslant 2x_N} |B_N^{(1)}(t+u) - B_N^{(1)}(t)| \geqslant \sqrt{Nx_N^3}/3\Big).\end{aligned} \tag{3.10}$$

Now, (3.1), (3.9) and an analogue of Lemma 1.1.1 of Csörgő and Révész [2] for a Brownian bridge (see Lemma A in the Appendix) allow us to estimate the right-hand side of (3.10) by

$$c_1 \exp\Big\{-c_2\sqrt{mnx_N^2}\Big\} + c_3 \exp\{-c_4 N x_N^2\} \\ + c_5 \exp\Big\{-c_6\sqrt{mnx_N^3}\Big\} + \frac{c_7}{x_N} \exp\{-c_8 N x_N^2\}.$$

This, the relation $1/x_N = o(\sqrt{N})$ and (3.6) imply (3.7). ∎

From Theorems 3.1 and 3.2 we get the first of our main results.

THEOREM 3.3. *If*

$$\nu_N^2 = \frac{mn}{N\psi^2\big(d(N)\big)\log^2 N} \to \infty, \qquad \frac{N}{\psi^4\big(d(N)\big)\log^3 N} \to \infty \tag{3.11}$$

as $N \to \infty$, *then*

$$|T_N^{0(1)} - \gamma_N^{(1)}|_{d(N)} \xrightarrow{\mathbf{P}} 0 \quad \text{and} \quad |T_N^{(1)} - \gamma_N^{(1)}|_{d(N)} \xrightarrow{\mathbf{P}} 0. \tag{3.12}$$

Consequently, finite-dimensional distributions of T_N *and* T_N^0 *converge weakly to that of* γ. *In particular, if* Φ *is an orthonormal system, then* $|T_N|_d^2 \xrightarrow{\mathcal{D}} \chi_d^2$ *for each fixed* $d \geqslant 1$, *where* χ_d^2 *denotes a random variable with chi-square distribution with* d *degrees of freedom.*

P r o o f. Let $\delta > 0$ be arbitrary. Applying (3.4) for $x_N = \delta/\psi\big(d(N)\big)$ we get

$$\mathbf{P}(|T_N^{0(1)} - \gamma_N^{(1)}|_{d(N)} \geqslant \delta) \leqslant C\exp\{-c\delta\nu_N \log N\}.$$

Putting $x_N^3 = \delta^2/N\psi^2\big(d(N)\big)$ into (3.7) we see that (3.6) is fulfilled and

$$\mathbf{P}(|T_N^{(1)} - T_N^{0(1)}|_{d(N)} \geqslant \delta) \leqslant C\exp\{-c\delta\Upsilon_N \log N\},$$

where $\Upsilon_N^3 = \min\big\{\nu_N^3,\ \delta N/\big(\psi^4\big(d(N)\big)\log^3 N\big)\big\}$. Combining the above two estimates gives (3.12). ∎

Using Theorems 3.1 and 3.2 one can also derive a moderate deviation theorem for the statistics T_N and T_N^0. First, we focus on the auxiliary statistic T_N^0.

LEMMA 3.2. *For every* $0 < \vartheta < 1$ *and every sequence* x_N *of positive numbers such that*

$$\frac{Nx_N^2}{\lambda_N d(N)\log N} \to \infty, \qquad \frac{Nx_N^{1-\vartheta}\psi\big(d(N)\big)}{\sqrt{mn}} \to 0$$

as $N \to \infty$, *we have*

$$(3.13)\quad P(|T_N^0|^2_{d(N)} \geqslant Nx_N^2) = \exp\Big\{ -\frac{1}{2\lambda_N} Nx_N^2 + O\Big(\frac{Nx_N^{2+\vartheta}}{\lambda_N}\Big) + O\big(d(N)\log Nx_N^2\big)\Big\},$$

where λ_N *is the largest eigenvalue of the* $d(N)$*-dimensional truncation of* Γ.

P r o o f. From (3.4), the triangle inequality and the expansion of the tail probability of a quadratic form of a Gaussian vector (cf., e.g., Gregory [6]) we have

$$\begin{aligned}
&\mathbf{P}(|T_N^{0(1)}|^2_{d(N)} \geqslant Nx_N^2)\\
&\leqslant \mathbf{P}\big(|\gamma_N^{(1)}|_{d(N)} \geqslant (1-x_N^{\vartheta})x_N\sqrt{N}\big) + \mathbf{P}\big(|T_N^{0(1)} - \gamma_N^{(1)}|_{d(N)} \geqslant x_N^{1+\vartheta}\sqrt{N}\big)\\
&\leqslant \exp\Big\{ -\frac{1}{2\lambda_N} Nx_N^2 + O\Big(\frac{Nx_N^{2+\vartheta}}{\lambda_N}\Big) + O\big(d(N)\log Nx_N^2\big)\Big\}\\
&\quad + C\exp\big\{ -cx_N^{1+\vartheta}\sqrt{mn}/\psi\big(d(N)\big)\big\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbf{P}(|T_N^{0(1)}|^2_{d(N)} \geqslant Nx_N^2)\\
&\qquad \geqslant \exp\Big\{ -\frac{1}{2\lambda_N} Nx_N^2 + O\Big(\frac{Nx_N^{2+\vartheta}}{\lambda_N}\Big) + O\big(d(N)\log Nx_N^2\big)\Big\}\\
&\qquad\quad - C\exp\big\{ -cx_N^{1+\vartheta}\sqrt{mn}/\psi\big(d(N)\big)\big\}.
\end{aligned}$$

Using now the assumption of the lemma, we get (3.13). ∎

Combining (3.7) and (3.13) we obtain a moderate deviation result for T_N.

THEOREM 3.4. *For every* $0 < \vartheta < 1/2$ *and every sequence* x_N *of positive numbers such that* $x_N \to 0$ *and*

$$(3.14)\quad \frac{Nx_N^2}{\lambda_N d(N)\log N} \to \infty, \qquad \frac{Nx_N^{1-\vartheta}\psi\big(d(N)\big)}{\sqrt{mn}} \to 0, \qquad \frac{\psi^2\big(d(N)\big)}{\lambda_N^{3/2}x_N^{2\vartheta-1}} \to 0$$

as $N \to \infty$, *we have*

$$(3.15)\quad P(|T_N|^2_{d(N)} \geqslant Nx_N^2) = \exp\Big\{ -\frac{1}{2\lambda_N} Nx_N^2 + O\Big(\frac{Nx_N^{2+\vartheta}}{\lambda_N}\Big) + O\big(d(N)\log Nx_N^2\big)\Big\},$$

where λ_N *is the largest eigenvalue of the* $d(N)$*-dimensional truncation of* Γ. *In particular*, *when* Φ *is an orthonormal system*, *then* $\lambda_N = 1$.

P r o o f. By (3.14) the assumptions of Theorem 3.2 and Lemma 3.2 are satisfied, and from Theorem 3.2, Lemma 3.2 and the triangle inequality we get

$$
\begin{aligned}
&P(|T_N|^2_{d(N)} \geqslant N x_N^2)\\
&\leqslant P\big(|T_N^0|^2_{d(N)} \geqslant (1-x_N^{\vartheta})^2 N x_N^2\big) + P\big(|T_N - T_N^0|_{d(N)} \geqslant x_N^{1+\vartheta}\sqrt{N}\big)\\
&\leqslant \exp\bigg\{-\frac{1}{2\lambda_N} N x_N^2 (1-x_N^{\vartheta})^2 + O\bigg(\frac{N x_N^{2+\vartheta}}{\lambda_N}\bigg) + O\big(d(N)\log N x_N^2\big)\bigg\}\\
&\quad + C\exp\big\{-cN x_N^{4(1+\vartheta)/3}/\psi_1^{4/3}\big(d(N)\big)\big\}\\
&\leqslant \exp\bigg\{-\frac{1}{2\lambda_N} N x_N^2 + O\bigg(\frac{N x_N^{2+\vartheta}}{\lambda_N}\bigg) + O\big(d(N)\log N x_N^2\big)\bigg\},
\end{aligned}
$$

where the last inequality holds true due to the third condition in (3.14). Similarly,

$$
\begin{aligned}
&P(|T_N|^2_{d(N)} \geqslant N x_N^2)\\
&\quad\geqslant P\big(|T_N^0|^2_{d(N)} \geqslant (1+x_N^{\vartheta})^2 N x_N^2\big) - P\big(|T_N - T_N^0|_{d(N)} \geqslant x_N^{1+\vartheta}\sqrt{N}\big)\\
&\quad\geqslant \exp\bigg\{-\frac{1}{2\lambda_N} N x_N^2 + O\bigg(\frac{N x_N^{2+\vartheta}}{\lambda_N}\bigg) + O\big(d(N)\log N x_N^2\big)\bigg\},
\end{aligned}
$$

which completes the proof of (3.15). ■

A particular case of Theorems 3.3 and 3.4, useful for applications, is stated below. It shows that (2.12) is not necessary for ensuring (3.12) and the expansion (3.15) of moderate deviation probabilities, too.

COROLLARY 3.1. *If* Φ *is an orthonormal system of absolutely continuous functions*, $d(N)\psi\big(d(N)\big) = O(\log^{\beta} N)$ *as* $N \to \infty$ *for some* $\beta \geqslant 0$, *and* $mn \sim N^{1+\alpha}$ *for some* $\alpha \in (0,1]$, *then* (3.12) *holds.*

If, in addition, $mn \sim N^{3/2+\alpha}$ *for some* $\alpha \in (0,1/2]$ *and* $x_N \sim N^{-\tau}$ *for* $\tau \in (0,1/2)$, *then, given* $0 < \vartheta < 1/2$, (3.15) *holds for* $\tau > (1-2\alpha)/(4-4\vartheta)$.

Moreover, if $\alpha \in [1/6,1/2]$, *then choosing* ϑ *sufficiently close to* $1/2$ *we have for every* $\tau \in (1/3,1/2)$

$$
P(|T_N|^2_{d(N)} \geqslant N x_N^2) = \exp\bigg\{-\frac{1}{2} N x_N^2 + o\big(x_N\sqrt{N}\big)\bigg\}. \tag{3.16}
$$

The proof of Theorem 3.4 is carried out by an application of Theorem 3.2 and Lemma 2.1. Assuming the boundedness of Φ' and applying the Lemma of Kallenberg [13], Ducharme and Ledwina [3] proved, by elementary but more involved calculations, a stronger result. Namely, they were able to get (3.16) for $\tau \in (1/4,1/2)$, speaking in terms of Corollary 3.1. We prove a theorem going towards such a result using the independence of $B_N^{(1)}$ and $B_N^{(2)}$ (cf. (2.14)) and some auxiliary results in place of Theorem 3.2 (see also Remark 5.1).

THEOREM 3.5. *Suppose* $\varphi_j' \in L_p[0,1]$ *for* $j = 1, 2, \ldots$ *and some* $p \in (1, \infty]$. *Then for every* $0 < \vartheta < 3/4$ *and every sequence* x_N *of positive numbers such that*

$$x_N^{(3/4)-\vartheta} \psi\big(d(N)\big) \to 0$$

and

$$\frac{N x_N^2}{\lambda_N d(N) \log N} \to \infty, \qquad \frac{N x_N^{1-\vartheta} \psi\big(d(N)\big) \sqrt{\log N}}{\sqrt{mn}} \to 0, \qquad \frac{\omega^2\big(d(N)\big)}{\lambda_N x_N^{2\vartheta - 2 + 1/p}} \to 0$$

as $N \to \infty$, *the expansion* (3.15) *holds*, *where we put* $1/p = 0$ *if* $p = \infty$, λ_N *is the largest eigenvalue of the* $d(N)$*-dimensional truncation of* Γ *and* $\omega^2(r) = \psi^2(r) + 5^{1-1/p} \max_{1 \leqslant j \leqslant r} \|\varphi_j'\|_p \sum_{j=1}^{r} \|\varphi_j'\|_1$.

The proof of Theorem 3.5 is given in Section 5. In the following corollary to Theorem 3.5 we present a counterpart of Corollary 3.1 under stronger assumptions on Φ.

COROLLARY 3.2. *If* Φ *is an orthonormal system of absolutely continuous functions such that* $\varphi_j' \in L_p[0,1]$ *for* $j = 1, 2, \ldots$ *and some* $p \in (1, \infty]$, $d(N)\omega\big(d(N)\big) = O(\log^\beta N)$ *as* $N \to \infty$ *for some* $\beta \geqslant 0$, $mn \sim N^{3/2+\alpha}$ *for some* $\alpha \in (0, 1/2]$ *and* $x_N \sim N^{-\tau}$ *for some* $\tau \in (0, 1/2)$, *then*, *given* $0 < \vartheta < \min\big(3/4, 1 - 1/(2p)\big)$, *the expansion* (3.15) *holds for* $\tau > (1 - 2\alpha)/(4 - 4\vartheta)$.

Moreover, *if* $\tau_0 \in \big[\max\big(2/7, p/(4p-1)\big), 1/3\big]$ *is chosen arbitrarily*, *then for every* $\alpha \in [3/2 - 4\tau_0, 1/2]$ *and every* $\tau \in (\tau_0, 1/2)$ *the expansion* (3.16) *holds.*

Observe that the second statement of Corollary 3.2 follows from the first one by choosing ϑ sufficiently close to $(1 - 2\tau_0)/(2\tau_0)$.

4. CONVERGENT ALTERNATIVES

Throughout this section we shall study the case (B), i.e. we shall assume that $F_N \neq G_N$ and that (2.10) and (2.11) are satisfied. Again, applying the KMT inequality we easily obtain the following lemma.

LEMMA 4.1. *For any sequence* δ_N *of positive numbers such that* $\delta_N \leqslant 1$ *and* $N \rho_N \delta_N^2 / \log N \to \infty$ *we have for sufficiently large* N

$$\mathbf{P}\bigg(\sup_t |\zeta_N^{(1)}(t) - B_N^{(1)}(t)| \geqslant \rho_N \delta_N \sqrt{\frac{N^3}{mn}}\bigg) \leqslant C \exp\{-cN\rho_N\delta_N^2\} \tag{4.1}$$

and

$$\mathbf{P}\bigg(\sup_t |\xi_N^{(2)}(t) - B_N^{(2)}(t)| \geqslant \rho_N \delta_N \sqrt{\frac{N^3}{mn}}\bigg) \leqslant C \exp\{-cN\rho_N\delta_N^2\}. \tag{4.2}$$

P r o o f. Since both inequalities can be proved similarly, we restrict ourselves only to (4.1). By (2.8) and (2.10) we have

$$\sup_t |F_N H_N^{-1}(t) - t| \leqslant \rho_N \sup_t \Big| \int_0^t a_N(u)du \Big| \leqslant \rho_N.$$

Hence, by (2.11), Lemma A of the Appendix, the KMT inequality and the relation

$$\begin{aligned} &\sup_t |\varepsilon'_m(t) - B'_m(t)| \\ &\qquad \leqslant \sup_t |e'_m(t) - B'_m(t)| + \sup_{0\leqslant t\leqslant 1-\rho_N} \sup_{0\leqslant u\leqslant \rho_N} |B'_m(t+u) - B'_m(t)|, \end{aligned}$$

we get for sufficiently large N

$$\begin{aligned} &\mathbf{P}\big(\sup_t |\varepsilon'_m(t) - B'_m(t)| \geqslant N\rho_N\delta_N/\sqrt{m}\big) \\ &\leqslant c_1 \exp\{-c_2 N\rho_N\delta_N\} + \frac{c_3}{\rho_N}\exp\{-c_4 N\rho_N\delta_N^2\} \leqslant C\exp\{-cN\rho_N\delta_N^2\} \end{aligned}$$

using the assumption of the lemma and $\delta_N \leqslant 1$. Similarly, we obtain the estimate for η''_n and B''_n. Combining these estimates, using (2.13), (2.14) and the relation $N/\sqrt{mn} = \sqrt{n/m} + \sqrt{m/n}$ we get (4.1). This completes the proof. ■

THEOREM 4.1. *For any ρ_N satisfying* (2.11) *and any $\kappa \in (0, 1/2]$ we have*

$$\mathbf{P}\bigg(|T_N^{0(1)} - \gamma_N^{(1)}|_{d(N)} \geqslant \psi\big(d(N)\big)\rho_N^\kappa \log N \sqrt{\frac{N^2}{mn}}\bigg) \leqslant C\exp\{-c\rho_N^{2\kappa-1}\log^2 N\}, \tag{4.3}$$

where γ is a Gaussian vector defined in (3.3) *and T_N^0 is given by* (2.6)*. In consequence*, *if*

$$\psi\big(d(N)\big)\rho_N^\kappa \log N\sqrt{\frac{N^2}{mn}} \to 0 \quad \textit{as } N \to \infty, \tag{4.4}$$

then

$$|T_N^{0(1)} - \gamma_N^{(1)}|_{d(N)} \xrightarrow{\mathbf{P}} 0.$$

P r o o f. We proceed similarly as in the proof of (3.4). To this end, we set $\delta_N = \rho_N^{\kappa-1}\log N/\sqrt{N}$ in (4.1). Then (4.3) follows from (4.1). ■

REMARK 4.1. Theorem 4.1 applied to $\kappa = 1/2$ and (2.11) imply that finite-dimensional distributions of T_N^0 converge weakly to the corresponding finite-dimensional distributions of γ provided $mn \sim N^{3/2+\alpha}$ for some $\alpha \in (0, 1/2]$ and $\rho_N = o(N^{\alpha-1/2}/\log^2 N)$. Observe that again (2.12) is not needed for weak convergence of T_N^0.

The next theorem is an analogue of Theorem 3.2 but stated for the case (B).

THEOREM 4.2. *For any positive sequence $\delta_N \to 0$ such that*

$$\frac{N\rho_N\delta_N^2}{\log N} \to \infty \quad \textit{and} \quad \frac{N^2\rho_N\delta_N^2}{mn} \to 0$$

as $N \to \infty$, we have for sufficiently large N

$$P\bigg(\Big|\int_0^1 \Phi\big(\mathcal{H}_N(t)\big)d\zeta_N(t) - T_N^0\Big|_{d(N)} \geqslant \psi\big(d(N)\big)\rho_N\delta_N/\sqrt{\frac{N^3}{mn}}\bigg) \leqslant C\exp\{-cN\rho_N\delta_N^2\} + C\exp\bigg\{-cN^2\delta_N\sqrt{\frac{\rho_N^3}{mn}}\bigg\}, \tag{4.5}$$

where $\psi(r)$ is given by (3.5).

Proof. We proceed similarly as in the proof of Theorem 3.2. Consider an event on $(\Omega, \mathcal{B}, \mathbf{P})$:

$$E_N = \big\{\sup_t |\xi_N^{(2)}(t)| \geqslant \delta_N\sqrt{N^3\rho_N/mn}\big\}. \tag{4.6}$$

Then from (4.2), (3.8) and the condition $\rho_N \to 0$ we get for sufficiently large N

$$\begin{aligned} \mathbf{P}(E_N) &\leqslant \mathbf{P}\big(\sup_t |\xi_N^{(2)}(t) - B_N^{(2)}(t)| \geqslant \rho_N\delta_N\sqrt{N^3/mn}\big) \\ &\quad + \mathbf{P}\big(\sup_t |B_N^{(2)}(t)| \geqslant (1-\sqrt{\rho_N})\delta_N\sqrt{N^3\rho_N/mn}\big) \\ &\leqslant c_1\exp\{-c_2N\rho_N\delta_N^2\} + c_3\exp\{-N^3\rho_N\delta_N^2/mn\} \\ &\leqslant C\exp\{-cN\rho_N\delta_N^2\}. \end{aligned} \tag{4.7}$$

From the equalities (2.8) and (2.9) it follows that for every $i = 1, 2, \ldots, N$ and $u \in [(i-1)/N, i/N]$ on the set E_N^c

$$|u - V_{(i)}^{(2)}| \leqslant \delta_N\sqrt{N^2\rho_N/mn} + 1/N = v_N, \tag{4.8}$$

which means that on the set E_N^c

$$\begin{aligned} &\max_{1\leqslant i\leqslant N} \sup_{u\in[(i-1)/N, i/N]} |\zeta_N^{(1)}(u) - \zeta_N^{(1)}(V_{(i)}^{(2)})| \\ &\leqslant 2\sup_{0\leqslant t\leqslant 1} |\zeta_N^{(1)}(t) - B_N^{(1)}(t)| + \sup_{0\leqslant t\leqslant 1-v_N} \sup_{0\leqslant u\leqslant v_N} |B_N^{(1)}(t+u) - B_N^{(1)}(t)|. \end{aligned}$$

Hence from Lemma 2.1 we obtain

$$(4.9)\quad \mathbf{P}\Big(\big|\int_0^1 \Phi\big(\mathcal{H}_N^{(2)}(t)\big)d\zeta_N^{(1)}(t) - T_N^{0(1)}\big|_{d(N)} \geqslant \psi\big(d(N)\big)\rho_N\delta_N\sqrt{N^3/mn}\Big)$$

$$\leqslant \mathbf{P}\big(\max_{1\leqslant i\leqslant N}\sup_{u\in[(i-1)/N,i/N]}|\zeta_N^{(1)}(u)-\zeta_N^{(1)}(V_{(i)}^{(2)})|\geqslant \rho_N\delta_N\sqrt{N^3/mn}\big)$$

$$\leqslant \mathbf{P}(E_N)+\mathbf{P}\left(\sup_t|\zeta_N^{(1)}(t)-B_N^{(1)}(t)|\geqslant \frac{1}{3}\rho_N\delta_N\sqrt{N^3/mn}\right)$$

$$+\mathbf{P}\left(\sup_{0\leqslant t\leqslant 1-v_N}\sup_{0\leqslant u\leqslant v_N}|B_N^{(1)}(t+u)-B_N^{(1)}(t)|\geqslant \frac{1}{3}\rho_N\delta_N\sqrt{N^3/mn}\right).$$

From (4.1), (4.7), the relation $mn/N^2 \leqslant 1/4$ and Lemma A of the Appendix the right-hand side of (4.9) can be estimated by

$$c_1\exp\{-c_2N\rho_N\delta_N^2\}+\frac{c_3}{v_N}\exp\Big\{-c_4N^2\delta_N\sqrt{\rho_N^3/mn}\Big\}$$

which proves (4.5). ∎

Now, denote by

$$(4.10)\qquad s_N=\sqrt{\frac{mn}{N}}\Big(\int_0^1\Phi(t)d[F_NH_N^{-1}(t)-G_NH_N^{-1}(t)]\Big)$$

$$=\rho_N\sqrt{\frac{mn}{N}}\int_0^1\Phi(t)dA_N(t)$$

an asymptotic expectation of the statistic T_N corresponding to its empirical counterpart

$$(4.11)\qquad \mathcal{S}_N=\rho_N\sqrt{\frac{mn}{N}}\Big(\int_0^1\Phi\big(\mathcal{H}_N(t)\big)dA_N(t)\Big)$$

(cf. (2.4)). Since A_N is absolutely continuous for every N, by Lemma 2.1 (cf. Remark 2.1) we have

$$(4.12)\quad \mathcal{S}_N-s_N=\rho_N\sqrt{\frac{mn}{N}}\sum_{i=1}^N\int_{(i-1)/N}^{i/N}\Phi'(u)\big(A_N(u)-A_N(V_{(i)})\big)du.$$

Using (4.12) we can prove that $\mathcal{S}_N$ and s_N are close together. Recall that each function a_N is bounded but may change with N.

LEMMA 4.2. *Suppose for some* $p\in[2,\infty]$

$$(4.13)\qquad \sup_N\|a_N\|_p=M<\infty.$$

Then for any positive sequence $\delta_N \to 0$ *such that* $N\rho_N\delta_N^2/\log N \to \infty$ *as* $N \to \infty$, *we have for sufficiently large* N

$$(4.14)\quad P\Big(|\mathcal{S}_N - s_N|_{d(N)} \geqslant 2M\psi\big(d(N)\big)\rho_N v_N^{1-1/p}\sqrt{\frac{mn}{N}}\Big) \leqslant C\exp\{-cN\rho_N\delta_N^2\},$$

where v_N *is given by* (4.8) *and for* $p = \infty$ *we put* $1/p = 0$.

P r o o f. By the Hölder inequality we get

$$|A_N(u) - A_N(V_{(i)})| \leqslant M|u - V_{(i)}|^{1-1/p}.$$

Hence and from (4.6), (4.8) and (4.12) we obtain for sufficiently large N

$$\begin{aligned} P\Big(|\mathcal{S}_N - s_N|_{d(N)} &\geqslant 2M\psi\big(d(N)\big)\rho_N v_N^{1-1/p}\sqrt{mn/N}\Big) \\ &\leqslant P\big(\max_{1\leqslant i\leqslant N}\sup_{u\in[(i-1)/N,i/N]}|A_N(u) - A_N(V_{(i)})| \geqslant 2Mv_N^{1-1/p}\big) \\ &\leqslant \mathbf{P}\big(\max_{1\leqslant i\leqslant N}\sup_{u\in[(i-1)/N,i/N]}|u - V_{(i)}^{(2)}| \geqslant 2^{p/(p-1)}v_N\big) \leqslant \mathbf{P}(E_N), \end{aligned}$$

where E_N is defined in (4.6). This proves (4.14). ■

Obviously, (4.13) holds trivially for $p = 2$ with $M = 1$ due to (2.10).

As $T_N = \int_0^1 \Phi\big(\mathcal{H}_N(t)\big)d\zeta_N(t) + \mathcal{S}_N$, combining (4.3) and (4.14) we obtain the main result of this section.

THEOREM 4.3. *If* (4.13) *holds for some* $p \in [2,\infty]$, *then for any* ρ_N *satisfying* (2.11) *and* $\kappa \in (0, 1/2]$ *such that*

$$(4.15)\quad \frac{\rho_N^{1-2\kappa}}{\log^2 N}\frac{mn}{N} \to \infty \quad \textit{and} \quad \Big(\rho_N\frac{mn}{N^2}\Big)\Big(\frac{\rho_N^{1-2\kappa}}{\log^2 N}\frac{mn}{N}\Big)^{1/p} \to 0$$

as $N \to \infty$, *we have*

$$(4.16)\quad P\Big(|T_N - s_N - T_N^0|_{d(N)} \geqslant \psi\big(d(N)\big)\rho_N^\kappa \log N\sqrt{\frac{N^2}{mn}}\Big) \leqslant C\exp\{-c\rho_N^{\kappa-1/2}\log^2 N\}.$$

If, *additionally*, (4.4) *holds*, *then*

$$(4.17)\quad |T_N - s_N - T_N^0|_{d(N)} \xrightarrow{P} 0.$$

P r o o f. Set $\delta_N = \rho_N^{\kappa-1} \log N/\sqrt{N}$ and

$$w_N = 2M\rho_N^{1-\kappa} v_N^{1-1/p} mn/(\sqrt{N^3}\log N),$$

where v_N is given by (4.8). Then $\delta_N \to 0$ and $w_N \to 0$ by (4.15), and the assumptions of Theorem 4.2 and Lemma 4.2 are satisfied. So, using the form of T_N (cf. (2.5)) and the triangle inequality we get

$$\begin{aligned}
&P\Big(|T_N - s_N - T_N^0|_{d(N)} \geqslant \psi\big(d(N)\big)N\rho_N^{\kappa}\log N/\sqrt{mn}\Big)\\
&\leqslant P\Big(\Big|\int_0^1 \Phi\big(\mathcal{H}(t)\big)d\zeta_N(t) - T_N^0\Big|_{d(N)} \geqslant (1-w_N)\psi\big(d(N)\big)N\rho_N^{\kappa}\log N/\sqrt{mn}\Big)\\
&\quad + P\Big(|\mathcal{S}_N - s_N|_{d(N)} \geqslant 2M\psi\big(d(N)\big)\rho_N v_N^{1-1/p}\sqrt{mn/N}\Big)\\
&\leqslant c_1\exp\{-c_2\rho_N^{2\kappa-1}\log^2 N\} + c_3\exp\big\{-c_4\rho_N^{\kappa+1/2}\log N\sqrt{N^3/mn}\big\},
\end{aligned}$$

which by (2.11) immediately implies (4.16). ∎

A straightforward application of Theorems 4.1 and 4.3 gives

$$|T_N^{(1)} - s_N - \gamma_N^{(1)}|_{d(N)} \xrightarrow{\mathbf{P}} 0 \tag{4.18}$$

provided (4.13) for some $p \in [2,\infty]$, (4.4) and (4.15) hold for some $\kappa \in (0,1/2]$. This, in turn, implies that finite-dimensional distributions of $T_N - s_N$ converge weakly to that of γ.

Among possible applications of Theorem 4.3 and (4.18) the following two corollaries seem to be interesting.

COROLLARY 4.1. *If* (4.13) *holds for some* $p \in [2,\infty]$ *and* $mn \sim N^{3/2+\alpha}$ *for some* $\alpha \in (0,1/2]$, *then* (4.18) *holds provided*

$$\psi^2\big(d(N)\big)N^{1/2-\alpha}\rho_N\log^2 N \to 0 \quad \textit{and} \quad N^{1/p}\rho_N \to 0.$$

In particular, for $p = \infty$ *and* $\alpha = 1/2$ (*i.e. when* (2.12) *is fulfilled*) *and for bounded* $d(N)$ *the last condition reduces to* (2.11).

In the next corollary we consider the opposite case, i.e. when (4.13) does not hold for any $p > 2$.

COROLLARY 4.2. *Suppose there exists* $0 < \delta < 1$ *such that* $N\rho_N^{3-\delta} \to 0$. *If*

$$mn \sim N^{3/2+\alpha}$$

with

$$\alpha \in \left(\frac{3-3\delta}{6-2\delta}, \frac{1}{2}\right] \quad \textit{and} \quad \psi^2\big(d(N)\big)(\log^2 N)N^{(3-3\delta)/(6-2\delta)} \to 0,$$

then (4.18) *holds.*

Corollary 4.1 follows from (4.16) by taking $\kappa = 1/2$ while Corollary 4.2 by taking $\kappa = \delta/2$.

Conditions (2.10) and (2.11) typically imply $|s_N|_{d(N)} \to \infty$. For such a case (4.16) can be formulated in a more convenient form.

COROLLARY 4.3. *If* (4.4) *and* (4.15) *are satisfied,* $|s_N|_{d(N)} \to \infty$ *and* $(s_N^T \Gamma s_N)/|s_N|^2_{d(N)} \to \sigma^2$, $\sigma > 0$, *where the product* $s_N^T \Gamma s_N$ *is calculated for* $d(N)$*-dimensional truncation of* s_N *and* Γ, *then*

$$\frac{|T_N|^2_{d(N)} - |s_N|^2_{d(N)}}{2|s_N|_{d(N)}} \xrightarrow{\mathcal{D}} N(0, \sigma).$$

Note that $\sigma = 1$ if Φ is an orthonormal system.

P r o o f. We have

$$\begin{aligned}(4.19)\quad \frac{|T_N|^2_{d(N)} - |s_N|^2_{d(N)}}{2|s_N|_{d(N)}} &= \frac{|(T_N - s_N) + s_N|^2_{d(N)} - |s_N|^2_{d(N)}}{2|s_N|_{d(N)}} \\ &= \frac{|T_N - s_N|^2_{d(N)}}{2|s_N|_{d(N)}} + \frac{s_N}{|s_N|_{d(N)}} \circ (T_N - s_N - T_N^0) \\ &\quad + \frac{s_N}{|s_N|_{d(N)}} \circ (T_N^0 - \gamma) + \frac{s_N}{|s_N|_{d(N)}} \circ \gamma,\end{aligned}$$

where $\circ$ denotes the Euclidean scalar multiplication in $R^{d(N)}$. The first three terms on the right-hand side of (4.19) converge in probability to zero due to (4.17) and (4.3), the boundedness of $s_N/|s_N|_{d(N)}$ and the assumption $|s_N|_{d(N)} \to \infty$ while the last converges to $N(0, \sigma)$ by assumption. ∎

5. PROOF OF THEOREM 3.5

We start with some auxiliary lemmas.

LEMMA 5.1. *Given* $N \geqslant 1$. *Let* $b_1, \ldots, b_N$ *be real numbers such that* $b_i + i/N \in [0, 1]$ *for every* $i = 1, \ldots, N$ *and*

$$(5.1)\qquad \max_{1 \leqslant i \leqslant N} |b_i| = b < \frac{1}{2}$$

and let $k_1, \ldots, k_N$ *be real numbers satisfying the condition*

$$(5.2)\qquad \max_{1 \leqslant r \leqslant N} \sum_{\{1 \leqslant i \leqslant N : |i - r| \leqslant 2Nb\}} |k_i| \leqslant K_0 b^{1/q}$$

for some $q \in [1, \infty]$ *and* $K_0 > 0$. *Then*

$$(5.3)\qquad E\left(\sum_{i=1}^N k_i \left[B\left(\frac{i}{N}\right) - B\left(\frac{i}{N} + b_i\right)\right]\right)^2 \leqslant (K_1^2 + K_0 K_1) b^{1+1/q},$$

where $B(t)$, $t \in [0, 1]$, *is the Brownian bridge and* $K_1 = \sum_{i=1}^N |k_i|$.

P r o o f. We have

$$E\bigg(\sum_{i=1}^{N} k_i\Big[B\Big(\frac{i}{N}\Big)-B\Big(\frac{i}{N}+b_i\Big)\Big]\bigg)^2\leqslant$$

$$\bigg|\sum_{r=1}^{N}\sum_{\{i:|i-r|\leqslant 2Nb\}} k_ik_rE\Big[B\Big(\frac{i}{N}\Big)-B\Big(\frac{i}{N}+b_i\Big)\Big]\Big[B\Big(\frac{r}{N}\Big)-B\Big(\frac{r}{N}+b_r\Big)\Big]\bigg|$$

$$+\bigg|\sum_{|i-r|>2Nb} k_ik_rE\Big[B\Big(\frac{i}{N}\Big)-B\Big(\frac{i}{N}+b_i\Big)\Big]\Big[B\Big(\frac{r}{N}\Big)-B\Big(\frac{r}{N}+b_r\Big)\Big]\bigg|.$$

By the Schwarz inequality, the relation $E\big(B(u)-B(t)\big)^2\leqslant|u-t|$, (5.1) and (5.2), the first term can be estimated by

$$b\sum_{r=1}^{N}\sum_{\{i:|i-r|\leqslant 2Nb\}}|k_i|\,|k_r|\leqslant K_0K_1b^{1+1/q}.$$

Since for the second sum we have $|i/N-r/N|>2b$ and both b_i, b_r are bounded by b, the corresponding intervals are disjoint and the covariance equals $-b_ib_r$. Using (5.1) we estimate the second term by $b^2\sum_{|i-r|>2Nb}|k_i||k_r|\leqslant K_1^2b^{1+1/q}$. Combining both estimates we get (5.3). ∎

Let us write $h(y)=\max\{\min(y,1),0\},\ y\in R$. Let $B(t)$, $B'(t)$ be independent Brownian bridges and $B_N^*(t)=\sqrt{N}\big[t-h\big(t-N^{-1/2}B'(t)\big)\big]$ be a truncation of B'. Then B and B^* are independent processes.

LEMMA 5.2. *Suppose $k_1,\dots,k_N$ are real numbers satisfying* (5.2) *for some $b\in[1/N,1/2)$ and set $Q^2=K_1^2+K_0K_1$. Then for any $x>0$*

$$P\bigg(\bigg|\sum_{i=1}^{N}k_i\Big[B\Big(\frac{i}{N}\Big)-B\Big(\frac{i}{N}-\frac{1}{\sqrt{N}}B_N^*\Big(\frac{i}{N}\Big)\Big)\Big]\bigg|\geqslant Qx\bigg)\leqslant 2\exp\{-2Nb^2\}+\exp\{-x^2b^{-1-1/q}/2\}. \tag{5.4}$$

P r o o f. Since $|B_N^*(t)|\leqslant|B'(t)|$ a.s. for every t, we have by (3.8)

$$P\big(\sup_t|B_N^*(t)|\geqslant b\sqrt{N}\big)\leqslant 2\exp\{-2Nb^2\}.$$

Put $g_1=\sum_{i=1}^{N}k_i\big[B(i/N)-B\big(i/N-N^{-1/2}B_N^*(i/N)\big)\big]$ and denote by μ the distribution of the vector $-N^{-1/2}\big(B_N^*(1/N),\dots,B_N^*\big((N-1)/N\big)\big)$. Note that the last summand of g_1 is equal to zero a.s. By the independence of B and B_N^* and the Fubini theorem we have

$$(5.5)\quad P\biggl(|g_1| \geqslant Qx, \max_{1\leqslant i\leqslant N}\biggl|B_N^*\Bigl(\frac{i}{N}\Bigr)\biggr| \leqslant b\sqrt{N}\biggr)$$
$$= \int_J P\biggl(\biggl|\sum_{i=1}^{N-1} k_i\biggl[B\Bigl(\frac{i}{N}\Bigr) - B\Bigl(\frac{i}{N}+b_i\Bigr)\biggr]\biggr| \geqslant Qx\biggr)\,d\mu(b_1,\ldots,b_{N-1}),$$

where $J = \{\max_{1\leqslant i\leqslant N-1}|b_i| \leqslant b\}$. By Lemma 5.1 the function under the integral in (5.5) is, on the set J, estimated by $\exp\{-x^2 b^{-1-1/q}/2\}$. So, (5.4) follows from the relation

$$\{|g_1| \geqslant Qx\}$$
$$\subset \biggl\{|g_1| \geqslant Qx,\ \max_{1\leqslant i\leqslant N}\biggl|B_N^*\Bigl(\frac{i}{N}\Bigr)\biggr| \leqslant b\sqrt{N}\biggr\} \cup \biggl\{\max_{1\leqslant i\leqslant N}\biggl|B_N^*\Bigl(\frac{i}{N}\Bigr)\biggr| > b\sqrt{N}\biggr\},$$

which completes the proof. ■

Now, let us put $k_{ij} = \int_{(i-1)/N}^{i/N}\varphi_j'(t)dt$ for $i = 1,\ldots,N$, $j = 1,2,\ldots$ If $\varphi_j' \in L_p[0,1]$ for some $p \in (1,\infty]$, then $k_{1j},\ldots,k_{Nj}$ satisfy (5.2) for any $b < 1/2$. Indeed, from the Hölder inequality we get for $r = 1,\ldots,N$

$$\sum_{\{i:|i-r|\leqslant 2Nb\}} |k_{ij}| \leqslant \int_{((r-1)/N-2b)\vee 0}^{(r/N+2b)\wedge 1} |\varphi_j'(t)|dt \leqslant \|\varphi_j'\|_p (5b)^{1/q},$$

where $q = 1/(1-1/p)$ or $q = 1$ if $p = \infty$, and it is sufficient to take $K_0 = 5^{1/q}\|\varphi_j'\|_p$.

Next, let $d(N) \geqslant 1$ and let us put $g = (g_1, g_2, \ldots)^T$ with

$$(5.6)\qquad g_j = \sum_{i=1}^{N} k_{ij}\left[B\Bigl(\frac{i}{N}\Bigr) - B\Bigl(\frac{i}{N} - \frac{1}{\sqrt{N}}B_N^*\Bigl(\frac{i}{N}\Bigr)\Bigr)\right].$$

LEMMA 5.3. *Suppose $\varphi_j' \in L_p[0,1]$ for $j = 1,2,\ldots$ and some $p \in (1,\infty]$. Then for every $0 < \vartheta < 1$, every $\delta_N > 0$ and every sequence x_N such that $x_N \to 0$ and $x_N \geqslant 1/N$ we have*

$$(5.7)\quad P\bigl(|g|_{d(N)} \geqslant \delta_N\sqrt{N}x_N^{1+\vartheta}\bigr)$$
$$\leqslant 2d(N)\exp\{-2Nx_N^2\} + d(N)\exp\biggl\{-\frac{\delta_N^2 N x_N^{2\vartheta+1/p}}{2\omega^2\bigl(d(N)\bigr)}\biggr\},$$

where

$$(5.8)\qquad \omega^2(r) = \sum_{j=1}^{r}\|\varphi_j'\|_1^2 + 5^{1/q}\max_{1\leqslant i\leqslant d(N)}\|\varphi_i'\|_p\sum_{j=1}^{r}\|\varphi_j'\|_1.$$

P r o o f. Applying Lemma 5.2 with $b = x_N$ and $x = \delta_N \sqrt{N} x_N^{1+\vartheta}/\omega\big(d(N)\big)$ we write the left-hand side of (5.7) as

$$P\bigg(\sum_{j=1}^{d(N)} g_j^2 \geqslant \sum_{j=1}^{d(N)} \delta_N^2 N x_N^{2+2\vartheta} \frac{\|\varphi_j'\|_1^2 + 5^{1/q}\|\varphi_j'\|_p\|\varphi_j'\|_1}{\omega^2\big(d(N)\big)}\bigg)$$
$$\leqslant \sum_{j=1}^{d(N)} P\bigg(g_j^2 \geqslant \delta_N^2 N x_N^{2+2\vartheta} \frac{\|\varphi_j'\|_1^2 + 5^{1/q}\|\varphi_j'\|_p\|\varphi_j'\|_1}{\omega^2\big(d(N)\big)}\bigg)$$
$$\leqslant \sum_{j=1}^{d(N)} \bigg[2\exp\{-2Nx_N^2\} + \exp\bigg\{-\frac{\delta_N^2}{2\omega^2\big(d(N)\big)} N x_N^{1+2\vartheta-1/q}\bigg\}\bigg],$$

which proves (5.7). ∎

Using Lemma 5.3 we are able to prove a stronger version of Theorem 3.2 under stronger assumptions on the system Φ.

PROPOSITION 5.1. *Let $\varphi_j' \in L_p[0,1]$ for $j = 1, 2, \ldots$ and some $p \in (1, \infty]$. Then for every $0 < \vartheta < 3/4$ and every sequence x_N of positive numbers such that $x_N \to 0$ and*

$$\frac{N x_N^2}{\log N} \to \infty, \qquad \frac{N x_N^{1-\vartheta} \psi\big(d(N)\big)\sqrt{\log N}}{\sqrt{mn}} \to 0, \qquad x_N^{(3/4)-\vartheta}\psi\big(d(N)\big) \to 0 \tag{5.9}$$

as $N \to \infty$, we have for sufficiently large N

$$P\big(|T_N - T_N^0|_{d(N)} \geqslant x_N^{1+\vartheta}\sqrt{N}\big) \leqslant C\exp\{-Nx_N^2\} + d(N)\exp\bigg\{-c\frac{N x_N^{2\vartheta+1/p}}{\omega^2\big(d(N)\big)}\bigg\}. \tag{5.10}$$

P r o o f. Using Lemma 2.1, for the versions of $\zeta_N^{(1)}$ and $\xi_N^{(2)}$ defined in (2.13) and the corresponding versions $T_N^{(1)}$ and $T_N^{0(1)}$ we can write

$$T_N^{(1)} - T_N^{0(1)} = \sum_{i=1}^{N} \int_{(i-1)/N}^{i/N} \Phi'(u)\bigg[\zeta_N^{(1)}(u) - \zeta_N^{(1)}\Big(\frac{i}{N}\Big)\bigg]du$$
$$+ \sum_{i=1}^{N} \int_{(i-1)/N}^{i/N} \Phi'(u)du\bigg[\zeta_N^{(1)}\Big(\frac{i}{N}\Big) - \zeta_N^{(1)}(V_{(i)}^{(2)})\bigg],$$

where, by (2.9), $V_{(i)}^{(2)} = i/N - N^{-1/2}\xi_N^{(2)}(V_{(i)}^{(2)})$ for $i = 1, \ldots, N$.

Set $v_N = N x_N^2 \sqrt{\log N}/\sqrt{mn}$ and write

$$E_{N1} = \{\sup_t |\xi_N^{(2)}(t)| \geqslant x_N\sqrt{N}\}, \qquad E_{N2} = \{\sup_t |\xi_N^{(2)}(t) - B_N^{(2)}(t)| \geqslant v_N\sqrt{N}\}$$

and

$$E_{N3} = \Big\{ \sup_{1\leqslant t\leqslant 1-x_N} \sup_{u\leqslant x_N} |B_N^{(2)}(t+u) - B_N^{(2)}(t)| \geqslant 2\sqrt{Nx_N^3} \Big\}.$$

Then from (3.9), (5.9), Lemma 3.1 or Lemma A of the Appendix it follows that $\mathbf{P}(E_{Nj}) \leqslant \exp\{-Nx_N^2\}$ for $j = 1,2,3$. Moreover, on the set $E_{N1}^c \cap E_{N2}^c \cap E_{N3}^c$ we have for $i = 1,\ldots,N$

$$\left| \left(\frac{i}{N} - \frac{1}{\sqrt{N}} B_N^{*(2)}\left(\frac{i}{N}\right)\right) - \left(\frac{i}{N} - \frac{1}{\sqrt{N}} \xi_N^{(2)}(V_{(i)}^{(2)})\right) \right| \leqslant 2v_N + 4x_N^{3/2}.$$

Hence for a version $g_N^{(1)}$ of g (cf. (5.6)) we have

$$\begin{aligned}
(5.11)\quad & \mathbf{P}\big(|T_N^{(1)} - T_N^{0(1)}|_{d(N)} \geqslant x_N^{1+\vartheta}\sqrt{N}\big) \leqslant \mathbf{P}(E_{N1}) + \mathbf{P}(E_{N2}) + \mathbf{P}(E_{N3}) \\
& + \mathbf{P}\Big(|g_N^{(1)}|_{d(N)} \geqslant \frac{1}{7} x_N^{1+\vartheta}\sqrt{N}\Big) \\
& + \mathbf{P}\Big(\sup_{1\leqslant t\leqslant 1-1/N} \sup_{u\leqslant 1/N} |B_N^{(1)}(t+u) - B_N^{(1)}(t)| \geqslant \frac{x_N^{1+\vartheta}\sqrt{N}}{7\psi\big(d(N)\big)}\Big) \\
& + \mathbf{P}\Big(\sup_t |\zeta_N^{(1)}(t) - B_N^{(1)}(t)| \geqslant \frac{x_N^{1+\vartheta}\sqrt{N}}{7\psi\big(d(N)\big)}\Big) \\
& + \mathbf{P}\Big(\sup_{1\leqslant t\leqslant 1-2v_N-4x_N^{3/2}} \sup_{u\leqslant 2v_N+4x_N^{3/2}} |B_N^{(1)}(t+u) - B_N^{(1)}(t)| \geqslant \frac{x_N^{1+\vartheta}\sqrt{N}}{7\psi\big(d(N)\big)}\Big).
\end{aligned}$$

The fourth term on the right-hand side of (5.11) is the main term. Other terms are remainders and using (5.9), Lemma 3.1 or Lemma A of the Appendix can be easily estimated by $\exp\{-Nx_N^2\}$. We estimate the main term by a straightforward application of Lemma 5.3 with $\delta_N = 1/7$ and obtain (5.10). ∎

Now, we are ready to complete the proof of Theorem 3.5. Since

$$\begin{aligned}
& P(|T_N|_{d(N)}^2 \geqslant Nx_N^2) \\
& \quad \leqslant P\big(|T_N^0|_{d(N)}^2 \geqslant (1-x_N^{\vartheta})^2 Nx_N^2\big) + P\big(|T_N - T_N^0|_{d(N)} \geqslant x_N^{1+\vartheta}\sqrt{N}\big),
\end{aligned}$$

applying Lemma 3.2 and Proposition 5.1 we get

$$\begin{aligned}
& P(|T_N|_{d(N)}^2 \geqslant Nx_N^2) \\
& \qquad \leqslant \exp\Big\{ -\frac{1}{2\lambda_N} Nx_N^2 + O\Big(\frac{Nx_N^{2+\vartheta}}{\lambda_N}\Big) + O\big(d(N)\log Nx_N^2\big)\Big\} \\
& \qquad + \exp\{-Nx_N^2\} + d(N)\exp\Big\{ -c\frac{Nx_N^{2\vartheta+1/p}}{\omega^2\big(d(N)\big)}\Big\}.
\end{aligned}$$

Similarly, we get

$$P(|T_N|^2_{d(N)} \geqslant Nx_N^2)$$
$$\geqslant \exp\Big\{-\frac{1}{2\lambda_N}Nx_N^2 + O\Big(\frac{Nx_N^{2+\vartheta}}{\lambda_N}\Big) + O\big(d(N)\log Nx_N^2\big)\Big\}$$
$$-\exp\{-Nx_N^2\} - d(N)\exp\Big\{-c\frac{Nx_N^{2\vartheta+1/p}}{\omega^2\big(d(N)\big)}\Big\}.$$

This implies (3.15) by the assumption of Theorem 3.5, and hence completes the proof of Theorem 3.5. ∎

REMARK 5.1. It can be easily seen that our reasoning can be modified so that ϑ cover the whole interval $(0, 1)$. For example, if we define

$$B_N^{**}(t) = \sqrt{N}\Big[t - h\Big(t - N^{-1/2}B'\Big(h\big(t - N^{-1/2}B'(t)\big)\Big)\Big)\Big]$$

instead of $B_N^*(t)$ (cf. Lemma 5.2 above), then we extend the range of ϑ from $(0, 3/4)$ to $(0, 7/8)$. We omit details to avoid complicated notation.

APPENDIX

The following lemma extends Lemma 1.1.1 of Csörgő and Révész [2] to the case of the Brownian bridge. We provide its proof for convenience of the reader.

LEMMA A. *For any* $0 < \delta < 1$ *there exists a constant* $C = C(\delta)$ *such that*

$$P\big(\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant u\leqslant h}|B(t+u) - B(t)| \geqslant y\sqrt{h}\big) \leqslant \frac{C}{h}\exp\{-y^2/(2+8\delta)\}$$

for every $y > 0$ *and* $h \leqslant \delta^2$.

Proof. We have

$$\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant u\leqslant h}|B(t+u) - B(t)|$$
$$\leqslant \sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant u\leqslant h}|W(t+u) - W(t)| + h|W(1)|,$$

where W is the Wiener process. Hence and by Lemma 1.1.1 of Csörgő and Révész [2] we get for $0 < v < 1$

$$P\big(\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant u\leqslant h}|B(t+u) - B(t)| \geqslant y\sqrt{h}\big)$$
$$\leqslant P\big(\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant u\leqslant h}|W(t+u) - W(t)| \geqslant (1-v)y\sqrt{h}\big) + P\Big(|W(1)| \geqslant \frac{vy}{\sqrt{h}}\Big)$$
$$\leqslant \frac{C}{h}\exp\Big\{-\frac{(1-v)^2y^2}{2+\delta}\Big\} + \exp\Big\{-\frac{v^2y^2}{2h}\Big\}.$$

Taking $v = 2\delta/(2\delta + \sqrt{4+2\delta})$ and using the assumptions of the lemma, we can estimate the above expression by $(C_1/h)\exp\{-y^2/(2+8\delta)\}$, which completes the proof. ■

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