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ASYMPTOTIC RESULTS FOR EXIT PROBABILITIES OF STOCHASTIC PROCESSES GOVERNED BY AN INTEGRAL TYPE RATE FUNCTION

BY

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Abstract. In this paper we present asymptotic results for exit probabilities of stochastic processes in the fashion of large deviations. The main result concerns stochastic processes which satisfy the large deviation principle with an integral type rate function. We also present results for exit probabilities of linear diffusions and particular growth processes, and we give two examples.

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1. INTRODUCTION

The theory of large deviations gives an asymptotic computation of small probabilities on exponential scale (see, e.g., [4] as a reference on this topic). The aim of this paper is to give asymptotic results, in the fashion of large deviations, for exit probabilities of stochastic processes from a given domain.

In Section 2 we prove an asymptotic result for exit probabilities of stochastic processes $\{X^\varepsilon : \varepsilon > 0\}$ which satisfy the large deviation principle with an integral type rate function. Integral type rate functions appear in some well-known results in literature as Mogulskii's theorem (see, e.g., [4], Theorem 5.1.2) and Theorem 1.2 in [3]. In the proof we combine asymptotic results for up-crossing and down-crossing probabilities; the result for up-crossing probabilities has some analogies with Lemma 1 in [9] for scaled partial sums processes. We also discuss the concept of *most likely path* for the exit (see, e.g., Lemma 4.2 in [8]; see also [11], p. 45).

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In Section 3 we prove an asymptotic result for exit probabilities of stochastic processes which can be derived from adequate transformations of time and state of other stochastic processes satisfying the large deviation principle with an integral type rate function (as happens for the results in Section 2). In particular, we show how this result can be applied to exit probabilities of linear diffusions and of growth processes. In this way we can recover some known results of Freidlin–Wentzell theory for small noise diffusions; see, e.g., [7] (Chapter 4, Section 1) for the connections with large deviations. Finally, in Section 4, we illustrate some consequences for the most likely paths, and we present two examples.

2. RESULTS BASED ON INTEGRAL TYPE RATE FUNCTIONS

We start by recalling some preliminaries on large deviations. A family $\{X^\varepsilon : \varepsilon > 0\}$ of Ω-valued random variables, where Ω is a topological space, satisfies the *large deviation principle* (LDP from now on) with rate function I if the function $I : \Omega \to [0, \infty]$ is lower semi-continuous; we have the lower bound

$$\liminf_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in G) \geqslant - \inf_{\omega \in G} I(\omega)$$

for all open sets G, and the upper bound

$$\limsup_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in F) \leqslant - \inf_{\omega \in F} I(\omega)$$

for all closed sets F. Moreover, a rate function I is said to be *good* if all its level sets $\big\{\{\omega \in \Omega : I(\omega) \leqslant \eta\} : \eta \geqslant 0\big\}$ are compact. In what follows we consider $\Omega = C[t_0, T]$ for $0 \leqslant t_0 < T < \infty$, i.e. the family of all continuous functions on $[t_0, T]$, equipped with the topology of the uniform convergence. Thus we consider families of continuous stochastic processes $\big\{X^\varepsilon = \{X^\varepsilon(t) : t \in [t_0, T]\} : \varepsilon > 0\big\}$.

The main result in this section is Theorem 2.1 which concerns exit probabilities of continuous stochastic processes $\{X^\varepsilon : \varepsilon > 0\}$ which satisfy the LDP with an integral type rate function. So we start with the definition of integral type rate function. Moreover, in view of the applications of Theorem 2.1 in the next section, the next Lemma 2.1 provides a class of examples where the LDP holds with an integral type rate function: references for this lemma are Theorems 5.6.12 and 4.2.13 in [4], together with Theorem 4.2.13 in [4] concerning the concept of exponential equivalence (see, e.g., Definition 4.2.10 in [4]).

DEFINITION 2.1 (Integral type rate function). A rate function $I_{J,x_0,t_0,T}$ is said to be an *integral type rate function* if it is defined by

$$I_{J,x_0,t_0,T}(f) = \begin{cases} \int\limits_{t_0}^{T} J\big(\dot f(t)\big)dt & \text{if } f \in AC[t_0, T] \text{ and } f(t_0) = x_0, \\ \infty & \text{otherwise,} \end{cases}$$

where $0 \leqslant t_0 < T < \infty$; $AC[t_0, T]$ is the family of all absolutely continuous functions on $[t_0, T]$; $\dot{f}$ is the almost everywhere derivative of $f \in AC[t_0, T]$; the function $J : \mathbb{R} \to [0, \infty)$ is convex and there exists a unique $\overline{x} \in \mathbb{R}$ such that $J(\overline{x}) = 0$.

LEMMA 2.1. *Let $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$ be arbitrarily fixed and let $J_{\mu,\sigma}$ be defined by $J_{\mu,\sigma}(x) = (x - \mu)^2/(2\sigma^2)$. Let B be a standard Brownian motion on $[t_0, T]$ for $0 \leqslant t_0 < T < \infty$, let $f : [t_0, T] \to \mathbb{R}$ be a continuous function, and assume that $x_0^\varepsilon \to x_0$ as $\varepsilon \to 0$. Then the family of stochastic processes $\big\{X^\varepsilon = \{X^\varepsilon(t) : t \in [t_0, T]\} : \varepsilon > 0\big\}$ defined by*

$$X^\varepsilon(t) = x_0^\varepsilon + \mu(t - t_0) + \varepsilon f(t) + \sqrt{\varepsilon}\sigma B(t - t_0)$$

satisfies the LDP with good integral type rate function $I_{J_{\mu,\sigma},x_0,t_0,T}$.

The proof of Theorem 2.1 will easily follow from Propositions 2.1 and 2.2 which concern up-crossing and down-crossing probabilities, respectively. In view of what follows, we introduce some symbols. Let $Z = \{Z(t) : t \in [t_0, T]\}$ be a continuous stochastic process, and consider the exit probability

$$\Psi_Z(t_0, T; b_{(-)}, b_{(+)}) = P\Big(\big\{\exists t \in [t_0, T] : Z(t) \notin \big(b_{(-)}(t), b_{(+)}(t)\big)\big\}\Big)$$

in the time interval $[t_0, T]$, where $b_{(-)}, b_{(+)} : [t_0, T] \to \mathbb{R}$ are two continuous barriers such that $b_{(-)}(t) < b_{(+)}(t)$ for all $t \in [t_0, T]$. Similarly, for a given continuous barrier $b : [t_0, T] \to \mathbb{R}$, we shall consider the up-crossing probability $\Psi_Z^\uparrow(t_0, T; b) = P\big(\{\exists t \in [t_0, T] : Z(t) \geqslant b(t)\}\big)$ and the down-crossing probability $\Psi_Z^\downarrow(t_0, T; b) = P\big(\{\exists t \in [t_0, T] : Z(t) \leqslant b(t)\}\big)$.

PROPOSITION 2.1. *Let $\big\{X^\varepsilon = \{X^\varepsilon(t) : t \in [t_0, T]\} : \varepsilon > 0\big\}$ be a family of continuous stochastic processes for $0 \leqslant t_0 < T < \infty$, which satisfies the LDP with an integral type rate function $I_{J,x_0,t_0,T}$. Furthermore, let $b : [t_0, T] \to \mathbb{R}$ be a continuous function such that*:

(H1 $\uparrow$): $x_0 < b(t_0)$;

(H2 $\uparrow$): $\inf\limits_{t_0 < t \leqslant T} (t - t_0) J\big((b(t) - x_0)/(t - t_0)\big)$ *is attained at some $t_* \in (t_0, T]$.*

Then

$$\lim_{\varepsilon \to 0} \varepsilon \log \Psi_{X^\varepsilon}^\uparrow(t_0, T; b) = -\inf_{t_0 < t \leqslant T} (t - t_0) J\left(\frac{b(t) - x_0}{t - t_0}\right).$$

P r o o f. We have to prove the upper bound

$$\limsup_{\varepsilon \to 0} \varepsilon \log \Psi_{X^\varepsilon}^\uparrow(t_0, T; b) \leqslant -\inf_{t_0 < t \leqslant T} (t - t_0) J\left(\frac{b(t) - x_0}{t - t_0}\right) \tag{2.1}$$

and the lower bound

$$\liminf_{\varepsilon \to 0} \varepsilon \log \Psi_{X^\varepsilon}^\uparrow(t_0, T; b) \geqslant -\inf_{t_0 < t \leqslant T} (t - t_0) J\left(\frac{b(t) - x_0}{t - t_0}\right). \tag{2.2}$$

We start with the proof of (2.1). Let us consider the closed set

$$\overline{E}_b = \{f \in C[t_0, T] : \text{there exists } t \in [t_0, T] \text{ such that } f(t) \geqslant b(t)\}.$$

Then we have

$$\limsup_{\varepsilon\to 0} \varepsilon \log \Psi^{\uparrow}_{X^\varepsilon}(t_0, T; b) \leqslant -\inf\{I_{J,x_0,t_0,T}(f) : f \in \overline{E}_b\} \tag{2.3}$$

by the upper bound for the closed sets in the LDP of $\{X^\varepsilon : \varepsilon > 0\}$. We have to compute the right-hand side in (2.3). We can restrict our attention to any absolutely continuous functions $f \in \overline{E}_b$ such that $f(t_0) = x_0$. Firstly, since $f(t_0) = x_0 < b(t_0)$ by (H1 $\uparrow$), there exists $t_f \in (t_0, T]$ such that $f(t_f) = b(t_f)$. Then we have

$$I_{J,x_0,t_0,T}(f) \geqslant \int_{t_0}^{t_f} J\big(\dot{f}(t)\big)dt \geqslant (t_f - t_0)J\left(\frac{b(t_f) - x_0}{t_f - t_0}\right) \tag{2.4}$$

by Jensen's inequality. Now consider the function $f_* : [t_0, T] \to \mathbb{R}$ defined by

$$f_*(t) = \begin{cases} x_0 + \dfrac{b(t_*) - x_0}{t_* - t_0}(t - t_0) & \text{if } t_0 \leqslant t \leqslant t_*, \\ b(t_*) + \overline{x}(t - t_*) & \text{if } t_* < t \leqslant T \text{ (and } t_* < T), \end{cases} \tag{2.5}$$

where t_* is as in (H2 $\uparrow$). Then, by (2.4) and (H2 $\uparrow$), we obtain

$$\begin{aligned} I_{J,x_0,t_0,T}(f) &\geqslant \inf_{t_0 < t \leqslant T} (t - t_0)J\left(\frac{b(t) - x_0}{t - t_0}\right) \\ &= (t_* - t_0)J\left(\frac{b(t_*) - x_0}{t_* - t_0}\right) = I_{J,x_0,t_0,T}(f_*), \end{aligned}$$

where the latter equality can be easily checked. In conclusion, since $f_* \in \overline{E}_b$, we have

$$\inf\{I_{J,x_0,t_0,T}(f) : f \in \overline{E}_b\} = \inf_{t_0 < t \leqslant T} (t - t_0)J\left(\frac{b(t) - x_0}{t - t_0}\right), \tag{2.6}$$

and (2.1) holds by (2.3) and (2.6).

Now we prove (2.2). Let us consider the open set

$$E_*^\circ = \{f \in C[t_0, T] : f(t_*) > b(t_*)\}.$$

Then we have

$$\begin{aligned} \liminf_{\varepsilon\to 0} \varepsilon \log \Psi^{\uparrow}_{X^\varepsilon}(t_0, T; b) &\geqslant \liminf_{\varepsilon\to 0} \varepsilon \log P\big(X^\varepsilon(t_*) > b(t_*)\big) \\ &\geqslant -\inf\{I_{J,x_0,t_0,T}(f) : f \in E_*^\circ\}, \end{aligned} \tag{2.7}$$

where the first inequality holds by construction, and the second one holds by the lower bound for the open sets in the LDP of $\{X^\varepsilon : \varepsilon > 0\}$. For $\eta > 0$ we define the function $f_*^{(\eta)}$ which is a slight modification of f_* presented above for the proof of (2.1):

$$f_*^{(\eta)}(t) = \begin{cases} x_0 + \dfrac{b(t_*) + \eta - x_0}{t_* - t_0}(t - t_0) & \text{if } t_0 \leqslant t \leqslant t_*, \\ b(t_*) + \eta + \overline{x}(t - t_*) & \text{if } t_* < t \leqslant T \text{ (and } t_* < T). \end{cases}$$

Then $f_*^{(\eta)} \in E_*^\circ$, and therefore

$$\inf\{I_{J,x_0,t_0,T}(f) : f \in E_*^\circ\} \leqslant I_{J,x_0,t_0,T}(f_*^{(\eta)}) = (t_* - t_0) J\left(\frac{b(t_*) + \eta - x_0}{t_* - t_0}\right).$$

Finally, we recall that J is continuous because it is a real finite-valued convex function defined on an open interval (see, e.g., [14], Theorem 3.2) and, letting η go to zero, we obtain

$$\begin{aligned} \text{(2.8)} \qquad \inf\{I_{J,x_0,t_0,T}(f) : f \in E_*^\circ\} &\leqslant (t_* - t_0) J\left(\frac{b(t_*) - x_0}{t_* - t_0}\right) \\ &= \inf_{t_0 < t \leqslant T} (t - t_0) J\left(\frac{b(t) - x_0}{t - t_0}\right). \end{aligned}$$

In conclusion, (2.2) holds by (2.7) and (2.8). ■

PROPOSITION 2.2. *Let* $\{X^\varepsilon = \{X^\varepsilon(t) : t \in [t_0, T]\} : \varepsilon > 0\}$ *be a family of continuous stochastic processes for* $0 \leqslant t_0 < T < \infty$, *which satisfies the LDP with an integral type rate function* $I_{J,x_0,t_0,T}$. *Furthermore, let* $b : [t_0, T] \to \mathbb{R}$ *be a continuous function such that*:

(H1 $\downarrow$): $b(t_0) < x_0$;

(H2 $\downarrow$): $\inf_{t_0 < t \leqslant T} (t - t_0) J\big((b(t) - x_0)/(t - t_0)\big)$ *is attained at some* $t_* \in (t_0, T]$.

Then

$$\lim_{\varepsilon \to 0} \varepsilon \log \Psi_{X^\varepsilon}^{\downarrow}(t_0, T; b) = -\inf_{t_0 < t \leqslant T} (t - t_0) J\left(\frac{b(t) - x_0}{t - t_0}\right).$$

Proof. It is similar to the proof of Proposition 2.1, and therefore omitted. ■

THEOREM 2.1. *Let* $\{X^\varepsilon = \{X^\varepsilon(t) : t \in [t_0, T]\} : \varepsilon > 0\}$ *be a family of continuous stochastic processes for* $0 \leqslant t_0 < T < \infty$, *which satisfies the LDP with an integral type rate function* $I_{J,x_0,t_0,T}$. *Furthermore, let* $b_{(-)}, b_{(+)} : [t_0, T] \to \mathbb{R}$ *be continuous functions such that*:

(H1): $b_{(-)}(t_0) < x_0 < b_{(+)}(t_0)$;

(H2): $\inf_{t_0<t\leqslant T}(t-t_0)J\big((b_{(\pm)}(t)-x_0)/(t-t_0)\big)$ *is attained at some* $t_*^{(\pm)} \in (t_0, T]$.

Then

$$\lim_{\varepsilon\to 0}\varepsilon\log\Psi_{X^\varepsilon}(t_0,T;b_{(-)},b_{(+)}) = -\min_{b\in\{b_{(+)},b_{(-)}\}}\left\{\inf_{t_0<t\leqslant T}(t-t_0)J\left(\frac{b(t)-x_0}{t-t_0}\right)\right\}.$$

P r o o f. Firstly we have

$$\Psi^{\downarrow}_{X^\varepsilon}(t_0,T;b_{(-)}),\Psi^{\uparrow}_{X^\varepsilon}(t_0,T;b_{(+)}) \leqslant \Psi_{X^\varepsilon}(t_0,T;b_{(-)},b_{(+)}),$$

and we obtain

$$\max\{\liminf_{\varepsilon\to 0}\varepsilon\log\Psi^{\downarrow}_{X^\varepsilon}(t_0,T;b_{(-)}),\liminf_{\varepsilon\to 0}\varepsilon\log\Psi^{\uparrow}_{X^\varepsilon}(t_0,T;b_{(+)})\} \leqslant \liminf_{\varepsilon\to 0}\varepsilon\log\Psi_{X^\varepsilon}(t_0,T;b_{(-)},b_{(+)}). \tag{2.9}$$

We also have

$$\Psi_{X^\varepsilon}(t_0,T;b_{(-)},b_{(+)}) \leqslant \Psi^{\downarrow}_{X^\varepsilon}(t_0,T;b_{(-)})+\Psi^{\uparrow}_{X^\varepsilon}(t_0,T;b_{(+)}),$$

and we get

$$\limsup_{\varepsilon\to 0}\varepsilon\log\Psi_{X^\varepsilon}(t_0,T;b_{(-)},b_{(+)}) = \max\{\limsup_{\varepsilon\to 0}\varepsilon\log\Psi^{\downarrow}_{X^\varepsilon}(t_0,T;b_{(-)}),\limsup_{\varepsilon\to 0}\varepsilon\log\Psi^{\uparrow}_{X^\varepsilon}(t_0,T;b_{(+)})\} \tag{2.10}$$

by Lemma 1.2.15 in [4]. Then, by (2.9) and (2.10), we complete the proof noting that we have

$$\lim_{\varepsilon\to 0}\varepsilon\log\Psi^{\uparrow}_{X^\varepsilon}(t_0,T;b_{(+)}) = -\inf_{t_0<t\leqslant T}(t-t_0)J\left(\frac{b_{(+)}(t)-x_0}{t-t_0}\right),$$
$$\lim_{\varepsilon\to 0}\varepsilon\log\Psi^{\downarrow}_{X^\varepsilon}(t_0,T;b_{(-)}) = -\inf_{t_0<t\leqslant T}(t-t_0)J\left(\frac{b_{(-)}(t)-x_0}{t-t_0}\right)$$

by Propositions 2.1 and 2.2, respectively. ■

REMARK 2.1. *The assumptions* (H2↑) *and* (H2↓) *in Propositions* 2.1 *and* 2.2 *and the assumption* (H2) *in Theorem* 2.1 *could be avoided by assuming that*

$$\lim_{x\to\infty}\frac{J(x)}{x}=\infty,\qquad \lim_{x\to-\infty}\frac{J(x)}{-x}=\infty \quad \textit{and} \quad \lim_{|x|\to\infty}\frac{J(x)}{|x|}=\infty,$$

respectively. If we assume that J is coercive, i.e. $\lim_{|x|\to\infty}J(x)/|x|=\infty$, *the rate function* $I_{J,x_0,t_0,T}$ *in Definition* 2.1 *is good*; *also*, *if* $I_{J,x_0,t_0,T}$ *is good*, *the function* J *must be coercive.*

We conclude this section with a brief discussion on the concept of *most likely path* for the exit, and we refer to the framework of Theorem 2.1. Let us consider the set $E_{b_{(-)},b_{(+)}} := E^{\uparrow}_{b_{(+)}} \cup E^{\downarrow}_{b_{(-)}}$, where $E^{\uparrow}_{b_{(+)}}$ is the set $\overline{E}_b$ in the proof of Proposition 2.1 with $b_{(+)}$ in place of b, and

$$E^{\downarrow}_{b_{(-)}} = \{f \in C[t_0, T] : \text{there exists } t \in [t_0, T] \text{ such that } f(t) \leqslant b_{(-)}(t)\}.$$

Note that $E_{b_{(-)},b_{(+)}}$ is closed because $E^{\uparrow}_{b_{(+)}}$ and $E^{\downarrow}_{b_{(-)}}$ are closed sets. Then $f_* \in E_{b_{(-)},b_{(+)}}$ is said to be a *most likely path* if

$$I_{J,x_0,t_0,T}(f_*) = \inf\{I_{J,x_0,t_0,T}(f) : f \in E_{b_{(-)},b_{(+)}}\}.$$

Note that if the rate function $I_{J,x_0,t_0,T}$ is good and

$$\inf\{I_{J,x_0,t_0,T}(f) : f \in E_{b_{(-)},b_{(+)}}\} < \infty,$$

then the infimum is attained because $E_{b_{(-)},b_{(+)}}$ is a closed set. In general, we do not have a unique most likely path and f_* is a most likely path if and only if it is defined by (2.5), where: $b \in \{b_{(+)}, b_{(-)}\}$ is such that

$$\begin{aligned}&\inf_{t_0<t\leqslant T}(t-t_0)J\left(\frac{b(t)-x_0}{t-t_0}\right)\\ &= \min\left\{\inf_{t_0<t\leqslant T}(t-t_0)J\left(\frac{b_{(+)}(t)-x_0}{t-t_0}\right), \inf_{t_0<t\leqslant T}(t-t_0)J\left(\frac{b_{(-)}(t)-x_0}{t-t_0}\right)\right\};\end{aligned}$$

$t_* = t_*^{(+)}$ if $b_* = b_{(+)}$ or $t_* = t_*^{(-)}$ if $b_* = b_{(-)}$, where $t_*^{(\pm)}$ are the values in (H2) in Theorem 2.1. Note that we have a unique most likely path f_* if and only if we have a unique choice of b_* and t_* as above.

It is interesting to remark that if f_* is the unique minimizing point in $E_{b_{(-)},b_{(+)}}$, then

$$\lim_{\varepsilon\to 0} P\big(X^\varepsilon \in A_\eta(f_*) | X^\varepsilon \in E_{b_{(-)},b_{(+)}}\big) = 0,$$

where $A_\eta(f_*) = \{g \in C[t_0, T] : \sup_{t\in[t_0,T]} |g(t) - f_*(t)| \geqslant \eta\}$ and $\eta > 0$. This can be checked following the lines of the proof of Lemma 4.2 in [8] (here we have some differences) and noting that Theorem 2.1 provides the limit

$$\lim_{\varepsilon\to 0} \varepsilon \log P(X^\varepsilon \in E_{b_{(-)},b_{(+)}}) = -I_{J,x_0,t_0,T}(f_*).$$

In conclusion, the unique minimizing function f_* has the following appealing interpretation: conditionally on the exit of the stochastic processes $\{X^\varepsilon : \varepsilon > 0\}$ for ε close to zero, with overwhelming probability this occurs *via* a path that is close to f_*.

3. APPLICATIONS FOR SOME FAMILIES OF DIFFUSIONS

In this section we prove an asymptotic result for exit probabilities of stochastic processes $\{Y^\varepsilon : \varepsilon > 0\}$ which can be derived from adequate transformations of time and state of other stochastic processes $\{X^\varepsilon : \varepsilon > 0\}$ satisfying the assumptions of Theorem 2.1 (more precisely, they are as in Lemma 2.1). In particular, we show how this result can be applied to exit probabilities of linear diffusions and of growth processes.

PROPOSITION 3.1. *We consider an open interval $\mathcal{X} \subset \mathbb{R}$, two continuous barriers $b_{(-)}, b_{(+)} : [t_0, T] \to \mathcal{X}$ such that $b_{(-)}(t) < b_{(+)}(t)$ for all $t \in [0, T]$, a family of continuous and strictly increasing functions $\{\mathcal{B}_{t,t_0} : t \in [t_0, T]\}$ such that $\mathcal{B}_{t,t_0} : \mathcal{X} \to \mathbb{R}$ for all $t \in [t_0, T]$, a continuous and strictly increasing function $\rho(\cdot; t_0) : [t_0, T] \to [0, \rho(T; t_0)]$ defined by $\rho(t; t_0) := \int_{t_0}^{t} \theta^2(v)dv$ for a square integrable function θ. Moreover, let $\{X^\varepsilon = \{X^\varepsilon(r) : r \in [0, \rho(T; t_0)]\} : \varepsilon > 0\}$ be defined by $X^\varepsilon(r) := x_0^\varepsilon + \varepsilon f(r) + \sqrt{\varepsilon}B(r)$, where $x_0^\varepsilon := \mathcal{B}_{t_0,t_0}(y_0^\varepsilon)$ for some y_0^ε such that $y_0^\varepsilon \to y_0 \in \big(b_{(-)}(t_0), b_{(+)}(t_0)\big)$, $f : [0, \rho(T; t_0)] \to \mathbb{R}$ is a continuous function and $\{B(r) : r \in [0, \rho(T; t_0)]\}$ is a standard Brownian motion. Finally, let $\{Y^\varepsilon = \{Y^\varepsilon(t) : t \in [t_0, T]\} : \varepsilon > 0\}$ be defined by $Y^\varepsilon(t) := \mathcal{B}_{t,t_0}^{-1}\big(X^\varepsilon\big(\rho(t; t_0)\big)\big)$. Then, if we set $\widehat{b}_{t_0}(t; b) := \mathcal{B}_{t,t_0}\big(b(t)\big)$, we have*

$$\lim_{\varepsilon \to 0} \varepsilon \log \Psi_{Y^\varepsilon}(t_0, T; b_{(-)}, b_{(+)}) = -u(y_0, t_0), \tag{3.1}$$

where

$$u(y_0, t_0) := \min_{b \in \{b_{(-)}, b_{(+)}\}} \inf_{t_0 < t \leqslant T} \frac{\big(\widehat{b}_{t_0}(t; b) - \mathcal{B}_{t_0,t_0}(y_0)\big)^2}{2\rho(t; t_0)}.$$

P r o o f. Firstly we note that $\{X^\varepsilon : \varepsilon > 0\}$ satisfies the assumptions of Lemma 2.1 with $\mu = 0$ and $\sigma = 1$, and with $[0, \rho(T; t_0)]$ in place of $[t_0, T]$. Moreover, for each fixed $\varepsilon > 0$, we have

$$\begin{aligned}
&\Psi_{Y^\varepsilon}(t_0, T; b_{(-)}, b_{(+)}) \\
&= P\Big(\Big\{\exists t \in [t_0, T] \colon \mathcal{B}_{t,t_0}^{-1}\Big(X^\varepsilon\big(\rho(t; t_0)\big)\Big) \notin \big(b_{(-)}(t), b_{(+)}(t)\big)\Big\}\Big) \\
&= P\Big(\Big\{\exists r \in [0, \rho(T; t_0)] \colon X^\varepsilon(r) \notin \Big(\widehat{b}_{t_0}\big(\rho^{-1}(r; t_0); b_{(-)}\big), \widehat{b}_{t_0}\big(\rho^{-1}(r; t_0); b_{(+)}\big)\Big)\Big\}\Big) \\
&= \Psi_{X^\varepsilon}\Big(0, \rho(T; t_0); \widehat{b}_{t_0}\big(\rho^{-1}(\cdot; t_0); b_{(-)}\big), \widehat{b}_{t_0}\big(\rho^{-1}(\cdot; t_0); b_{(+)}\big)\Big).
\end{aligned}$$

Then, by Theorem 2.1, we obtain

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \varepsilon \log \Psi_{Y^\varepsilon}(t_0, T; b_{(-)}, b_{(+)}) \\
&\qquad = - \min_{b \in \{b_{(-)}, b_{(+)}\}} \inf_{0 < r \leqslant \rho(T; t_0)} r J_{0,1}\left(\frac{\widehat{b}_{t_0}\big(\rho^{-1}(r; t_0); b\big) - x_0}{r}\right),
\end{aligned}$$

where $x_0 := \mathcal{B}_{t_0,t_0}(y_0)$ and $J_{0,1}$ is as in Lemma 2.1. Noting that

$$\inf_{0<r\leqslant\rho(T;t_0)} rJ_{0,1}\left(\frac{\widehat{b}_{t_0}\big(\rho^{-1}(r;t_0);b\big)-x_0}{r}\right) = \inf_{t_0<t\leqslant T} \frac{\big(\widehat{b}_{t_0}(t;b)-\mathcal{B}_{t_0,t_0}(y_0)\big)^2}{2\rho(t;t_0)},$$

we complete the proof. ■

Linear diffusion with additive noise. Let $\big\{Y^\varepsilon=\{Y^\varepsilon(t)\colon t\in[t_0,T]\}\colon \varepsilon>0\big\}$ be defined by

$$\begin{aligned} dY^\varepsilon(t) &= \big(\mu_1(t)Y^\varepsilon(t)+\mu_0(t)\big)dt+\sqrt{\varepsilon}\sigma(t)dB(t),\\ Y^\varepsilon(t_0) &= y_0^\varepsilon, \end{aligned}$$

where $y_0^\varepsilon \to y_0$ as $\varepsilon\to 0$. We assume that

$$\mu_1(\cdot), \mu_0(\cdot)\exp\Big(-\int_{t_0}^{\cdot}\mu_1(w)dw\Big) \text{ and } \Big(\sigma(\cdot)\exp\Big(-\int_{t_0}^{\cdot}\mu_1(w)dw\Big)\Big)^2$$

are integrable functions on $[t_0,T]$, and that $\sigma(\cdot)$ is positive. By a known result in the literature (see, e.g., [10], Chapter 4, Section 4, equation (4.3)), we can check that $\{Y^\varepsilon : \varepsilon>0\}$ is as in Proposition 3.1, where

$$\mathcal{X}=\mathbb{R}, \quad x_0^\varepsilon = y_0^\varepsilon, \quad f(r)=0, \quad \theta(v)=\sigma(v)\exp\Big(-\int_{t_0}^{v}\mu_1(w)dw\Big),$$

$$\mathcal{B}_{t,t_0}^{-1}(\widehat{b}) = \exp\Big(\int_{t_0}^{t}\mu_1(v)dv\Big)\Big(\widehat{b}+\int_{t_0}^{t}\mu_0(v)\exp\Big(-\int_{t_0}^{v}\mu_1(w)dw\Big)dv\Big).$$

Linear diffusion with multiplicative noise. Let $\big\{Y^\varepsilon=\{Y^\varepsilon(t): t\in[t_0,T]\} : \varepsilon>0\big\}$ be defined by

$$\begin{aligned} dY^\varepsilon(t) &= \mu(t)Y^\varepsilon(t)dt+\sqrt{\varepsilon}Y^\varepsilon(t)\sigma(t)dB(t),\\ Y^\varepsilon(t_0) &= y_0^\varepsilon, \end{aligned}$$

where $y_0^\varepsilon \to y_0\in(0,\infty)$ as $\varepsilon\to 0$; here we consider $\varepsilon>0$ small enough to have $y_0^\varepsilon\in(0,\infty)$. We assume that $\mu(\cdot)$ and $\sigma^2(\cdot)$ are integrable functions on $[t_0,T]$, and that $\sigma(\cdot)$ is positive. By a known result in the literature (see, e.g., [10], Chapter 4, Section 4, equation (4.10)), we can check that $\{Y^\varepsilon : \varepsilon>0\}$ is as in Proposition 3.1, where

$$\mathcal{X}=(0,\infty), \quad x_0^\varepsilon=\log y_0^\varepsilon, \quad f(r)=-r/2, \quad \theta(v)=\sigma(v),$$

$$\mathcal{B}_{t,t_0}^{-1}(\widehat{b}) = \exp\Big(\widehat{b}+\int_{t_0}^{t}\mu(v)dv\Big).$$

Growth processes. Let $\{Y^\varepsilon = \{Y^\varepsilon(t) : t \in [t_0, T]\} : \varepsilon > 0\}$ be defined by

$$dY^\varepsilon(t) = \big(\alpha Y^\varepsilon(t) - \beta Y^\varepsilon(t)\log Y^\varepsilon(t)\big)dt + \sqrt{\varepsilon}\sigma Y^\varepsilon(t)dB(t),$$
$$Y^\varepsilon(t_0) = y_0^\varepsilon,$$

where $y_0^\varepsilon \to y_0 \in (0,\infty)$ as $\varepsilon \to 0$, and $\alpha, \beta, \sigma \in (0,\infty)$; here we consider $\varepsilon > 0$ small enough to have $y_0^\varepsilon \in (0,\infty)$. Let F_ε be the function defined by

$$F_\varepsilon(t,y) := \frac{e^{\beta t}}{\sqrt{\varepsilon}\sigma}\left(\log y + \frac{\varepsilon\sigma^2 - 2\alpha}{2\beta}\right)$$

and, by using Itō's formula (see, e.g., Theorem 4.1.2 in [12]) for $F_\varepsilon\big(t, Y^\varepsilon(t)\big)$, we get

$$\begin{aligned} dF_\varepsilon\big(t, Y^\varepsilon(t)\big) &= \beta\frac{e^{\beta t}}{\sqrt{\varepsilon}\sigma}\left(\log Y^\varepsilon(t) + \frac{\varepsilon\sigma^2 - 2\alpha}{2\beta}\right)dt \\ &\quad + \frac{e^{\beta t}}{\sqrt{\varepsilon}\sigma}\frac{1}{Y^\varepsilon(t)}dY^\varepsilon(t) + \frac{e^{\beta t}}{2\sqrt{\varepsilon}\sigma}\left(-\frac{1}{(Y^\varepsilon(t))^2}\right)\big(\sqrt{\varepsilon}\sigma Y^\varepsilon(t)\big)^2 dt. \end{aligned}$$

Then we have

$$\begin{aligned} dF_\varepsilon\big(t, Y^\varepsilon(t)\big) &= \frac{e^{\beta t}}{\sqrt{\varepsilon}\sigma}\left(\beta\log Y^\varepsilon(t) + \frac{\varepsilon\sigma^2}{2} - \alpha\right)dt \\ &\quad + \frac{e^{\beta t}}{\sqrt{\varepsilon}\sigma}\big(\alpha - \beta\log Y^\varepsilon(t)\big)dt + e^{\beta t}dB(t) - \frac{e^{\beta t}\sqrt{\varepsilon}\sigma}{2}dt \\ &= e^{\beta t}dB(t) \end{aligned}$$

and, by taking into account the initial condition $F_\varepsilon\big(t_0, Y^\varepsilon(0)\big) = F_\varepsilon(t_0, y_0^\varepsilon)$, we obtain

$$F_\varepsilon\big(t, Y^\varepsilon(t)\big) = \int_{t_0}^{t} e^{\beta v}dB(v) + F_\varepsilon\,(t_0, y_0^\varepsilon).$$

Thus

$$\frac{e^{\beta t}}{\sqrt{\varepsilon}\sigma}\left(\log Y^\varepsilon(t) + \frac{\varepsilon\sigma^2 - 2\alpha}{2\beta}\right) = \int_{t_0}^{t} e^{\beta v}dB(v) + \frac{e^{\beta t_0}}{\sqrt{\varepsilon}\sigma}\left(\log y_0^\varepsilon + \frac{\varepsilon\sigma^2 - 2\alpha}{2\beta}\right)$$

and, with some easy computations, one can check that $\{Y^\varepsilon : \varepsilon > 0\}$ is as in Proposition 3.1, where

$$\mathcal{X} = (0,\infty), \quad x_0^\varepsilon = e^{\beta t_0}\log y_0^\varepsilon, \quad f(r) = -\frac{\sigma^2}{2\beta}\Big(\exp\big(\beta\rho^{-1}(r;t_0)\big) - \exp(\beta t_0)\Big),$$
$$\theta(v) = \sigma e^{\beta v}, \quad \mathcal{B}_{t,t_0}^{-1}(\widehat{b}) = \exp\left(e^{-\beta t}\left(\widehat{b} + (e^{\beta t} - e^{\beta t_0})\frac{\alpha}{\beta}\right)\right).$$

4. MOST LIKELY PATHS AND TWO EXAMPLES

In this section we illustrate the relationship between the most likely paths for the exit of $\{Y^\varepsilon : \varepsilon > 0\}$ and of $\{X^\varepsilon : \varepsilon > 0\}$ as in Section 3. Furthermore, we present two examples. Example 4.1 concerns the exit probabilities of stochastic processes from a tubular neighborhood of the limit deterministic trajectory Y^0, i.e. the solution of the deterministic equation obtained by taking the noise term equal to zero; this is of interest in economics where Y^0 is the equilibrium trajectory (results of this kind can be found in [2] with applications to the issue of the efficiency of financial market; see also [13] and [15] with applications in macroeconomics). Example 4.2 concerns the exit probabilities of geometric Brownian motions from a domain with positive exponential barriers; in this case we have an extension of the content of the Remark just after Theorem 4.1 in [5] because we have non-constant barriers.

We start with some preliminaries. The limit (3.1) can be proved for exit probabilities of family of possibly n-dimensional diffusions $\{Y^\varepsilon : \varepsilon > 0\}$ under suitable assumptions. This is what happens for Theorem 4.1 in [5] where we have

$$\begin{aligned} &dY^\varepsilon(t) = m\big(Y^\varepsilon(t), t\big)dt + \sqrt{\varepsilon}\sqrt{a\big(Y^\varepsilon(t)\big)}dB(t), \\ &Y^\varepsilon(t_0) =: y_0^\varepsilon = y_0 \end{aligned}$$

and $\{Y^\varepsilon : \varepsilon > 0\}$ satisfies the LDP (as $\varepsilon \to 0$) with good rate function $I_{y_0,t_0,T}$ defined by

(4.1)
$$I_{y_0,t_0,T}(f) = \begin{cases} \int_{t_0}^{T} \mathcal{L}\big(t, f(t), \dot f(t)\big)dt & \text{if } f \in AC[t_0, T] \text{ and } f(t_0) = y_0, \\ \infty & \text{otherwise,} \end{cases}$$

where

$$\mathcal{L}(t, y, \dot y) = \frac{\big(\dot y - m(y,t)\big)^2}{2a(y)};$$

in such a case we have

(4.2)
$$u(y_0, t_0) = \inf\big\{I_{y_0,t_0,T}(f) : f(t) \notin \big(b_{(-)}(t), b_{(+)}(t)\big) \text{ for some } t \in (t_0, T]\big\}.$$

Note that in general $I_{y_0,t_0,T}$ is not an integral type rate function and the minimization problem (4.2) can be solved by using standard techniques of calculus of variations. If we specialize Theorem 4.1 in [5] to one-dimensional diffusions ($n = 1$) as in this paper, $b_{(-)}$ and $b_{(+)}$ have to be constant barriers (this is not the case of Proposition 3.1). Here we allow a slight generalization on the initial condition because we have $y_0^\varepsilon \to y_0$ as $\varepsilon \to 0$ instead of $y_0^\varepsilon = y_0$.

Now, arguing as in Section 2, we say that f_* is a most likely path if it attains the infimum in (4.2), i.e., if $f_*(t) \notin \big(b_{(-)}(t), b_{(+)}(t)\big)$ for some $t \in (t_0, T]$ and

$$I_{y_0,t_0,T}(f_*) = \inf\big\{I_{y_0,t_0,T}(f) : f(t) \notin \big(b_{(-)}(t), b_{(+)}(t)\big) \text{ for some } t \in (t_0, T]\big\}.$$

It is known (see, e.g., [6], Chapter 1, Corollary 3.1) that f_* is a solution of Euler's equation

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0, \tag{4.3}$$

where $\mathcal{L}$ is as in (4.1); moreover, if both sides of the equation are evaluated at $\big(t, y^*(t), \dot{y}^*(t)\big)$ and $\dot{y}^*(t)$ has a discontinuity at some point, then the equation (4.3) is satisfied by right-hand and left-hand derivatives. In detail, we have

$$\mathcal{L}(t, y, \dot{y}) = \frac{\big(\dot{y} - \big(\mu_1(t)y + \mu_0(t)\big)\big)^2}{2\sigma^2(t)}$$

for the linear diffusions with additive noise,

$$\mathcal{L}(t, y, \dot{y}) = \frac{\big(\dot{y} - \mu(t)y\big)^2}{2\sigma^2(t)y^2}$$

for the linear diffusions with multiplicative noise, and

$$\mathcal{L}(t, y, \dot{y}) = \frac{\big(\dot{y} - (\alpha y - \beta y \log y)\big)^2}{2\sigma^2 y^2}$$

for the growth processes.

Finally, by referring to the framework of Proposition 3.1, we illustrate an alternative method to find a most likely path f_* for each family of diffusions $\{Y^\varepsilon : \varepsilon > 0\}$ in this section (linear diffusions or growth processes). We set $f_*(\cdot) = \mathcal{B}^{-1}_{\cdot,t_0}\big(g_*\big(\rho(\cdot; t_0)\big)\big)$, where $g_*(\cdot)$ is a most likely path for $\{X^\varepsilon : \varepsilon > 0\}$, i.e.

$$g_*(r) = \begin{cases} \mathcal{B}_{t_0,t_0}(y_0) + \dfrac{\widehat{b}_{t_0}\big(\rho^{-1}(r_*; t_0); b_*\big) - \mathcal{B}_{t_0,t_0}(y_0)}{r_*} r & \text{if } r \in [0, r_*], \\ \widehat{b}_{t_0}\big(\rho^{-1}(r_*; t_0); b_*\big) & \text{if } r \in \big(r_*, \rho(T; t_0)\big] \end{cases}$$

for suitable choices of $b_* \in \{b_{(+)}, b_{(-)}\}$ and $r_* \in \big(0, \rho(T; t_0)\big]$. Then, for $t_* = \rho^{-1}(r_*; t_0)$, we have

$$\begin{aligned} f_*(t) &= \mathcal{B}^{-1}_{t,t_0}\Big(g_*\big(\rho(t; t_0)\big)\Big) \\ &= \begin{cases} \mathcal{B}^{-1}_{t,t_0}\left(\mathcal{B}_{t_0,t_0}(y_0) + \dfrac{\widehat{b}_{t_0}(t_*; b_*) - \mathcal{B}_{t_0,t_0}(y_0)}{\rho(t_*; t_0)}\rho(t; t_0)\right) & \text{if } t \in [t_0, t_*], \\ \mathcal{B}^{-1}_{t,t_0}\big(\widehat{b}_{t_0}(t_*; b_*)\big) & \text{if } t \in (t_*, T]. \end{cases} \end{aligned}$$

For instance, in the case of linear diffusions with additive noise, we have

$$f_*(t)=\begin{cases}\exp\big(\int\limits_{t_0}^{t}\mu_1(v)dv\big)\cdot\Big(y_0+\dfrac{\widehat{b}_{t_0}(t_*;b_*)-y_0}{\rho(t_*;t_0)}\rho(t;t_0)\\ \qquad\qquad +\int\limits_{t_0}^{t}\mu_0(v)\exp\big(-\int\limits_{t_0}^{v}\mu_1(w)dw\big)dv\Big) & \text{if } t\in[t_0,t_*],\\ \exp\big(\int\limits_{t_0}^{t}\mu_1(v)dv\big)\\ \times\Big(\widehat{b}_{t_0}(t_*;b_*)+\int\limits_{t_0}^{t}\mu_0(v)\exp\big(-\int\limits_{t_0}^{v}\mu_1(w)dw\big)dv\Big) & \text{if } t\in(t_*,T].\end{cases}$$

It is easy to see that $f_*(t_0)=y_0$ and $f_*(t_*)=b_*(t_*)$. Moreover, for each family of diffusions $\{Y^\varepsilon:\varepsilon>0\}$ in this section, we can check that f_* is a solution of Euler's equation (4.3) and, if $\dot{f}_*$ has a discontinuity at some point (this could happen at $t=t_*$), then the equation (4.3) is satisfied by right-hand and left-hand derivatives. This shows that $f_*(\cdot)=\mathcal{B}^{-1}_{\cdot,t_0}\big(g_*\big(\rho(\cdot;t_0)\big)\big)$ is a most likely path for $\{Y^\varepsilon:\varepsilon>0\}$.

EXAMPLE 4.1. We consider the framework of Proposition 3.1 with $b_{(\pm)}=Y^0\pm\delta$ for some $\delta>0$. Then, by taking into account that

$$\widehat{b}_{t_0}(t;b_{(\pm)})=\mathcal{B}_{t,t_0}\big(\mathcal{B}^{-1}_{t,t_0}(x_0)\pm\delta\big),$$

we have

$$u(y_0,t_0)=\min_{a\in\{1,-1\}}\inf_{t_0<t\leqslant T}\frac{\Big(\mathcal{B}_{t,t_0}\big(\mathcal{B}^{-1}_{t,t_0}(x_0)+a\delta\big)-x_0\Big)^2}{2\rho(t;t_0)}.$$

In particular, if we deal with the linear diffusions with additive noise, the last equality takes the form

$$u(y_0,t_0)=\inf_{t_0<t\leqslant T}\frac{\delta^2\exp\big(-2\int_{t_0}^{t}\mu_1(v)dv\big)}{2\int_{t_0}^{t}\sigma^2(v)\exp\big(-2\int_{t_0}^{v}\mu_1(w)dw\big)dv}$$

because

$$\mathcal{B}_{t,t_0}(b)=b\exp\big(-\int\limits_{t_0}^{t}\mu_1(v)dv\big)-\int\limits_{t_0}^{t}\mu_0(v)\exp\big(-\int\limits_{t_0}^{v}\mu_1(w)dw\big)dv.$$

Moreover, if μ_0, μ_1 and σ are constant functions with $\mu_1\neq0$, we get the limit value in Proposition 2 in [2], i.e.

$$u(y_0,t_0)=\inf_{t_0<t\leqslant T}\frac{\mu_1\delta^2\exp\big(-2\mu_1(t-t_0)\big)}{\sigma^2\Big(1-\exp\big(-2\mu_1(t-t_0)\big)\Big)}=\frac{\mu_1\delta^2\exp\big(-2\mu_1(T-t_0)\big)}{\sigma^2\Big(1-\exp\big(-2\mu_1(T-t_0)\big)\Big)}.$$

EXAMPLE 4.2. We consider geometric Brownian motions, i.e. linear diffusions with multiplicative noise where μ and σ are constant functions. Moreover we consider $b_{(\pm)}(t) = r_{(\pm)}e^{\gamma t}$ (for $t \in [0,T]$) for $r_{(+)}, r_{(-)} > 0$ and $\gamma \in \mathbb{R}$ (note that we have positive constant barriers with $\gamma = 0$). Finally, we assume that $r_{(-)}e^{\gamma t_0} < y_0 < r_{(+)}e^{\gamma t_0}$. Then, by taking into account that

$$\widehat{b}_{t_0}(t; b_{(\pm)}) = \log(r_{(\pm)}e^{\gamma t}) - \mu(t - t_0),$$

we have $u(y_0, t_0) = \min_{r \in \{r_{(+)}, r_{(-)}\}} u(y_0, t_0; r)$, where

$$u(y_0, t_0; r) := \inf_{t_0 < t \leqslant T} \frac{\big(\log r + \gamma t - \mu(t - t_0) - \log y_0\big)^2}{2\sigma^2(t - t_0)}.$$

We have two cases. Firstly, if $\mu = \gamma$, we get

$$u(y_0, t_0; r) = \inf_{t_0 < t \leqslant T} \frac{(\log r + \mu t_0 - \log y_0)^2}{2\sigma^2(t - t_0)} = \frac{(\log r + \mu t_0 - \log y_0)^2}{2\sigma^2(T - t_0)}.$$

Secondly, if $\mu \neq \gamma$, we obtain

$$u(y_0, t_0; r) = \begin{cases} \dfrac{(\log r + \gamma T - \mu(T - t_0) - \log y_0)^2}{2\sigma^2(T - t_0)} & \text{if } T - t_0 < \dfrac{|\log r - \log y_0 + \gamma t_0|}{|\gamma - \mu|}, \\ \dfrac{(\gamma - \mu)^2}{\sigma^2}\left(\dfrac{|\log r - \log y_0 + \gamma t_0|}{|\gamma - \mu|} + \dfrac{\log r - \log y_0 + \gamma t_0}{\gamma - \mu}\right) & \\ & \text{if } T - t_0 \geqslant \dfrac{|\log r - \log y_0 + \gamma t_0|}{|\gamma - \mu|}. \end{cases}$$

In particular, if $T - t_0 \geqslant |\log r - \log y_0 + \gamma t_0||\gamma - \mu|^{-1}$, we have

$$u(y_0, t_0; r) = \begin{cases} 0 & \text{if } \dfrac{\log r - \log y_0 + \gamma t_0}{\gamma - \mu} < 0, \\ \dfrac{2(\gamma - \mu)}{\sigma^2}(\log r - \log y_0 + \gamma t_0) & \text{if } \dfrac{\log r - \log y_0 + \gamma t_0}{\gamma - \mu} > 0. \end{cases}$$

In both the cases, for $D = \{(y,t) \in [0,T) \times \mathbb{R} : b_{(-)}(t) < y < b_{(+)}(t)\}$, one can check that $u(\cdot, \cdot; r)$ is a classical solution of the equation

$$-\frac{\partial}{\partial t_0}u(y_0, t_0; r) - \mu y_0 \frac{\partial}{\partial y_0}u(y_0, t_0; r) + \frac{1}{2}\sigma^2 y_0^2\left(\frac{\partial}{\partial y_0}u(y_0, t_0; r)\right)^2 = 0 \quad \text{on } D,$$

$$u(y, t; r) = 0 \quad \text{if } y \in \{b_{(+)}(t), b_{(-)}(t)\} \text{ for some } t \in [0,T),$$

$$u(y, t; r) \to +\infty \quad \text{as } t \uparrow T, \text{ for } y \in \big(b_{(-)}(T), b_{(+)}(T)\big).$$

Finally, one can check that $u(\cdot,\cdot) = \min_{r\in\{r_{(+)},r_{(-)}\}} u(\cdot,\cdot;r)$ is a viscosity solution of the same PDE because it is the minimum between two classical solutions. This can be checked by referring to the concept of viscosity solution (see, e.g., [1], pp. 4 and 5).

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