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FAST APPROXIMATION OF SOLUTIONS OF SDE'S WITH OBLIQUE REFLECTION ON AN ORTHANT

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Abstract. We consider the discrete "fast" penalization scheme for SDE's driven by general semimartingale on orthant $\mathbb{R}_+^d$ with oblique reflection.

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1. INTRODUCTION

Suppose we have a d-dimensional semimartingale $Z = (Z^1, \ldots, Z^d)^T$, a Lipschitz continuous function $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, and a nonnegative $d \times d$ matrix Q with zeros on the diagonal and spectral radius $\rho(Q)$ strictly less than one. Consider a d-dimensional stochastic differential equation (SDE) on orthant $\mathbb{R}_+^d$ with oblique reflection:

$$X_t = X_0 + \int_0^t \sigma(X_{s-})dZ_s + (1 - Q^T)K_t, \quad t \in \mathbb{R}_+. \tag{1.1}$$

The equation of type (1.1) was introduced by Harrison and Reiman [9]. Later it was discussed by Dupuis and Ishi [5]. Czarkowski and Słomiński [3] introduced a numerical scheme for approximation of solution of SDE (1.1). In this paper we will define a new numerical scheme.

Throughout the paper we assume $\rho_t^n = \max\{i/n; i \in \mathbb{N} \cup \{0\}, i/n \leqslant t\}$ and $Z_t^{(n)}$ is a discretization of Z, i.e. $Z_t^{(n)} = Z_{\rho_t^n}$, $\to_{\mathcal{P}}$ denotes convergence in probability, $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ means the space of càdlàg function $y : \mathbb{R}_+ \to \mathbb{R}^d$, $\Delta y_t = y_t - y_{t-}$ and $\omega_{1/n}(y, [0, t])$ denotes the modulus of continuity of y on $[0, t]$.

Let us define the function $[z]^+ = \max\{z, 0\}$ for $z \in \mathbb{R}$ and, by analogy, the function $[z]^+ = ([z^1]^+, \ldots, [z^d]^+)^T$ for $z = (z^1, \ldots, z^d)^T \in \mathbb{R}^d$. We will use the norm $\|Q\| = \max_{1 \leqslant i \leqslant d} \sum_{j=1}^d q_{ij}$.

In the simplest form our new numerical scheme is given in Section 3 (see (3.3)):

$$x^n_{(i+1)/n} = x^n_{i/n} + \Delta y_{(i+1)/n} + (I - Q^T)[-x^n_{i/n} - \Delta y_{(i+1)/n}]^+.$$

In Section 3 we prove also convergence in some topology for the càdlàg and continuous function y (see Theorem 3.1 and Corollary 3.1). We show that our scheme satisfies the Lipschitz property for the càdlàg function. In Section 4, we use the scheme for SDE driven by a semimartingale Z_t. In Section 5, we prove that

$$E \sup_{s \leqslant t} |X^n_s - X_s|^{2p} = \mathcal{O}\left(\left(\frac{\ln n}{n}\right)^p\right)$$

for diffusion X_t.

The Appendix includes a description of some properties of Π_Q projection on the orthant $\mathbb{R}^d_+$, which connects this paper with [3].

2. THE SKOROKHOD PROBLEM ON AN ORTHANT

Let Q be a nonnegative matrix with zeros on the diagonal and spectral radius $\rho(Q) < 1$ and let $y \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ with $y_0 \in \mathbb{R}^d_+$. Following Harrison and Reiman [9] a pair $(x, k) \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^{2d})$ is called a *solution to the Skorokhod problem*

$$x_t = y_t + (I - Q^T)k_t, \quad t \in \mathbb{R}_+, \tag{2.1}$$

on $\mathbb{R}^d_+$ associated with y, if (2.1) is satisfied and

$x_t \in \mathbb{R}^d_+$, $t \in \mathbb{R}_+$,

k^j is nondecreasing, $k^j_0 = 0$ and $\int_0^t x^j_s \, dk^j_s = 0$ for $j = 1, \ldots, d,\ t \in \mathbb{R}_+$.

REMARK 2.1 ([3], Theorem 1).

1. For every $y \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ with $y_0 \in \mathbb{R}^d_+$ there exists a unique solution (x_t, k_t) of the Skorokhod problem.

2. If additionally $\|Q\| < 1$, then k_t satisfies the equation

$$k_t = F(k)_t, \tag{2.2}$$

where

$$F(u)_t = \sup_{s \leqslant t} [Q^T u_s - y_s]^+.$$

In this paper, like in [9] and [3], we make a technical assumption that

$$\|Q\| < 1. \tag{2.3}$$

3. FAST APPROXIMATION SCHEME

Let (x, k) be a solution to the Skorokhod problem for $y \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ with $y_0 \in \mathbb{R}^d_+$.

For every $n \in \mathbb{N}$ we define the approximations (x^n, k^n) of (x, k):

$$\begin{cases} k^n_0 = 0, \quad x^n_0 = y_0, \\ k^n_{(i+1)/n} = [Q^T k^n_{i/n} - y_{(i+1)/n}]^+ \vee k^n_{i/n}, \\ x^n_{(i+1)/n} = y_{(i+1)/n} + (I - Q^T) k^n_{(i+1)/n}, \\ k^n_t = k^n_{i/n}, \;\; x^n_t = x^n_{i/n} \text{ for } t \in \big[i/n, (i+1)/n\big). \end{cases} \tag{3.1}$$

REMARK 3.1. We can write another but equivalent form of k^n, x^n. Note that for every $n \in \mathbb{N}$, $i \in \mathbb{N} \cup \{0\}$:

$$\begin{aligned} k^n_{(i+1)/n} &= [(Q^T - I)k^n_i - y_{(i+1)/n}]^+ + k^n_{i/n} \qquad (3.2)\\ &= [-(x^n_{i/n} + \Delta y_{(i+1)/n})]^+ + k^n_{i/n}, \end{aligned}$$

$$x^n_{(i+1)/n} = x^n_{i/n} + \Delta y_{(i+1)/n} + (I - Q^T)[-x^n_{i/n} - \Delta y_{(i+1)/n}]^+, \tag{3.3}$$

where $\Delta y_{(i+1)/n} = y_{(i+1)/n} - y_{i/n}$.

Formulas (3.1) and (3.3) are equivalent, but (3.3) looks better and is simpler to calculate. The form (3.3) can be used in computer simulations.

REMARK 3.2. We can see that k^n_t satisfies the equations

$$k^n_t = F^n(k^{n,(n-)})_t, \tag{3.4}$$

where $F^n(u)_t = \sup_{s \leqslant t}[Q^T u_s - y^{(n)}_s]^+$, $u^{(n-)}_t = u_{(i-1)/n}$, $t \in \big[i/n, (i+1)/n\big)$.

The next two theorems describe some properties of scheme (3.1). In Theorem 3.1 we estimate a "distance" between a function x and its approximation x^n (and k and k^n), and in Theorem 3.2 we prove the Lipschitz property for our scheme.

THEOREM 3.1. *There exists a constant $\mathcal{C} > 0$ depending only on Q such that for every $y \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ with $y_0 \in \mathbb{R}^d_+, t \in \mathbb{R}_+$:*

$$\sup_{s \leqslant t} |x^n_s - x_s| + \sup_{s \leqslant t} |k^n_s - k_s| \leqslant \mathcal{C}\omega_{1/n}(y, [0, t]). \tag{3.5}$$

Proof. Since $\sup_{s \leqslant t} |x^n_s - x_s| < \|Q\| \sup_{s \leqslant t} |k^n_s - k_s| + \omega_{1/n}(y, [0, t])$, we estimate only the first term, i.e. $\sup_{s \leqslant t} |k^n_s - k_s|$.

We assume that (2.3) is satisfied, i.e. $\|Q\| < 1$.

From Remarks 2.1 and 3.2 we obtain

$$\begin{aligned}\sup_{s\leqslant t}|k^n_s-k_s| &= \sup_{s\leqslant t}|F^n(k^{n,(n-)})_s-F(k)_s|\\ &\leqslant \sup_{s\leqslant t}|F^n(k^{n,(n-)})_s-F^n(k^n)_s|+\sup_{s\leqslant t}|F^n(k^n)_s-F(k^n)_s|\\ &\quad+\sup_{s\leqslant t}|F(k^n)_s-F(k)_s|\\ &= I^1_t+I^2_t+I^3_t.\end{aligned}$$

Now we estimate every part separately:

$$\begin{aligned}I^1_t &= \sup_{s\leqslant t}|F^n(k^{n,(n-)})_s-F^n(k^n)_s|\leqslant \|Q\|\max_{i/n\leqslant t}|k^n_{(i-1)/n}-k^n_{i/n}|\\ &\leqslant \|Q\|^2\max_{i/n\leqslant t}|k^n_{(i-2)/n}-k^n_{(i-1)/n}|+\|Q\|\max_{i/n\leqslant t}|y_{(i-1)/n}-y_{i/n}|\\ &\leqslant \frac{\|Q\|}{1-\|Q\|}\omega_{1/n}(y,[0,t]),\\ I^2_t &= \sup_{s\leqslant t}|F^n(k^n)_s-F(k^n)_s|\\ &\leqslant \sup_{s\leqslant t}|[Q^Tk^n_s-y^{(n)}_s]^+-[Q^Tk^n_s-y_s]^+|\\ &\leqslant \sup_{s\leqslant t}|y^{(n)}_s-y_s|=\omega_{1/n}(y,[0,t]),\\ I^3_t &= \sup_{s\leqslant t}|F(k^n)_s-F(k)_s|\leqslant \|Q\|\sup_{s\leqslant t}|k^n_s-k_s|.\end{aligned}$$

Consequently, we have

$$\sup_{s\leqslant t}|k^n_s-k_s|\leqslant \frac{\|Q\|}{1-\|Q\|}\omega_{1/n}(y,[0,t])+\omega_{1/n}(y,[0,t])+\|Q\|\sup_{s\leqslant t}|k^n_s-k_s|.$$

So we can calculate the value of the constant $\mathcal{C}$ as follows:

$$\sup_{s\leqslant t}|k^n_s-k_s|\leqslant \frac{1}{(1-\|Q\|)^2}\omega_{1/n}(y,[0,t]).\quad\blacksquare$$

COROLLARY 3.1. *For every $y\in\mathbb{C}(\mathbb{R}_+,\,\mathbb{R}^d)$ with $y_0\in\mathbb{R}^d_+$ we have*

$$\sup_{s\leqslant t}|x^n_s-x_s|+\sup_{s\leqslant t}|k^n_s-k_s|\to 0.$$

THEOREM 3.2. *There exists a constant $\mathcal{C}>0$ depending only on Q such that for every $y^1,y^2\in\mathbb{D}(\mathbb{R}_+,\,\mathbb{R}^d)$, $y^1_0,y^2_0\in\mathbb{R}^d_+$:*

$$\sup_{s\leqslant t}|k^{1,n}_s-k^{2,n}_s|+\sup_{s\leqslant t}|x^{1,n}_s-x^{2,n}_s|\leqslant \mathcal{C}\sup_{s\leqslant t}|y^1_s-y^2_s|.$$

P r o o f. As in Theorem 3.1 we need only to prove the first of the examined terms. The second can be obtained from (3.1):

$$\begin{aligned}\sup_{s\leqslant t}|k_s^{1,n}-k_s^{2,n}| &= \sup_{s\leqslant t}|F^n(k^{1,n,(n-)})_s - F^n(k^{2,n,(n-)})_s|\\ &= \sup_{s\leqslant t}|[Q^T k_s^{1,n,(n-)} - y_s^{1,(n)}]^+ - [Q^T k_s^{2,n,(n-)} - y_s^{2,(n)}]^+|\\ &\leqslant \|Q\| \max_{i/n\leqslant t}|k^{1,n}_{(i-1)/n} - k^{2,n}_{(i-1)/n}| + \max_{i/n\leqslant t}|y^1_{i/n} - y^2_{i/n}|\\ &\leqslant \|Q\|\sup_{s\leqslant t}|k_s^{1,n}-k_s^{2,n}| + \sup_{s\leqslant t}|y_s^1 - y_s^2|\end{aligned}$$

and

$$\sup_{s\leqslant t}|k_s^{1,n}-k_s^{2,n}| \leqslant \frac{1}{1-\|Q\|}\sup_{s\leqslant t}|y_s^1-y_s^2|. \quad \blacksquare$$

We obtain an easy corollary:

COROLLARY 3.2. *There exists a constant* $\mathcal{C} > 0$ *such that for every* $y \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ *with* $y_0 \in \mathbb{R}^d_+$:

$$k_t^n \leqslant \mathcal{C}\sup_{s\leqslant t}|y_s| < +\infty.$$

From previous theorems we can obtain convergence for continuous functions. For a càdlàg function, we expect some problems with convergence in points of discontinuity.

LEMMA 3.1. *Assume that* $y \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ *with* $y_0 \in \mathbb{R}^d_+$ *has the form*

$$y_t = \sum_{i=0}^{+\infty} y_{t_i}\mathbf{1}_{[t_i,t_{i+1})}(t), \tag{3.6}$$

where $0 = t_0 < t_1 < \ldots$ *Then*

$$x_t^n \to x_t \quad \text{as } n\to\infty \tag{3.7}$$

for $t \neq t_i$, $i \in \mathbb{N}$, *where* (x_t, k_t) *is a solution of the Skorokhod problem for* y_t.

P r o o f. We prove the lemma by induction. It is well known that if y is of the form (3.6), then

$$x_t = \begin{cases} y_0, & t\in[0,t_1),\\ \Pi_Q(x_{t_{i-1}} + \Delta y_{t_i}), & t\in[t_i,t_{i+1}),\ i\in\mathbb{N}.\end{cases}$$

1. For $t \in [0, t_1)$ the assertion is satisfied by definition.

2. By (3.3) we have:

$$x^n_{(i+1)/n} = \begin{cases} x^n_{i/n} + \Delta y_{(i+1)/n} + (I - Q^T)[-x^n_{i/n} - \Delta y_{(i+1)/n}]^+ \\ \qquad\qquad\qquad\qquad \text{for } i \text{ such that } i/n \leqslant t_i < (i+1)/n, \\ x^n_{i/n} + (I - Q^T)[-x^n_{i/n}]^+ \quad \text{for } i \text{ such that } t_i < i/n < t_{i+1}. \end{cases}$$

The second part, the sequence $x_{i/n}$ between jumps looks like z_i (by (6.2) in the Appendix) for a starting point $z_0 = (x_{t_i} + \Delta y_{t_i})$.

Now, from Corollary 6.1 we have

$$\lim_{n\to+\infty} x^n_t = \Pi_Q(x_{t_i} + \Delta y_{t_i}) \quad \text{for } t \in (t_i, t_{i+1}). \quad \blacksquare$$

In the next example we show that (3.7) cannot be straightened to the convergence in the Skorokhod topology J_1.

EXAMPLE 3.1. Let $d = 2$,

$$Q = \begin{Bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{Bmatrix} \quad \text{and} \quad y_t = \begin{cases} (0,0)^T, & t < 1, \\ (-1,-1)^T, & t \geqslant 1. \end{cases}$$

The following functions are the solution of the Skorokhod problem:

$$x_t = (0,0)^T,\ t \in \mathbb{R}_+, \quad \text{and} \quad k_t = \begin{cases} (0,0)^T, & t < 1, \\ (2,2)^T, & t \geqslant 1. \end{cases}$$

Now, we use scheme (3.1) for these functions and try to find the limit of (x^n, k^n) when n tends to infinity. Then we obtain

$$k^n_t = \begin{cases} (0,0)^T, & t < 1, \\ (1,1)^T, & t \in [1, 1+1/n), \\ (2 - 1/2^i, 2 - 1/2^i)^T, & t \in \big[1 + i/n, 1 + (i+1)/n\big),\ \ i \in \mathbb{N}, \end{cases}$$

and

$$x^n_t = \begin{cases} (0,0)^T, & t < 1, \\ \big(-\frac{1}{2}, -\frac{1}{2}\big)^T, & t \in [1, 1+1/n), \\ (-1/2^{i+1}, -1/2^{i+1})^T, & t \in \big[1 + i/n, 1 + (i+1)/n\big),\ \ i \in \mathbb{N}. \end{cases}$$

Since $\sup_{t\leqslant 2} |x^n_t| = \frac{1}{2}$, we have the solution $x^n \not\to x$ in J_1.

Using the notation of jump point, we can obtain another type of problems. For example, for jump at $t_1 = \frac{1}{3}$ the limit $\lim_{n\to+\infty} x^n_{t_1}$ does not exist.

Classical topology J_1 is too strong in order to obtain convergence for our scheme. There exists a topology S that is weaker than J_1, which is obviously weaker than the uniform topology. The S topology has been introduced by Jakubowski in [11] and in the next papers (e.g. [12]) good criteria of convergence in topology S were given. From the point of view of computer simulation and numerical methods convergence in the S topology is sufficient.

From Lemma 2.14 in [11] we obtain

COROLLARY 3.3. *If y satisfies the assumption of Lemma 3.1, then*

$$(x^n, k^n) \to (x, k) \text{ in } \big(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{2d}), S\big). \tag{3.8}$$

The following theorem can be generalized for càdlàg functions.

THEOREM 3.3. *If $y \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ and $y_0 \in \mathbb{R}^d_+$, then*

$$(x^n, k^n) \to (x, k) \text{ in } \big(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{2d}), S\big). \tag{3.9}$$

Proof. For all $y \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ and all $\epsilon > 0$ there exists $y^\epsilon \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ satisfying the assumption of Lemma 3.1 so that $\sup_{s \leqslant t} |y^\epsilon_s - y_s| \leqslant \epsilon$.

Let $t_0 = 0$ and

$$t_{i+1} = \inf\{s > t_i : |y_s - y_{t_i}| \geqslant \epsilon\}.$$

Then

$$y^\epsilon_t = y_{t_i}, \quad t \in [t_i, t_{i+1}).$$

Let the pair (x^ϵ, k^ϵ) be a solution of the Skorokhod problem for y^ϵ. Then from Lemma 3.1 we get

$$(x^{\epsilon,n}, k^{\epsilon,n}) \to (x^\epsilon, k^\epsilon) \text{ in } \big(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{2d}), S\big).$$

To prove the assertion we need to show that

$$\lim_{\epsilon \to 0} \sup_n \sup_{s \leqslant t} |k^{\epsilon,n}_s - k^n_s| = 0.$$

From Theorem 3.2 we have

$$\sup_{s \leqslant t} |k^{\epsilon,n}_s - k^n_s| \leqslant \mathcal{C} \sup_{s \leqslant t} |y^{\epsilon(n)}_s - y^{(n)}_s| \leqslant \mathcal{C}\epsilon. \quad \blacksquare$$

REMARK 3.3. If $y \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ and $y_0 \in \mathbb{R}^d_+$, then $x^n \to x$ for continuity point of y and $\{x^n\}$ is relatively S-compact.

4. FAST APPROXIMATION SCHEME FOR SDE

Let Z be an $(\mathcal{F}_t)$-adapted semimartingale. Let us recall that the pair (X, K) of $(\mathcal{F}_t)$-adapted processes is said to be a *strong solution* of (1.1) if (X, K) is a solution to the Skorokhod problem associated with the semimartingale

$$Y_t = X_0 + \int_0^t \sigma(X_{s-})dZ_s, \quad t \in \mathbb{R}_+. \tag{4.1}$$

REMARK 4.1. If σ is Lipschitz continuous, then there exists a unique strong solution to the SDE (1.1).

Using formulas (3.1) we can define a "fast" scheme for SDE:

$$\begin{aligned} X_0^n &= X_0, \quad K_0^n = 0, \\ K_{(i+1)/n}^n &= \big[Q^T K_{i/n}^n - \big(X_{i/n}^n + \sigma(X_{i/n}^n)(Z_{(i+1)/n} - Z_{i/n})\big)\big]^+ \vee K_{i/n}^n, \\ X_{(i+1)/n}^n &= X_{i/n}^n + \sigma(X_{i/n}^n)(Z_{(i+1)/n} - Z_{i/n}) + (1 - Q^T)K_{(i+1)/n}^n, \\ (X_t^n, K_t^n) &= (X_{i/n}^n, K_{i/n}^n), \quad t \in \big[i/n, (i+1)/n\big). \end{aligned}$$

LEMMA 4.1. *Assume that there exist stoping times $\{\tau_i\} \subset \mathbb{R}_+$ such that $0 = \tau_0 < \tau_1 < \ldots$ and $\{Z_i\} \subset \mathbb{R}^d$. If Z is a semimartingale such that $Z_t = Z_i$ for $t \in [\tau_i, \tau_{i+1})$, $i \in \mathbb{N} \cup \{0\}$, then*

$$X_t^n \to X_t \quad \textit{for } t \neq t_i. \tag{4.2}$$

Proof. We define

$$X_t = \begin{cases} X_0, & t \in [0, \tau_1), \\ \Pi_Q\big(X_{\tau_{i-1}} + \sigma(X_{\tau_{i-1}})\Delta Z_{\tau_i}\big), & t \in [\tau_i, \tau_{i+1}),\ i \in \mathbb{N}. \end{cases}$$

The rest of the proof is the same as for Lemma 3.1. We need only to change $\Delta y_{(i+1)/n}^{(n)}$ by $\sigma(X_{\tau_{i-1}})\Delta Z_{\tau_i}$. ■

THEOREM 4.1. *Assume that σ is Lipschitz continuous. Then*

$$(X^n, K^n) \underset{\mathcal{P}}{\to} (X, K) \textit{ in } \big(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{2d}), S\big). \tag{4.3}$$

Proof. As in Theorem 3.3, for all $\epsilon > 0$ we construct a "piecewise constant" martingale Z^ϵ. It follows that for all $\epsilon > 0$ there exists Z^ϵ such that

$$\sup_{s \leqslant t} |Z_s - Z_s^\epsilon| \leqslant \epsilon.$$

Let the pair $(X^{\epsilon n}, K^{\epsilon n})$ be a solution of the Skorokhod problem for the semimartingale

$$Y_t = X_0 + \int_0^t \sigma(X_{s-})dZ_s^\epsilon, \quad t \in \mathbb{R}_+.$$

From Lemma 4.1 we have the convergence

$$(X^{\epsilon n}, K^{\epsilon n}) \to (X^{\epsilon}, K^{\epsilon}) \text{ in } \big(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{2d}), S\big).$$

To complete the proof we need to show that, for all $\eta > 0$,

$$\lim_{\epsilon\to 0} \limsup_{n} P(\sup_{s\leqslant t} |X_s^{\epsilon n} - X_s^n| > \eta) = 0.$$

Using Theorem 3.2, we obtain

$$\begin{aligned}
\sup_{s\leqslant t} |X_s^{\epsilon n} - X_s^n| &\leqslant \mathcal{C} \sup_{s\leqslant t} |Y_s^{\epsilon n} - Y_s^n| \\
&= \mathcal{C} \sup_{s\leqslant t} \Big| \int_0^s \sigma(X_{u-}^n) dZ_u^{(n)} - \int_0^s \sigma(X_{u-}^{\epsilon n}) dZ_u^{\epsilon,(n)} \Big| \\
&\leqslant \mathcal{C} \sup_{s\leqslant t} \Big| \int_0^s \sigma(X_{u-}^n) dZ_u^{(n)} - \int_0^s \sigma(X_{u-}^{\epsilon n}) dZ_u^{(n)} \Big| \\
&\quad + \mathcal{C} \sup_{s\leqslant t} \Big| \int_0^s \sigma(X_{u-}^{\epsilon n}) dZ_u^{(n)} - \int_0^s \sigma(X_{u-}^{\epsilon n}) dZ_u^{\epsilon,(n)} \Big| \\
&= \mathcal{C} \sup_{s\leqslant t} \Big| \int_0^s \big(\sigma(X_{u-}^n) - \sigma(X_{u-}^{\epsilon n})\big) dZ_u^{(n)} \Big| + \mathcal{C} \sup_{s\leqslant t} |H_s^{\epsilon,n}|,
\end{aligned}$$

$$\begin{aligned}
H_t^{\epsilon,n} &= \int_0^t \sigma(X_{s-}^{\epsilon n}) dZ_s^{(n)} - \int_0^t \sigma(X_{s-}^{\epsilon n}) dZ_s^{\epsilon,(n)} \\
&= \int_0^t \sigma(X_{s-}^{\epsilon n}) d(Z_s^{(n)} - Z_s^{\epsilon,(n)}) \\
&= \sigma(X_t^{\epsilon n})(Z_t^{(n)} - Z_t^{\epsilon,(n)}) - \int_0^t (Z_{s-}^{(n)} - Z_{s-}^{\epsilon,(n)}) d\sigma(X_s^{\epsilon n}) \\
&\quad - [\sigma(X_t^{\epsilon n}), (Z_t^{(n)} - Z_t^{\epsilon,(n)})].
\end{aligned}$$

Obviously,

$$[\sigma(X_t^{\epsilon n}), (Z_t^{(n)} - Z_t^{\epsilon,(n)})] \leqslant \big([\sigma(X_t^{\epsilon n})]\big)^{1/2} \big([(Z_t^{(n)} - Z_t^{\epsilon,(n)})]\big)^{1/2}$$

By definition, $X_t^{\epsilon n}$ has the form

$$X_t^{\epsilon n} = X_0 + \int_0^t \sigma(X^{\epsilon_{s-}^n}) dZ_s^{\epsilon,n} + (1 - Q^T) K_t^{\epsilon n}.$$

So, we can write the inequalities

$$\sup_{s\leqslant t}|X_t^{\epsilon n}| \leqslant |X_0| + \sup_{s\leqslant t}\Big|\int_0^t \sigma(X^{\epsilon_u - n})dZ_u^{\epsilon,n}\Big| + (1-Q^T)\sup_{s\leqslant t}|K^{\epsilon_s^n}|$$
$$\leqslant |X_0| + 2\mathcal{C}\sup_{s\leqslant t}\Big|\int_0^t \sigma(X^{\epsilon_{u-}^n})dZ_u^{\epsilon,n}\Big|.$$

From Gronwall's lemma it follows that $\{\sup|X^{\epsilon^n}|\}$ is bounded for σ satisfying the Lipschitz condition. So, if $\{\sigma(|X^{\epsilon^n}|)\}$ is bounded in probability, then

$$\int_0^t \sigma(X^{\epsilon_s - n})dZ_s^{\epsilon,n}$$

satisfies UT condition.

Because $\{K^{\epsilon^n}\}$ is bounded in probability, this means that it also satisfies UT. $\{X^{\epsilon^n}\}$ satisfies UT as a sum of two processes that satisfy UT. So, $\sigma(X^{\epsilon^n})$ satisfies UT for $\sigma \in C^2$. ■

5. FAST APPROXIMATION SCHEME FOR DIFFUSION

Consider SDE with reflection on $\mathbb{R}_+^d$ of the form

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + (1-Q^T)K_t, \tag{5.1}$$

where W is a d-dimensional Wiener process, and $b\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$.

REMARK 5.1. If b and σ are Lipschitz continuous, then there exists a unique strong solution of the SDE (5.1).

Let us define

$$X_0^n = X_0, \quad K_0^n = 0,$$
$$K_{(i+1)/n}^n = \big[Q^T K_{i/n}^n - \big(X_{i/n}^n + b(X_{i/n}^n)n^{-1} + \sigma(X_{i/n}^n)(W_{(i+1)/n} - W_{i/n})\big)\big]^+ \vee K_{i/n}^n,$$
$$X_{(i+1)/n}^n = X_{i/n}^n + b(X_{i/n}^n)n^{-1} + \sigma(X_{i/n}^n)(W_{(i+1)/n} - W_{i/n}) + (1-Q^T)K_{(i+1)/n}^n,$$
$$(X_t^n, K_t^n) = (X_{i/n}^n, K_{i/n}^n), \qquad t \in \big[i/n, (i+1)/n\big).$$

We can see that X^n satisfies the equation

$$X_t^n = X_0^n + \int_0^t b(X_{s-}^n)d\rho_s^n + \int_0^t \sigma(X_{s-}^n)dW_s^{(n)} + (1-Q^T)K_t^n. \tag{5.2}$$

THEOREM 5.1. *Let the assumptions of Remark* 5.1 *be satisfied and let* (X, K) *be a strong solution to the SDE* (5.1). *Then for every* $p \in \mathbb{N}$

$$E \sup_{s \leqslant t} |X_s^n - X_s|^{2p} = \mathcal{O}\Big(\Big(\frac{\ln n}{n}\Big)^p\Big). \tag{5.3}$$

First we prove the following lemma:

LEMMA 5.1. *Under the assumptions as in Theorem* 5.1 *we obtain*

$$\sup_n E \sup_{s \leqslant t} |X_s^n|^{2p} < +\infty. \tag{5.4}$$

Proof. We have

$$\sup_{s \leqslant t} |X_s^n - X_0^n| \leqslant \mathcal{C} \sup_{s \leqslant t} \Big| \int_0^s \sigma(X_{u-}^n) dW_u^{(n)} + \int_0^s b(X_{u-}^n) d\rho_u^n \Big|, \tag{5.5}$$

which implies

$$\sup_{s \leqslant t} |X_s^n - X_0^n|^{2p} \leqslant 2\mathcal{C} \sup_{s \leqslant t} \Big| \int_0^s \sigma(X_{u-}^n) dW_u^{(n)} \Big|^{2p} + 2\mathcal{C} \sup_{s \leqslant t} \Big| \int_0^s b(X_{u-}^n) d\rho_u^n \Big|^{2p}.$$

Now, because b and σ are Lipschitz, we have

$$\begin{aligned} E \sup_{s \leqslant t} |X_s^n - X_0^n|^{2p} &\leqslant 2\mathcal{C}E\Big(\int_0^t \sigma(X_{s-}^n) dW_s^{(n)} \Big)^{2p} + 2\mathcal{C}E\Big(\int_0^t |b(X_s^n)| ds \Big)^{2p} \\ &\leqslant 2\mathcal{C}E \int_0^t \sigma^{2p}(X_{s-}^n) d\rho_s^n + 2\mathcal{C}E \int_0^t b^{2p}(X_{s-}^n) d\rho_s^n \\ &\leqslant \mathcal{C}E \int_0^t \big((X_{s-}^n)^{2p} + 1\big) d\rho_s^n \\ &\leqslant \mathcal{C}\Big(1 + \int_0^t E \sup_{u \leqslant s} |X_u^n - X_0^n|^{2p} ds\Big). \end{aligned}$$

Thus, from Gronwall's lemma we have the assertion. ■

Proof of Theorem 5.1. By definition we have

$$\begin{aligned} X_t^n - X_t = &\int_0^t \big(\sigma(X_{s-}^n) - \sigma(X_{s-})\big) dW_s^{(n)} \\ &+ \int_0^t \big(b(X_{s-}^n) - b(X_{s-})\big) d\rho_s^n + (1 - Q^T)(K_t^n - K_t). \end{aligned}$$

Because b and σ are Lipschitz, we get

$$E\sup_{s\leqslant t}|X_s^n - X_s|^{2p} \leqslant 2\mathcal{C}\big(E\sup_{s\leqslant t}|K_s^n - K_s|^{2p} + \int_0^t E\sup_{u\leqslant s}|X_u^n - X_u|^{2p}ds\big).$$

From Gronwall's lemma we obtain

$$E\sup_{s\leqslant t}|X_s^n - X_s|^{2p} \leqslant 2\mathcal{C}E\sup_{s\leqslant t}|K_s^n - K_s|^{2p}.$$

In the same way we can prove that

$$E\sup_{s\leqslant t}|X_s^n|^2 \leqslant \mathcal{C}E\sup_{s\leqslant t}|K_s^n|^2.$$

Since

$$K_t^n = \sup_{s\leqslant t}\big[Q^T K^{n,(n-)} - \big(X_0^n + \int_0^s b(X_{u-}^n)d\rho_u^n + \int_0^s \sigma(X_{u-}^n)dW_u^{(n)}\big)\big]^+$$

and

$$K_t = \sup_{s\leqslant t}\big[Q^T K_s - \big(X_0^n + \int_0^s b(X_{u-})d\rho_u + \int_0^s \sigma(X_{u-})dW_u\big)\big]^+,$$

we have

$$\begin{aligned}
K_t^n - K_t &= \sup_{s\leqslant t}\big[Q^T K_s^{n,(n-)} - \big(X_0^n + \int_0^s b(X_{u-}^n)d\rho_u^n + \int_0^s \sigma(X_{u-}^n)dW_u^{(n)}\big)\big]^+ \\
&\quad - \sup_{s\leqslant t}\big[Q^T K_s^n - \big(X_0^n + \int_0^s b(X_{u-}^n)d\rho_u^n + \int_0^s \sigma(X_{u-}^n)dW_u^{(n)}\big)\big]^+ \\
&\quad + \sup_{s\leqslant t}\big[Q^T K_s^n - \big(X_0^n + \int_0^s b(X_{u-}^n)d\rho_u^n + \int_0^s \sigma(X_{u-}^n)dW_u^{(n)}\big)\big]^+ \\
&\quad - \sup_{s\leqslant t}\big[Q^T K_s^n - \big(X_0 + \int_0^s b(X_{u-})d\rho_u + \int_0^s \sigma(X_{u-})dW_u\big)\big]^+ \\
&\quad + \sup_{s\leqslant t}\big[Q^T K_s^n - \big(X_0 + \int_0^s b(X_{u-})d\rho_u + \int_0^s \sigma(X_{u-})dW_u\big)\big]^+ \\
&\quad - \sup_{s\leqslant t}\big[Q^T K_s - \big(X_0 + \int_0^s b(X_{u-})d\rho_u + \int_0^s \sigma(X_{u-})dW_u\big)\big]^+ \\
&= I_t^1 + I_t^2 + I_t^3.
\end{aligned}$$

Now we estimate every part separately:

$$I_t^1 \leqslant \sup_{s\leqslant t}|K_s^{n,(n-)} - K_s^n| \leqslant \sup_{s\leqslant t}|\sigma(X_{s-}^n)(W_s - W_s^{(n)}) + b(X_{s-}^n)(s-\rho_s^n)|,$$

$$I_t^2 \leqslant \sup_{u\leqslant s}\Big|\int_0^s \big(b(X_{u-}^n) - b(X_{u-})\big)d\rho_u^n + \int_0^s \big(\sigma(X_{u-}^n) - \sigma(X_{u-})\big)dW_u^{(n)}\Big|,$$

$$I_t^3 \leqslant \sup_{s\leqslant t}|K_s^n - K_s|$$

and we have

$$\begin{aligned} E\sup_{s\leqslant t}|K_s - K_s^n|^{2p} &\leqslant \mathcal{C}\Big(E\sup_{s\leqslant t}|W_s - W_s^{(n)}|^{2p} + \int_0^s E\sup_{u\leqslant s}|K_u - K_u^n|^{2p}du\Big) \\ &\leqslant \mathcal{C}E\big(\omega_{1/n}(W,[0,t])\big)^{2p} \\ &= \mathcal{O}\bigg(\Big(\frac{\ln n}{n}\Big)^p\bigg). \quad \blacksquare \end{aligned}$$

6. APPENDIX. Π_Q PROJECTION

Finding the projection π on the domain D is the standard technique to obtain a solution of the Skorokhod problem. In [3] we define a projection on the orthant $\mathbb{R}_+^d$ as follows:

REMARK 6.1. $\Pi_Q : \mathbb{R}^d \to \mathbb{R}_+^d$ is defined by

$$\Pi_Q(z) = z + (I - Q^T)\bar{r},$$

where $\bar{r}$ satisfies the equation $\bar{r} = [Q^T\bar{r} - z]^+$.

In that definition, we have to find the fixed point $\bar{r}$. Typically, we use the approximation sequence of $\bar{r}$ and $\bar{z}$:

$$\begin{cases} \bar{r}_0 = 0, \\ \bar{z}_0 = z, \\ \bar{r}_{n+1} = [Q^T\bar{r}_n - z]^+, \quad n \in \mathbb{N}\cup\{0\}, \\ \bar{z}_{n+1} = z + (I - Q^T)\bar{r}_{n+1}, \quad n \in \mathbb{N}\cup\{0\}. \end{cases} \tag{6.1}$$

It is easy to see that

$$\lim_{n\to+\infty}\bar{r}_n = \bar{r} \quad \text{and} \quad \lim_{n\to+\infty}\bar{z}_n = \Pi_Q(z).$$

Using simple calculations, we can obtain an equivalent formula for $\bar{r}_{n+1}$:

REMARK 6.2. We have

$$\bar{r}_{n+1} = [Q^T\bar{r}_n - z]^+ = \big[- \big(z + (I - Q^T)\bar{r}_n\big) + \bar{r}_n\big]^+ = [\bar{z}_n + \bar{r}_n]^+.$$

Now we define another sequence starting from the same point:

$$\begin{aligned} z_0 &= z, \\ z_{n+1} &= z_n + (I - Q^T)[-z_n]^+, \quad n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{6.2}$$

That sequence looks like that in our scheme used for a constant function $y_t = z$ ($\Delta y_t = 0$).

Once again simple calculations lead to obtaining an equivalent formula:

REMARK 6.3. We have

$$z_{n+1} = z_n + (I - Q^T)[-z_n]^+ = z + (I - Q^T)\sum_{i=0}^{n}[-z_i]^+.$$

Sequences z_n and $\bar{z}_n$ look different, but in fact they are only different representations of the same sequence.

LEMMA 6.1. *For every $z \in \mathbb{R}^d$ and for all $n \in \mathbb{N} \cup \{0\}$ we have*

$$z_n = \bar{z}_n. \tag{6.3}$$

P r o o f. The proof will be done by induction. For $n = 0$ we have

$$z_0 = z = \bar{z}_0.$$

Now assume that $z_i = \bar{z}_i$ for $i = 0, \ldots, n$. Then from (6.1) and Remark 6.2 we have

$$\bar{r}_i = \bar{r}_{i-1} + [-\bar{z}_{i-1}]^+ \tag{6.4}$$

for $i = 0, \ldots, n$.

Now we check

$$\begin{aligned} \bar{z}_{n+1} - z_{n+1} &= z + (I - Q^T)\bar{r}_{n+1} - z_n + (I - Q^T)\,[-z_n]^+ \\ &= \bar{z}_n + (I - Q^T)\,([-\bar{z}_n + \bar{r}_i]^+ - \bar{r}_n) - z_n + (I - Q^T)\,[-z_n]^+ \\ &= (\bar{z}_n - z_n) + (I - Q^T)\,([-\bar{z}_n + \bar{r}_n]^+ - \bar{r}_n - [-z_n]^+) \\ &= (I - Q^T)\,([-\bar{z}_n + \bar{r}_n]^+ - \bar{r}_n - [-\bar{z}_n]^+). \end{aligned}$$

Let us define

$$R_n^j = [-\bar{z}_n^j + \bar{r}_n^j]^+ - \bar{r}_n^j - [-\bar{z}_n^j]^+, \quad j = 1, \ldots, d. \tag{6.5}$$

It is easy to show that if $\bar{z}_n^j \leqslant 0$, then $R_n^j = 0$. To complete the proof we need to check whether $R_n^j = 0$ when $\bar{z}_n^j > 0$.

Without loss of generality we can assume that $j = 1$. Then

$$\begin{aligned}\bar{z}_n^1 &= z_n^1\\ &= z_{n-1}^1 + [-z_{n-1}^1]^+ - q_{21}[-z_{n-1}^2]^+ + \ldots - q_{d1}[-z_{n-1}^d]^+\\ &\leqslant z_{n-1}^1 + [-z_{n-1}^1]^+\\ &= \bar{z}_{n-1}^1 + [-\bar{z}_{n-1}^1]^+.\end{aligned}$$

Consequently, if $\bar{z}_n^1 > 0$, then $\bar{z}_{n-1}^1 > 0$. In the same way we can prove that $\bar{z}_i^1 > 0$ for $i = n-1, \ldots, 0$. If $\bar{z}_0^1 > 0$, then $\bar{r}_1^1 = 0$, and by (6.4) we have $\bar{r}_n^1 = 0$. Thus $R_n^1 = 0$. ■

COROLLARY 6.1. *We have*

$$\lim_{n\to+\infty} z_n = \Pi_Q(z).$$

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