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ON MULTIPLE POISSON STOCHASTIC INTEGRALS AND ASSOCIATED MARKOV SEMIGROUPS

BY

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Abstract. Multiple stochastic integrals (m.s.i.)

$$q^{(n)}(f) = \int_{X^n} f(x_1, \ldots, x_n) q(dx_1) \ldots q(dx_n), \quad n = 1, 2, \ldots,$$

with respect to the centered Poisson random measure $q(dx)$, $\mathrm{E}[q(dx)] = 0$, $\mathrm{E}[(q(dx))^2] = m(dx)$, are discussed, where (X, m) is a measurable space. A "diagram formula" for evaluation of products of (Poisson) m.s.i. as sums of m.s.i. is derived. With a given contraction semigroup A_t, $t \geqslant 0$, in $L^2(X)$ we associate a semigroup $\Gamma(A_t)$, $t \geqslant 0$, in $L^2(\Omega)$ by the relation

$$\Gamma(A_t) q^{(n)}(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n) = q^{(n)}(A_t f_1 \hat{\otimes} \ldots \hat{\otimes} A_t f_n)$$

and prove that $\Gamma(A_t)$, $t \geqslant 0$, is Markov if and only if A_t, $t \geqslant 0$, is doubly sub-Markov; the corresponding Markov process can be described as time evolution (with immigration) of the (infinite) system of particles, each moving independently according to A_t, $t \geqslant 0$.

0. Introduction. It is well known that the analysis of the structure of $L^2(\Omega)$-spaces arising from the Gaussian and the Poissonian white noises has certain common features, the main one being the existence of an orthogonal system of "polynomials" ("orthogonal polynomial chaos") defined by means of multiple stochastic integrals (m.s.i.). In the Gaussian case, such integrals were first discussed by Wiener [15] and Ito [4] (on this ground called also *Wiener-Ito integrals*), and in the Poissonian case by Ito [5]. M.s.i. of both types have been applied to deal with non-linear problems in engineering (see, e.g., [16], [9], [10]), while "Gaussian" m.s.i. appeared to play a major role in many areas of mathematical physics (e.g., quantum field theory [11], statistical physics [1], [12], statistical turbulence [8], etc.). This physical interest led to a number of remarkable mathematical results about (Gaussian) m.s.i. some of which have no analogues yet in the Poissonian theory. In particular, we are interested in the properties of the semigroup $\Gamma(A_t)$, $t \geqslant 0$, acting in $L^2(\Omega)$, which is defined for a given contraction semigroup A_t, $t \geqslant 0$, in $L^2(X)$ by the relation

$$\Gamma(A_t)\,q^{(n)}(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n) = q^{(n)}(A_t f_1 \hat{\otimes} \ldots \hat{\otimes} A_t f_n),$$

where

$$q^{(n)}(f) = \int_{X^n} f(x_1, \ldots, x_n)\, q(dx_1) \ldots q(dx_n)$$

denotes m.s.i. with respect to the noise (Gaussian or Poissonian) $q(dx)$ on X with $\mathrm{E}[q(dx)] = 0$, $\mathrm{E}[(q(dx))^2] = m(dx)$, and m is a σ-finite measure on a measurable space $(X, \mathscr{B}(X))$. In the Gaussian case the semigroup $\Gamma(A_t)$ enjoys two remarkable properties: first, $\Gamma(A_t)$ is positivity preserving and actually a Markov semigroup in $L^2(\Omega)$ for any contraction semigroup A_t, $t \geqslant 0$, in $L^2(X)$, and second, $\Gamma(A_t)$ admits some L^p-estimates, known as "hypercontractivity estimates", which are fundamental in constructive quantum field theory [11]. In this paper we are mainly concerned with the Poissonian analogue of the first property; it turns out that, in the Poissonian case, $\Gamma(A_t)$ is Markov if and only if A_t is doubly sub-Markov, which means – roughly speaking – that A_t and the dual (semigroup) A_t^* are positivity preserving and $\max(A_t 1, A_t^* 1) \leqslant 1$, $t \geqslant 0$ (Theorem 5.1). The Markov process associated with $\Gamma(A_t)$ can be interpreted as time evolution of the (infinite) system of unit masses (particles), distributed initially at $t = 0$ in X according to the Poisson law with mean $m(dx)$, such that each particle evolves independently according to A_t, $t \geqslant 0$, with immigration at random moments of time of new independent identically behaving particles. One hopes that this result can provide a better understanding of the probabilistic sense of the corresponding Markov process in the Gaussian case as the Gaussian noise can be approximated by suitably normalized Poissonian ones and it is reasonable to expect the corresponding approximation of $\Gamma(A_t)$.

Apart from the semigroup $\Gamma(A_t)$ we discuss also some properties of Poisson m.s.i., in particular an alternative definition of m.s.i. which is close to the well-known definition of Gaussian m.s.i. by means of "Wick polynomials" [2], [11] (Section 1), the relation between Poisson m.s.i. and Charlier polynomials (Section 2), the interpretation of Poisson m.s.i. as multiple integrals with respect to random point measure (Section 4), and a "diagram formula" for evaluation of products of m.s.i. as sums of m.s.i. (see [6] for a particular case and [1] for an analogous formula for Gaussian m.s.i.). For other discussion of Poisson m.s.i. and related topics we refer to [3], [6], [9], and [10].

1. Poisson m.s.i.: definition and basic properties. Denote by $M(X)$ the set of all σ-finite measures m on a measurable space $(X, \mathscr{B}(X))$. Given $m \in M(X)$, write $L^p(X) = L^p(X, m)$ for the space of all measurable functions $f: X \to C^1$ such that

$$\int_X |f|^p \, dm < +\infty.$$

By a *Poisson random measure* (r.m.) on X with intensity $m \in M(X)$ we mean the integer-valued random measure $p = p(A)$, $A \in \mathscr{B}(X)$, defined on a probability space $(\Omega, \mathscr{F}, P)$ such that, for any $n \geqslant 1$ and any non-intersecting $A_1, \ldots, A_n \in \mathscr{B}(X)$, $p(A_1), \ldots, p(A_n)$ are independent and, for each $A \in \mathscr{B}(X)$ such that $m(A) < +\infty$, $p(A)$ is distributed according to the Poisson law with mean $m(A)$:

$$P\big(p(A) = k\big) = e^{-m(A)}\big(m(A)\big)^k/k!, \qquad k = 0, 1, \ldots$$

A random signed measure $q = q(A)$, $q(A) = p(A) - m(A)$, $A \in \mathscr{B}(X)$, will be called the *centered Poisson r.m.* It is well known that for any $m \in M(X)$ the Poisson r.m. with intensity m exists.

Assume that the σ-algebra $\mathscr{B}(X)$ *contains points of* X, i.e., for every $x \in X$, $\{x\} \in \mathscr{B}(X)$. Denote by $M'(X) \subset M(X)$ the set of diffuse measures on X: $m \in M'(X)$ if $m(\{x\}) = 0$ for every $x \in X$. In this paper we discuss m.s.i. with respect to the Poisson r.m. with diffuse intensities, as a rule.

Let $(\Delta)_k$, $k = 1, 2, \ldots$, be a monotone (i.e. $(\Delta)_k \subset (\Delta)_{k+1}$) sequence of (countable) partitions of X by measurable sets Δ such that

$$\max_{\Delta \in (\Delta)_k} m(\Delta) \to 0 \qquad (k \to \infty). \tag{1.1}$$

A complex-valued function $f = f(x_1, \ldots, x_n)$, $x_1, \ldots, x_n \in X$, is said to be *simple* if

(a) f is symmetric (i.e. invariant with respect to all permutations of its arguments $x_1, \ldots, x_n$),

(b) f is constant on subsets

$$D \subset X^n = \underbrace{X \times \ldots \times X}_{n}$$

of the form $D = \Delta_1 \times \ldots \times \Delta_n$ ("quasi-intervals"), $\Delta_1, \ldots, \Delta_n \in (\Delta)_k$ for some $k = 1, 2, \ldots$, and f vanishes but on a finite number of such D's,

(c) f vanishes on "diagonals": $f(x_1, \ldots, x_n) = 0$ if $x_i = x_j$ for some $i \neq j$, $i, j = 1, \ldots, n$.

Denote by $L^2(X^n)$ the Hilbert space of all symmetric functions $f: X^n \to C^1$ such that

$$\|f\|_n = \Big(\int_{X^n} |f(x_1, \ldots, x_n)|^2 \, m(dx_1) \ldots m(dx_n)\Big)^{1/2} < +\infty,$$

while $L_0^2(X^n)$ stands for the set of all simple f's. Clearly, $L_0^2(X^n)$ is a linear dense subset of $L^2(X^n)$.

For any $f \in L^2(X^n)$ which equals $f^{\Delta_1,\ldots,\Delta_n}$ on $\Delta_1 \times \ldots \times \Delta_n \subset X^n$, $\Delta_1, \ldots, \Delta_n \in (\Delta)_k$, set

$$q^{(n)}(f) = \sum_{\Delta_1,\ldots,\Delta_n \in (\Delta)_k} f^{\Delta_1,\ldots,\Delta_n} q(\Delta_1)\ldots q(\Delta_n). \tag{1.2}$$

A Poisson m.s.i.

$$q^{(n)}(f) = \int_{X^n} f(x_1, \ldots, x_n) q(dx_1)\ldots q(dx_n)$$

is defined for every $f \in L^2(X^n)$ as the square-mean limit of integral sums $q^{(n)}(f_j)$ of the form (1.2), where $(f_j)_{j=1}^{\infty}$ is a sequence of simple functions convergent to f in $L^2(X^n)$ as $j \to \infty$, and has the following properties:

(q1) $q^{(n)}(f) \in L^2(\Omega)$;

(q2) $\mathrm{E}[q^{(n)}(f)] = 0$;

(q3) $\mathrm{E}[(q^{(n)}(f))^2] = n!\,\|f\|_n^2$;

(q4) if $f \in L^2(X^n)$, $g \in L^2(X^m)$, and $n \neq m$, then

$$\mathrm{E}[q^{(n)}(f)\overline{q^{(m)}(g)}] = 0$$

($\bar{u}$ denotes the complex conjugate of $u \in C^1$).

Properties (q1)-(q4) can be easily verified for simple functions and then extended to the general case by (q3). It follows from (q3) that the definition of $q^{(n)}(f)$ does not depend on a particular choice of the sequence of simple functions convergent to f in $L^2(X^n)$ as well as of the sequence of monotone partitions of X satisfying (1.1).

Set $L^2(X^0) = C^1$, $q^{(0)}(f) = f \in C^1$. It is known [5] that Poisson m.s.i. constitute a complete orthogonal system in $L^2(\Omega)$ ([1]): any random variable (r.v.) $\xi \in L^2(\Omega)$ can be uniquely expanded in series of m.s.i. convergent in $L^2(\Omega)$:

$$\xi = \sum_{n=0}^{\infty} q^{(n)}(f_n), \quad f_n \in L^2(X^n),\ n = 0, 1, \ldots \tag{1.3}$$

As $L^2(X^n)$ can be identified with the n-tuple symmetric tensor product $(\hat{\otimes} L^2(X))^n$, $L^2(\Omega)$ is unitary equivalent to the direct sum (the Fock space)

$$\bigoplus_{n=0}^{\infty} (\hat{\otimes} L^2(X))^n \equiv \exp\{L^2(X)\}$$

with the norm

$$\|f\| = \Big(\sum_{n=0}^{\infty} \|f_n\|_n^2/n!\Big)^{1/2}, \quad f = (f_0, f_1, \ldots) \in \exp\{L^2(X)\},$$

([1]) Here and in the sequel $L^2(\Omega)$ stands for the set of all (complex) square integrable r.v.'s, measurable with respect to the σ-algebra $\sigma(p(A), A \in \mathscr{B}(X))$, generated by the Poisson r.m. p.

the unitary mapping i: $\exp\{L^2(X)\} \to L^2(\Omega)$ being given by

$$(1.4) \qquad i(f) = \sum_{n=0}^{\infty} q^{(n)}(f_n)/n!, \quad f = (f_0, f_1, \ldots) \in \exp\{L^2(X)\}.$$

Poisson m.s.i. can be defined also in a more abstract manner which does not involve integral sums, similarly as Wick polynomials are defined in the Gaussian case (see [2] and [11]). Let the linear Poisson process indexed by $L^2(X)$ be given, i.e. a (generalized) random field $q = q(f), f \in L^2(X)$, with the characteristic functional

$$(1.5) \quad \mathrm{E}[\exp\{i\mathrm{Re}\,q(f)u\}] = \exp\{\int_X (e^{i\mathrm{Re}f(x)u} - 1 - i\mathrm{Re}f(x)u)m(dx)\}, \quad f \in L^2(X),\ u \in C^1$$

(in other words, $q(A) = q(1_A)$, $1_A(x) = 1$ if $x \in A$, $1_A(x) = 0$ otherwise, $A \in \mathscr{B}(X)$ is a Poisson r.m. on X with intensity m and $q(f) = \int_X f(x)q(dx) = q^{(1)}(f)$). Let $\mathscr{E}$ be a linear dense subspace of $L^2(X)$ such that $\mathscr{E} \subset L^p(X)$ for any $p \geqslant 2$. Sums of products of $q(f)$'s with $f \in \mathscr{E}$,

$$\sum_{k=1}^{r} q(f_{k1}) \ldots q(f_{kn_k}), \quad f_{k1}, \ldots, f_{kn_k} \in \mathscr{E},$$

$n_k \leqslant n$, $k = 1, \ldots, r$, will be called *polynomials of degree* n, $n = 1, 2, \ldots$, while constants $c \in L^2(\Omega)$ will be called *polynomials of degree* 0. Let Γ_n consist of all polynomials of degree n and their $L^2(\Omega)$-limits, and denote by $\Gamma_{[n]}$ the orthogonal complement (in $L^2(\Omega)$):

$$\Gamma_{[n]} = \Gamma_n \ominus \Gamma_{n-1}, \quad n = 1, 2, \ldots$$

Set

$$(1.6) \qquad :q(f_1) \ldots q(f_n) := \mathrm{proj}\,(q(f_1) \ldots q(f_n) | \Gamma_{[n]}).$$

For any $f_1, \ldots, f_n \in L^2(X)$, write

$$(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n)(x_1, \ldots, x_n) = \mathrm{sym}\, f(x_1) \ldots f(x_n),$$

where sym means symmetrization in $x_1, \ldots, x_n$ and

$$(\otimes f)^n = f \hat{\otimes} \underset{n}{\ldots} \hat{\otimes} f.$$

PROPOSITION 1.1. *For any* $f_1, \ldots, f_n \in \mathscr{E}$,

$$(1.7) \qquad :q(f_1) \ldots q(f_n): = q^{(n)}(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n).$$

We prove this statement in Section 3. Proposition 1.1 implies that $q^{(n)}(f)$ can be defined by means of (1.7) on a dense subset of $L^2(X^n)$ and then extended to general f's by (q3). However, (q3) does not readily follow from (1.6). Note also that the right-hand side of (1.7) is well defined for any $f_1, \ldots, f_n \in L^2(X)$

while the left-hand one is not such in general, as the product $q(f_1)\ldots q(f_n)$ may not be square integrable. In this context the following problem remains open: suppose $f_1, \ldots, f_n$ (belonging to $L^2(X)$) are such that $q(f_1)\ldots q(f_n) \in L^2(\Omega)$; is it true that $q(f_1)\ldots q(f_n) \in \Gamma_n$ and (1.7) holds?

2. Charlier polynomials. Poisson m.s.i. are related to Charlier polynomials in such a way as Hermite polynomials are related to Gaussian m.s.i. (see [3], [6], [9]); however, the analogy is not complete. In this section we study this relation which will also appear to be useful later (Section 6). Although many results discussed below are known, we prefer to provide an independent and self-contained exposition. As was noted in [9], there is no generally accepted set of definitions and notation for Charlier polynomials; our definition follows that of [6] and differs from it by the factor $n!$.

As in [6], we define *Charlier polynomials* $j_n(x; \lambda)$, $n = 0, 1, \ldots$, of discrete argument $x = -\lambda, -\lambda+1, \ldots$, where $\lambda > 0$ is a parameter, by means of their generating function

$$J(z, x, \lambda) \equiv \sum_{n=0}^{\infty} z^n j_n(x; \lambda)/n! = (1+z)^{x+\lambda} e^{-z\lambda}, \tag{2.1}$$

where the series is convergent to the right-hand side for any $\lambda > 0$, $x = -\lambda$, $-\lambda+1, \ldots$, and $z \in C^1$. We have $j_0(x; \lambda) = 1$, $j_1(x; \lambda) = x$, $j_2(x; \lambda) = x^2 - x - \lambda$. Charlier polynomials are related to the (centered) Poisson distribution with mean $\lambda > 0$ by the formula

$$j_n(x; \lambda) = (-1)^n \lambda^n j^{-1}(x; \lambda) D^n j(x; \lambda), \tag{2.2}$$

where $j(x; \lambda) = e^{-\lambda} \lambda^{x+\lambda}/(x+\lambda)!$, $D^n = DD^{n-1}$, $Df(x) = f(x) - f(x-1)$, $D^0 f = f$. Relation (2.2) can be verified as follows. Denote by $\tilde{j}_n(x; \lambda)$ the right-hand side of (2.2) and check that it gives correct values of $j_n(x; \lambda)$ for $n = 0, 1, 2$. By induction we verify the recurrence relation

$$\tilde{j}_{n+1}(x; \lambda) = (x-n)\tilde{j}_n(x; \lambda) - \lambda n \tilde{j}_{n-1}(x; \lambda), \qquad n = 1, 2, \ldots \tag{2.3}$$

The same relation is satisfied by the polynomials $j_n(x; \lambda)$ with the generating function (2.1), which follows from the identity

$$(1+z)\,\partial J/\partial z = (x - \lambda z) J. \tag{2.4}$$

This proves (2.2).

To discuss the relation between Charlier polynomials and Poisson m.s.i., let $A_1, \ldots, A_k$ be measurable subsets of X, let $i_1, \ldots, i_k$ be non-negative integers, $i_1 + \ldots + i_k = n$, and set

$$A_1^{i_1} \times \ldots \times A_k^{i_k} = (A_1 \times \underset{i_1}{\ldots} \times A_1) \times \ldots \times (A_k \times \underset{i_k}{\ldots} \times A_k) \in \mathscr{B}(X^n) \equiv \mathop{\mathsf{X}}_{i=1}^{n} \mathscr{B}(X).$$

If sym $1_{A_1^{i_1}\times\ldots\times A_k^{i_k}}\in L^2(X^n)$, set

$$q^{(n)}(A_1^{i_1}\times\ldots\times A_k^{i_k})=q^{(n)}(\mathrm{sym}\, 1_{A_1^{i_1}\times\ldots\times A_k^{i_k}}).$$

PROPOSITION 2.1 (cf. [3] and [6]). *Let $A_1,\ldots,A_k$ be pairwise disjoint measurable subsets of X, $m(A_i)<+\infty$, $i=1,\ldots,k$, and let $i_1,\ldots,i_k$ be non-negative integers, $i_1+\ldots+i_k=n$. Then*

$$q^{(n)}(A_1^{i_1}\times\ldots\times A_k^{i_k})=j_{i_1}\big(q(A_1);\, m(A_1)\big)\ldots j_{i_k}\big(q(A_k);\, m(A_k)\big). \tag{2.5}$$

Let us prove first

PROPOSITION 2.2. (i) *For any $f\in L^2(X)$, $e^{q(f)}\in L^2(\Omega)$ is equivalent to*

$$\int\limits_{\mathrm{Re}f(x)>1}\exp\{2\,\mathrm{Re}f(x)\}\, m(dx)<+\infty. \tag{2.6}$$

(ii) *For any $h\in L^2(X^n)$, $n=1,2,\ldots$, and $f\in L^2(X)$ satisfying* (2.6),

(2.7)

$$\mathrm{E}\,[e^{q(f)}\overline{q^{(n)}(h)}]=\mathrm{E}\,[e^{q(f)}]\int\limits_{X^n}\prod_{j=1}^{n}(e^{f(x_j)}-1)\,\overline{h(x_1,\ldots,x_n)}\, m(dx_1)\ldots m(dx_n).$$

Proof. (i) If $f\in L^2(X)$ is simple, then $e^{q(f)}\in L^2(\Omega)$ and

$$\mathrm{E}\,[\exp\{q(f)\}]=\exp\{\int\limits_X F(f)\,dm\}, \tag{2.8}$$

where $F(f)(x)=e^{f(x)}-1-f(x)$. Observe that, for $f\in L^2(X)$, formula (2.6) is equivalent to

$$\int\limits_X |F(f)|\,dm<+\infty.$$

For any $f\in L^2(X)$, there exists a sequence $(f_j)_{j=1}^{\infty}$ of simple functions convergent to f in $L^2(X)$ such that

$$\mathrm{Re}\, f_j(x)\leqslant \mathrm{Re}\, f(x),\qquad j=1,2,\ldots \tag{2.9}$$

Now, if f satisfies (2.6), then by (2.8) and the above observation $e^{q(f_j)}$ is a Cauchy sequence in $L^2(\Omega)$ which converges to $e^{q(f)}$ in probability. Therefore, $e^{q(f)}\in L^2(\Omega)$ and (2.8) holds.

Conversely, let $f\in L^2(X)$ be such that (2.6) is not true; we want to prove that $e^{q(f)}$ is not in $L^2(\Omega)$. In fact, it suffices to prove this for f real and such that $f>1$ m-a.e. as

$$q(f)=q(f1_{(f>1)})+q(f1_{(f\leqslant 1)})\equiv q(f')+q(f''),$$

where the last two integrals are independent and $\exp\{q(f'')\}\in L^2(\Omega)$ by the discussion above. As

$$\int\limits_X f\,dm\leqslant\int\limits_X f^2\,dm<+\infty\qquad (f>1),$$

the assumption $e^{q(f)} \in L^2(\Omega)$ implies $e^{p(f)} \in L^2(\Omega)$, where $p(f) = q(f) + \int f\,dm$. Consider a sequence $(f_j)_{j=1}^{\infty}$, $0 \leqslant f_j \uparrow f$ $(j \to \infty)$; then

$$p(f_j) \leqslant p(f) \quad \text{and} \quad \lim_{j\to\infty} \mathrm{E}[\exp\{2p(f_j)\}] = +\infty,$$

which yields a contradiction.

(ii) By (i) it suffices to prove (2.7) for simple f and h equal to f^{Δ} and $h^{\Delta_1,\ldots,\Delta_n}$ on quasi-intervals Δ and $\Delta_1 \times \ldots \times \Delta_n$, respectively, $\Delta, \Delta_1, \ldots, \Delta_n \in (\Delta)_k$. Then

$$\text{(2.10)} \quad \mathrm{E}[\exp\{q(f)\}\overline{\tilde{q}^{(n)}(h)}]$$

$$= \sum_{\Delta_1,\ldots,\Delta_n} \mathrm{E}[\exp\{\sum_{\Delta} f^{\Delta} q(\Delta)\} q(\Delta_1)\ldots q(\Delta_n)]\, \overline{h^{\Delta_1,\ldots,\Delta_n}}$$

$$= \sum_{\Delta_1,\ldots,\Delta_n} \mathrm{E}[\exp\{\sum_{\Delta \neq \Delta_1,\ldots,\Delta_n} f^{\Delta} q(\Delta)\}] \prod_{j=1}^{n} \mathrm{E}[\exp\{f^{\Delta_j} q(\Delta_j)\} q(\Delta_j)]\, \overline{h^{\Delta_1,\ldots,\Delta_n}}.$$

We have

$$\mathrm{E}[\exp\{f^{\Delta} q(\Delta)\} q(\Delta)] = \frac{\partial}{\partial a} \mathrm{E}[\exp\{(f^{\Delta}+a) q(\Delta)\}]|_{a=0}$$

$$= \frac{\partial}{\partial a} \exp\{m(\Delta)(e^{f^{\Delta}+a} - 1 - (f^{\Delta}+a))\}\big|_{a=0}$$

$$= \exp\{m(\Delta)(e^{f^{\Delta}} - 1 - f^{\Delta})\}(e^{f^{\Delta}} - 1)\, m(\Delta),$$

which together with (2.8) implies (2.7).

Corollary 2.1. *For any $f \in L^2(X)$ such that $e^{q(f)} \in L^2(\Omega)$, we have*

$$\text{(2.11)} \quad e^{q(f)} = \sum_{n=0}^{\infty} \exp\{\int_X F(f)\,dm\}\, q^{(n)}\big((\otimes(e^f - 1))^n\big)/n!.$$

Proof of Proposition 2.1. It suffices to verify that

$$\text{(2.12)} \quad \sum_{i_1,\ldots,i_k=0}^{\infty} \mathrm{E}[\exp\{iq(f)\}\, q^{(i_1+\ldots+i_k)} \times$$

$$\times (A_1^{i_1} \times \ldots \times A_k^{i_k})]\, z_1^{i_1} \ldots z_k^{i_k}/i_1! \ldots i_k!$$

$$= \sum_{i_1,\ldots,i_k=0}^{\infty} \mathrm{E}[\exp\{iq(f)\} j_{i_1}(q(A_1);\, m(A_1)) \ldots$$

$$\ldots j_{i_k}(q(A_k);\, m(A_k))]\, z_1^{i_1} \ldots z_k^{i_k}/i_1! \ldots i_k!$$

for any real $f \in L^2(X)$, $z_1, \ldots, z_k \in R^1$, $|z_i| < 1$. Denote by S the right-hand side of (2.12). By (2.1), we have

$$S = \mathrm{E}\left[\exp\{iq(f)\} \prod_{j=1}^{k} (1+z_j)^{q(A_j)+m(A_j)} \exp\{-z_j m(A_j)\}\right]$$
$$= \mathrm{E}\left[\exp\{q(\tilde{f})\}\right] \exp\Big\{\sum_{j=1}^{k} \big(\ln(1+z_j)-z_j\big) m(A_j)\Big\},$$

where

$$\tilde{f} = \sum_{j=1}^{k} \ln(1+z_j)\cdot 1_{A_j} - if.$$

By (2.8), we obtain

$$S = \mathrm{E}\left[\exp\{iq(f)\}\right] \exp\Big\{\sum_{j=1}^{k} z_j \int_{A_j} (e^{if}-1)\,dm\Big\}. \tag{2.13}$$

The left-hand side of (2.12) can be evaluated by means of (2.7) and the result is also (2.13).

COROLLARY 2.2. *Under conditions and notation of Proposition* 2.1,

$$q^{(n)}(A_1^{i_1} \times \ldots \times A_k^{i_k}) = q^{(i_1)}(A_1^{i_1}) \ldots q^{(i_k)}(A_k^{i_k}).$$

Contrary to Hermite polynomials in the Gaussian case, Charlier polynomials do not constitute a complete orthonormal system in $L^2(\Omega)$ (see [3], [9]), which is easily understandable as only very special Poisson m.s.i. can be expressed in terms of Charlier polynomials. As linear combinations of m.s.i. of the form $q^{(n)}(\Delta_1^{i_1} \times \ldots \times \Delta_k^{i_k})$, $\Delta_1, \ldots, \Delta_k \in (\Delta)_r$, $i_1, \ldots, i_k = 0, 1, \ldots, i_1+\ldots+i_k = n$, $n = 0, 1, \ldots$, $r = 1, 2, \ldots$, are dense in $L^2(\Omega)$, by orthogonalization one can construct an orthonormal basis in $L^2(\Omega)$ made of linear combinations of (multivariate) Charlier polynomials. Somewhat unexpectedly it turned out that in each subspace

$$\Gamma'_{[n]} \equiv \{q^{(n)}(f): f \in L^2(X^n)\}, \qquad n = 1, 2, \ldots,$$

there exists an orthonormal basis made of linear combinations of a finite number $d = d(n)$ of Charlier polynomials, where (most likely) $d(n) = O(n)$ ($n \to \infty$). Let us describe such a basis in the case $n = 2$.

Assume that $m(\Delta)$ are equal for any $\Delta \in (\Delta)_r$, $r = 1, 2, \ldots$, and that every $\Delta \in (\Delta)_r$ splits into two "intervals" Δ^+, $\Delta^- \in (\Delta)_{r+1}$. Introduce r.v.'s

$$\xi_0^{(r)}(\Delta_1, \Delta_2) = \begin{cases} q^{(2)}(\Delta_1 \times \Delta_2) & \text{if } r = 1, \\ 0 & \text{if } r \geqslant 2, \end{cases}$$

$$\xi_1^{(r)}(\Delta_1, \Delta_2) = q^{(2)}(\Delta_1^+ \times \Delta_2^+) - q^{(2)}(\Delta_1^- \times \Delta_2^-),$$

$$\xi_2^{(r)}(\Delta_1, \Delta_2) = \begin{cases} q^{(2)}(\Delta_1^+ \times \Delta_2^-) - q^{(2)}(\Delta_1^- \times \Delta_2^+) & \text{if } \Delta_1 \neq \Delta_2, \\ 0 & \text{if } \Delta_1 = \Delta_2, \end{cases}$$

$$\xi_3^{(r)}(\Delta_1, \Delta_2) = q^{(2)}(\Delta_1 \times \Delta_2) - 2q^{(2)}(\Delta_1^+ \times \Delta_2^-) - 2q^{(2)}(\Delta_1^- \times \Delta_2^+).$$

We leave to the reader the verification of the fact that the system

$$\{\xi_j^{(r)}(\Delta_1, \Delta_2): \Delta_1, \Delta_2 \in (\Delta)_r, r = 1, 2, \ldots, j = 0, 1, 2, 3\}$$

is complete and orthogonal in $\Gamma'_{[2]}$. Note that all elements of this system can be expressed by means of linear combinations of at most 3 Charlier polynomials.

3. Products of Poisson m.s.i.: a "diagram formula". It is well known [1] that products of Gaussian m.s.i. can be most conveniently expressed as linear combinations of m.s.i. by means of the so-called "diagrams". In the case of Poisson m.s.i. there exists an analogous diagram formalism under the assumption that no arbitrary products of Poisson m.s.i. can be expanded in such a way (as they need not be square integrable), and the "diagrams" are somewhat more complicated (see below).

Let $f_1, \ldots, f_k$ be symmetric functions depending on $n_1, \ldots, n_k$ variables $x \in X$, respectively. Write

$$F(x_1, \ldots, x_N) = f_1(x_1, \ldots, x_{n_1}) \ldots f_k(x_{n_1+\ldots+n_{k-1}+1}, \ldots, x_N), \tag{3.1}$$

where $N = n_1 + \ldots + n_k$. By a *diagram over (grouped) variables*

$$(x_1, \ldots, x_{n_1}), (x_{n_1+1} \ldots, x_{n_1+n_2}), \ldots, (x_{n_1+\ldots+n_{k-1}+1}, \ldots, x_N) \tag{3.2}$$

or, shortly, a *diagram* we mean a graph γ connecting variables $x_1, \ldots, x_N$ arranged in groups such that

(a) γ connects only variables which enter different groups (brackets) (or belong to different functions $f_1, \ldots, f_k$),

(b) every variable is directly connected with at most two other variables.

More precisely, if the variables $x_1, \ldots, x_N$ are considered as vertices of the graph γ, we say that x_i and x_j are *directly connected* or that the pair (x_i, x_j) is *directly connected* if there is a branch of γ which connects x_i and x_j; and we say that x_i and x_j are *indirectly connected* or just *connected* if there exist $y_1, \ldots, y_m \subset \{x_1, \ldots, x_N\}$ such that the pairs $(x_i, y_1), (y_1, y_2), \ldots, (y_m, x_j)$ are directly connected. According to (a), variables entering the same group cannot be (indirectly) connected by a diagram. Denote by $\{\gamma\}$ the set of all diagrams (over a given set of grouped variables).

With every diagram $\gamma \in \{\gamma\}$ and every function F defined by (3.1) we associate a formal sum F^γ of symmetric functions depending on different number of variables $x \in X$ in the following way. If $\gamma \in \{\gamma\}$ is empty (i.e., the corresponding graph is empty), we set $F^\gamma = \operatorname{sym} F$. If $\gamma \in \{\gamma\}$ is not empty, denote by $\gamma(1), \ldots, \gamma(r)$ the connected components of γ which connect variables $(x_j, j \in T_1), \ldots, (x_j, j \in T_r)$ respectively, where $T_1, \ldots, T_r$ are disjoint subsets of $\{1, \ldots, N\}$. Introduce the linear operator $D^{\gamma(s)}$, $s = 1, \ldots, r$, which transforms a function $G(x_1, \ldots, x_N)$ into a (formal) sum of two functions G_1 and G_2, where G_1 is obtained from G by replacing the variables $(x_j, j \in T_s)$

(connected by the component $\gamma(s)$) by a single new one which we denote, e.g., by $\tilde{x}_s$, while G_2 equals the integral of G_1 with respect to $m(d\tilde{x}_s)$. Set

$$F^\gamma = \text{sym } D^{\gamma(r)} \ldots D^{\gamma(1)} F, \tag{3.3}$$

where symmetrization involves all variables, the new $\tilde{x}_1, \ldots, \tilde{x}_r$ as well as the old (unconnected) x_j, $j \in \{1, \ldots, N\} \setminus \bigcup_{s=1}^{r} T_s$, which altogether can be enumerated in each term of F in a more convenient fashion. For example, let

$$f_1 = f_1(x_1, x_2), \quad f_2 = f_2(x_1, x_2, x_3), \quad f_3 = f_3(x_1),$$

$x_1, x_2, x_3 \in X$,

$$F(x_1, \ldots, x_6) = f_1(x_1, x_2) f_2(x_3, x_4, x_5) f_3(x_6).$$

Consider the following diagram γ consisting of two connected components:

$$(x_1, x_2), (x_3, x_4, x_5), (x_6).$$

Then

$$F^\gamma = \text{sym } D^{\gamma(2)} D^{\gamma(1)} F = \text{sym } D^{\gamma(2)} \big(f_1(\tilde{x}_1, x_2) f_2(\tilde{x}_1, x_4, x_5) f_3(x_6) +$$

$$+ \int_X f_1(\tilde{x}_1, x_2) f_2(\tilde{x}_1, x_4, x_5) f_3(x_6) m(d\tilde{x}_1)\big)$$

$$= \text{sym } \big(f_1(x_1, x_2) f_2(x_1, x_3, x_2) f_3(x_2) + \int_X f_1(x_1, x) f_2(x_1, x_2, x) f_3(x) m(dx)$$

$$+ \int_X f_1(x, x_1) f_2(x, x_2, x_1) f_3(x_1) m(dx) +$$

$$+ \int_X \int_X f_1(x, y) f_2(x, x_1, y) f_3(y) m(dx) m(dy)\big).$$

Finally, let us introduce the operator $q[\cdot]$ which maps formal sums F^γ into sums of Poisson m.s.i. Let

$$H = \sum_{j=1}^{k} H_j, \quad \text{where } H_j \in L^2(X^{n_j}),\ j = 1, \ldots, k;$$

then

$$q[H] = \sum_{j=1}^{k} q^{(n_j)}(H_j). \tag{3.4}$$

Write $|F|^\gamma$ for the formal sum (3.3), where F is replaced by $|F| = |F(x_1, \ldots, x_N)|$. We have

PROPOSITION 3.1. *Let functions $f_j \in L^2(X^{n_j})$, $j = 1, \ldots, k$, be such that, for every diagram $\gamma \in \{\gamma\}$ over variables (3.2), every function $H = H(x_1, \ldots, x_s)$ which enters the formal sum $|F|^\gamma$ is in $L^2(X^s)$. Then $q^{(n_1)}(f_1) \ldots q^{(n_k)}(f_k) \in L^2(\Omega)$ and*

$$q^{(n_1)}(f_1) \ldots q^{(n_k)}(f_k) = \sum_{\gamma \in \{\gamma\}} q[F^\gamma]. \tag{3.5}$$

Remark 3.1. This result is not very satisfactory as it provides only sufficient conditions on $f_1, \ldots, f_k$ under which $q^{(n_1)}(f_1)\ldots q^{(n_k)}(f_k)$ is in $L^2(\Omega)$ and can be expanded in series of m.s.i. with integrands determined by some given rule. Although necessary conditions for this are not known even in the case $n_1 = \ldots = n_k = 1$ (see Proposition 1.1 and the discussion at the end of Section 1), there remains an open problem whether $|F|^\gamma$ in Proposition 3.1 can be replaced by F^γ, as the right-hand side of (3.5) in such a case is still well-defined.

Before proving Proposition 3.1, let us consider two examples where more explicit formulae can be obtained from (3.5).

Example 1. Let $k = 2$, $n_1 = n$, and $n_2 = 1$. Every diagram γ over variables $(x_1, \ldots, x_n)$, (x_{n+1}) connects x_{n+1} with some x_j, $j = 1, \ldots, n$ (we write $\gamma = \gamma(j)$) and $\{\gamma\} = \{\gamma(0), \gamma(1), \ldots, \gamma(n)\}$, where $\gamma(0)$ is empty diagram. Now, $F^{\gamma(0)} = F = \mathrm{sym}\, f_1 \otimes f_2$ and

$$F^{\gamma(j)} = \mathrm{sym}\; D^{\gamma(j)} F = \mathrm{sym}\, f_1(x_1, \ldots, x_j, \ldots, x_n) f_2(x_j) + \\ + \int_X f_1(x_1, \ldots, x_j, \ldots, x_n) f_2(x_j)\, m(dx_j) \equiv F_j' + F_j''.$$

If f_1 and f_2 are in $L^2(X^n)$ and $L^2(X)$, respectively, this implies that $|F|_j'' \in L^2(X^{n-1})$. Assume, in addition, that $|F|_j' \in L^2(X^n)$, $j = 1, \ldots, n$, i.e., that $f_1(x_1, \ldots, x_n) f_2(x_1)$ is in $L^2(X^n)$. According to Proposition 3.1, we have

$$q^{(n)}(f_1)\, q^{(1)}(f_2) = n\big(q^{(n)}(F_1') + q^{(n-1)}(F_1'')\big) + q^{(n+1)}(F). \tag{3.6}$$

Formula (3.6) under the same assumptions on f_1 and f_2 was obtained earlier by Kabanov [6], Theorem 2.

Example 2. Let $n_1 = \ldots = n_k = 1$ and $N = k$. If $\gamma \in \{\gamma\}$ consists of connected components $\gamma(1), \ldots, \gamma(r)$ which connect variables $(x_j, j \in T_1)$, $\ldots, (x_j, j \in T_r)$, respectively, then

$$D^{\gamma(r)} \ldots D^{\gamma(1)} F = \prod_{j \in K^c} f_j(x_j) \prod_{i=1}^{r} \Big(F_{T_i}(\tilde{x}_i) + \int_X F_{T_i}\, dm\Big),$$

where $K = \{1, \ldots, k\}$, $F_T(x) = \prod_{j \in T} f_j(x)$, $T \subseteq K$, $K^c = K \setminus \bigcup_{s=1}^{r} T_s$.

It is easy to see that the assumptions of Proposition 3.1 are fulfilled if and only if

$$f_i \in L^2(X),\ i = 1, \ldots, k, \qquad F_T \in L^p(X),\ p = 1, 2, \text{ for every } T \subseteq K. \tag{3.7}$$

By Hölder's inequality

$$\int |h_1 \ldots h_k| \leqslant \prod_{j=1}^{k} \Big(\int |h_j|^k\Big)^{1/k},$$

we infer that a sufficient condition for (3.7) is of the form

(3.8) $$f_j \in L^2(X) \cap L^{2k}(X), \quad j = 1, \ldots, k.$$

For any disjoint subsets $T_1, \ldots, T_r \subseteq K$ and any $S \subseteq \{1, \ldots, r\}$, $r = 0, \ldots, k$, set

$$l = k - \sum_{j=1}^{r} |T_j| + |S|, \quad K^c = K \setminus \bigcup_{j=1}^{r} T_j, \quad S^c = \{1, \ldots, r\} \setminus S,$$

and

(3.9) $$f_{T_1, \ldots, T_r; S} = \operatorname{sym} \prod_{j \in K^c} f_j(x_j) \prod_{j \in S} F_{T_j}(\tilde{x}_j) \prod_{j \in S^c} F_{T_j} dm$$

($|A|$ denotes the number of elements in A). According to Proposition 3.1, under conditions (3.7) we have

(3.10) $$\prod_{j=1}^{k} q(f_j) = \sum_{r=0}^{k} \sum_{T_1, \ldots, T_r \subseteq K} \sum_{S \subseteq \{1, \ldots, r\}} q^{(l)}(f_{T_1, \ldots, T_r; S}),$$

the second sum being taken over all mutually disjoint non-empty subsets $T_1, \ldots, T_r$ of K such that $|T_i| \geqslant 2$.

Now we come back to the proof of Proposition 3.1. The idea of the proof is, roughly spoken, the same as in the Gaussian case (see [1]): one has to justify the possibility to replace "products of $q(dt)$'s on diagonals" $(q(dt))^n$, $n \geqslant 2$, by some other differentials, in our case by $q(dt) + m(dt) = p(dt)$. (The reason for such a replacement is intuitively clear and it can be verified that using such a formal procedure one arrives at formula (3.4).) Due to this fact and in order to avoid cumbersome notation we present a somewhat abbreviated version of the proof.

Proof of Proposition 3.1. It is sufficient to discuss the case $k = 2$, i.e., the product of two m.s.i. $q^{(n)}(f)$ and $q^{(m)}(g)$. In this case our diagrams are identical with "Gaussian" diagrams as they connect only pairs of variables. Consider two sequences of simple functions convergent to f and g in the corresponding L^2-spaces and expand the products of the corresponding integral sums just as in the Gaussian case ([1], p. 17). It is not difficult to see that for the proof it suffices to show that

$$S \equiv \sum_{\Delta_1, \ldots, \Delta_l \in (\Delta)_p} h_p^{\Delta_1, \ldots, \Delta_l} \big(q(\Delta_1)^2 \ldots q(\Delta_k)^2 q(\Delta_{k+1}) \ldots q(\Delta_l) -$$
$$- (q(\Delta_1) + m(\Delta_1)) \ldots (q(\Delta_k) + m(\Delta_k)) q(\Delta_{k+1}) \ldots q(\Delta_l) \big) \to 0 \quad (p \to \infty)$$

in $L^2(\Omega)$, where $h_p \in L^2(X^l)$, $p = 1, 2, \ldots$, is a sequence of simple functions equal to $h_p^{\Delta_1, \ldots, \Delta_l}$ on $\Delta_1 \times \ldots \times \Delta_l$, $\Delta_1, \ldots, \Delta_l \in (\Delta)_p$, and convergent to some $h \in L^2(X^l)$ in $L^2(X^l)$ as $p \to \infty$. (Note that S corresponds to a diagram over $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_m)$ which connects k pairs of variables in the two groups,

$k \leqslant \min(n, m)$, $l = m+n-k$.) According to the assumptions of Proposition 3.1, we can assume without loss of generality that

(3.11)
$$\int_{X^{l-s}} \Big(\int_{X^s} |h_p(x_1, \ldots, x_s, \ldots, x_l)|\, m(dx_1)\ldots m(dx_s)\Big)^2 m(dx_{s+1})\ldots m(dx_l) \leqslant C$$

for every s $(0 \leqslant s \leqslant k)$, where $C < +\infty$ does not depend on p and $s \leqslant k$. Assume also that $m(\Delta) = \mu = \mu(p)$ for every $\Delta \in (\Delta)_p$. Now, S can be rewritten as

(3.12)
$$S \equiv \sum_{\Delta_1,\ldots,\Delta_l} h_p^{\Delta_1,\ldots,\Delta_l} \sum_{R\neq\emptyset} \prod_{j\in R} \varrho_1(\Delta_j) \prod_{j\in K\setminus R} \varrho_2(\Delta_j) \prod_{j\in L\setminus K} \varrho_3(\Delta_j),$$

where

$$\varrho_1(\Delta) = q(\Delta)^2 - q(\Delta) - \mu,\quad \varrho_2(\Delta) = q(\Delta)+\mu,\quad \varrho_3(\Delta) = q(\Delta),$$

$K = \{1, \ldots, k\}$, $L = \{1, \ldots, l\}$, and the sum $\sum_{R\neq\emptyset}$ is taken over all non-empty subsets $R \subseteq K$. We have

(3.13)
$$\mathrm{E}[|S|^2] = \sum_{\Delta_1,\ldots,\Delta_l;\Delta_1',\ldots,\Delta_l'} h_p^{\Delta_1,\ldots,\Delta_l}\, \overline{h_p^{\Delta_1',\ldots,\Delta_l'}} \times$$
$$\times \sum_{\substack{R\neq\emptyset\\ R'\neq\emptyset}} \mathrm{E}\Big[\prod_{j\in R} \varrho_1(\Delta_j) \prod_{j\in K\setminus R} \varrho_2(\Delta_j) \prod_{j\in L\setminus K} \varrho_3(\Delta_j) \prod_{j\in R'} \varrho_1(\Delta_j') \prod_{j\in K\setminus R'} \varrho_2(\Delta_j') \prod_{j\in L\setminus K} \varrho_3(\Delta_j')\Big].$$

Denote the last expectation in (3.13) by $d_{RR'}(\Delta_1, \ldots, \Delta_l; \Delta_1', \ldots, \Delta_l')$. Consider the sum

(3.14)
$$\sum_{(r)} \equiv \sum_{(r)} h_p^{\Delta_1,\ldots,\Delta_l}\, \overline{h_p^{\Delta_1',\ldots,\Delta_l'}}\, d_{RR'}(\Delta_1, \ldots, \Delta_l; \Delta_1', \ldots, \Delta_l')$$

taken over all collections $(\Delta_1, \ldots, \Delta_l)$ and $(\Delta_1', \ldots, \Delta_l')$ in which r quasi-intervals Δ coincide (regardless of their positions), $0 \leqslant r \leqslant l$. In (3.14), $d_{RR'}(\Delta_1, \ldots, \Delta_l; \Delta_1', \ldots, \Delta_l')$ does not vanish only if those quasi-intervals in the collections $(\Delta_1, \ldots, \Delta_l)$ and $(\Delta_1', \ldots, \Delta_l')$ which are not common to both of them enter ϱ_2 as $\mathrm{E}[\varrho_1(\Delta)] = \mathrm{E}[\varrho_3(\Delta)] = 0$ and if in every collection $(\Delta_1, \ldots, \Delta_l)$ quasi-intervals $\Delta_1, \ldots, \Delta_l$ are different (h_p vanishes on diagonals). Thus $d_{RR'}(\Delta_1, \ldots, \Delta_l; \Delta_1', \ldots, \Delta_l') \neq 0$ implies $r > l-k$. But then there exists at least one $\Delta \in (\Delta_1, \ldots, \Delta_l) \cap (\Delta_1', \ldots, \Delta_l')$ which enters ϱ_1 (once or twice), as R and R' are non-empty. If $\mu = m(\Delta) \leqslant 1$, then

$$0 \leqslant \mathrm{E}[\varrho_i(\Delta)\varrho_j(\Delta)] \leqslant \begin{cases} \mu^2 & \text{if } i = 1 \text{ or } j = 1,\\ 2\mu & \text{if } i \neq 1 \text{ and } j \neq 1,\end{cases} \quad i, j = 1, 2, 3.$$

Consequently, if $\mu \leqslant 1$ and $r > l-k$, then

(3.15)
$$\Big|\sum_{(r)}\Big| \leqslant C_1 \sum_{(r)} |h_p^{\Delta_1,\ldots,\Delta_l}\, h_p^{\Delta_1',\ldots,\Delta_l'}|\, \mu^{2(l-r)} \mu^{r-1} \mu^2$$
$$\leqslant C_1\mu \sum_{\Delta_1,\ldots,\Delta_r} \Big(\sum_{\Delta_{r+1},\ldots,\Delta_l} |h_p^{\Delta_1,\ldots,\Delta_l}|\, \mu^{l-r}\Big)^2 \mu^r,$$

where C_1 does not depend on p while

$$\sum_{(r)} = 0 \quad \text{if } r \leqslant l-k.$$

Since the last double sum in (3.15) is just the integral (3.11), where $s = l-r$, bounded by C, we obtain $\mathrm{E}[|S|^2] \to 0$ as $p \to \infty$.

Proof of Proposition 1.1. Set

$$\Gamma'_{[n]} = \{\xi = q^{(n)}(f): f \in L^2(X^n)\} \quad \text{and} \quad \Gamma'_n = \bigoplus_{k=0}^{n} \Gamma'_{[k]}.$$

Observe that

$$\Gamma_n \subseteq \Gamma'_n, \quad n = 0, 1, \ldots \tag{3.16}$$

In fact, if $f_1, \ldots f_n \in \mathscr{E}$, then, by Proposition 3.1 and Example 2, $q(f_1)\ldots q(f_n)$ can be written as the sum of m.s.i. of order less than or equal to n, i.e.,

$$q(f_1)\ldots q(f_n) = q^{(n)}(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n) + \sum_{j<n} q^{(j)}(h_j), \tag{3.17}$$

where $h_j \in L^2(X^j)$, $j = 0, \ldots, n-1$. Therefore (3.16) holds. As $\mathscr{E}$ is dense in $L^2(X)$, $\Gamma'_1 = \Gamma_1$. Assume that $\Gamma'_j = \Gamma_j$, $j < n$. Then, by (3.17),

$$\Gamma_{[n]} = \Gamma_n \ominus \Gamma_{n-1} \supseteq \bigvee_{f_1, \ldots, f_n \in \mathscr{E}} \{q^{(n)}(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n)\} \equiv D_n, \tag{3.18}$$

where $\bigvee_{\alpha \in A} \{\xi_\alpha\}$ denotes the Hilbert subspace of $L^2(\Omega)$, spanned by the r.v. ξ_α, $\alpha \in A$. Again, as $\mathscr{E}$ is dense in $L^2(X)$, (3.18) implies $D_n = \Gamma'_{[n]} \subseteq \Gamma_{[n]}$. Thus $\Gamma_n = \Gamma'_n$, $\Gamma_{[n]} = \Gamma'_{[n]}$, $n = 0, 1, \ldots$, and (1.6) holds.

4. Multiple integrals with respect to random point measure. It is sometimes useful to interpret Poisson m.s.i. as (multiple) integrals with respect to random point measure (or point process) [7]. Assume that the Poisson r.m. $p = p(A)$, $A \in \mathscr{B}(X)$, with intensity $m \in M(X)$ can be written as the (infinite) sum of unit masses at random points (²) $\tau_1, \tau_2, \ldots$, i.e.,

$$p = \sum_{i=1}^{\infty} \delta_{\tau_i},$$

where $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ otherwise. Write $\{\tau\} = \{\tau_1, \tau_2, \ldots\}$. Let $K_r \uparrow X$ $(r \to \infty)$, where $K_r \in \mathscr{B}(X)$ and $m(K_r) < +\infty$ for every $m \geqslant 1$. For any $n = 1, 2, \ldots$ and any measurable function

$$f = f(x_1, \ldots, x_n): X^n \to C^1$$

(²) This can be done if $(X, \mathscr{B}(X))$ satisfies some mild topological assumptions [7] or if $(X, \mathscr{B}(X))$ is an arbitrary measurable space by the appropriate choice of the probability space $(\Omega, \mathscr{F})$ [14].

the sum

$$\sum_{\substack{x_1,\ldots,x_n\in\{\tau\}\cap K_r\\ x_i\neq x_j, i\neq j}} f(x_1, \ldots, x_n) \tag{4.1}$$

is a well-defined r.v. as the number of τ's in K_r is finite with probability 1 for any $r = 1, 2, \ldots$

PROPOSITION 4.1. *Let K_r, $r \geqslant 1$, be as above and let $f \in L^2(X^n) \cap L^1(X^n)$, $n = 1, 2, \ldots$ Then*

$$q^{(n)}(f) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!(n-k)!} \sum_{\substack{x_1,\ldots,x_k\in\{\tau\}\\ x_i\neq x_j, i\neq j}} f^{(k)}(x_1, \ldots, x_k), \tag{4.2}$$

where

$$f^{(k)}(x_1, \ldots, x_k) = \int_{X^{n-k}} f(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) m(dx_{k+1}) \ldots m(dx_n),$$

$k = 0, \ldots, n-1$, $f^{(n)} = f$, and the interior sum in (4.2) *is defined as the limit in $L^1(\Omega)$ of finite sums* (4.1) *with f replaced by $f^{(k)}$ as $r \to \infty$.*

Proof. If $f \in L_0^2(X^k)$ is simple and equal to $f^{\Delta_1,\ldots,\Delta_k}$ on $\Delta_1 \times \ldots \times \Delta_k$, then

$$p^{(k)}(f) = \sum_{\substack{x_1,\ldots,x_k\in\{\tau\}\\ x_i\neq x_j, i\neq j}} f(x_1, \ldots, x_k) = \sum_{\Delta_1,\ldots,\Delta_k} f^{\Delta_1,\ldots,\Delta_k} p(\Delta_1) \ldots p(\Delta_k), \tag{4.3}$$

which implies

$$\mathrm{E}[|p^{(n)}(f)|] \leqslant \|f\|_{L^1(X^n)}. \tag{4.4}$$

Therefore, $p^{(n)}(f)$ can be extended by $L^1(\Omega)$-continuity to an arbitrary $f \in L^1(X^n)$ preserving (4.4). In particular,

$$p^{(n)}(f) = \lim_{r\to\infty} p^{(n)}(f \cdot 1_{(K_r)^n}) = \lim_{r\to\infty} \sum_{\substack{x_1,\ldots,x_n\in\{\tau\}\cap K_r\\ x_i\neq x_j, i\neq j}} f(x_1, \ldots, x_n)$$

in $L^1(\Omega)$, and the right-hand side of (4.2) is well defined. By the linearity of both sides of (4.2) in f, it suffices to establish (4.2) for simple $f = \mathrm{sym}\, 1_{\Delta_1\times\ldots\times\Delta_n}$, where $\Delta_1, \ldots, \Delta_n$ are mutually disjoint. Then

$$\begin{aligned} q^{(n)}(f) &= q(\Delta_1) \ldots q(\Delta_n) = (p(\Delta_1) - m(\Delta_1)) \ldots (p(\Delta_n) - m(\Delta_n)) \\ &= \sum_{k=0}^{n} (-1)^{n-k} \sum_{|\alpha|=k} \prod_{i\in\alpha} p(\Delta_i) \prod_{j\notin\alpha} m(\Delta_j), \end{aligned} \tag{4.5}$$

where the second sum is taken over all subsets α of $\{1, \ldots, n\}$ such that $|\alpha| = k$. From (4.5) and (4.3) we obtain easily (4.2).

Let R be an automorphism of (X, m), i.e., a measurable 1-1 mapping of X onto X preserving the measure m. For $g \in L^2(X^n)$, $n = 1, 2, \ldots$, introduce

$$(Rg)(x_1, \ldots, x_n) = g(Rx_1, \ldots, Rx_n).$$

Clearly, R is a unitary operator in $L^2(X^n)$. Set $Rg = g$, $g \in L^2(X^0) = C^1$, and $Rf = (Rf_0, Rf_1, \ldots)$ for $f = (f_0, f_1, \ldots) \in \exp\{L^2(X)\}$. The following statement was proved to be useful in construction of new classes of self-similar random fields by means of Poisson m.s.i. [13]:

PROPOSITION 4.2. *For any $f \in \exp\{L^2(X)\}$, $f = (f_0, f_1, \ldots)$, the r.v.'s*

$$i(f) = \sum_{n=0}^{\infty} q^{(n)}(f_n)/n! \quad \textit{and} \quad i(Rf) = \sum_{n=0}^{\infty} q^{(n)}(Rf_n)/n!$$

are identically distributed.

Proof. It suffices to prove this fact for $f = (f_0, f_1, \ldots)$ such that all $f_0, f_1, \ldots$ are simple and all but a finite number of them are zero. By Proposition 4.1, we have

$$i(Rf) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n-k} C_n^k p^{(n)}\big((Rf)_n^{(k)}\big)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n-k} C_n^k (R^{-1}p)^{(n)}(f_n^{(k)}),$$

where $R^{-1}p = \sum_{j=1}^{\infty} \delta_{R\tau_j}$ is a Poisson r.m. identically distributed with p.

5. The operator $\Gamma(A)$. Let A be a *contraction* in $L^2(X)$, i.e., a continuous linear operator with norm less than or equal to 1. There exists a unique contraction $\Gamma(A)$ in $L^2(\Omega)$ such that

$$\Gamma(A) q^{(n)}(f_1 \hat{\otimes} \ldots \hat{\otimes} f_n) = q^{(n)}(Af_1 \hat{\otimes} \ldots \hat{\otimes} Af_n), \tag{5.1}$$

$f_1, \ldots, f_n \in L^2(X)$, $n = 1, 2, \ldots$ (we set $\Gamma(A) q^{(0)}(f) = q^{(0)}(f)$). An analogical operator in the Gaussian case plays an important role in quantum field theory, in connection with the second quantization, and exhibits two remarkable properties mentioned in the Introduction, namely: (a) it is positivity preserving in $L^2(\Omega)$ and (b) it is hypercontractive in $L^p(\Omega)$-spaces, $p \geqslant 1$ (see, e.g., [11] for details). In this section we establish the Poissonian analogue of (a) which is different from the Gaussian one.

Let $(K, \mathscr{B}(K))$ be a measurable space with a σ-finite measure m and let A be a bounded linear operator in $L^2(K)$. We say that A is *positivity preserving* and write $A \nabla 0$ if $f \geqslant 0$ m-a.e. implies $Af \geqslant 0$ m-a.e. (3). We call $A \nabla 0$ *sub-Markov* if $Af(x) \leqslant 1$ for any $f \leqslant 1$, and *Markov* if

$$\sup_n Af_n(x) = 1$$

(3) In the sequel we omit the phrase "m-a.e." (m-almost everywhere).

for an increasing sequence $f_n \in L^2(K)$, $0 \leqslant f_n \uparrow 1$. Finally, we say that A is *doubly sub-Markov* (resp. *doubly Markov*) if both A and A^* (A^* is the dual of A in $L^2(K)$) are sub-Markov (resp. Markov).

THEOREM 5.1. *$\Gamma(A)$ is positivity preserving if and only if A is doubly sub-Markov (in the latter case $\Gamma(A)$ is also doubly Markov).*

Remark 5.1. It follows from Riesz-Thorin theorem (see, e.g., Theorem 1.13 in [11]) that every doubly sub-Markov operator is a contraction.

Remark 5.2. We prove Theorem 5.1 by a probabilistic argument which at the same time reveals the probabilistic sense of the Markov semigroup $\Gamma(A^n)$, $n = 0, 1, \ldots$ An alternative analytic proof reduces to the computation of the kernel of $\Gamma(A)$ in the finite-dimensional case, which is more cumbersome.

Proof of Theorem 5.1. Sufficiency. Let A be doubly sub-Markov. We prove that $\Gamma(A) \nabla 0$.

Identify $(\Omega, \mathscr{F})$ with the space of all random point measures on X (see Section 4). Set $p(A)(\omega) = \omega(A)$, $A \in \mathscr{B}(X)$. Given a measure $m \in M(X)$, denote by P_m the probability on $(\Omega, \mathscr{F})$ such that $p = p(A)$, $A \in \mathscr{B}(X)$, is a Poisson r.m. on X with intensity m.

Every sub-Markov operator A in $L^2(X)$ determines (sub-Markov) transition kernel

$$P(x, B) = P^{(A)}(x, B) = \lim_{r \to \infty} A \cdot 1_{B \cap K_r}(x)$$

(K_r were introduced in Section 4), $0 \leqslant P(x, B) \leqslant 1$, $x \in X$, $B \in \mathscr{B}(X)$. Set $P^*(x, B) = P^{(A^*)}(x, B)$.

Define a Markov process w_t, $t = 0, 1, \ldots$, with the state space $(\Omega, \mathscr{F})$ as the time evolution of the initial point measure $w_0 \in \Omega$ such that every atom x of w_0 evolves in time $t = 0, 1, \ldots, \zeta$ (ξ is the life time) according to the transition law $P(x, B)$ independently of others, and the initial distribution of w_0 is P_m. More exactly, if w_0 is fixed and written in the form

$$w_0 = \sum_{j=1}^{\infty} \delta_{x_j(0)},$$

then

$$w_t = \sum_{j \in \{j: \zeta_j \geqslant t\}} \delta_{x_j(t)},$$

where $x_j = (x_j(t), t = 0, 1, \ldots, \zeta_j)$, $j = 1, 2, \ldots$, are independent Markov processes having the same transition function $P(x, B)$. Write $\tilde{w}_t$, $t = 0, 1, \ldots$, for the analogical process with the initial distribution $P_{\tilde{m}}$, $\tilde{m}(dx) = \varrho(x)\, m(dx)$, and $\varrho(x) = 1 - P^*(x, X)$ instead of P_m. Finally, let $\tilde{w}_t^{(k)}$, $k = 1, 2, \ldots$, $t = 0, 1, \ldots$, be independent copies of $\tilde{w}_t$ which are also independent of w_t, and set

$$r_t = w_t + \sum_{j=1}^{t} \tilde{w}_{t-j}^{(j)}, \quad t = 1, 2, \ldots, \qquad r_0 = w_0. \tag{5.2}$$

Clearly, r_t is a homogeneous Markov process with the state space Ω, which can be interpreted as time evolution of a Poisson point measure w_0 during which every unit mass evolves independently within its life time according to the transition function $P(x, B)$, with the consequent immigration at the moments of time $t = 1, 2, \ldots$ of identically behaving independent Poisson point measures with initial distributions $P_{\tilde{m}}$.

Let $F \in L^2(\Omega)$. We claim that

$$\Gamma(A)F = \mathrm{E}[F(r_1)| r_0], \tag{5.3}$$

which implies $\Gamma(A) \nabla 0$. Let L denote the linear space of functions $F: \Omega \to C^1$ of the form

$$F(\omega) = \sum_{j=1}^{n} a_j \exp\{-p(f_j)\}, \quad n = 1, 2, \ldots, \quad a_1, \ldots, a_n \in C^1, \tag{5.4}$$

where $f_1 \geqslant 0, \ldots, f_n \geqslant 0$ are simple functions. As L is dense in $L^2(\Omega)$, it suffices to prove (5.3) for $F \in L$. By the linearity of both sides of (5.3) in F, it is enough to consider the case $n = 1$, $a_1 = 1$, $f_1 = f$. By (2.11), we have

$$F = \exp\{\int_X (e^{-f}-1)\,dm\} \sum_{n=0}^{\infty} q^{(n)}\big((\otimes (e^{-f}-1))^n\big)/n! \tag{5.5}$$

and

(5.6)

$$\Gamma(A)F = \exp\{\int_X (e^{-f}-1)\,dm\} \sum_{n=0}^{\infty} q^{(n)}\big((\otimes A(e^{-f}-1))^n\big)/n!$$
$$= \exp\{p(\ln[A(e^{-f}-1)+1])\} \exp\{\int_X (e^{-f}-1)\,dm - \int_X A(e^{-f}-1)\,dm\}$$

(all the integrals in (5.5) and (5.6) are well defined). On the other hand, we obtain

$$\mathrm{E}[F(r_1)| r_0] = \mathrm{E}_{\tilde{m}}[\exp\{-p(f)\}]\,\mathrm{E}_m[\exp\{-\sum_{j \in \{j: \zeta_j \geqslant 1\}} f(x_j(1))\}| w_0] \tag{5.7}$$
$$= \exp\{\int_X (e^{-f}-1)\,d\tilde{m}\} \prod_{j=1}^{\infty} \big(\int_X P(x_j, dy) e^{-f(y)} + (1-P(x_j, X))\big)$$
$$= \exp\{\int_X (e^{-f}-1)\,\varrho\,dm\} \exp\{\sum_{j=1}^{\infty} \ln[A(e^{-f}-1)(x_j)+1]\}$$
$$= \exp\{\int_X (e^{-f}-1)\,\varrho\,dm\} \exp\{p(\ln[A(e^{-f}-1)+1])\},$$

where E_m denotes the expectation with respect to P_m. By the definition of ϱ,

$$\int_X (e^{-f}-1)\,\varrho\,dm = \int_X (e^{-f}-1)\,dm - \int_X A(e^{-f}-1)\,dm,$$

which proves (5.3).

From the general definition (5.1) of $\Gamma(A)$ and the uniqueness of expansion (1.3) it follows that $\Gamma(A)\cdot 1 = 1$, i.e., $\Gamma(A)$ is Markov, and

$$\Gamma(A)^* = \Gamma(A^*), \tag{5.8}$$

which implies that $\Gamma(A)^*$ is also Markov. This completes the proof of the sufficiency of Theorem 5.1, including the statement in brackets.

Necessity. Let A be a contraction in $L^2(X)$ but not positivity preserving. Then there exist a function $f \geqslant 0$, $f \in L^2(X) \cap L^1(X)$, and $\varepsilon > 0$ such that $m(B_\varepsilon) > \varepsilon$, where $B_\varepsilon = \{x \in X:\ Af(x) < -\varepsilon\}$. Let ξ be an r.v. and c a constant. Then $\xi = q(f)+c \geqslant 0$ if and only if $c \geqslant \int_X f\,dm$. As

$$\Gamma(A)\xi \leqslant -\varepsilon p(B_\varepsilon) + \int_{X\setminus B_\varepsilon} Af(x)\,q(dx)+c,$$

we have

$$P(\Gamma(A)\xi < 0) \geqslant P(p(B_\varepsilon) \geqslant N)\,P\Big(\int_{X\setminus B_\varepsilon} Af(x)\,q(dx)+c \leqslant \varepsilon N\Big). \tag{5.9}$$

If $N = N(\varepsilon)$ is sufficiently large, then the right-hand side of (5.9) is strictly positive. This proves the necessity of the condition $A \nabla 0$.

Denote by $\{A\}_n$ the set of all contractions $A \nabla 0$, A: $L^2(X) \to L^2(X)$, such that A maps $\mathscr{E}_n$ into itself, where $\mathscr{E}_n$ is a finite-dimensional subspace of simple functions:

$$\mathscr{E}_n = \{f = \sum_{k=1}^{n} f_k \cdot 1_{\Delta_k}:\ f_1, \ldots, f_n \in C^1\},$$

$\Delta_1, \ldots, \Delta_n$ being mutually disjoint and $m(\Delta_i) = \mu$, $i = 1, \ldots, n$. Identify $\mathscr{E}_n$ with C^n and $A_n \equiv A|\mathscr{E}_n$ (A restricted to $\mathscr{E}_n$), $A \in \{A\}_n$, with the matrix (a_{ij}), $i, j = 1, \ldots, n$. Clearly, $A_n \nabla 0$ is equivalent to $a_{ij} \geqslant 0$, $i, j = 1, \ldots, n$, and A_n is sub-Markov if and only if $(A_n 1)_i \leqslant 1$, $i = 1, \ldots, n$, where $1 = (1, \ldots, 1) \in R^n$.

Denote by $L^2(Z_+^n)$ the set of all functions F: $Z_+^n \to C^1$, $Z_+^n = (Z_+)^n$, $Z_+ = \{0, 1, \ldots\}$, such that

$$\mathrm{E}\,[|F(p)|^2] < +\infty, \qquad (p) = (p(\Delta_1), \ldots, p(\Delta_n)).$$

For any $F \in L^2(Z_+^n)$, $A \in \{A\}_n$, $\Gamma(A)F(p)$ is again $\sigma(p)$-measurable and defines a function $\Gamma(A)F \in L^2(Z_+^n)$ by

$$(\Gamma(A)F)(p) = \Gamma(A)(F(p)). \tag{5.10}$$

We prove below in the Appendix that for any contraction A, $A \in \{A\}_n$, and any $w = (w_1, \ldots, w_n) \in C^n$ we have

$$\Gamma(A)\Big[\prod_{i=1}^{n} (1+w_i)^{z_i}\Big] = \prod_{i=1}^{n} \big(1+(A_n w)_i\big)^{z_i} \exp\{\mu(w_i-(A_n w)_i)\} \tag{5.11}$$

(for $w_1 > -1, \ldots, w_n > -1$ and sub-Markov A this follows also from (5.6)). Set

$$F_w = F_w(z_1, \ldots, z_n) = (1+w)^{z_1+\ldots+z_n}, \quad w \in \mathbf{R}^1.$$

By (5.11), we have

$$[\Gamma(A)F_w](z_1, \ldots, z_n) = \prod_{i=1}^{n} \left(1+(A_n 1)_i w\right)^{z_i} \exp\{\mu w(1-(A_n 1)_i)\}.$$

Note that $F_w \geqslant 0$ (resp. $\Gamma(A)F_w \geqslant 0$) if and only if $1+w \geqslant 0$ (resp. $1+ +w(A_n 1)_i \geqslant 0$, $i = 1, \ldots, n$). Therefore, the relation $\Gamma(A) \nabla 0$ for operators $A \in \{A\}_n$ implies that $A_n = A|\mathscr{E}_n$ is sub-Markov.

Define operators $I_n\colon L^2(X) \to \mathscr{E}_n$ by

$$(I_n f)(x) = \begin{cases} \int_{\Delta_i} f\,dm/m(\Delta_i) & \text{if } x \in \Delta_i,\ i = 1, \ldots, n, \\ 0 & \text{otherwise} \end{cases}$$

and set $B^{(n)} = I_n A$. It can be shown that both operators A and $\Gamma(A)$ can be approximated strongly by $B^{(n)}$ and $\Gamma(B^{(n)})$, respectively, by taking the space $\mathscr{E}_n$ "sufficiently large". Moreover, if all operators $B^{(n)}|\mathscr{E}_n$ are sub-Markov, it can be shown that A is also sub-Markov. Now, as $\Gamma(B^{(n)}) = \Gamma(I_n)\Gamma(A)$ and I_n is doubly sub-Markov, $\Gamma(A) \nabla 0$ implies $\Gamma(B^{(n)}) \nabla 0$. Since $B^{(n)} \in \{A\}_n$, using the assumption that $\Gamma(A) \nabla 0$ but A is not sub-Markov we get a contradiction.

Finally, by (5.8), $\Gamma(A) \nabla 0$ implies $\Gamma(A^*) \nabla 0$, i.e., A^* is also sub-Markov.

Remark 5.3. Let A_t, $t \geqslant 0$, be a continuous time contraction semigroup in $L^2(X)$. Then from Theorem 5.1 it follows that the corresponding semigroup $\Gamma(A_t)$, $t \geqslant 0$, in $L^2(\Omega)$ is Markov if and only if all operators A_t, $t \geqslant 0$, are doubly sub-Markov. In this case, we can also define the analogical continuous time Markov process r_t, $t \geqslant 0$, with the state space Ω, with immigration, and with Poisson measure P_m as the invariant measure.

Let us present the formula for the transition function of the process r_t, $t \geqslant 0$, in the simplest case where $X = \{x\}$, $m(\{x\}) = m$, $A_t = e^{-ct}$, $c > 0$. Set $\Omega = \{0, 1, \ldots\}$ and

$$\Gamma(e^{-ct})f(i) = \sum_{j=0}^{\infty} f(j)\Gamma_c(t, i, j), \quad i \in \Omega.$$

Then

$$\Gamma_c(t, i, j) = \exp\{m(e^{-ct}-1)\}(1-e^{-ct})^{i-j} e^{-ctj} \times$$
$$\times \sum_{k=\max(0,j-i)}^{j} m^k (1-e^{-ct})^{2k} e^{ctk} C_i^{j-k}/k!.$$

Thus r_t is a stationary birth and death process with invariant Poisson measure $e^{-m}m^i/i!$, $i \in \Omega$, and transition rates

$$q_i^+ = q_{i,i+1} = cm, \quad q_i^- = q_{i,i-1} = ci, \quad i = 0, 1, \ldots,$$
$$q_{ij} = \partial \Gamma_c(t, i, j)/\partial t|_{t=0},$$

invariant also with respect to time reversion.

6. Appendix. Proof of relation (5.11). By the definition of $\Gamma(A)$ and (1.6), we have

$$(6.1) \quad \Gamma(A){:}\, q(\Delta_1)^{k(1)}\ldots q(\Delta_n)^{k(n)}{:} = {:}(\sum_{j=1}^n a_{j1}q(\Delta_j))^{k(1)}\ldots(\sum_{j=1}^n a_{jn}q(\Delta_j))^{k(n)}{:}$$
$$= \sum_{(r)} \prod_{j=1}^n \frac{k(j)!}{r(j,1)!\ldots r(j,n)!} a_{1j}^{r(j,1)}\ldots a_{nj}^{r(j,n)}{:}\, q(\Delta_1)^{t(1)}\ldots q(\Delta_n)^{t(n)}{:}$$

for any integers $k(1) \geq 0, \ldots, k(n) \geq 0$, where the sum $\sum_{(r)}$ is taken over all non-negative integers $r(i,j)$, $i, j = 1, \ldots, n$, such that $r(j,1)+\ldots+r(j,n) = k(j)$, and $t(j) = r(1,j)+\ldots+r(n,j)$, $j = 1, \ldots, n$. Introduce "shifted Charlier polynomials" $j_k'(x) = j_k(x-\mu;\, \mu)$, $x = 0, 1, \ldots$, $\mu = m(\Delta_j)$. By (6.1) and (2.5), we get

$$(6.2) \quad \Gamma(A) j'_{k(1)}(x_1)\ldots j'_{k(n)}(x_n)$$
$$= \sum_{(r)} \prod_{j=1}^n \frac{k(j)!}{r(j,1)!\ldots r(j,n)!} a_{1j}^{r(j,1)}\ldots a_{nj}^{r(j,n)} j'_{t(1)}(x_1)\ldots j'_{t(n)}(x_n).$$

As

$$(6.3) \qquad \prod_{i=1}^n (1+w_i)^{x_i} = \prod_{i=1}^n \sum_{k(i)=0}^{\infty} w_i^{k(i)} j'_{k(i)}(x_i) e^{\mu w_i}/k(i)!$$

and the right-hand side is convergent in $L^2(Z_+^n)$ for any $w_1, \ldots, w_n \in C^1$, by (6.2) we obtain

$$(6.4) \quad \Gamma(A) \prod_{i=1}^n (1+w_i)^{x_i} = \sum_{k(1),\ldots,k(n)=0}^{\infty} w_1^{k(1)}\ldots w_n^{k(n)} \exp\{\mu(w_1+\ldots+w_n)\} \times$$
$$\times \sum_{(r)} \prod_{j=1}^n \frac{k(j)!}{r(j,1)!\ldots r(j,n)!} a_{1j}^{r(j,1)}\ldots a_{nj}^{r(j,n)} j'_{t(1)}(x_1)\ldots j'_{t(n)}(x_n) \equiv \sigma.$$

Replace now summation in (6.4) over $k(1), \ldots, k(n) \geq 0$ by summation over integers $t(1), \ldots, t(n) \geq 0$; this yields

$$(6.5) \quad \sigma \exp\{-\mu(w_1+\ldots+w_n)\}$$
$$= \sum_{t(1),\ldots,t(n)=0}^{\infty} j'_{t(1)}(x_1)\ldots j'_{t(n)}(x_n) \sum_{[r]} \prod_{i=1}^n \frac{(a_{i1}w_1)^{r(1,i)}\ldots(a_{in}w_n)^{r(n,i)}}{r(1,i)!\ldots r(n,i)!}$$
$$= \sum_{t(1),\ldots,t(n)=0}^{\infty} \frac{j'_{t(1)}(x_1)\ldots j'_{t(n)}(x_n)}{t(1)!\ldots t(n)!} \prod_{i=1}^n (a_{i1}w_1+\ldots+a_{in}w_n)^{t(i)},$$

where the sum $\sum_{[r]}$ is taken over all integers $r(\cdot, \cdot)$ such that

$$r(1, 1)+\ldots+r(n, 1) = t(1),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$r(1, n)+\ldots+r(n, n) = t(n).$$

Now, (6.5) and (6.3) results in (5.11).

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