PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 3, Fasc. 2 (1984), p. 205-215

INEQUALITIES BETWEEN INTEGRALS OF p-STABLE SYMMETRIC MEASURES ON BANACH SPACES

BY

WERNER LINDE AND PETER MATHÉ (JENA)

Abstract. Let μ and ν be symmetric Gaussian probability measures on a Banach space E and let E' be the dual of E. Then, as is well known, the inequality

$$\int_E |\langle x, a\rangle|^2 \, d\mu(x) \leqslant \int_E |\langle x, a\rangle|^2 \, d\nu(x) \quad \text{for all } a \in E'$$

implies

$$\int_E \|x\|^2 \, d\mu(x) \leqslant \int_E \|x\|^2 \, d\nu(x).$$

If we replace Gaussian measures by p-stable ones $(0 < p < 2)$, the property does not hold. Thus we consider the class $\mathscr{A}_p$ of such Banach spaces, where a generalization to the p-stable case is true. Furthermore, we give relations of $\mathscr{A}_p$ to some other classes of Banach spaces and we get also inclusion properties of $\mathscr{A}_p$, $0 < p < 2$. Recently, similar classes of Banach spaces have been investigated by Mandrekar, Thang, Tien, and Weron.

Given a real number p, $0 < p \leqslant 2$, we investigate Banach spaces E having the following property:

If $0 < r < p$, then there exists a constant $c \geqslant 1$ such that the inequality

$$(*) \qquad \int_E |\langle x, a\rangle|^r \, d\mu(x) \leqslant \int_E |\langle x, a\rangle|^r \, d\nu(x)$$

for all $a \in E'$ (E' is the dual of E) implies

$$\int_E \|x\|^r \, d\mu(x) \leqslant c^r \int_E \|x\|^r \, d\nu(x)$$

for all p-stable symmetric Radon measures μ and ν on E.

It is well known (cf. [2]) that every Banach space has this property in the

Gaussian case, i.e., for $p = 2$. But there are examples of Banach spaces where such a constant does not exist for any $p < 2$, e.g., L_q, $2 < q \leqslant \infty$.

We characterize Banach spaces having this property in terms of inequalities of sums $\sum_{i=1}^{n} x_i\theta_i$, where $x_1, \ldots, x_n \in E$ and $\theta_1, \theta_2, \ldots$ is an independent sequence of standard p-stable real random variables.

Using ideas of [10] we prove that the property becomes stronger if p is lessened and that there are examples of Banach spaces satisfying such an inequality for some $p > 1$, but not for any $q < p$.

Finally, we show that every Banach space having this property must be of cotype 2 in the sense of [9] whenever $0 < p < 2$.

It is not known whether or not the above-mentioned property coincides with the property "stable cotype p" in the sense of [10], $0 < p < 2$.

1. Notation and definitions. E always denotes a real Banach space, E' its dual, and unless otherwise stated p is a real number with $0 < p \leqslant 2$. The set of all p-stable symmetric Radon measures on E will be denoted by $R_p(E)$. Let us recall that a symmetric Radon measure μ is *p-stable* if its characteristic function (c.f.) $\hat{\mu}$ can be written as

$$\hat{\mu}(a) = \exp(-\|Ta\|^p), \qquad a \in E',$$

where T is an operator from E' into some L_p.

Given $\mu, \nu \in R_p(E)$ we write

$$\mu < \nu$$

provided that ($*$) holds for some (each) r with $0 < r < p$, which is equivalent to

$$\hat{\nu}(a) \leqslant \hat{\mu}(a), \qquad a \in E',$$

or

$$\mu\{x \in E; |\langle x, a\rangle| \geqslant 1\} \leqslant \nu\{x \in E; |\langle x, a\rangle| \geqslant 1\}, \qquad a \in E'.$$

By an *E-valued random variable* (r.v.) we mean a strongly measurable mapping φ from a probability space (Ω, P) into E. Its distribution dist(φ) is defined by

$$\operatorname{dist}(\varphi)(B) := P\{\varphi(\omega) \in B\},$$

where B is a Borel subset of E. Then dist (φ) defines a Radon measure on E.

In general, L_p means $L_p(\Omega, P)$ unless otherwise stated. For fixed p we denote by $\theta_1, \theta_2, \ldots$ a sequence of independent real r.v.'s with c.f. $\exp(-|t|^p)$.

A Banach space E is of *stable type* p (cf. [11]) if there is a constant $c \geqslant 0$ such that

$$\Big\{\mathrm{E}\Big\|\sum_{i=1}^{n} x_i\theta_i\Big\|^r\Big\}^{1/r} \leqslant c\Big\{\sum_{i=1}^{n} \|x_i\|^p\Big\}^{1/p}$$

for some r with $0<r<p$ and all $x_1, \ldots, x_n \in E$ (E means the expectation).

E is said to be of *cotype* 2 if

$$\{\sum_{i=1}^{n} \|x_i\|^2\}^{1/2} \leqslant c\{\mathrm{E}\|\sum_{i=1}^{n} x_i\gamma_i\|^2\}^{1/2}$$

for some $c \geqslant 0$ and all $x_1, \ldots, x_n \in E$, where $\gamma_1, \gamma_2, \ldots$ is an independent sequence of standard Gaussian r.v.'s.

2. The space $\Lambda_p(E', L_p)$. Here we want to recall some definitions and results of [4] which will be used in the sequel.

$\Lambda_p(E', L_p)$ denotes the set of all operators T from E' into L_p for which $\exp(-\|Ta\|^p)$ is the c.f. of a Radon measure μ_T. Clearly, $\mu_T \in R_p(E)$ whenever $T \in \Lambda_p(E', L_p)$.

If we put

$$\lambda_r(T) := \{\int_E \|x\|^r d\mu_T(x)\}^{1/r}, \quad 0<r<p,$$

the space $\Lambda_p(E', L_p)$ becomes a complete normed $(1 \leqslant r < p)$ (resp. quasi-normed $(0 \leqslant r < 1)$) space (cf. [4]).

For the $\tau_c(E', E)$-topology on E' generated by the compact subsets of E, it is known that each operator $T \in \Lambda_p(E', L_p)$ is continuous with respect to $\tau_c(E', E)$ and the norm (quasi-norm) topology on L_p.

3. The class $\mathscr{A}_p$. Given p with $0<p\leqslant 2$, $\mathscr{A}_p$ denotes the class of all Banach spaces E having the following property:

For some (each) r with $0<r<p$ there exists a constant $c \geqslant 1$ such that for all $\mu, \nu \in R_p(E)$ with $\mu < \nu$ the estimation

$$\int_E \|x\|^r d\mu(x) \leqslant c^r \int_E \|x\|^r d\nu(x)$$

holds.

As already mentioned, $\mathscr{A}_2$ is the class of all Banach spaces.

For later purposes we reformulate the above definition in terms of operators in $\Lambda_p(E', L_p)$:

A Banach space E belongs to $\mathscr{A}_p$ if for some (each) r with $0<r<p$ there exists a constant $c \geqslant 1$ such that for all $T, S \in \Lambda_p(E', L_p)$ with $\|Ta\| \leqslant \|Sa\|$, $a \in E'$, the estimation

$$\lambda_r(T) \leqslant c\lambda_r(S)$$

is valid.

Remark. Theorem 4 of [4] shows that the definition is independent of the special choice of the number r.

THEOREM 1. *Suppose $\|Ta\| \leqslant \|Sa\|$, $a \in E'$, implies $T \in \Lambda_p(E', L_p)$ whenever $S \in \Lambda_p(E', L_p)$. Then E belongs to $\mathscr{A}_p$.*

Proof. Assume that $E \notin \mathscr{A}_p$. Then there are operators T_n and S_n in $\Lambda_p(E', L_p(\Omega_n, P_n))$ such that $\|T_n a\| \leqslant \|S_n a\|$, $a \in E'$, while $\lambda_r(T_n) \geqslant 1$ and $\lambda_r(S_n) \leqslant 2^{-n}$, $n = 1, 2, \ldots$, for some $r < p$. Without loss of generality (taking disjoint unions) we may assume $T_n, S_n \in \Lambda_p(E', L_p)$ and

$$\|\sum_{n=1}^{m} T_n a\|^p = \sum_{n=1}^{m} \|T_n a\|^p, \quad \|\sum_{n=1}^{m} S_n a\|^p = \sum_{n=1}^{m} \|S_n a\|^p, \quad m = 1, 2, \ldots$$

Using

$$\|T_n a\| \leqslant \|S_n a\| \leqslant c_{rp}^{-1} \lambda_r(S_n) \|a\| \leqslant c_{rp}^{-1} \cdot 2^{-n} \|a\|$$

(cf. [4], $c_{rp} := \{\mathrm{E}|\theta_1|^r\}^{1/r}$), we infer that the operators

$$T = \sum_{n=1}^{\infty} T_n \quad \text{and} \quad S = \sum_{n=1}^{\infty} S_n$$

exist. Moreover, $\|Ta\| \leqslant \|Sa\|$, $a \in E'$, and because of the completeness of $\Lambda_p(E', L_p)$ with respect to λ_r the operator S belongs to $\Lambda_p(E', L_p)$. Now, by assumption, $T \in \Lambda_p(E', L_p)$, i.e.,

$$\exp(-\|Ta\|^p) = \exp\left(-\sum_{n=1}^{\infty} \|T_n a\|^p\right)$$

is the c.f. of a Radon measure on E.

If ξ_n is an independent sequence of E-valued r.v.'s with c.f. $\exp(-\|T_n a\|^p)$, then by the Ito-Nisio theorem (cf. [3]) there exists an r.v. ξ such that

$$\mathrm{E}\|\xi - \sum_{n=1}^{m} \xi_n\|^r \to 0 \quad \text{as } m \to \infty$$

Since

$$\lambda_r(T_n) = \{\mathrm{E}\|\xi_n\|^r\}^{1/r},$$

this contradicts $\lambda_r(T_n) \geqslant 1$, proving the theorem.

Corollary 1. *If E is of stable cotype p, $0 < p < 2$, in the sense of* [10], *then E belongs to $\mathscr{A}_p$.*

Remark. We do not know whether or not the converse of Corollary 1 also holds.

Our next aim is to prove the converse of Theorem 1 under an additional property of E. But for this purpose we need

Lemma 1. *Let A_i, $i \in I$, be a generalized sequence of operators in E such that*

$$\sup_{i \in I} \|A_i\| < \infty \quad \textit{and} \quad \lim A_i x = x$$

uniformly on compact subsets of E. If $T \in \Lambda_p(E', L_p)$, then

$$\lim_{i \in I} \lambda_r(T - TA_i') = 0.$$

Proof. Let $\mu = \mu_T$ be the Radon measure on E generated by T and put

$$d = \sup_{i \in I} \|A_i\|.$$

Given $\varepsilon > 0$ we choose a compact subset $K \subseteq E$ such that

$$\int_{E \setminus K} \|x\|^r d\mu(x) \leqslant \varepsilon^r (2(1+d)^r)^{-1}.$$

Then we find an element $i_0 \in I$ with

$$\sup_{x \in K} \|A_i x - x\|^r \leqslant \varepsilon^r/2$$

whenever $i > i_0$. Thus

$$\lambda_r(T - TA_i') = \{\int_E \|x - A_i x\|^r d\mu(x)\}^{1/r}$$
$$\leqslant \{(1+d)^r \int_{E \setminus K} \|x\|^r d\mu(x) + \int_K \|x - A_i x\|^r d\mu(x)\}^{1/r} \leqslant \varepsilon$$

provided $i > i_0$. This proves Lemma 1.

THEOREM 2. *Let E be a Banach space in $\mathscr{A}_p$ having the metric approximation property* (*m.a.p.*) (cf. [7] for the definition). *Then for any operator T from E' into L_p and $S \in \Lambda_p(E', L_p)$ the inequality $\|Ta\| \leqslant \|Sa\|$, $a \in E'$, implies $T \in \Lambda_p(E', L_p)$.*

Proof. By assumption there exists a generalized sequence A_i, $i \in I$, of operators of finite rank in E such that

$$\sup_{i \in I} \|A_i\| \leqslant 1 \quad \text{and} \quad \lim A_i x = x$$

uniformly on compact subsets of E. From Lemma 1 we get

$$\lim \lambda_r(S - SA_i') = 0.$$

Consequently, since

$$\|T(A_i' - A_j')a\| \leqslant \|S(A_i' - A_j')a\|, \quad a \in E', i, j \in I,$$

there exists an $i_0 \in I$ such that $\lambda_r(TA_i' - TA_j') < \varepsilon$ whenever $i, j > i_0$ for given $\varepsilon > 0$. By the completeness of $\Lambda_p(E', L_p)$ with respect to λ_r, the generalized sequence TA_i' converges to some operator $T_0 \in \Lambda_p(E', L_p)$. It remains to prove that $T = T_0$.

Since S is $\tau_c(E', E)$-continuous, so is T, which follows from the inequality $\|Ta\| \leqslant \|Sa\|$, $a \in E'$. Therefore, for each $a \in E'$ the generalized sequence $TA_i'a$ converges to Ta. On the other hand, $TA_i'a$ converges also to $T_0 a$ because

$$\|TA_i'a - T_0 a\| \leqslant c_{rp}^{-1} \lambda_r(TA_i' - T_0)\|a\|,$$

which proves that $T = T_0$.

Remark. Theorems 1 and 2 combined together give the following:

If E has the m.a.p., then E belongs to $\mathscr{A}_p$ *iff for every* $T \in L(E', L_p)$ *and* $S \in \Lambda_p(E', L_p)$ *the inequality* $\|Ta\| \leqslant \|Sa\|$, $a \in E'$, *implies* $T \in \Lambda_p(E', L_p)$.

For the last property we refer to [14] or [8].

A careful examination of the proof of Theorem 2 shows that we only used the estimation in the definition of $\mathscr{A}_p$ for the operators $T(A_i' - A_j')$ and $S(A_i' - A_j')$ which are of finite rank. This proves the following

THEOREM 3. *Let E be a Banach space having the m.a.p. Then E belongs to* $\mathscr{A}_p$ *iff for some r with* $0 < r < p$ *there exists a constant* $c \geqslant 1$ *such that for all finite-dimensional subspaces* $F \subseteq E$ *and all* $\mu, \nu \in R_p(F)$ *with* $\mu \prec \nu$ *the estimation*

$$\int_F \|x\|^r d\mu(x) \leqslant c^r \int_F \|x\|^r d\nu(x)$$

holds.

Remark. If a Banach space E has the m.a.p., then from Theorem 3 it follows that the property that E belongs to $\mathscr{A}_p$ depends in fact only on finite-dimensional subspaces of E.

Now, we give some examples of Banach spaces in $\mathscr{A}_p$. The following theorem is implied also by Corollary 1 and [10].

THEOREM 4. *If* $1 \leqslant q \leqslant 2$, *then every* $\mathscr{L}_q$*-space in the sense of* [6] *belongs to* $\mathscr{A}_p$, $0 < p \leqslant 2$.

Proof. Using Theorem 3 it suffices to consider measures $\mu, \nu \in R_p(l_q^m)$ with $\mu \prec \nu$. If $\varrho_1, \varrho_2, \ldots$ denotes a sequence of independent q-stable random variables, i.e., their c.f. is $\exp(-|t|^q)$, then for $0 < r < \min(p, q)$ we get

$$\begin{aligned}\int_{l_q^m} \|x\|^r d\mu(x) &= \int_{l_q^m} \Big(\sum_{i=1}^m |\langle x, e_i\rangle|^q\Big)^{r/q} d\mu(x) \\ &= c_{rq}^{-r} \int_{l_q^m} \mathrm{E}\Big|\sum_{i=1}^m \langle x, e_i\rangle \varrho_i\Big|^r d\mu(x) \\ &\leqslant c_{rq}^{-r} \mathrm{E} \int_{l_q^m} \Big|\Big\langle x, \sum_{i=1}^m e_i \varrho_i\Big\rangle\Big|^r d\nu(x) = \int_{l_q^m} \|x\|^r d\nu(x),\end{aligned}$$

where $e_1, \ldots, e_m$ are the unit vectors of l_q^m.

Our next aim is to restrict the set of measures in the definition of the class $\mathscr{A}_p$. More precisely, we show that it suffices to investigate measures $\mu \in R_p(E)$ which can be written as

$$\mu = \operatorname{dist}\Big(\sum_{i=1}^n x_i \theta_i\Big), \qquad x_1, \ldots, x_n \in E.$$

Note that even if E is finite dimensional, not every measure $\mu \in R_p(E)$, $0 < p < 2$, can be written in this way.

THEOREM 5. *Suppose E has the m.a.p. Then E belongs to* $\mathscr{A}_p$ *iff for some r with*

$0 < r < p$ there exists a constant $c \geqslant 1$ such that for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in E$ with

$$\sum_{i=1}^{n} |\langle x_i, a\rangle|^p \leqslant \sum_{i=1}^{n} |\langle y_i, a\rangle|^p, \qquad a \in E',$$

the estimation

$$\mathrm{E}\Big\|\sum_{i=1}^{n} x_i\theta_i\Big\|^r \leqslant c^r \mathrm{E}\Big\|\sum_{i=1}^{n} y_i\theta_i\Big\|^r$$

is valid.

Proof. Of course, the condition of Theorem 5 is necessary even without any further assumption. To prove that it is sufficient we use Theorem 3. Let $F \subseteq E$ be finite dimensional and let T and S be operators in $\Lambda_p(F', L_p)$ such that $\|Tb\| \leqslant \|Sb\|$ for all $b \in F'$. We may assume S to be injective. Otherwise, we have to take a subspace of F. Given $\varepsilon > 0$ we put

$$\delta = \inf\{(1+\varepsilon)\|Sb\| - \|Tb\|;\ b \in F',\ \|b\| = 1\}.$$

From the inequality $\|Tb\| \leqslant \|Sb\|$ and the compactness of the unit sphere of F' we get $\delta > 0$. Next, we approximate T and S by operators $T_m, S_m \in \Lambda_p(F', L_p)$ such that

$$\|T_m b\|^p = \sum_{i=1}^{n_m} |\langle x_i^m, b\rangle|^p, \qquad \|S_m b\|^p = \sum_{i=1}^{n_m} |\langle y_i^m, b\rangle|^p, \qquad b \in F',$$

$x_1^m, \ldots, x_{n_m}^m, y_1^m, \ldots, y_{n_m}^m \in F$. Since $\delta > 0$, we find a natural number m_0 with

$$\|T_m b\| \leqslant (1+\varepsilon)\|S_m b\|, \qquad b \in F',$$

whenever $m \geqslant m_0$. Thus, by assumption and the special form of T_m and S_m, we get

$$\lambda_r(T_m) \leqslant c(1+\varepsilon)\lambda_r(S_m).$$

It remains to prove that

$$\lim \lambda_r(T_m) = \lambda_r(T) \quad \text{and} \quad \lim \lambda_r(S_m) = \lambda_r(S)$$

Let μ_m, μ be the sequence of measures in $R_p(F)$ generated by T_m and T, respectively. Then, since $\dim F < \infty$, μ_m converges weakly to μ. On the other hand, the function $x \to \|x\|^r$ from the Banach space F into the real numbers is uniformly integrable with respect to $\{\mu_m\}$. This follows, e.g., by the results of [1] ($0 < r < p$!). Consequently,

$$\lim \lambda_r(T_m) = \lim \Big\{\int_F \|x\|^r d\mu_m(x)\Big\}^{1/r} = \Big\{\int_F \|x\|^r d\mu(x)\Big\}^{1/r} = \lambda_r(T),$$

which completes the proof of the theorem.

4. Inclusion properties of $\mathscr{A}_p$. As the main result in this section we get the inclusion $\mathscr{A}_q \subseteq \mathscr{A}_p$ for $0 < q \leqslant p \leqslant 2$. For this purpose we construct a mapping from $\Lambda_p(E', L_p)$ into $\Lambda_q(E', L_q)$. The main ideas of this construction can be found in [10].

LEMMA 2. *Assume $0 < q < p \leqslant 2$. Then there exists a probability measure α on $[0, \infty)$ such that*

(1) $\exp(-u^q) = \int_0^\infty \exp(-vu^p)\, d\alpha(v)$, $u \in [0, \infty)$,

(2) $\alpha\{0\} = 0$,

(3) *for positive s the integral $\int_0^\infty v^s\, d\alpha(v)$ is finite iff $s < q/p$.*

Proof. By Schönberg's theorem (cf. [2]), for $t = 2q/p$, $0 < t < 2$, there exists a measure α on $[0, \infty)$ such that

$$\exp(-w^t) = \int_0^\infty \exp(-vw^2)\, d\alpha(v), \qquad w \in [0, \infty).$$

Replacing w by $u^{p/2}$ we obtain (1).

Putting $u = 0$ we get $1 = \alpha([0, \infty))$, i.e., α is a probability measure.

Property (2) can be shown by taking the limit as $u \to \infty$ on both sides of (1) and using the Lebesgue theorem.

Finally, the integral

$$\int_0^\infty \frac{1 - \exp(-u^t)}{u^{1+2s}}\, du$$

is finite iff $0 < s < t/2 = q/p$. But this integral is equal to

$$\int_0^\infty v^s\, d\alpha(v) \int_0^\infty \frac{1 - \exp(-w^2)}{w^{1+2s}}\, dw,$$

which proves (3).

THEOREM 6. *Let T be an operator from E' into L_p. Then the following statements are equivalent:*

(1) $T \in \Lambda_p(E', L_p)$.

(2) *For one (each) real number q with $0 < q < p$ the function $\exp(-\|Ta\|^q)$ is the c.f. of a Radon measure ν_q on E $(\nu_q \in R_q(E))$. Moreover, if $0 < r < q < p$, then there exists a constant $c(r, q, p)$ (independent of E and T) such that*

$$\lambda_r(T) = c(r, q, p) \{\int_E \|x\|^r\, d\nu_q(x)\}^{1/r}.$$

Proof. It is shown in [4] that (2) implies (1) if (2) is satisfied for one $q < p$.

Now we choose $T \in \Lambda_p(E', L_p)$ and our aim is to show that $\exp(-\|Ta\|^q)$ defines the c.f. of a Radon measure on E. We put

$$\nu_q(B) := \int_0^\infty \mu(v^{-1/p} B)\, d\alpha(v),$$

where $B \subseteq E$ is a Borel subset, $\mu \in R_p(E)$ is defined by T, and α denotes the probability measure on $[0, \infty)$ constructed in Lemma 2. Clearly, ν_q is a Radon measure on E (it is σ-additive and concentrated on the separable support of μ). Next we calculate the c.f. of ν_q:

$$\hat{\nu}_q(a) = \int_E \exp(i\langle x, a\rangle)\, d\nu_q(x) = \int_E \int_0^\infty \exp(i\langle v^{1/p} x, a\rangle)\, d\alpha(v)\, d\mu(x)$$
$$= \int_0^\infty \hat{\mu}(v^{1/p} a)\, d\alpha(v) = \int_0^\infty \exp(-v\|Ta\|^p)\, d\alpha(v) = \exp(-\|Ta\|^q).$$

This proves (2).

Now, if $0 < r < q$, we get

$$\Big\{\int_E \|x\|^r\, d\nu_q(x)\Big\}^{1/r} = \Big\{\int_E \int_0^\infty v^{r/p}\, d\alpha(v)\, \|x\|^r\, d\mu(x)\Big\}^{1/r}$$
$$= \lambda_r(T) \Big\{\int_0^\infty v^{r/p}\, d\alpha(v)\Big\}^{1/r}.$$

The last integral is finite because $s = r/p < q/p$. Putting

$$c(r, q, p) = \Big\{\int_0^\infty v^{r/p}\, d\alpha(v)\Big\}^{-1/r},$$

we obtain the result.

The next corollary was shown in [12]. It is an immediate consequence of Theorem 6 in the case $p = 2$.

Corollary 2. *Let R be an operator from E' into E which is positive ($\langle Ra, a\rangle \geq 0$, $a \in E'$) and symmetric ($\langle Ra, b\rangle = \langle Rb, a\rangle$, $a, b \in E'$). Then R is a Gaussian covariance, i.e., $\exp(-\langle Ra, a\rangle)$ is the c.f. of a Radon measure, iff for some (each) q with $0 < q < 2$ the function $\exp(-\langle Ra, a\rangle^{q/2})$ is the c.f. of a Radon measure on E.*

Corollary 3. *If $0 < q \leq p \leq 2$, then $\mathcal{A}_q \subseteq \mathcal{A}_p$.*

Proof. Fix $E \in \mathcal{A}_q$ and $S, T \in \Lambda_p(E', L_p)$ with $\|Ta\| \leq \|Sa\|$. Since this implies $\|Ta\|^q \leq \|Sa\|^q$, we infer from Theorem 6 that there exist $T_1, S_1 \in \Lambda_q(E', L_q)$ with $\|T_1 a\| = \|Ta\|^q$, $\|S_1 a\| = \|Sa\|^q$, and for $r < q \leq p$

$$\lambda_r(T) = c(r, q, p)\, \lambda_r(T_1), \qquad \lambda_r(S) = c(r, q, p)\, \lambda_r(S_1).$$

Hence $E \in \mathcal{A}_q$ and $\|Ta\| \leq \|Sa\|$ imply

$$\|T_1 a\| \leq \|S_1 a\| \quad \text{and} \quad \lambda_r(T_1) \leq c\lambda_r(S_1).$$

Consequently, $E \in \mathcal{A}_p$.

Our next aim is to prove that the inclusion $\mathscr{A}_q \subseteq \mathscr{A}_p$ is strict whenever $p > 1$ and $0 < q < p$. Here we also use the example constructed in [10].

The following two lemmas are proved by exactly the same arguments as used in [10] and [15], respectively.

LEMMA 3. *For* $1 \leqslant s < p \leqslant 2$, $E \in \mathscr{A}_p$ *implies* $l_s(E) \in \mathscr{A}_p$, *where* $l_s(E)$ *is defined as*

$$l_s(E) = \{(x_i) \subseteq E;\ \sum_{i=1}^{\infty} \|x_i\|^s < \infty\}.$$

LEMMA 4. *Assume* $1 \leqslant p < 2$ *and let* $E \in \mathscr{A}_p$. *If, moreover, E is of stable type p, then E is isomorphic to some subspace of* $L_p(\nu)$, *where* ν *is arbitrary.*

Remark. The proof depends mainly on a result of Lindenstrauss and Pełczyński [6].

THEOREM 7. *Let t and s be real numbers such that* $1 < t < s \leqslant 2$. *Then* $l_s(l_t) \in \mathscr{A}_p \setminus \mathscr{A}_q$ *provided that* $0 < q < t < s < p \leqslant 2$.

Proof. It is well known that $l_s(l_t)$ is of stable type r whenever $r < t$. Now, if $1 \leqslant r < t$ and $q < r$, then $l_s(l_t)$ can belong neither to $\mathscr{A}_r$ nor to $\mathscr{A}_q$, since $l_s(l_t)$ is not isomorphic to a subspace of $L_r(\nu)$ (cf. [10] and Lemma 4). On the other hand, we have $l_s(l_t) \in \mathscr{A}_p$ because of Lemma 3 and $l_t \in \mathscr{A}_p$.

Finally, we want to prove that the property "$E \in \mathscr{A}_p$" is rather strong if $0 < p < 2$. More precisely, we show that in this case E has to be of cotype 2. Particularly, this implies $L_q \notin \mathscr{A}_p$, $2 < q \leqslant \infty$ and $0 < p < 2$.

THEOREM 8. *If* $0 < p < 2$ *and* $E \in \mathscr{A}_p$, *then E is of cotype* 2.

Proof. Given p with $0 < p < 2$ we put $q := 2/(2-p)$. Let $x_1, \ldots, x_n \in E$ be arbitrary and let $\beta_1, \ldots, \beta_n$ be real numbers such that

$$\sum_{i=1}^{n} |\beta_i|^q \leqslant 1.$$

Then we define $\mu, \nu \in R_p(E)$ by

$$\hat{\mu}(a) = \exp\Big(-\sum_{i=1}^{n} |\beta_i|\, |\langle x_i, a\rangle|^p\Big), \quad \hat{\nu}(a) = \exp\Big(-\big(\sum_{i=1}^{n} |\langle x_i, a\rangle|^2\big)^{p/2}\Big).$$

Using Hölder's inequality we get $\mu \prec \nu$, which implies

$$\Big\{\mathrm{E}\Big\|\sum_{i=1}^{n} |\beta_i|^{1/p} x_i \theta_i\Big\|^r\Big\}^{1/r} \leqslant c\,\Big\{\int_E \|x\|^r\, d\nu(x)\Big\}^{1/r}.$$

But the right-hand side equals

$$c\,c(r, p, 2)^{-1}\Big\{\mathrm{E}\Big\|\sum_{i=1}^{n} x_i \gamma_i\Big\|^r\Big\}^{1/r}$$

(see Theorem 6) and the left-hand side can be estimated by

$$\Big(\sum_{i=1}^{n} |\beta_i|\, \|x_i\|^p\Big)^{1/p} \leqslant c'\Big(\mathrm{E}\Big\|\sum_{i=1}^{n} |\beta_i|^{1/p} x_i \theta_i\Big\|^r\Big)^{1/r}.$$

Here $\gamma_1, \ldots, \gamma_n$ denotes an independent sequence of standard Gaussian r.v.'s. Now, taking the supremum over all $\beta_1, \ldots, \beta_n$,

$$\sum_{i=1}^{n} |\beta_i|^q \leqslant 1,$$

we get

$$\Big(\sum_{i=1}^{n} \|x_i\|^2\Big)^{1/2} \leqslant c''\Big(\mathrm{E}\Big\|\sum_{i=1}^{n} x_i\gamma_i\Big\|^r\Big)^{1/r} \leqslant c''\Big(\mathrm{E}\Big\|\sum_{i=1}^{n} x_i\gamma_i\Big\|^2\Big)^{1/2}.$$

Thus, E is of cotype 2.

References

[1] A. de Acosta, *Asymptotic behaviour of stable measures*, Ann. Probability 5 (1977), p. 494-499.

[2] A. Badrikian et S. Chevet, *Mesures cylindriques, espaces de Wiener et fonctions aléatoires gaussiennes*, Lecture Notes in Math. 379 (1974).

[3] K. Ito and M. Nisio, *On the convergence of sums of independent Banach space valued random variables*, Osaka J. Math. 5 (1968), p. 35-48.

[4] W. Linde, *Operators generating stable measures on Banach spaces*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 60 (1982), p. 171-184.

[5] — V. Mandrekar and A. Weron, *p-stable measures and absolutely summing operators*, Lecture Notes in Math. 828 (1980), p. 167-178.

[6] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications*, Studia Math. 29 (1968), p. 275-326.

[7] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Vol. I. *Sequence spaces*, Berlin–Heidelberg–New York 1978.

[8] V. Mandrekar and A. Weron, *α-stable characterization of Banach spaces* $(1 < \alpha < 2)$, preprint, 1979.

[9] B. Maurey, *Espaces de cotype p*, $0 < p \leqslant 2$, Sém. Maurey-Schwartz, Exp. 7, 1972/73.

[10] D. Mouchtari, *Spaces of cotype p* $(0 \leqslant p \leqslant 2)$, Theor. Probability Appl. 24 (1980), p. 105-118.

[11] G. Pisier, *Type de espaces normes*, C. R. Acad. Sci. Paris 276 (1973), p. 1673-1676.

[12] A. Račkauskas, *Remark on stable laws on Banach spaces*, Litovsk. Mat. Sb. 19 (1979), p. 161-165.

[13] D. H. Thang and N. Z. Tien, *Mapping of stable cylindrical measures in Banach spaces*, preprint, 1980.

[14] N. Z. Tien, and A. Weron, *Banach spaces related to α-stable measures*, Lecture Notes in Math. 828 (1980), p. 309-317.

[15] A. Tortrat, *Lois $e(\lambda)$ dans les espaces vectorielles et lois stables*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 37 (1976), p. 175-182.

Sektion Mathematik
Friedrich-Schiller-Universität Jena, GDR

Received on 9. 10. 1980