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FREE BOUNDARY PROBLEM FOR CONTROLLED STOCHASTIC DIFFERENTIAL EQUATIONS

BY

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Abstract. This article is concerned with an optimal stopping problem of controlled stochastic differential equations. The value function up to t provides a unique solution of the free boundary problem of parabolic type (Theorem 1, a little variant of [6]). On the other hand, the value function provides a non-linear semigroup (Proposition 3) and the Cauchy problem of this generator will be considered (corollaries and remarks on uniqueness).

1. Introduction. In this paper we treat the optimal stopping problem of controlled stochastic differential equations. Let Γ be a compact convex subset of R^k, called a *control region*. Let $B(t)$, $t \geqslant 0$, be an n-dimensional Brownian motion on a probability space (Ω, F, P). Put $F_t = \sigma_t(B)$ to be the σ-field generated by $B(s)$, $s \leqslant t$. An F_t-progressible measurable Γ-valued process is called an *admissible control.* $\mathfrak{A}$ denotes the totality of admissible controls. Let α and γ be a symmetric $n \times n$ matrix valued function and an n-vector valued function defined on $R^n \times \Gamma$, respectively. We assume that both are bounded and satisfy

$$(1.1) \qquad |h(x, u)-h(x', u')| \leqslant K|x-x'|+\varrho(|u-u'|),$$

where K is a positive constant and ϱ is a continuous function with $\varrho(0) = 0$.

Consider the following controlled stochastic differential equation for $U \in \mathfrak{A}$:

$$(1.2) \qquad dX(t) = \alpha(X(t), U(t))dB(t)+\gamma(X(t), U(t))dt, \quad X(0) = x.$$

By (1.1), equation (1.2) has the unique solution $X(t) = X(t, x, U)$, called the *response* for U. Let m be the totality of F_t-stopping times and put $m(T) = \{\tau \wedge T;\ \tau \in m\}$ (1).

(1) $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$.

For $U \in \mathfrak{A}$ and $\tau \in m$ the gain is given by

$$(1.3)\quad V(\tau, x, \varphi, U) = \mathrm{E}_x\Big[\int_0^\tau \exp\{-\int_0^s c(X(t), U(t))dt\} f(X(s), U(s))ds + $$
$$+ \exp\{-\int_0^\tau c(X(t), U(t))dt\} \varphi(X(\tau))\Big],$$

where $X(t) = X(t, x, U)$. We assume that $c \geqslant 0$ and f are bounded and satisfy (1.1).

Let C be a Banach lattice of the totality of bounded and uniformly continuous functions on R^n, with usual order and supremum norm. Put

$$(1.4)\qquad V(t, x, \varphi) = \sup_{U \in \mathfrak{A}, \tau \in m(t)} V(\tau, x, \varphi, U).$$

Then $V(t, \cdot, \varphi) \in C$ whenever $\varphi \in C$. Moreover, the operator $V(t)$, defined by

$$(1.5)\qquad V(t)\varphi(x) = V(t, x, \varphi),$$

is a monotone contraction semigroup on C. Putting

$$G\varphi = \sup_{u \in \Gamma} L^u\varphi - c^u\varphi + f^u \text{ (2)},$$

where L^u is the generator of the response for the constant control $u \in \Gamma$, i.e.

$$L^u = \frac{1}{2}\sum_{i,j}\sum_k \alpha_{ik}(x, u)\alpha_{kj}(x, u)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \gamma(x, u)\frac{\partial}{\partial x_i},$$

we express the generator $\mathfrak{G}$ of $V(t)$, $t \geqslant 0$, by the formula

$$(1.6)\qquad \mathfrak{G}\varphi = 0 \vee G\varphi \quad \text{for smooth } \varphi.$$

The value function $V(t, x, \varphi)$ is related to the free boundary problem. The following theorem will be proved in Section 3:

THEOREM 1. *We assume that the smooth condition is satisfied, that is α, γ, c, f, and φ are twice continuously differentiable in x and their first derivatives are bounded and satisfy* (1.1) *with possibly another K and ϱ. Moreover, the second derivatives are bounded and satisfy*

$$|h(x, u) - h(x', u')| \leqslant \tilde{K}|x - x'|^{\Delta} + \tilde{\varrho}(|u - u'|),$$

where $\tilde{K}$ and Δ are positive constants and $\tilde{\varrho}$ is a continuous function with $\tilde{\varrho}(0) = 0$. Suppose α is uniformly positive definite.

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(2) $h^u(x) = h(x, u)$.

Then $V(t, x, \varphi) \in W^{1,2}_{p,\mathrm{loc}}$ ([1], [7]) *for any large* p *and* $V(t, x, \varphi)$ *satisfies the following free boundary problem*:

$$W \geqslant \varphi \quad \text{in } [0, \infty) \times R^n, \tag{1.7}$$

$$GW \leqslant \partial W/\partial t \ \text{a.e.} \quad \text{in } (0, \infty) \times R^n, \tag{1.8}$$

$$(W - \varphi)(\partial W/\partial t - GW) = 0 \ \text{a.e} \quad \text{in } (0, \infty) \times R^n, \tag{1.9}$$

$$W(0, \cdot) = \varphi \quad \text{in } R^n. \tag{1.10}$$

Furthermore, a solution of (1.7)-(1.10) *is unique in* $W^{1,2}_{2n+2,\mathrm{loc}}$.

COROLLARY. *Under the same conditions as in Theorem* 1, V *satisfies the following Bellman equation*:

$$\begin{aligned} &\frac{\partial V}{\partial t} = GV \vee 0 \ \text{a.e} \quad \text{in } (0, \infty) \times R^n, \\ &V(0, \cdot) = \varphi \quad \text{in } R^n. \end{aligned} \tag{1.11}$$

If $\inf_{x,u} c(x, u) > 0$, then $v = \lim_{t \uparrow \infty} V(t)\varphi$ exists in C and

$$v(x) = \sup_{U \in \mathfrak{A}, \tau \in m} V(\tau, x, \varphi, U).$$

The limit function v will be considered in Section 4.

2. Semigroups associated with optimization. We summarize non-linear semigroups on C which are related to optimization problems of controlled stochastic differential equations [2]. Let $\tilde{\Gamma}$ be a compact subset of R^k. Let $\mathfrak{A}_N$ denote the totality of a $\tilde{\Gamma}$-valued F_t-progressible measurable process U such that

$$U(t) = U(k/2^N) \quad \text{for } t \in [k/2^N, (k+1)/2^N),\ k = 0, 1, \ldots$$

Put

$$\tilde{\mathfrak{A}} = \bigcup_{N=1}^{\infty} \mathfrak{A}_N.$$

We call $U \in \tilde{\mathfrak{A}}$ a *switching control.*

For a constant time t we define

$$Q(t, x, \varphi) = \sup_{U \in \tilde{\mathfrak{A}}} V(t, x, \varphi, U). \tag{2.1}$$

Then we can easily see that $Q(t, \cdot, \varphi) \in C$ whenever $\varphi \in C$. Hence we can define the operator $Q(t)$ by

$$Q(t)\varphi(x) = Q(t, x, \varphi). \tag{2.2}$$

Put

$$C^2 = \left\{\varphi \in C;\ \frac{\partial \varphi}{\partial x_i}, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \in C,\quad i, j = 1, \ldots, n\right\}$$

and

$$T^u(t) = H^u(t) + \int_0^t H^u(s) f^u(s)\, ds,$$

where $H^u(t)$ is the transition semigroup of the response for $u \in \Gamma$ with killing rate c^u. Hence $T^u(t)$ is the semigroup with generator $A^u\varphi = L^u\varphi - c^u\varphi + f^u$.

PROPOSITION 1. *$Q(t)$, $t \geq 0$, is a monotone contraction semigroup* (3) *on C, whose generator G is expressed by*

(2.3) $$G\varphi = \sup_{u \in \Gamma} L^u\varphi - c^u\varphi + f^u \quad \text{for } \varphi \in C^2.$$

Moreover:

(2.4) $$T^u(t)\varphi \leqslant Q(t)\varphi \quad \text{for all } u, t, \varphi.$$

(2.5) Minimum property. *If a semigroup $\Lambda(t)$, $t \geqslant 0$, satisfies $T^u(t)\varphi \leqslant \Lambda(t)\varphi$ for all u, t, φ, then $\Lambda(t)\varphi \geqslant Q(t)\varphi$.*

PROPOSITION 2. *Suppose that α, γ, c, f, and φ satisfy the smoothness condition of Theorem 1. Suppose that there exists $\tilde{u} \in \tilde{\Gamma}$ such that $\alpha(\cdot, \tilde{u})$ is uniformly positive definite. Then $Q(t, x, \varphi) \in W^{1,2}_{p,\mathrm{loc}}$ for any large p. Moreover, $L^{\tilde{u}}Q - c^{\tilde{u}}Q + f^{\tilde{u}} - \partial Q/\partial t$ is essentially bounded on $[0, T] \times R^n$ for any $T > 0$.*

When $\tilde{\Gamma}$ is convex, any admissible control can be approximated by a switching control, i.e.

$$\sup_{U \in \tilde{\mathfrak{A}}} V(t, x, U, \varphi) = \sup_{U \in \mathfrak{A}} V(t, x, U, \varphi).$$

PROPOSITION 3. *$V(t)$ of (1.5) is a monotone contraction semigroup on C, whose generator $\mathfrak{G}$ is expressed by*

(2.6) $$\mathfrak{G}\varphi = 0 \vee G\varphi \quad \text{for } \varphi \in C^2.$$

Moreover:

(2.7) $$Q(t)\varphi \leqslant V(t)\varphi \quad \text{and} \quad \varphi \leqslant V(t)\varphi \quad \text{for all } t, \varphi.$$

(2.8) Minimum property. *If a semigroup $\Lambda(t)$ instead of $V(t)$ satisfies (2.7), then $V(t)\varphi \leqslant \Lambda(t)\varphi$ for all t, φ.*

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(3) The semigroup property means the so-called *Bellman principle*.

Now we prove the former half of Theorem 1. Put $\tilde{\Gamma} = \Gamma \cup \{d\}$, where $d \notin \Gamma$. Assume that $\alpha(x, d)$, $\gamma(x, d)$, $c(x, d)$, and $f(x, d)$ are equal to 0. Hence $T^d(t)$ is the identity. In this case, (2.5) and (2.8) imply $V(t) = Q(t)$. Therefore, it follows from Proposition 2 that $V(t, x, \varphi) \in W^{1,2}_{p,\mathrm{loc}}$, and $L^u V - c^u V + f^u - \partial V/\partial t$ is essentially bounded in $[0, T] \times R^n$ for any $T > 0$.

3. Proof of Theorem 1. For the proof of the latter half of Theorem 1, we apply the same method as in [1]. By the definition of $V(t, x, \varphi)$, (1.7) and (1.10) are clear.

Fix T arbitrarily and put $W_T(t, x) = W(t, x) = V(T-t)\varphi(x)$. Then (2.4) and (2.7) imply

(3.1) $$W(t, x) = V(s)V(T-t-s)\varphi(x) \geqslant T^u(s)W(t+s, x).$$

Moreover, appealing to the end of Section 2, we see that $W \in W^{1,2}_{p,[0,T]\times S}$ for any compact S of R^n. Let $\xi(t)$, $t \geqslant 0$, be the response for a constant control $u \in \Gamma$. Since $\alpha(\cdot, u)$ is uniformly positive definite, Ito's formula holds for a function of $W^{1,2}_{2n+2\mathrm{loc}}$ (see [7]). Hence, using (3.1), we have

$$W(t, x) \geqslant \mathrm{E}_x\Big[\int_0^s \exp\Big\{-\int_0^\theta c(\xi(z), u)\,dz\Big\} f(\xi(z), u)\,d\theta +$$
$$+\exp\Big\{-\int_0^s c(\xi(z), u)\,dz\Big\} W(t+s, \xi(s))\Big].$$

So, putting

(3.2)
$$F_T(t, y, u) = F(t, y, u) = \frac{\partial W}{\partial t}(t, y) + L^u W(t, y) - c(y, u)W(t, y) + f(y, u),$$

we get

$$0 \geqslant \mathrm{E}_x \int_0^s \exp\Big\{-\int_0^\theta c(\xi(z), u)\,dz\Big\} F(t+\theta, \xi(\theta), u)\,d\theta.$$

Integrating both sides from 0 to $T-t$ with measure $\mu^2 e^{-\mu s} ds$, we have

(3.3)
$$0 \geqslant \mathrm{E}_x \int_0^{T-t} \mu \exp\Big\{-\mu \exp\Big[\Big(-\int_0^\theta c(\xi(z), u)\,dz\Big)\ln\theta\Big] F(t+\theta, \xi(\theta), u)\,d\theta -$$
$$-\mu e^{-\mu(T-t)}\mathrm{E}_x \int_0^t \exp\Big\{-\int_0^\theta c(\xi(z), u)\,dz\Big\} F(t+\theta, \xi(\theta), u)\,d\theta.$$

Since F is essentially bounded and $\alpha(\cdot, u)$ is uniformly positive definite, the second term of (3.3) tends to 0 as $\mu \uparrow \infty$. On the other hand, the first term tends to $F(t, x, u)$ in $L_p([0, T] \times S)$ for any compact S of R^n. Therefore,

(3.4) $$F(t, x, u) \leqslant 0 \text{ a.e. } \quad \text{in } [0, T] \times R^n.$$

Letting $\partial W/\partial x_i$ and $\partial^2 W/\partial x_i \partial x_j$ independent of u, we obtain (1.8) from (3.4) by the continuity with respect to u.

For (1.9) we use the random stopping. Let $\mathfrak{R}$ be the totality of non-negative and bounded F_t-progressible measurable processes. $r \in \mathfrak{R}$ gives the following random stopping:

$$P(\text{stop at}\,(t, t+dt) \mid \text{non-stopping before } t) = r(t) \exp\left\{-\int_0^t r(s)\,ds\right\} dt.$$

For $r \in \mathfrak{R}$ and $U \in \mathfrak{A}$, the gain is given by

$$J(T, x, \varphi, U, r) = \int_0^T I(t, x, \varphi, U) r(t) \exp\left\{-\int_0^t r(s)\,ds\right\} dt + \tag{3.5}$$
$$+ I(T, x, \varphi, U) \exp\left\{-\int_0^T r(s)\,ds\right\},$$

where

$$I(t, x, \varphi, U) = \int_0^t \exp\left\{-\int_0^s c(X(\theta), U(\theta))\,d\theta\right\} f(X(s), U(s))\,ds +$$
$$+ \exp\left\{-\int_0^t c(X(\theta), U(\theta))\,d\theta\right\} \varphi(X(t)),$$

and $X(t) = X(t, x, U)$. In the same way as in [1] we can show that

$$V(T)\varphi(x) = \sup_{U \in \mathfrak{A}, r \in \mathfrak{R}} \mathrm{E}\, J(T, x, \varphi, U, r). \tag{3.6}$$

Using again Ito's formula, we obtain (3.6) in the form

(3.7)

$$0 = \sup_{U \in \mathfrak{A}, r \in \mathfrak{R}} \Big[\mathrm{E}_x \int_0^T \exp\left\{-\int_0^t [c(X(s), U(s)) + r(s)]\,ds\right\} F(t, X(t), U(t))\,dt +$$
$$+ \mathrm{E}_x \int_0^T \exp\left\{-\int_0^t [c(X(s), U(s)) + r(s)]\,ds\right\} r(t)\big(\varphi(X(t)) -$$
$$- W(t, X(t))\big)\,dt\Big].$$

The set

$$D_\varepsilon(T) = \{(t, x) \in [0, T] \times R^n;\ V(T-t)\varphi(x) \leqslant \varphi(x) + \varepsilon\}$$

is closed. Thus for $(0, y) \notin D_\varepsilon(T)$ there exist positive $\tilde{d}$ and d such that $[0, \tilde{d}] \times S(y, d) \subset D_\varepsilon(T)^c$, where $S(y, d)$ is the sphere with origin y and radius d. By virtue of (1.7) and (1.8) each term of (3.7) is non-positive, so we can replace T by $T \wedge \tau$.

Let $x \in S(y, d)$ and let (U_k, r_k), $k = 1, 2, \ldots$, be an approximate optimal sequence. Since

$$(3.8) \qquad \varphi(X(t)) - W(t, X(t)) < -\varepsilon \quad \text{for } t < \tau,$$

we get, as $k \uparrow \infty$,

$$(3.9) \qquad \mathrm{E}_x \int_0^{T \wedge \tau_k} \exp\{-\int_0^t r_k(s)\,ds\} r_k(t)\,dt \to 0$$

and

$$(3.10) \qquad \mathrm{E}_x \int_0^{T \wedge \tau_k} F(t, X_k(t), U_k(t))\,dt \to 0,$$

where X_k is the response for U_k. Since α and γ are bounded, for $x \notin S(y, d)$ we can choose $c = c(x) > 0$ so that

$$\inf_U P_x(\tau > c) > 0.$$

Therefore, by (3.9), we see that

$$(3.11) \qquad \exp\{-\int_0^{T \wedge \tau_k} r_k(s)\,ds\} \to 1 \text{ in } P \quad \text{as } k \to \infty.$$

Let $M = M_{k,\lambda}$ be the set of all (A, R) which satisfy the following conditions:

(3.12) A is an $n \times n$ symmetric matrix valued F_t-progressible measurable process with

$$|A(t, \omega)| \leqslant K \quad \text{and} \quad \sum_{i,j=1}^{n} A_{ij}(t, \omega)\theta_i\theta_j \geqslant \lambda|\theta|^2 \quad \text{for any } \omega, t, \theta.$$

(3.13) R is an F_t-progressible measurable n-dimensional process with $|R(t, \omega)| \leqslant K$ for any ω, t.

Put

$$W(t, x;\, h, \mu, K, \lambda, d) = \sup_{(A,R) \in M} \mathrm{E} \int_0^{(T-t) \wedge \tau} e^{-\mu s} h\Big(t+s, x + \int_0^s A\,dB + \int_0^s R\,d\theta\Big)\,ds,$$

where τ is the hitting time for $\partial S(0, d)$. Then for any $\mu \geqslant (1+2K)/n\lambda^2$ and $p \geqslant 2n+2$ we have

$$(3.14) \qquad \|W\|_{L_p([0,T] \times S(0,d))} \leqslant \frac{\tilde{K}}{\mu} \|h\|_{L_p([0,T] \times S(0,d))}$$

with $\tilde{K} = \tilde{K}(n, K, \lambda)$.

Putting $\sup_{u\in\Gamma} F_T(t, x, u) = M_T(t, x)$, we infer from (1.8) that for any $T>0$

$$M_T(t, x) \leqslant 0 \text{ a.e.} \quad \text{in } (0, T)\times R^n,$$

Therefore, by (3.10), we have

$$\sup_{(A,R)\in M} \mathrm{E} \int_0^{T\wedge\tau} M_T\Big(t, x+\int_0^t AdB+\int_0^t Rd\theta\Big)dt = 0. \tag{3.15}$$

Replacing T by $T-s$, we obtain

$$\sup_{(A,R)\in M} \mathrm{E} \int_0^{(T-s)\wedge\tau} M_{T-s}\Big(t, x+\int_0^t AdB+\int_0^t Rd\theta\Big)dt = 0. \tag{3.16}$$

From the definition of M_T we get $M_T(t+s, x) = M_{T-s}(t, x)$. Hence

$$\sup_{(A,R)\in M} \mathrm{E} \int_0^{(T-s)\wedge\tau} M_T\Big(t+s, x+\int_0^t AdB+\int_0^t Rd\theta\Big)dt = 0. \tag{3.17}$$

Since $M_T \leqslant 0$ a.e. and A is positive definite, we have

$$\sup_{(A,R)\in M} \mathrm{E} \int_0^{(\tilde{d}-s)\wedge\tau} \mu e^{-\mu t} M_T\Big(t+s, x+\int_0^t AdB+\int_0^t Rd\theta\Big)dt = 0. \tag{3.18}$$

As $\mu\to\infty$, we get $M_T=0$ a.e. by virtue of (3.14); namely,

$$\frac{\partial}{\partial t} V(t)\varphi = GV(t)\varphi \text{ a.e.} \quad \text{in } (T-\tilde{d}, T)\times S(y, d); \tag{3.19}$$

clearly, $[0, \tilde{d}+\theta)\times S(y, d) \subset D_\varepsilon(T+\theta)^c$ for any $\theta>0$. Hence (3.19) holds in $(T-\tilde{d}, T+\theta)\times S(y, d)$. As $\varepsilon\downarrow 0$, we get (1.9). This completes the proof.

Now we prove the uniqueness of solution. Let $W\in W^{1,2}_{2n+2,\mathrm{loc}}$ be a solution. Fix $T>0$ arbitrarily and put $v(t, x) = W(T-t, x)$. We can choose a Γ-valued Borel function $u(t, x)$ so that

$$L^{u(t,x)}v - c^{u(t,x)}v(t, x) + f^{u(t,x)}(x) = Gv(t, x). \tag{3.20}$$

Moreover, the stochastic differential equation

$$d\xi(t) = \alpha\big(\xi(t), u(t, \xi(t))\big)dB(t) + \gamma\big(\xi(t), u(t, \xi(t))\big)dt,$$
$$\xi(0) = x$$

has a weak solution ([5], [7]). Put

$$D = \{(t, x)\in[0, T]\times R^n;\ v(t, x) = \varphi(x)\}.$$

Then $\partial v/\partial t + Gv = 0$ a.e. in D^c. Let ξ be a weak solution and τ the hitting time for D of the process $(t, \xi(t))$. By the definition of D, we have $\tau \leqslant T$.

Since $v \in W^{1,2}_{p,\mathrm{loc}}$, putting $U(t) = u(t, \xi(t))$ and using Ito's formula we obtain

$$(3.21)\qquad \mathrm{E}\big[\exp\{-\int_0^\tau c(\xi(s), U(s))\,ds\}\, v(\tau, \xi(\tau)) - v(0, \xi(T))\big]$$

$$= \mathrm{E}\int_0^\tau \exp\{-\int_0^t c(\xi(s), U(s))\,ds\}\big(\partial v/\partial t(t, \xi(t)) + L^{u(t)} v(t, \xi(t))\big)\,dt.$$

For $T \geqslant \tau$, $v(T \wedge \tau, \xi(T \wedge \tau))$ is $\varphi(\xi(T \wedge \tau))$. Hence (3.21) turns out

$$(3.22)\qquad W(T, x) = \mathrm{E}\big[\exp\{-\int_0^\tau c(\xi(s), U(s))\,ds\}\,\varphi(\xi(\tau)) +$$

$$+ \int_0^\tau \exp\{-\int_0^t c(\xi(s), U(s))\,ds\}\, f(\xi(t), U(t))\,dt\big]$$

and $\tau = \tau \wedge T$.

On the other hand, for any Γ-valued Brownian B-non-anticipative process U, we consider the controlled stochastic differential equation

$$dX(t) = \alpha(X(t), U(t))\,dB(t) + \gamma(X(t), U(t))\,dt,$$

$$X(0) = x.$$

Then for any $\sigma_t(X, B)$-stopping time τ with $\tau \leqslant T$ we have

$$(3.23)\qquad W(T, x) \geqslant \mathrm{E}\big[\exp\{-\int_0^\tau c(X(s), U(s))\,ds\}\,\varphi(X(\tau)) +$$

$$+ \int_0^\tau \exp\{-\int_0^t c(X(s), U(s))\,ds\}\, f(X(t), U(t))\,dt\big].$$

Hence (3.22) and (3.23) imply the uniqueness of the solution W; namely

$$(3.24)\qquad W(T, x) = \sup_{U, \tau \leqslant T} \mathrm{E}_x\big[\exp\{-\int_0^\tau c(X(s), U(s))\,ds\}\,\varphi(X(\tau)) +$$

$$+ \int_0^\tau \exp\{-\int_0^t c(X(s), U(s))\,ds\}\, f(X(t), U(t))\,dt\big].$$

Remark. The Markovian policy u (i.e., a Γ-valued Borel function) defined by (3.21) and the hitting time for D give an optimal policy.

Proof of the Corollary. Since a solution is unique, (3.24) means that $W(T, x)$ is increasing in T. Put

$$D = \{(t, x) \in [0, \infty) \times R^n;\ W(t, x) = \varphi(x)\}.$$

Then we have

$$\partial W/\partial t = \begin{cases} GW \text{ a.e.} & \text{in } D^c, \\ 0 \text{ a.e.} & \text{in } D. \end{cases}$$

Hence

$$\frac{\partial W}{\partial t}\left(\frac{\partial W}{\partial t}-GW\right)=0 \text{ a.e.} \tag{3.25}$$

Since $\partial W/\partial t$ and $\partial W/\partial t-GW$ are non-negative, from (3.25) we obtain $\partial W/\partial t = 0 \vee GW$ a.e. This completes the proof of the Corollary.

Remarks on uniqueness of solution of (1.11). *Suppose that there exists a compact set $E \subset R^n$ such that* supp $\varphi \subset E$ *and* supp $f(\cdot, u) \subset E$ *for any* $u \in \Gamma$. *Put*

$$C_0(or\ L_{\infty,0}) = \{\psi \in C(or\ L_\infty);\ \lim_{|x|\to\infty} |\psi(x)| = 0\}$$

and

$$D = \{\psi \in C_0 \cap W^2_{n+1,\text{loc}};\ 0 \vee G\psi \in L_{\infty,0}\}.$$

Then

(i) $V(t,\cdot)\in C_0$ *and* $0 \vee G\varphi \in L_{\infty,0}$ *for almost all* t;

(ii) *the operator* $0 \vee G$: $D \to L_{\infty,0}$ *is dissipative*;

(iii) V *is a unique solution of* (1.11) *if* $0 \vee GV(t,\cdot)$ *is almost separably valued; namely, there exists a null set* $N \subset [0, \infty)$ *such that* $\{0 \vee G(V,\cdot), t \in [0,\infty) \setminus N\}$ *is a separable set of* $L_{\infty,0}$.

For example, since C_0 is separable, V is a unique solution if $0 \vee GV(t,\cdot)\in C_0$ for almost all t.

Proof. (i) is easy by the routine method.

(ii) We recall the proof of [4]. For φ, $\psi \in D$,

$$J(\varepsilon) = \operatorname*{ess\,inf}_{|x-\tilde{x}|\leqslant\varepsilon} \big(0 \vee G\varphi(x) - 0 \vee G\psi(x)\big) \leqslant 0 \vee \operatorname*{ess\,inf}_{|x-\tilde{x}|\leqslant\varepsilon}\big(G\varphi(x)-G\psi(x)\big)$$

$$\leqslant 0 \vee \operatorname*{ess\,inf}_{|x-\tilde{x}|\leqslant\varepsilon} L^{u(x)}(\varphi-\psi)(x),$$

where $u(x)$ is a Borel function such that

$$L^{u(x)}\varphi(x) = \sup_{u\in\Gamma} L^u\varphi(x) = G\varphi(x).$$

Since $L^{u(x)}$ is a uniformly elliptic operator with bounded coefficients, Bony's maximum principle [3] implies that if $\varphi-\psi$ has a positive maximum at $\tilde{x}$, then

$$\operatorname*{ess\,inf}_{|x-\tilde{x}|\leqslant\varepsilon} L^{u(x)}(\varphi-\psi)x \leqslant 0 \quad \text{for any } \varepsilon > 0.$$

Hence $J(\varepsilon) \leqslant 0$ for any $\varepsilon > 0$. Consequently,

$$\lim_{\varepsilon\downarrow 0} J(\varepsilon) \leqslant 0,$$

which completes the proof.

(iii) Since $0 \vee GV(t, x)$ is a bounded Borel function of (t, x), $\int 0 \vee GV(t, x)\mu(dx)$ is a Borel function of t for any $\mu \in L'_\infty(R^n)$, i.e., $0 \vee GV(t, \cdot)$ is weakly measurable. Hence, by the Pettis theorem, the condition of (iii) implies that $0 \vee GV(t, \cdot)$ is strongly measurable. Consequently,

$$V(t, x) = \int_0^t 0 \vee GV(s, x)\,ds$$

can be regarded as a Bochner integral. This means that $V(t, \cdot)$ is strongly differentiable and

$$\frac{dV}{dt}(t, \cdot) = 0 \vee GV(t, \cdot) \quad \text{for almost all } t.$$

Therefore, by (ii) we obtain (iii).

4. Limit function. Suppose $\inf_{x,u} c(x, u) > 0$. Then

$$v = \lim_{t \uparrow \infty} V(t)\varphi$$

exists in C and v is the least $Q(t)$-superharmonic majorant of φ; namely,

(4.1) $\varphi \leqslant v$ and $Q(t)v \leqslant v$ for any t,

(4.2) if $\tilde{v} \in C$ satisfies (4.1), then $v \leqslant \tilde{v}$.

Since v is the value of the optimal stopping problem, v is related to the free boundary problem. Now we recall the following

THEOREM 2. *Under the same conditions as in Theorem* 1, $v \in W^2_{p,\mathrm{loc}}$ *for any large* p *and* v *satisfies the following free boundary problem*:

(4.3) $$\varphi \leqslant v \quad \textit{in } R^n,$$

(4.4) $$Gv \leqslant 0 \textit{ a.e.} \quad \textit{in } R^n,$$

(4.5) $$(v-\varphi)Gv = 0 \textit{ a.e.} \quad \textit{in } R^n.$$

Moreover, (4.3)-(4.5) *has the unique solution in* $W^2_{2n+2,\mathrm{loc}}$.

Since (4.4) is equivalent to $0 \vee Gv = 0$ a.e., we have

COROLLARY. v *satisfies the following Bellman equation with* (4.3):

(4.6) $$0 \vee Gv = 0 \textit{ a.e.} \quad \textit{in } R^n.$$

Equation (4.6) with (4.3) has many solutions. For example, we replace f by $f+k$, where k is a positive constant. Then its optimal value $\tilde{v}$ satisfies (4.3) and

(4.7) $$G\tilde{v}+k \leqslant 0 \text{ a.e.} \quad \text{in } R^n.$$

Since k is positive, $\tilde{v}$ is a solution of (4.6). Equation (4.6) means that v is $Q(t)$-superharmonic. Thus, the solution of (4.3)-(4.5) is the minimum one of (4.3) and (4.6).

References

[1] V. Barbu, *Nonlinear semigroups and differential equations in Banach space*, Noodhoff 1976.

[2] A. Bensoussan et J. L. Lions, *Applications des inéquations variationelles en contrôle stochastique*, Dunod 1978.

[3] J. M. Bony, *Principle du maximum dans les espaces de Sobolev*, C. R. Acad. Sci. (Paris) 265 (1967), p. 333-336.

[4] L. C. Evans and A. Friedman, *Optimal stochastic switching and the Dirichlet problem for the Bellman equation*, Trans. Amer. Math. Soc. 253 (1979), p. 365-389.

[5] N. V. Krylov, *On Ito's stochastic integral equations*, Theor. Probability Appl. 14 (1969), p. 330-336.

[6] — *The control of the solution of a stochastic integral equation*, ibidem 17 (1972), p. 114-131.

[7] — *Some estimates of the probability density of a stochastic integral*, Math. USSR-Izv. 8 (1974), p. 233-254.

[8] — *Controlled diffusion processes*, Appl. Math. Springer, 1980 (in Russian: 1977).

[9] M. Nisio, *Stochastic control theory*, ISI Lect. Note, Macmillan, India, 1981.

[10] A. N. Shiryayev, *Optimal stopping rules*, Appl. Math. Springer, 1977 (in Russian: 1976).

[11] K. Yoshida, *Functional analysis*, Springer, 1980.

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