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SOME RESULTS ON THE SUBSAMPLING FOR φ-MIXING PERIODICALLY STRICTLY STATIONARY TIME SERIES

BY

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Abstract. The article deals with the special subclass of φ-mixing periodically correlated (PC) time series and the estimation of autocovariance through Fourier coefficients. The aim is to investigate whether the subsampling of the autocovariance estimator is consistent. It is shown that the consistency holds for frequencies $\lambda = 0$ and $\lambda = \pi$. Theoretical reasoning is supplemented with a simulation study.

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1. INTRODUCTION

Periodically correlated (PC) time series constitute mathematical models for repeatable random phenomena. Therefore, they have a vast spectrum of possible applications in many fields, like climatology, economy, or signal processing (see e.g. [2], [4], [9]).

We will define a special class of PC time series, i.e. periodically strictly stationary time series (PSS) and we will be interested in time-domain statistical inference within this class. The inference is based on Fourier representation of the autocovariance function. It would be very desirable to construct confidence intervals for the estimator of Fourier coefficients. We could then perform different tests about the model. One of the most important is testing stationarity (some Fourier coefficients are equal to zero), i.e. whether the model is really periodic. Let us mention two situations when such a test is crucial.

At present, $GARCH$ (generalized autoregressive conditional heteroscedasticity) models are quite popular in the analysis of financial time series. It seems that they capture the most important features of the behavior of financial

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market, where there are periods with high and low volatility. Although the *GARCH* model incorporates conditional heteroscedasticity (changes in the conditional variance), the series must be unconditionally stationary. Therefore, if we are not sure whether the data are seasonal or stationary, the test for stationarity should be used before applying *GARCH* or other models. The other situation is when using the *PARMA* (periodic autoregressive moving average) model. At the beginning we identify the order of the model and then estimate the parameters by means of the maximum likelihood or moment estimators (see [2]). Next, the residuals are computed. Provided that the model was properly chosen, the residuals should be stationary. Therefore, stationarity test determines the correctness of the selected model. Let us add here that there is still a lack of efficient tests of stationarity within non-linear time series models.

Another example of the test is when we want to determine the period of *PSS* or *PC* time series. Our guess can be verified by testing hypothesis that only the specific Fourier coefficients are non-zero. Certainly, the same as for the stationarity test, the pointwise value of the estimator is almost useless here. Testing is impossible without knowing the estimator distribution or, at least, its approximation.

It is well known in literature (see [5] and [8]) that natural estimator of Fourier coefficients has an asymptotic normal distribution. However, it is not applicable in practice because of complicated form of the covariance matrix. Therefore, we would like to propose resampling-type method of obtaining the confidence intervals for the estimator.

Section 2 covers essential definitions and notation as well as the characterization of periodically strictly stationary time series. Section 3 contains the main result regarding the consistency of subsampling. Section 4 presents the outcome of simulation study and further considerations. Section 5 summarizes the article.

2. THEORETICAL BACKGROUND

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{X(t): t \in \boldsymbol{Z}\}$ be a time series, i.e. $X(t)$ is a random variable for each $t \in \boldsymbol{Z}$. At the beginning, we will provide the concise definitions of all time series classes that are under study.

DEFINITION 2.1 (Brockwell and Davis [3]). The time series $\{X(t): t \in \boldsymbol{Z}\}$ is *stationary in the strict sense* (*SSS*) if, for any $k \in \boldsymbol{N}$, any $t_1, \ldots, t_k \in \boldsymbol{Z}$, and any $h \in \boldsymbol{Z}$,

$$\big(X(t_1), \ldots, X(t_k)\big) \stackrel{d}{=} \big(X(t_1+h), \ldots, X(t_k+h)\big).$$

Stationarity means that the structure is invariant in time. The following classes exhibit periodicity in distribution and second order structure.

DEFINITION 2.2. The time series $\{X(t): t \in \boldsymbol{Z}\}$ is called *periodically strictly stationary* (*PSS*) if there exists a smallest number $T \in \boldsymbol{N}$, called a *period*, such that for any $k \in \boldsymbol{N}$, any $t_1, \ldots, t_k \in \boldsymbol{Z}$,

$$\big(X(t_1), \ldots, X(t_k)\big) \stackrel{d}{=} \big(X(t_1+T), \ldots, X(t_k+T)\big).$$

DEFINITION 2.3 (Bloomfield et al. [2]). The time series $\{X(t): t \in \boldsymbol{Z}\}$ is *periodically correlated* (*PC*) if there exists a smallest number $T \in \boldsymbol{N}$, called a *period*, such that for any $t, \tau \in \boldsymbol{Z}$

(i) $\mu_X(t) := E\big(X(t)\big) = \mu_X(t+T)$;

(ii) $B_X(t, \tau) := \operatorname{Cov}\big(X(t), X(t+\tau)\big) = B_X(t+T, \tau)$.

It is easy to see that $SSS \subset PSS$ and, under the condition that the second moment exists, $PSS \subset PC$. Below, we provide and prove the theorem that characterizes some PSS models.

THEOREM 2.1. *Assume that the time series* $\{X(t): t \in \boldsymbol{Z}\}$ *has the structure*

$$X(t) = F\big(Z(t), f(t)\big), \tag{2.1}$$

where the function f is periodic with period $T \in \boldsymbol{N}$, the time series $\{Z(t)\}$ is strictly stationary (SSS), and the function $F\big(\cdot, f(t)\big): \boldsymbol{R} \to \boldsymbol{R}$ is measurable for any t. Then the series $\{X(t)\}$ is PSS with period T.

Proof. Take any $k \in \boldsymbol{N}$ and any $t_1, \ldots, t_k \in \boldsymbol{Z}$. By strict stationarity of $\{Z(t)\}$, we may write

$$\big(Z(t_1), \ldots, Z(t_k)\big) \stackrel{d}{=} \big(Z(t_1+T), \ldots, Z(t_k+T)\big).$$

Since the function $F\big(\cdot, f(t)\big)$ is measurable, we get immediately

$$\Big(F\big(Z(t_1), f(t_1)\big), \ldots, F\big(Z(t_k), f(t_k)\big)\Big) \stackrel{d}{=} \Big(F\big(Z(t_1+T), f(t_1)\big), \ldots, F\big(Z(t_k+T), f(t_k)\big)\Big).$$

But, for any $i = 1, \ldots, k$, we have $f(t_i) = f(t_i+T)$, so

$$\Big(F\big(Z(t_1), f(t_1)\big), \ldots, F\big(Z(t_k), f(t_k)\big)\Big) \stackrel{d}{=} \Big(F\big(Z(t_1+T), f(t_1+T)\big), \ldots, F\big(Z(t_k+T), f(t_k+T)\big)\Big),$$

which implies $\big(X(t_1), \ldots, X(t_k)\big) \stackrel{d}{=} \big(X(t_1+T), \ldots, X(t_k+T)\big)$. ■

From the above we infer, for example, that the models

$$X(t) = f(t) + Z(t) \quad \text{and} \quad X(t) = f(t)Z(t)$$

are PSS provided that $\{Z(t)\}$ is SSS and the function f is periodic. However, the structure from Theorem 2.1 does not cover all possible PSS models. It is easy to see that the process defined by the equality $X(t) = Z\big(t+f(t)\big)$ is PSS but cannot be presented in the form (2.1).

If we do not assume any particular model of *PSS* (or *PC*) time series, then the statistical inference concerns the estimation of autocovariance function $B_X(t, \tau)$ and is based on its Fourier representation (see [6])

$$B_X(t, \tau) = \sum_{\lambda \in \Lambda_\tau} a(\lambda, \tau) e^{i\lambda t},$$

where

$$a(\lambda, \tau) = \frac{1}{T} \sum_{t=1}^{T} B_X(t, \tau) e^{-i\lambda t}$$

and the set

$$\Lambda_\tau = \{\lambda: a(\lambda, \tau) \neq 0\} \subset \{(2\pi k)/T: k = 0, \ldots, T-1\}$$

is finite. In this way the inference regarding a function is transformed into the inference regarding the complex numbers $a(\lambda, \tau)$. With no loss of generality we may assume that $\tau > 0$. Then the natural estimator ([7], [8]) of $a(\lambda, \tau)$ for the series $\{X(t)\}$ observed for $t = 1, \ldots, n$ (n is the sample size) takes the form

$$\hat{a}_n(\lambda, \tau) = \frac{1}{n} \sum_{t=1}^{n-\tau} X(t+\tau) X(t) e^{-i\lambda t}.$$

Assume that the series is φ-mixing, i.e. the observations that are far away become independent. For convenience of the reader we state the precise definition of this property below.

Definition 2.4 (Billingsley [1]). For the time series $\{X(t): t \in \mathbf{Z}\}$ we define the corresponding *φ-mixing function* by

$$\sup_{t \in \mathbf{Z}} \sup_{\substack{A \in \mathscr{F}_X(-\infty, t) \\ B \in \mathscr{F}_X(t+s, \infty)}} |P(A \cap B) - P(A) P(B)| \leqslant \varphi_X(s) P(A),$$

where $s \in \mathbf{N}$ and $\mathscr{F}_X(t_1, t_2)$ stands for the σ-algebra generated by $\{X(t): t_1 \leqslant t \leqslant t_2\}$. The process $\{X(t)\}$ is called *φ-mixing* if $\varphi_X(s) \to 0$ for $s \to \infty$.

Under some additional assumptions on the moments of $X(t)$, it is known that the estimator $\hat{a}_n(\lambda, \tau)$ is strongly consistent and asymptotically normal ([5], [7], [8]), i.e.

$$\sqrt{n} \left(\hat{a}_n(\lambda, \tau) - a(\lambda, \tau)\right) \xrightarrow{d} \mathcal{N}(0, \Sigma), \tag{2.2}$$

where

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}.$$

However, the matrix Σ depends on higher moments of $X(t)$ and has a very complicated form ([5], [7]). Therefore, such an asymptotic result cannot be easily applied in statistical inference. In the next section we would like to propose and examine subsampling as a method of approximating this distribution.

3. SUBSAMPLING AND ITS CONSISTENCY FOR *PSS* TIME SERIES

Let $(X(1), \ldots, X(n))$ be a sample from *PC* time series $\{X(t): t \in \mathbf{Z}\}$. For $t = 1, 2, \ldots, n-b+1$ let us put

$$\hat{a}_{n,b,t}(\lambda, \tau) = \frac{1}{b} \sum_{j=t}^{t-1+b-\tau} X(j+\tau) X(j) e^{-i\lambda(j-t+1)}$$

for the estimator $\hat{a}_n$ computed over the subsample $(X(t), \ldots, X(t+b-1))$ of length $b = b(n) \to \infty$, $b/n \to 0$. Thus, we obtain $n-b+1$ replications of the estimator and the empirical distribution of the real part of the root statistics in the form

$$L_{n,b}(x) = \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \mathbf{1}\{\sqrt{b}\,\mathrm{Re}(\hat{a}_{n,b,t}(\lambda, \tau) - \hat{a}_n(\lambda, \tau)) \leqslant x\}.$$

Note that this subsampling procedure can be efficiently computed, but an essential question is whether it is consistent, i.e. whether the quantiles from $L_{n,b}$ converge to the quantiles of the asymptotic distribution (see [10]). In order to verify consistency, we will use the sufficient condition formulated for a general case in [10] (Theorem 4.2.1, p. 103), that is

$$\text{(3.1)} \qquad \sqrt{b}\,\mathrm{Re}\,(\hat{a}_{n,b,t_b}(\lambda, \tau) - a(\lambda, \tau)) \xrightarrow{d} \mathcal{N}(0, \sigma_{11}) \quad \text{for any sequence } \{t_b\},$$

where σ_{11} is an appropriate element of the matrix Σ in (2.2). Notice that the sequence of starting points $\{t_b\}$ may converge to infinity; therefore, the above condition is much more stronger than (2.2). The following theorem is the main result regarding subsampling consistency.

THEOREM 3.1. *Let the real time series $\{X(t): t \in \mathbf{Z}\}$ satisfy the following conditions:*

(i) *$\{X(t)\}$ is PSS with period T;*

(ii) *$\{X(t)\}$ is φ-mixing with $\varphi_X^{1/2} \in l^1$;*

(iii) $E|X(t)|^4 < M_4 < \infty$.

Then the subsampling for the estimator $\hat{a}_n(\lambda, \tau)$ is consistent for $\lambda = 0$ and, if $a(\pi, \tau) = 0$, for $\lambda = \pi$.

Proof. Firstly, we will prove the following lemma.

LEMMA 3.1. *Under the assumptions of Theorem 3.1, we have for any sequence of positive integers $\{t_b\}$*

$$\text{(3.2)} \qquad \sqrt{b}\left(\frac{1}{b} \sum_{j=t_b-t_{b,T}}^{t_b-t_{b,T}-1+b-\tau} X(j+\tau) X(j) e^{-i\lambda j} - a(\lambda, \tau)\right) \xrightarrow{d} \mathcal{N}_2(0, \Sigma),$$

where Σ was defined in (2.2), $t_{b,T} = \lfloor (t_b-1)/T \rfloor T$ and the number T is a period.

Proof. To simplify the notation, let

$$l_b = t_b - t_{b,T} - 1 + b,$$

and notice that $t_b - t_{b,T} \in \{1, \ldots, T\}$. Therefore,

$$\sqrt{b}\left(\frac{1}{b}\sum_{j=t_b-t_{b,T}}^{t_b-t_{b,T}-1+b-\tau} X(j+\tau)X(j)e^{-i\lambda j} - a(\lambda, \tau)\right)$$

$$= \frac{\sqrt{l_b}}{\sqrt{b}}\sqrt{l_b}\left(\frac{1}{l_b}\sum_{j=1}^{l_b-\tau} X(j+\tau)X(j)e^{-i\lambda j} - a(\lambda, \tau)\right)$$

$$-\frac{1}{\sqrt{b}}\sum_{j=1}^{t_b-t_{b,T}-1} X(j+\tau)X(j)e^{-i\lambda j} + \frac{t_b-t_{b,T}-1}{\sqrt{b}}a(\lambda, \tau)$$

$$= \frac{\sqrt{l_b}}{\sqrt{b}}R_{b,t_b}(\lambda, \tau) + S_{b,t_b}(\lambda, \tau) + T_{b,t_b}(\lambda, \tau).$$

It is easy to see that $l_b/b \to 1$ and, by (2.2), $R_{b,t_b} \xrightarrow{d} \mathcal{N}_2(0, \Sigma)$. For the second term we have

$$E|S_{b,t_b}(\lambda, \tau)| \leqslant \frac{T-1}{\sqrt{b}}\sqrt{M_4} \to 0 \quad \text{as } b \to \infty,$$

which means that $S_{b,t_b} \xrightarrow{L_1} 0$. The non-random term T_{b,t_b} converges to 0 as well. By Slutsky's lemma we see that (3.2) holds. ■

Take any sequence of positive integers $\{t_b\}$. We will be checking whether the condition (3.1) is satisfied. Recalling that we have restricted our considerations to *PSS* series, we have

$$(3.3) \quad \big(X(t_b+\tau), \ldots, X(t_b-1+b), X(t_b), \ldots, X(t_b-1+b-\tau)\big)$$
$$\stackrel{d}{=} \big(X(t_b-t_{b,T}+\tau), \ldots, X(l_b), X(t_b-t_{b,T}), \ldots, X(l_b-\tau)\big).$$

Therefore, we can shift indices by $-t_{b,T}$ and obtain

$$\mathrm{Re}\big(\hat{a}_{n,b,t_b}(\lambda, \tau)\big) = \frac{1}{b}\sum_{j=t_b}^{t_b-1+b-\tau} X(j+\tau)X(j)\cos\big(\lambda(j-t_b+1)\big)$$

$$= \frac{1}{b}\sum_{j=t_b-t_{b,T}}^{l_b-\tau} X(j+t_{b,T}+\tau)X(j+t_{b,T})\cos\big(\lambda(j-t_b+1+t_{b,T})\big)$$

$$\stackrel{d}{=} \frac{1}{b}\sum_{j=t_b-t_{b,T}}^{l_b-\tau} X(j+\tau)X(j)\cos\big(\lambda(j-t_b+1+t_{b,T})\big)$$

$$= \cos\big(\lambda(t_b-1-t_{b,T})\big)\left(\frac{1}{b}\sum_{j=t_b-t_{b,T}}^{l_b-\tau} X(j+\tau)X(j)\cos(\lambda j)\right)$$

$$+\sin\big(\lambda(t_b-1-t_{b,T})\big)\Bigg(\frac{1}{b}\sum_{j=t_b-t_{b,T}}^{l_b-\tau} X(j+\tau)X(j)\sin(\lambda j)\Bigg)$$

$$=\cos\big(\lambda(t_b-1-t_{b,T})\big)\,\Phi_b(\lambda,\tau)-\sin\big(\lambda(t_b-1-t_{b,T})\big)\,\Psi_b(\lambda,\tau),$$

where

$$\Phi_b(\lambda,\tau)=\frac{1}{b}\sum_{j=t_b-t_{b,T}}^{l_b-\tau} X(j+\tau)X(j)\cos(\lambda j),$$

$$\Psi_b(\lambda,\tau)=\frac{1}{b}\sum_{j=t_b-t_{b,T}}^{l_b-\tau} X(j+\tau)X(j)\sin(\lambda j).$$

In order to simplify the notation let us put $\lambda_{b,T}=\lambda(t_b-1-t_{b,T})$. We may write

$$\begin{aligned}
R_{n,b,t}(\lambda,\tau) &= \sqrt{b}\,\mathrm{Re}\big(\hat{a}_{n,b,t_b}(\lambda,\tau)-a(\lambda,\tau)\big)\\
&\overset{d}{=} \sqrt{b}\,\big(\cos(\lambda_{b,T})\,\Phi_b(\lambda,\tau)-\sin(\lambda_{b,T})\,\Psi_b(\lambda,\tau)-\mathrm{Re}\big(a(\lambda,\tau)\big)\big)\\
&= \cos(\lambda_{b,T})\sqrt{b}\,\big(\Phi_b(\lambda,\tau)-\mathrm{Re}\big(a(\lambda,\tau)\big)\big)\\
&\quad -\sin(\lambda_{b,T})\sqrt{b}\,\big(\Psi_b(\lambda,\tau)-\mathrm{Im}\big(a(\lambda,\tau)\big)\big)\\
&\quad +\sqrt{b}\,\big(\cos(\lambda_{b,T})\,\mathrm{Re}\big(a(\lambda,\tau)\big)-\mathrm{Re}\big(a(\lambda,\tau)\big)-\sin(\lambda_{b,T})\,\mathrm{Im}\big(a(\lambda,\tau)\big)\big).
\end{aligned}$$

Now, take sequences of the form $t_b^h=1+h+bT$, where $h\in\{0,\ldots,T-1\}$. Then $t_b^h-1-t_{b,T}^h=h$, where $t_{b,T}^h=\lfloor(t_b^h-1)/T\rfloor T$. By Lemma 3.1 and the Cramer–Wold theorem, we infer that the random part of $R_{n,b,t}(\lambda,\tau)$ tends weakly to a normal distribution, i.e.

$$\cos(\lambda h)\sqrt{b}\,\big(\Phi_b(\lambda,\tau)-\mathrm{Re}\big(a(\lambda,\tau)\big)\big)$$
$$-\sin(\lambda h)\sqrt{b}\,\big(\Psi_b(\lambda,\tau)-\mathrm{Im}\big(a(\lambda,\tau)\big)\big)\xrightarrow{d}\mathcal{N}(0,\sigma_h^2),$$

where

$$\sigma_h^2=\cos^2(\lambda h)\,\sigma_{11}+\sin^2(\lambda h)\,\sigma_{22}-2\sin(\lambda h)\cos(\lambda h)\,\sigma_{12},$$

and σ_{11}, σ_{22}, σ_{12} are components of Σ as before. Since the rest of $R_{n,b,t}$ is non-random and does not influence the variance, it is necessary for the condition (3.1) that, for any $h\in\{0,\ldots,T-1\}$,

$$\sigma_h^2=\sigma_{11}.$$

This is equivalent to

$$\cos^2(\lambda h)\,\sigma_{11}+\sin^2(\lambda h)\,\sigma_{22}-2\sin(\lambda h)\cos(\lambda h)\,\sigma_{12}=\sigma_{11}$$

and, further, to

$$2\sin(\lambda h)\cos(\lambda h)\,\sigma_{12}=\sin^2(\lambda h)(\sigma_{22}-\sigma_{11}).$$

Assuming that $\sin(\lambda h) \neq 0$ and $\sigma_{12} \neq 0$ we get

$$\operatorname{ctg}(\lambda h) = \frac{\sigma_{22}-\sigma_{11}}{2\sigma_{12}}.$$

Observe that the right-hand side of the above equality does not depend on h and the equality can be satisfied only due to periodicity of the cotangent function. However, it allows to exclude many λ possible to satisfy condition (3.1). Note that, including the *PSS* subclass, for *PC* time series we have

$$a\left(\frac{2\pi(k+nT)}{T}, \tau\right) = a\left(\frac{2\pi k}{T}, \tau\right), \quad \hat{a}_n\left(\frac{2\pi(k+nT)}{T}, \tau\right) = \hat{a}_n\left(\frac{2\pi k}{T}, \tau\right)$$

for $n \in \mathbf{Z}$. This justifies why we may restrict to $\lambda \in [0, 2\pi)$. Let us denote the non-random term of $R_{n,b,t}(\lambda, \tau)$ by

$$N_R = \cos(\lambda_{b,T}) \operatorname{Re}(a(\lambda, \tau)) - \operatorname{Re}(a(\lambda, \tau)) - \sin(\lambda_{b,T}) \operatorname{Im}(a(\lambda, \tau))$$

and consider the following cases:

1. $T = 2$, $\sigma_{12} \neq 0$, $\sin(\lambda) \neq 0$.

Condition (3.1) might be satisfied only if $\operatorname{ctg}(\lambda) = (\sigma_{11}-\sigma_{22})/2\sigma_{12}$ and $N_R = 0$. Contrarily, $R_{n,b,t}(\lambda, \tau)$ is not convergent or has not appropriate asymptotic variance. However, this is a very restrictive case with rigid conditions on Σ, λ and $a(\lambda, \tau)$.

2. $T \geqslant 3$, $\sigma_{12} \neq 0$, $\sin(\lambda h) \neq 0$ for all $h \in \{1, \ldots, T-1\}$.

The equality $\operatorname{ctg}(\lambda) = \operatorname{ctg}(2\lambda)$ implies that $2\lambda - \lambda = l\pi$ for some $l \in \mathbf{Z}$, which means that $\sin(\lambda) = 0$ and yields a contradiction.

3. $T \geqslant 3$, $\sigma_{12} \neq 0$, there exists the smallest $h_0 \in \{1, \ldots, T-1\}$ such that $\sin(\lambda h_0) = 0$ and there exists $h \in \{1, \ldots, T-1\}$ such that $\sin(\lambda h) \neq 0$.

If $h_0 \geqslant 3$, the equality $\operatorname{ctg}(\lambda) = \operatorname{ctg}(2\lambda)$ must be satisfied. Then, as in the previous case, $\lambda = l\pi$ and $\sin(\lambda h) = 0$ for all $h \in \{0, \ldots, T-1\}$, which yields a contradiction. If $h_0 = 1$, then $\sin(\lambda) = 0$, so $\lambda = 0$ or $\lambda = \pi$, but then $\sin(\lambda h) = 0$ for all h. If $h_0 = 2$, we have $\sin(2\lambda) = 0$, which gives $\lambda = 0$, $\lambda = \pi/2$, $\lambda = \pi$ or $\lambda = 3\pi/2$ (remember that $\lambda \in [0, 2\pi)$). For $\lambda = 0$ and $\lambda = \pi$, we have $\sin(\lambda h) = 0$ for all h, so a contradiction. If $\lambda = \pi/2$ or $\lambda = 3\pi/2$, then, for h such that $\sin(\lambda h) \neq 0$, $\sigma_h^2 = \sigma_{22}$. Therefore, for those two values of λ, (3.1) might be satisfied only if $\sigma_{11} = \sigma_{22}$ and $N_R = 0$.

4. $\sigma_{12} = 0$, there exists $h \in \{1, \ldots, T-1\}$ such that $\sin(\lambda h) \neq 0$.

By the condition $\sigma_h^2 = \sigma_{11}$ we infer that the equality $\sigma_{11} = \sigma_{22}$ must be true. Assuming additionally that $N_R = 0$, we see that (3.1) might be satisfied.

5. $\sin(\lambda h) = 0$ for all $h \in \{1, \ldots, T-1\}$.

It means that $\lambda \in \{0, \pi\}$. For $\lambda = 0$ we have

$$\sqrt{b} \operatorname{Re}(\hat{a}_{n,b,t_b}(\lambda, \tau) - a(\lambda, \tau)) = \sqrt{b}\left(\Phi_b(\lambda, \tau) - \operatorname{Re}(a(\lambda, \tau))\right) \xrightarrow{d} \mathcal{N}(0, \sigma_{11}),$$

so (3.1) is satisfied without further restrictions. For $\lambda = \pi$

$$\sqrt{b}\,\mathrm{Re}\big(\hat{a}_{n,b,t_b}(\pi, \tau) - a(\pi, \tau)\big) = (-1)^{t_b - 1 - t_{b,T}}\sqrt{b}\,\big(\Phi_b(\pi, \tau) - \mathrm{Re}\big(a(\pi, \tau)\big)\big)$$
$$+\sqrt{b}\,\big((-1)^{t_b - 1 - t_{b,T}} - 1\big)\,\mathrm{Re}\big(a(\pi, \tau)\big).$$

Hence, the weak convergence to the desired normal distribution holds if and only if $\mathrm{Re}\big(a(\pi, \tau)\big) = 0$.

While estimating $a(\lambda, \tau)$ we do not know Σ, so we should avoid making *a priori* assumptions regarding its form. The two cases about which we proved that the subsampling is consistent are: $\lambda = 0$ and $\lambda = \pi$, if only $a(\pi, \tau) = 0$. Let us add that by applying identical reasoning to the imaginary part of the estimator we get the analogous necessary condition

$$\sin^2(\lambda h)\,\sigma_{11} + \cos^2(\lambda h)\,\sigma_{22} + 2\sin(\lambda h)\cos(\lambda h)\,\sigma_{12} = \sigma_{22}.$$

Therefore, the cases for the imaginary part are the same and it does not provide new consistent values of λ. Lastly, notice that, for $\lambda = 0$ and $\lambda = \pi$, the imaginary part of the estimator and parameter is equal to zero, so we may say that the consistency concerns the whole estimator. ■

COROLLARY 3.1. *The sufficient condition* (3.1) *is not satisfied apart from the cases listed in Theorem* 3.1.

We just proved the consistency for the particular case ($\lambda = 0$, $\lambda = \pi$). However, the condition (3.1), in general, does not solve the problem of consistency for the parameter $a(\lambda, \tau)$. In the next section we will try to approach the question about consistency by means of simulations.

4. SIMULATION STUDY

In this section, we would like to check how the subsampling algorithm performs in practice. We will simulate time series from the model that satisfies all assumptions of Theorem 3.1 and then compute subsampling replications of the estimator $\hat{a}_n(\lambda, \tau)$. As subsampling consistency is equivalent to convergence of $L_{n,b}(\cdot)$ to c.d.f. of $\mathcal{N}(0, \sigma_{11})$, we will study the histogram based on the values of $\sqrt{b}\,\mathrm{Re}\big(\hat{a}_{n,b,t}(\lambda, \tau) - \hat{a}_n(\lambda, \tau)\big)$, where $t = 1, \ldots, n-b+1$, and combine it with density function of a normal distribution. It will be also very interesting to run the procedure for the cases in which we were able neither to prove nor disprove the consistency.

Consider the model

$$X(t) = f(t)\,Z(t),$$

where the modulating function $f(t) = \sin\big((2\pi/T)\,t\big)$ is periodic with period

T and $\{Z(t): t = 1, \ldots, n\}$ is moving average process of the form

$$Z(t) = \frac{1}{\sqrt{2q+1}} \sum_{j=t-q}^{t+q} U(j),$$

where $q \in N$, $\big(U(1-q), \ldots, U(n+q)\big)$ is an i.i.d. sample. Asssume that $E\big(U(j)\big) = 0$ and $E\big(U^4(j)\big) < \infty$. It is easy to see that the process $\{Z(t)\}$, and thus the process $\{X(t)\}$, is $2q$-dependent, which implies that they are φ-mixing. Moreover, $\{Z(t)\}$ is strictly stationary, so we can apply Theorem 2.1 to see that $\{X(t)\}$ is *PSS*. Therefore, all assumptions of Theorem 3.1 are satisfied and subsampling is consistent for particular cases $\lambda = 0$ and, if $a(\pi, \tau) = 0$, for $\lambda = \pi$. Recall that for *PC* processes $a(\lambda, \tau) = 0$ for $\lambda \neq 2\pi k/T$. Therefore, if T is an odd integer, we are sure that $a(\pi, \tau) = 0$. The model introduced above is simple enough so as we are able to compute $a(\lambda, \tau)$ analytically. It is easy to check that the shifted covariance for the moving average process takes form of the 'tent' function

$$B_Z(t, \tau) = B_Z(\tau) = \sigma^2 \left(1 - \frac{1}{2q+1}|\tau|\right) \mathbf{1}_{[-2q-1, 2q+1]}(\tau),$$

where $\sigma^2 = E\big(U^2(j)\big)$. Hence, for the process $\{X(t)\}$ we have

$$B_X(t, \tau) = \sigma^2 f(t+\tau) f(t) \left(1 - \frac{1}{2q+1}|\tau|\right) \mathbf{1}_{[-2q-1, 2q+1]}(\tau).$$

Now, for any τ and $\lambda_k = (2\pi k)/T$, we may write

$$\begin{aligned} a(\lambda_k, \tau) &= \frac{1}{T} \sum_{t=1}^{T} B_X(t, \tau) \exp(-i\lambda_k t) \\ &= \frac{\sigma^2}{T} \left(1 - \frac{1}{2q+1}|\tau|\right) \mathbf{1}_{[-2q-1, 2q+1]}(\tau) \sum_{t=1}^{T} f(t+\tau) f(t) \exp(i\lambda_k t). \end{aligned}$$

It is very desirable in our simulations to know the exact value of the parameter. Unfortunately, the asymptotic variance of the estimator cannot be easily calculated, even for small values of q and τ.

To draw Figures 1 and 2 we simulated the sequence $\{U(j)\}$ from uniform distribution on the interval $[-1, 1]$ and took $q = 3$ and $T = 7$. Such a model could be used in situations when the observations are collected daily (e.g. the number of clients of the supermarket per day), the pattern with some random fluctuations repeats weakly, and the process has memory no longer than one week. The convergence in limit theorems for dependent data is slow, so we decided to take quite large sample sizes ($n = 500$, 1000, 7000) and $b = n^{3/5}$. Subsampling consistency means that the empirical distribution of the subsampling replications should become normal. However, we do not use the tests for normality (e.g. the Wilks–Shapiro test) for at least two reasons. Firstly, though we build empirical distribution from subsampling replications, they are not i.i.d. observations. Secondly, we do not expect that subsampling replications are normal but that they are approximately normal. For large samples normality tests do not tolerate even small disturbances from normality. Therefore, apart from visual assessment of histograms and comparison with normal density functions with sample mean and sample variance, we must content ourselves with some more rough methods of indicating convergence to the limit normal distribution. We mean here computing statistics like sample mean, sample skewness and sample kurtosis. Since every normal distribution has the skewness equal to zero and kurtosis equal to three, estimators of them based on the subsampling sample should tend to those values. The estimator of the mean should tend to zero.

Figure 1 depicts the situation when $a(\lambda, \tau) \neq 0$ and compares the case of proven subsampling consistency (left column, $\lambda = 0$) with that in which we cannot say anything about the consistency (right column, $\lambda = 4\pi/T$). The histograms on the right do not resemble normal distribution at all and become more strange as n increases. Moreover, they do not become zero-centered. The histograms on the left, though for a small sample have quite high skewness, become more normal for large samples. Their mean is very close to zero as well. We would like to add that for some choices of periodic function f, the histograms for $\lambda = 4\pi/T$ were quite normal but never have mean tending to zero. Such a performance makes us worry about the consistency in the case when $a(\lambda, \tau) \neq 0$ and $\lambda \neq 0$.

Figure 2 depicts the situation when $a(\lambda, \tau) = 0$ and compares the case of proven subsampling consistency (left column, $\lambda = \pi$) with that in which we cannot say anything about the consistency (right column, $\lambda = 3$). In both cases we observe convergence to a normal distribution. Both means behave well, so the question is about the variances. Since both frequencies are close to each other, the asymptotic variances should be also close. The sample variance for $\lambda = 3$ is a bit smaller but decreases slower.

Concluding the simulations we may hypothesize that subsampling is not consistent for $\lambda \neq 0$ for which $a(\lambda, \tau) \neq 0$. However, it may be consistent for those values of λ for which $a(\lambda, \tau) = 0$. If it were true, we would be able to test a hypothesis $H_0\colon a(\lambda, \tau) = 0$, as for the test we need the distribution of $\hat{a}_n(\lambda, \tau)$ under H_0.

At the end, we would like to provide an interesting application of our theoretical result. It enables, by means of quantiles of $L_{n,b}(\cdot)$, to test the problem

$$H_0\colon a(\pi, \tau) = 0,$$
$$H_1\colon a(\pi, \tau) \neq 0.$$

If we reject H_0, it means that with high probability $a(\pi, \tau) \neq 0$. It implies that $\pi = 2\pi k/T$ and $T = 2k$, so T is an even number. Consequently, we obtain the consistent test for evenness of T.

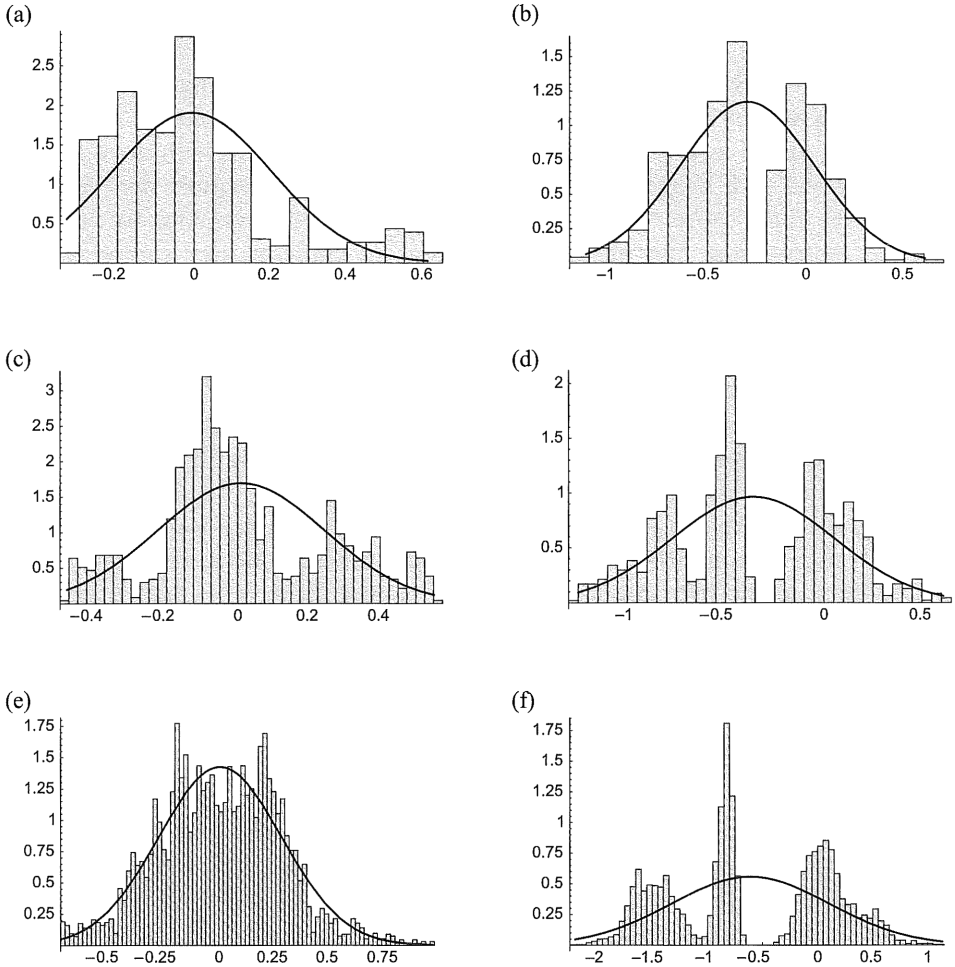

FIGURE 1. Comparison of subsampling performance for $\lambda = 0$ (left column) and $\lambda = 4\pi/T$ (right column). Analytical values of parameters are $\mathrm{Re}(a(0, 1)) = 0.0890$ and $\mathrm{Re}(a(4\pi/T, 1)) = 0.0445$

Values of the estimators

Parts of the figure	Mean	Variance	Skewness	Kurtosis
(a) $\mathrm{Re}(\hat{a}_{500}(0, 1)) = 0.0900$	−0.0053	0.0438	1.0625	3.8259
(b) $\mathrm{Re}(\hat{a}_{500}(4\pi/T, 1)) = 0.0445$	−0.2975	0.1159	0.0023	2.3162
(c) $\mathrm{Re}(\hat{a}_{1000}(0, 1)) = 0.0875$	0.0092	0.0553	0.3153	2.6735
(d) $\mathrm{Re}(\hat{a}_{1000}(4\pi/T, 1)) = 0.0437$	−0.3587	0.1708	−0.0014	2.1039
(e) $\mathrm{Re}(\hat{a}_{7000}(0, 1)) = 0.0864$	0.0061	0.0785	0.0798	3.0092
(f) $\mathrm{Re}(\hat{a}_{7000}(4\pi/T, 1)) = 0.0426$	−0.6114	0.5126	−0.0009	1.7918

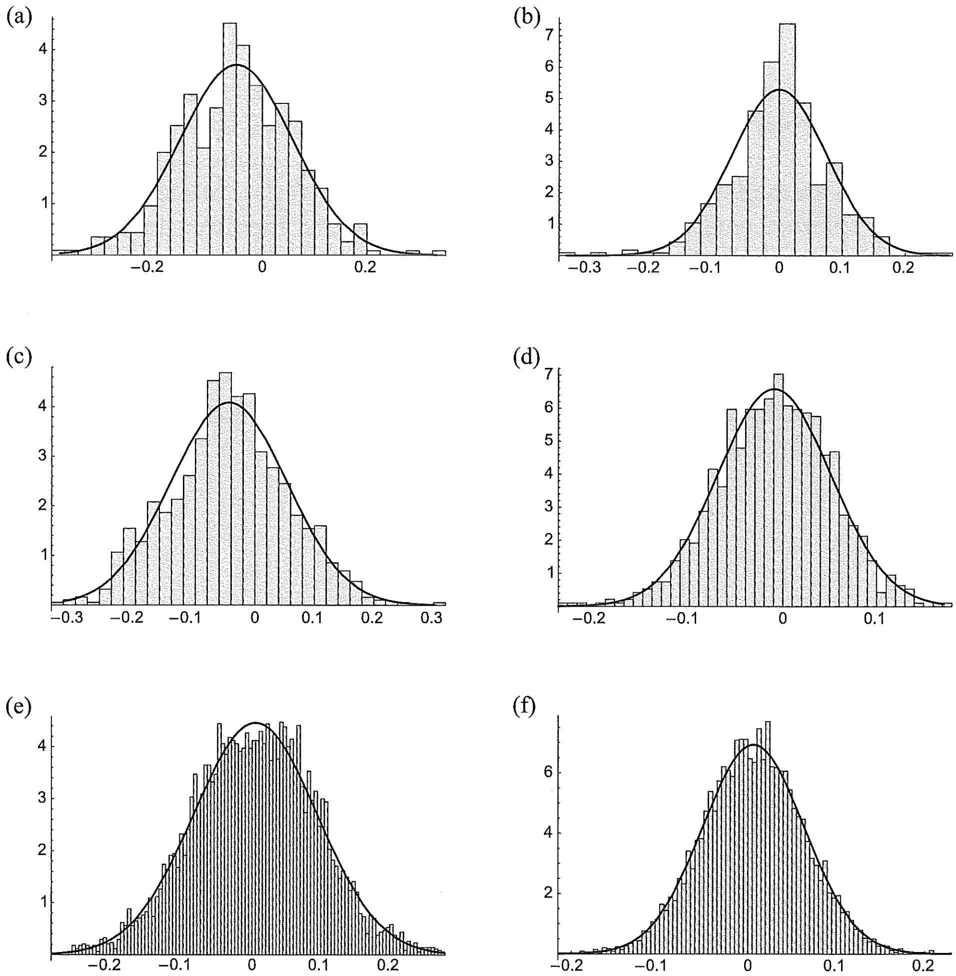

FIGURE 2. Comparison of subsampling performance for $\lambda = \pi$ (left column) and $\lambda = 3$ (right column). Analytical values of parameters are $\mathrm{Re}(a(\pi, 1)) = 0$ and $\mathrm{Re}(a(3, 1)) = 0$

Values of the estimators

Parts of the figure	Mean	Variance	Skewness	Kurtosis
(a) $\mathrm{Re}(\hat{a}_{500}(\pi, 1)) = 0.0077$	-0.0494	0.0115	0.0036	3.1612
(b) $\mathrm{Re}(\hat{a}_{500}(3, 1)) = -0.0001$	0.0000	0.0056	-0.2910	4.5279
(c) $\mathrm{Re}(\hat{a}_{1000}(\pi, 1)) = 0.0054$	-0.0435	0.0095	0.0182	2.8340
(d) $\mathrm{Re}(\hat{a}_{1000}(3, 1)) = -0.0001$	-0.0093	0.0036	-0.1973	3.2503
(e) $\mathrm{Re}(\hat{a}_{7000}(\pi, 1)) = -0.0007$	0.0049	0.0080	-0.0032	2.9327
(f) $\mathrm{Re}(\hat{a}_{7000}(3, 1)) = -0.0004$	0.0061	0.0033	0.0079	3.1387

5. CONCLUSIONS

We have presented the classes of *PC* and *PSS* time series with related statistical problem in time-domain. Since the asymptotic distribution of the autocovariance estimator is not applicable in practice, we must look for other statistical method to construct confidence intervals for it. We expected that a good solution could be subsampling. Unfortunately, the procedure does not occur to be universally consistent. Another idea could be to modify the subsampling procedure (e.g. nonoverlapping blocks), to modify the estimator (e.g. $|\hat{a}_n(\lambda, \tau)|$) or other resampling-type methods, for example MBB (moving blocks bootstrap). These ideas are under the current research of the author.

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