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# OCCUPATION TIME FLUCTUATIONS OF POISSON AND EQUILIBRIUM FINITE VARIANCE BRANCHING SYSTEMS

BY

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*Abstract.* Functional limit theorems are presented for the rescaled occupation time fluctuation process of a critical finite variance branching particle system in $\boldsymbol{R}^d$ with symmetric $\alpha$-stable motion starting off from either a standard Poisson random field or from the equilibrium distribution for intermediate dimensions $\alpha < d < 2\alpha$. The limit processes are determined by sub-fractional and fractional Brownian motions, respectively.

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## 1. INTRODUCTION

Consider a system of particles in $\boldsymbol{R}^d$ starting off at time $t = 0$ from a certain distribution (standard Poisson and equilibrium fields are investigated in this paper). They evolve independently, moving according to a symmetric $\alpha$-stable Lévy process and undergoing finite variance branching at rate $V$ ($V > 0$). We obtain functional limit theorems for the rescaled occupation time fluctuations of this system when $\alpha < d < 2\alpha$. This is an extension of Theorem 2 in [7] where the starting distribution is a Poisson field and the branching law is critical and binary.

**1.1. Branching law.** In [4], [7], and [8] the law of branching is critical and binary. In this paper an extended model is investigated. The particles branch according to the law given by a moment generating function $F$. The function $F$ fulfills two requirements:

1. $F'(1) = 1$, which means that the law is critical (the expected number of particles spawning from one particle is 1);
2. $F''(1) < +\infty$, which states that the second moment exists.

(Note that the branching law in [7] is given by $F(s) = \frac{1}{2}(1+s^2)$ and obviously fulfills the two requirements.) Although constraints imposed on $F$ are not very restrictive and quite natural (so that the class of the branching laws satisfying them is broad), still there remain other interesting cases to be investigated. One of them is the class of branching laws in the domain of attraction of the $(1+\beta)$-stable law, i.e., the moment generating function is

$$F(s) = s + \frac{1}{1+\beta}(1-s)^{1+\beta},$$

the case studied in [5] and [6]. A remarkable feature of the latter case is that the limit processes are stable ones and not Gaussian as it occurs in the finite variance case.

**1.2. Equilibrium distribution.** Another concept naturally related to particle systems is an equilibrium distribution. It has been shown that in certain circumstances the system converges to the equilibrium distribution [11]. It is both an interesting and important question whether the theorems shown by Bojdecki et al. still hold in the case when the equilibrium state is taken as the initial condition. A conjecture in [4] states that the temporal structure of the limit is given by fractional Brownian motion. It is of interest to notice that the limit is different from the one in the case of the system starting off from the Poisson field (where temporal structure is sub-fractional Brownian motion). We study behavior of the system for a branching law given by $F$. But there is still a broad area of further studies. No attempt has been made to develop more general theory concerning systems with a general starting distribution (or a large class of distributions).

**1.3. General concepts and notation.** Let us denote by $N_t^{\mathrm{Poiss}}$ and $N_t^{\mathrm{eq}}$ the empirical processes for the system starting off from the Poisson field with Lebesgue intensity measure and the equilibrium, respectively. For a measurable set $A \subset \boldsymbol{R}^d$, $N_t^{\mathrm{Poiss}}(A)$ and $N_t^{\mathrm{eq}}(A)$, respectively, are the numbers of particles of the system in the set $A$ at time $t$. Note that they are measure-valued processes but we will consider them as processes with values in $\mathcal{S}'$ (the space of tempered distributions) because this space has good analytical properties.

The equilibrium distribution is defined by

$$\lim_{t\to+\infty} N_t^{\mathrm{Poiss}} = N_{\mathrm{eq}},$$

where the limit is understood in weak sense. The Laplace functional of the equilibrium distribution is given by

$$\boldsymbol{E}\exp\{-\langle N_{\mathrm{eq}}, \varphi\rangle\} = \exp\Big\{\langle\lambda, e^{-\varphi}-1\rangle + V\int_0^\infty \langle\lambda, H(j(\cdot, s))\rangle\, ds\Big\}, \tag{1.1}$$

where

$$j(x, l) := \boldsymbol{E}\exp(-\langle N_l^x, \varphi\rangle), \tag{1.2}$$

$H(s) = F(s)-s$, $\varphi\colon \boldsymbol{R}^d \to \boldsymbol{R}_+$, $\varphi \in \mathscr{L}^1(\boldsymbol{R}^d) \cap C(\boldsymbol{R}^d)$, and $j$ satisfies the integral equation

$$j(x, l) = \mathscr{T}_l e^{-\varphi}(x) + V\int_0^l \mathscr{T}_{l-s} H\big(j(\cdot, s)\big)(x)\, ds.$$

This equation can be obtained in the same way as (2.4) in [11]. Note that in [11] the function $\varphi$ is continuous with compact support. We approximate $\varphi \in \mathscr{L}^1$ using functions $\varphi_n$ with compact support $\varphi_n \nearrow \varphi$. Using the Lebesgue monotone convergence theorem it is easy to obtain the above equations for $\varphi$ ($H$ is decreasing because of the criticality of the branching law).

For an empirical process $N_t$ the rescaled occupation time fluctuation process is defined by

$$X_T(t) = \frac{1}{F_T}\int_0^{Tt} (N_s - EN_s)\, ds, \qquad t \geqslant 0, \tag{1.3}$$

where $T > 0$ and $F_T$ is a suitable norming. We are interested in the weak functional limit of $X_T$ when time is accelerated (i.e., $T$ tends to $\infty$).

The $\alpha$-stable process starting from $x$ will be denoted by $\eta_t^x$, its semigroup by $\mathscr{T}_t$, and its infinitesimal operator by $\Delta_\alpha$. The Fourier transform of $\mathscr{T}_t$ is

$$\widehat{\mathscr{T}_t\varphi}(z) = \exp(-t|z|^\alpha)\,\hat{\varphi}(z). \tag{1.4}$$

For brevity let us put

$$K = \frac{V\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)\,h(h-1)}, \tag{1.5}$$

where

$$h = 3 - d/\alpha \tag{1.6}$$

(in this paper we always assume that $\alpha < d < 2\alpha$, so $h > 1$) and

$$M = F''(1). \tag{1.7}$$

We will now introduce two centered Gaussian processes. One of them is a sub-fractional Brownian motion with parameter $h$ with the covariance function $C_h$,

$$C_h(s, t) = s^h + t^h - \tfrac{1}{2}[(s+t)^h + |s-t|^h], \tag{1.8}$$

and the second one is a fractional Brownian motion with parameter $h$ and the covariance function $c_h$,

$$c_h(s, t) = \tfrac{1}{2}(s^h + t^h - |s-t|^h). \tag{1.9}$$

**1.4. Space-time method.** The space-time method is a very convenient technique for investigating the weak convergence in the $C\big([0, \tau], \mathscr{S}'(\boldsymbol{R}^d)\big)$ space.

It was developed by Bojdecki et al. and can be found in [3]. If $X = \big(X(t)\big)_{t\in[0,\tau]}$ is a continuous $\mathcal{S}'(\boldsymbol{R}^d)$-valued process, we define a random element $\tilde{X}$ of $\mathcal{S}'(\boldsymbol{R}^{d+1})$ by

$$(1.10) \qquad \langle \tilde{X}, \Phi\rangle = \int_0^\tau \langle X(t), \Phi(\cdot, t)\rangle\, dt,$$

where $\Phi \in \mathcal{S}(\boldsymbol{R}^{d+1})$. In order to prove that $X_T$ converges weakly to $X$ in $C\big([0, \tau], \mathcal{S}'(\boldsymbol{R}^d)\big)$ it suffices to show that

$$\langle \tilde{X}_T, \Phi\rangle \Rightarrow \langle \tilde{X}, \Phi\rangle \quad \text{for all } \Phi \in \mathcal{S}(\boldsymbol{R}^{d+1})$$

and that the family $X_T$ is tight.

## 2. CONVERGENCE THEOREMS

We will present two theorems. In the first of them (which is a direct extension of Theorem 2.2 in [7]) we study the occupation time fluctuation process for the branching system starting off from the Poisson field with Lebesgue intensity measure (denoted by $\lambda$) with the branching law given by a moment generating function as described in Section 1.1. The result is very similar to the one obtained in Theorem 2.2 of [7] – namely, the limit process is the same up to constants.

THEOREM 2.1. *Assume that $\alpha < d < 2\alpha$ and let $X_T$ be the occupation time fluctuation process defined by* (1.3) *for the branching system $N^{\mathrm{Poiss}}$, and $F_T = T^{(3-d/\alpha)/2}$. Then $X_T \Rightarrow X$ in $C\big([0, \tau], \mathcal{S}'(\boldsymbol{R}^d)\big)$ as $T \to +\infty$ for any $\tau > 0$, where $\big(X(t)\big)_{t\geqslant 0}$ is a centered $\mathcal{S}'$-valued Gaussian process with the covariance function*

$$(2.1) \qquad \mathrm{Cov}\big(\langle X(s), \varphi\rangle, \langle X(t), \psi\rangle\big) = KM\langle\lambda, \varphi\rangle\langle\lambda, \psi\rangle C_h(s, t),$$

*where $\varphi, \psi \in \mathcal{S}(\boldsymbol{R}^d)$.*

The second theorem concerns the case where the system starts from the equilibrium distribution. As mentioned hereinabove, the theorem is interesting because the limit has a different time structure from the one in Theorem 2.2 of [7] and Theorem 2.1.

THEOREM 2.2. *Assume that $\alpha < d < 2\alpha$ and let $X_T$ be the occupation time fluctuation process defined by* (1.3) *for the branching system $N^{\mathrm{eq}}$, and $F_T = T^{(3-d/\alpha)/2}$. Then $X_T \Rightarrow X$ in $C\big([0, \tau], \mathcal{S}'(\boldsymbol{R}^d)\big)$ as $T \to +\infty$ for any $\tau > 0$, where $\big(X(t)\big)_{t\geqslant 0}$ is a centered Gaussian process with the covariance function*

$$(2.2) \qquad \mathrm{Cov}\big(\langle X(s), \varphi\rangle, \langle X(t), \psi\rangle\big) = KM\langle\lambda, \varphi\rangle\langle\lambda, \psi\rangle c_h(s, t),$$

*where $\varphi, \psi \in \mathcal{S}(\boldsymbol{R}^d)$.*

Remark 2.1. The limit processes above can be represented as follows: For Theorem 2.1

$$X = (MK)^{1/2} \lambda \beta^h$$

and for Theorem 2.2

$$X = (MK)^{1/2} \lambda \xi^h,$$

where $\beta^h$ and $\xi^h$ are sub-fractional and fractional Gaussian processes, respectively, defined in Section 1.3. In both cases the limit process $X$ has a trivial spatial structure (Lebesgue measure), whereas the time structure is complicated, with long range dependence.

Remark 2.2. The occupation time fluctuation processes of particle systems form an area that receives a lot of research attention. We would like to mention some other related work. Firstly, the case of non-branching systems has been studied in [7], Theorem 2.1. The result is analogous both to Theorems 2.1 and 2.2 because the Poisson field is the equilibrium distribution for the system. The limit process is essentially the same as in Theorem 2.2. For the critical $d = 2\alpha$ and large dimensions $d > 2\alpha$, there is no long range dependence and the results can be found in [8]. In [2] the fluctuations of the occupation time of the origin are studied for a critical binary branching random walks on the $d$-dimensional lattice, $d \geqslant 3$, including also the equilibrium case. The convergence results are analogous to those in [7] and [8] and in this paper, but the proofs are substantially different. A similar model with $\alpha = 2$ was investigated in [9] (i.e. with particles moving according to Brownian motion).

## 3. PROOFS

The main idea used in both of the proofs is to study the Laplace functional of a process given by the space-time method. The Fourier transform is used for this purpose. This is similar to the method in [7]. In the case of Theorem 2.1 the proof follows the same principle as Theorem 2.2 in [7]. The moment generating function can be represented by using the Taylor expansion and the following two statements need to be proved. Firstly, one has to check that the method used in [7] can still be applied. Secondly, it needs to be shown that terms of order higher than 2 play no role in the limit. The proof of Theorem 2.2 requires more work. The Laplace formula contains a function that is a solution of a differential equation. This makes the computations more cumbersome. Some expressions in this proof had to be examined more carefully than in Theorem 2.1. It should be noted that Theorem 2.2 covers all branching laws described in Section 1.1.

Now we introduce some notation and facts used further on.

For a generating function $F$ we define

$$G(s) = F(1-s) - 1 + s. \tag{3.1}$$

The following fact describes basic properties of $G$ which are straightforward consequences of the properties of $F$.

FACT 3.1. 1. $G(0) = F(1)-1 = 0$.

2. $G'(0) = -F'(1)+1 = 0$ *since* $F'(1) = 1$.

3. $G''(0) = F''(1) < +\infty$.

4. $G(v) = (M/2)v^2+g(v)v^2$, *where $M$ is defined by* (1.7) *and* $\lim_{v\to 0} g(v) = 0$.

The next simple fact will be useful in proving some inequalities.

FACT 3.2. $G(v) \geqslant 0$ *for* $v \in [0, 1]$.

Proof. The property $F''(1-v) \geqslant 0$ is an obvious consequence of the fact that all of the coefficients in the expansion of $F''$ are non-negative and $1-v \in [0, 1]$. We have $G''(v) = F''(1-v) \geqslant 0$. We also know that $G'(0) = 0$, so $G'(v) \geqslant 0$ for $v \in [0, 1]$. The proof is complete since $G(0) = 0$ and $G$ is non-decreasing. ∎

The existence of the second moment of the moment generating function $F$ implies also that $G$ is comparable with the function $v^2$.

FACT 3.3. *We have*

$$\sup_{v\in[0,1]} \frac{G(v)}{v^2} < +\infty.$$

Proof. Since both $G(v)$ and $v^2$ are continuous, we only have to check that the limit of the quotient at $v = 0$ is finite. This becomes obvious when we recall Taylor's expansion of $G(v)$ from Fact 3.1, property 4. ∎

Let us now introduce some notation used throughout the rest of the paper. $\Phi$ will denote a positive function from $\mathcal{S}(\boldsymbol{R}^{d+1})$. The Lemma in Section 3.2 of [7] explains why without loss of generality it can be assumed that $\Phi \geqslant 0$. We put

$$\Psi(x, s) = \int_s^1 \Phi(x, t)\,dt, \qquad \Psi_T(x, s) = \frac{1}{F_T}\Psi\left(x, \frac{s}{T}\right).$$

To make computations less cumbersome we will sometimes assume that $\Phi$ is of the form $\Phi(x, t) = \varphi(x)\psi(t)$ for $\varphi \in \mathcal{S}(\boldsymbol{R}^d)$, $\psi \in \mathcal{S}(\boldsymbol{R})$, and hence

$$\Psi_T(x, t) = \varphi_T(x)\chi_T(t), \tag{3.2}$$

where

$$\varphi_T(x) = \frac{1}{F_T}\varphi(x), \qquad \chi(t) = \int_t^1 \psi(s)\,ds, \qquad \chi_T = \chi\left(\frac{t}{T}\right).$$

Notice that $\varphi \geqslant 0$, $\chi \geqslant 0$ as $\Phi \geqslant 0$.

Let us introduce now an important function which will appear as a part of the Laplace functional of the occupation time fluctuation processes:

$$v_\Psi(x, r, t) = 1-\boldsymbol{E}\exp\left\{-\int_0^t \langle N_s^x, \Psi(\cdot, r+s)\rangle\,ds\right\},$$

where $N_s^x$ denotes the empirical measure of the particle system with the initial condition $N_0^x = \delta_x$. Let us note here that due to the fact that $\Psi \geqslant 0$ we have $v_\Psi \in [0, 1]$. We also write

$$n_\Psi(x, r, t) = \int_0^t \mathcal{T}_{t-s} \Psi(\cdot, r+t-s)(x)\, ds. \tag{3.3}$$

For simplicity of the notation, we write

$$v_T(x, r, t) = v_{\Psi_T}(x, r, t), \tag{3.4}$$

$$n_T(x, r, t) = n_{\Psi_T}(x, r, t), \tag{3.5}$$

$$v_T(x) = v_T(x, 0, T), \tag{3.6}$$

$$n_T(x) = n_T(x, 0, T), \tag{3.7}$$

when no confusion can arise.

Now we obtain an integral equation for $v$ which will play a crucial role in the next proofs. Note that similar computations can be found also in [12].

LEMMA 3.1. *$v_\Psi$ satisfies the equation*

$$\begin{aligned} &v_\Psi(x, r, t) \\ &= \int_0^t \mathcal{T}_{t-s}\big[\Psi(\cdot, r+t-s)\big(1-v_\Psi(\cdot, r+t-s, s)\big) - VG\big(v_\Psi(\cdot, r+t-s, s)\big)\big](x)\, ds. \end{aligned} \tag{3.8}$$

Proof. Firstly let us investigate

$$w(x, r, t) \equiv w_\Psi(x, r, t) = E\exp\Big(-\int_0^t \langle N_s^x, \Psi(\cdot, r+s)\rangle\, ds\Big) = 1 - v_\Psi(x, r, t).$$

We assume $\Psi \geqslant 0$; hence we have $w(x, r, t) \in [0, 1]$. By conditioning on the time of the first branching we obtain the following equation:

$$\begin{aligned} w(x, r, t) &= e^{-Vt} E\Big(-\int_0^t \Psi(\eta_s^x, r+s)\, ds\Big) \\ &\quad + V\int_0^t e^{-Vs} E\exp\Big(-\int_0^s \Psi(\eta_u^x, r+u)\, du\Big) F\big(w(\eta_s^x, r+s, t-s)\big), \end{aligned}$$

where $t \geqslant 0$, $r \geqslant 0$. Using the Feynman–Kac formula one can obtain the following equation for $w$ (for details see (3.13)–(3.17) in [7]):

$$\begin{cases} \dfrac{\partial}{\partial t} w(x, r, t) = \Big(\Delta_\alpha + \dfrac{\partial}{\partial r} - \Psi(x, r)\Big) w(x, r, t) + V\big[F\big(w(x, r, t)\big) - w(x, r, t)\big], \\ w(x, r, 0) = 1. \end{cases}$$

Since $v(x, r, t) = v_\Psi(x, r, t) = 1 - w_\Psi(x, r, t)$, $v$ satisfies the equation

$$\begin{cases} \dfrac{\partial}{\partial t} v(x, r, t) = \left(\Delta_\alpha + \dfrac{\partial}{\partial r}\right) v(x, r, t) + \Psi(x, r)\big(1 - v(x, r, t)\big) - VG\big(v(x, r, t)\big), \\ v(x, r, 0) = 0. \end{cases}$$

Its integral version is (3.8) (note that, in [7], $G(t) = \frac{1}{2}t^2$). Then we obtain

$$v(x, r, t) = \int_0^t \mathcal{T}_{t-s}\big[\Psi(\cdot, r+t-s)\big(1 - v(\cdot, r+t-s, s)\big) - VG\big(v(x, r+t-s, t)\big)\big](x)\, ds. \ \blacksquare$$

FACT 3.4. *We have*

$$v_\Psi(x, r, t) \leqslant n_\Psi(x, r, t). \tag{3.9}$$

Proof. This is a direct consequence of the equation (3.8), the fact that $1 \geqslant v \geqslant 0$ and Fact 3.2. $\blacksquare$

FACT 3.5. *For the system $N_t^{\mathrm{Poiss}}$ the covariance function is given by*

$$\begin{aligned} \operatorname{Cov}(\langle N_u^{\mathrm{Poiss}}, \varphi\rangle, \langle N_v^{\mathrm{Poiss}}, \psi\rangle) \\ = \langle \lambda, \varphi \mathcal{T}_{v-u} \psi\rangle F''(1) \cdot V \int_0^u \langle \lambda, \varphi \mathcal{T}_{u+v-2r} \psi\rangle\, dr, \quad u \leqslant v, \end{aligned} \tag{3.10}$$

*where* $\varphi, \psi \in \mathcal{S}(\boldsymbol{R}^d)$.

The proof of the fact follows from a simple computation which can be carried on using formula (3.14) of [10], so we omit it.

### 3.1. Proof of Theorem 2.1

**3.1.1.** Tightness. The first step required to establish the weak convergence is to prove tightness of $X_T$. By the Mitoma theorem [14], it is sufficient to show tightness of the real processes $\langle X_T, \phi\rangle$ for all $\phi \in \mathcal{S}(\boldsymbol{R}^d)$. This can be done by using a criterion from [1], Theorem 12.3. Detailed examination of the proof in [7] reveals that only the covariance function of the $N_t^{\mathrm{Poiss}}$ is needed ([7], Section 3.1). One can see that the covariance function (3.10) is essentially the same as for the binary branching. Hence the proof from [7] still holds for the new family of processes.

**3.1.2.** The Laplace functional. The second step uses the space-time method. According to (1.10) we define $\tilde{X}_T$ (from now on $\tau = 1$). To establish the convergence we use the Laplace functional. By the Poisson initial condition we have (this equation is the same as (3.10) in [7])

$$E \exp\{-\langle \tilde{X}_T, \Phi\rangle\} = \exp\Big\{\int_{\boldsymbol{R}^d} \int_0^T \Psi_T(x, s)\, ds\, dx\Big\} \exp\Big\{\int_{\boldsymbol{R}^d} -v_T(x, 0, T)\, dx\Big\}. \tag{3.11}$$

Now we make similar computations to (3.21)–(3.23) in [7]. By combining (3.11) and (3.8) we obtain

$$E\exp\{-\langle\tilde{X}_T, \Phi\rangle\}$$

$$= \exp\{\int_{R^d}\int_0^T \Psi_T(x, T-s)v_T(x, T-s, s) + VG(v_T(x, T-s, s))\,dsdx\}.$$

The last expression can be rewritten as

(3.12) $$E\exp\{-\langle\tilde{X}_T, \Phi\rangle\} = \exp\{V(I_1(T)+I_2(T))+I_3(T)\},$$

where

$$I_1(T) = \int_0^T\int_{R^d}\frac{M}{2}(\int_0^s \mathcal{T}_u\Psi_T(\cdot, T+u-s)(x)\,du)^2\,dxds,$$

(3.13)

$$I_2(T) = \int_0^T\int_{R^d}\left[G(v_T(x, T-s, s)) - \frac{M}{2}(\int_0^s \mathcal{T}_u\Psi_T(\cdot, T+u-s)(x)\,du)^2\right]dxds,$$

$$I_3(T) = \int_0^T\int_{R^d}\Psi_T(x, T-s)v_T(x, T-s, s)\,dxds.$$

To complete the proof we have to compute limits as $T\to+\infty$. We claim

(3.14) $$I_1(T) \to \frac{MK}{2V}\int_0^1\int_0^1\int_{R^d}\int_{R^d}\Phi(x, t)\Phi(y, s)\,dxdyC_h(s, t)\,dsdt,$$

$$I_2(T)\to 0,\qquad I_3(T)\to 0.$$

Combining (3.12) with the above limits we obtain

(3.15)

$$\lim_{T\to+\infty} E\exp\{-\langle\tilde{X}_T, \Phi\rangle\} = \exp\left\{\frac{MK}{2}\int_0^1\int_0^1\int_{R^d}\int_{R^d}\Phi(x, t)\Phi(y, s)\,dxdyC_h(s, t)\,dsdt\right\};$$

hence the limit process $X_T$ is a Gaussian process with covariance (2.1).

**3.1.3.** Convergence proofs. $I_1(T)$ does not depend on $F$, so it can be evaluated in the same way as (3.32)–(3.34) in [7].

Let us now deal with $I_3(T)$. Using (3.9) we obtain

$$I_3(T) \leqslant \int_0^T\int_{R^d}\Psi_T(x, T-s)\int_0^s \mathcal{T}_u\Psi_T(\cdot, T-u)\,dudxds$$

$$\leqslant \frac{C}{F_T^2}\int_0^T\int_{R^d}\varphi(x)\int_0^s \mathcal{T}_u\varphi(x)\,dudxds.$$

Now the rest of the proof goes along the same lines as in [7].

We will turn to $I_2(T)$ which is a little more intricate. Combining (3.13) and property 4 from Fact 3.1 we get

$$I_2(T) = \int_0^T \int_{R^d} \left[\frac{M}{2}\left[v_T(\ldots)^2 - \Big(\int_0^s \mathcal{T}_u \Psi_T(\cdot,\, T+u-s)(x)\,du\Big)^2\right] + g\big(v_T(\ldots)\big)v_T(\ldots)^2\right] dxds$$
$$= \frac{M}{2} I_2'(T) + I_2''(T),$$

where

(3.16)
$$I_2'(T) = \int_0^T \int_{R^d} v_T(x,\, T-s,\, s)^2 - \Big(\int_0^s \mathcal{T}_u \Psi_T(\cdot,\, T+u-s)(x)\,du\Big)^2 dxds,$$
$$I_2''(T) = \int_0^T \int_{R^d} g\big(v_T(x,\, T-s,\, s)\big)v_T(x,\, T-s,\, s)^2\, dxds.$$

By inequality (3.9) we have

$$0 \leqslant -I_2'(T) = \int_0^T \int_{R^d} \left[\big(n_T(x,\, T-s,\, s)\big)^2 - \big(v_T(x,\, T-s,\, s)\big)^2\right].$$

Combining (3.8) and (3.3) yields

$$0 \leqslant n_T(x,\, T-s,\, s) - v_T(x,\, T-s,\, s)$$
$$= \int_0^s \mathcal{T}_{s-u}\left[\Psi_T(\cdot,\, T-u)v_T(\cdot,\, T-u,\, u) + VG\big(v_T(\cdot,\, T-u,\, u)\big)\right](x)\,du = (*).$$

We have $\mathcal{T}_s \Psi \geqslant 0$ for $\Psi \geqslant 0$, which is a direct consequence of the fact that $\mathcal{T}$ is the semigroup of a Markov process. By Fact 3.3 we have $c(F)$ such that

$$G(v) \leqslant \frac{c(F)}{2} v^2.$$

Hence

$$(*) \leqslant \int_0^s \mathcal{T}_{s-u}\left[\Psi_T(\cdot,\, T-u)v_T(\cdot,\, T-u,\, u) + c(F)\frac{V}{2} v_T(\cdot,\, T-u,\, u)^2\right](x)\,du$$
$$\leqslant \max\big(1,\, c(F)\big)\int_0^s \mathcal{T}_{s-u}\left[\Psi_T(\cdot,\, T-u)v_T(\cdot,\, T-u,\, u) + \frac{V}{2} v_T(\cdot,\, T-u,\, u)^2\right](x)\,du$$
$$\leqslant \max\big(1,\, c(F)\big)\int_0^s \mathcal{T}_{s-u}\left[\Psi_T(\cdot,\, T-u)n_T(\cdot,\, T-u,\, u) + \frac{V}{2} n_T(\cdot,\, T-u,\, u)^2\right](x)\,du.$$

Except of the constant $c(F)$ the last expression does not depend on $F$.

Next we consider

$$n_T(x,\, T-s,\, s) + v_T(x,\, T-s,\, s) \leqslant 2n_T(x,\, T-s,\, s) \leqslant 2\int_0^s \mathcal{T}_{s-u}\Psi(\cdot,\, T-u)(x)\,du.$$

The rest of the proof goes along the lines of the proof of inequalities (3.39)–(3.42) in [7], and hence we acquire $I'_2(T) \to 0$.

Before proving the convergence of $I''_2(T)$ we state two facts:

FACT 3.6. $n_T(x, T-s, s) \to 0$ *in uniformly* $x \in R^d$, $s \in [0, T]$, *as* $T \to +\infty$.

Proof. We have

$$\begin{aligned} n_T(x, T-s, s) &= \int_0^s \mathcal{T}_{s-u} \Psi_T(\cdot, T-u)\, du \\ &= \frac{1}{F_T} \int_0^s \mathcal{T}_{s-u} \varphi(x) \chi\left(\frac{T-u}{T}\right) du \\ &\leqslant \frac{C}{F_T} \int_0^{+\infty} \mathcal{T}_u \varphi(x)\, du = \frac{C_1}{F_T} \int_{R^d} \frac{\varphi(y)}{|x-y|^{d-\alpha}} dy \leqslant \frac{C_2}{F_T} \to 0. \end{aligned}$$

The last line contains the definition of the potential operator of the semigroup $\mathcal{T}_t$ which is bounded with respect to $x$ (this can be found in [13], Lemma 5.3). ∎

FACT 3.7. *The following convergence holds*:

$$\int_0^T \int_{R^d} v_T(x, T-s, s)^2 \to c'(\Psi) \quad \text{as } T \to +\infty.$$

Proof. One easily checks that

$$2\frac{I_1(T)}{M} + I'_2(T) = \int_0^T \int_{R^d} v_T(x, T-s, s)^2\, dx ds.$$

Hence the result follows from (3.14) and the convergence $I'_2(T) \to 0$ as $T \to 0$. ∎

It is now easy to prove the convergence of $I''_2$. From Fact 3.1, property 4, we know that for given $\varepsilon > 0$ we can choose $\delta$ such that, for all $x \in (-\delta, \delta)$, $|g(x)| \leqslant \varepsilon$. Fact 3.6 provides us with $T_0$ such that, for all $T \geqslant T_0$, $n_T(x, T-s, s) < \delta$. Combining this with (3.9) we obtain, for all $T \geqslant T_0$, $g\big(v_T(x, T-s, s)\big) \leqslant \varepsilon$. Hence for $T > T_0$ we get

$$|I''_2(T)| \leqslant \varepsilon \int_0^T \int_{R^d} v_T^2(x, T-s, s)\, dx ds \to \varepsilon c'(\Psi).$$

Since $\varepsilon$ was chosen arbitrary, we have the convergence $I''_2(T) \to 0$, and hence also $I_2(T) \to 0$ as $T \to +\infty$.

Thus we obtained the limits for $I_1, I_2$ and $I_3$ and the proof of Theorem 2.1 is completed.

### 3.2. Proof of Theorem 2.2

**3.2.1.** Tightness. We begin by claiming that the family $\{X_T\}_{T>0}$ is tight. Close examination of Section 3.1 in [7] reveals that only the covariance function of the underlying system is significant for the proof. By (3.16) in [4] we know that the covariance function of the branching system is of the same form as the covariance function of the non-branching system with the Poisson initial condition. From this we conclude that $X_T$ is tight.

**3.2.2.** The Laplace functional for $\tilde{X}_T$. We consider $\tilde{X}_T$ defined by (1.10). Using (1.3) and interchanging the order of integration we obtain

$$\langle \tilde{X}_T, \Phi\rangle = \frac{T}{F_T}\Big[\int_0^1 \langle N_{Ts}, \Psi(\cdot, s)\rangle ds - \langle \lambda, \int_0^1 \Psi(\cdot, s) ds\rangle\Big].$$

To prove the convergence of $\tilde{X}_T$ to $\tilde{X}$ we will use its Laplace functional

$$(3.17)\qquad E\exp\{-\langle \tilde{X}_T, \Phi\rangle\}$$
$$= \exp\Big\{\int_{R^d}\int_0^T \Psi_T(x, t)\,dt\,dx\Big\} E\exp\Big\{-\int_0^T \langle N_s, \Psi_T(\cdot, s)\rangle ds\Big\}.$$

It is easy to check that

$$(3.18)\qquad E\Big(\exp\Big\{-\int_0^T \langle N_s, \Psi_T(\cdot, s)\rangle ds\Big\}\,\Big|\, N_0 = \mu\Big) = \exp\{\langle \mu, \ln w_T\rangle\},$$

where

$$w_T(x) = E\exp\Big\{-\int_0^T \langle N_s^x, \Psi_T(\cdot, s)\rangle ds\Big\}.$$

Now we check that $0 \leqslant -\ln(w_T)$ is integrable. For $T$ big enough, by Fact 3.6 and inequality (3.9) we have $0 \leqslant v_T \leqslant c < 1$. Hence there exists a constant $C$ such that we have $-\ln(w_T) = -\ln(1-v_T) \leqslant Cv_T \leqslant Cn_T$. A trivial verification shows that $n_T \in \mathscr{L}^1(R^d)$, so by (1.1) and (3.18) we obtain

$$E\exp\Big\{-\int_0^T \langle N_s, \Psi_T(\cdot, s)\rangle ds\Big\} = E\Big(E\Big(\exp\Big\{-\int_0^T \langle N_s, \Psi_T(\cdot, s)\rangle ds\Big\}\,\Big|\, N_0\Big)\Big)$$
$$= \exp\Big\{\langle \lambda, w_T - 1\rangle + V\int_0^{+\infty} \langle \lambda, H(W_T(\cdot, s))\rangle ds\Big\},$$

where $W_T$ satisfies the equation

$$W_T(x, l) = \mathscr{T}_l w_T(x) + V\int_0^l \mathscr{T}_{l-s} H(W_T(\cdot, s))(x)\,ds.$$

It will be a bit easier to deal with $V_T(x, l) = 1 - W_T(x, l)$. The equations have the form (let us recall that $G$ is defined by (3.1))

(3.19)

$$E\exp\Big\{-\int_0^T \langle N_s, \Psi_T(\cdot, s)\rangle ds\Big\} = \exp\Big\{\langle \lambda, -v_T\rangle + V\int_0^{+\infty} \langle \lambda, G(V_T(\cdot, s))\rangle ds\Big\},$$

and

(3.20) $$V_T(x, l) = \mathcal{T}_l v_T(x) - V\int_0^l \mathcal{T}_{l-s} G\big(V_T(\cdot, s)\big)(x)\,ds,$$

$W_T$ is defined by (1.2) with $\varphi(x) = -\ln w_T(x)$ ($w_T \in [0, 1]$, hence $\varphi$ is positive). One can easily see that the definition implies that $W_T \in [0, 1]$. Consequently, $V_T \in [0, 1]$, which together with Fact 3.2 yields $G(V_T) \geqslant 0$. Hence we obtain the inequality

(3.21) $$V_T(x, l) \leqslant \mathcal{T}_l v_T(x) \quad \text{for all } x \in \boldsymbol{R}^d,\ l \geqslant 0.$$

Combining (3.17) and (3.19) we obtain

$$E\exp\{-\langle \tilde{X}_T, \Phi\rangle\} = \exp\Big\{\int_{\boldsymbol{R}^d}\int_0^T \Psi_T(x, t)\,dtdx\Big\}\exp\Big\{-\int_{\boldsymbol{R}^d} v_T(x)\,dx\Big\}$$
$$\times\exp\Big\{V\int_0^{+\infty}\int_{\boldsymbol{R}^d} G\big(V_T(x, t)\big)\,dxdt\Big\} = A(T)\cdot B(T),$$

where

$$A(T) = \exp\Big\{\int_{\boldsymbol{R}^d}\int_0^T \Psi_T(x, t)\,dtdx\Big\}\exp\Big\{-\int_{\boldsymbol{R}^d} v_T(x)\,dx\Big\},$$

$$B(T) = \exp\Big\{V\int_0^{+\infty}\int_{\boldsymbol{R}^d} G\big(V_T(x, t)\big)\,dxdt\Big\}.$$

Let us note that $A$ is the same as (3.11) in the first proof, hence we know that its limit is given by (3.15).

**3.2.3.** Limit of $B$. To complete the proof, the limit $\lim_{T\to+\infty} B(T)$ has to be calculated. It suffices to consider

(3.22) $$\int_0^{+\infty}\int_{\boldsymbol{R}^d} G\big(V_T(x, t)\big)\,dxdt.$$

Using Fact 3.1, property 4, we split it in the following way:

$$\int_0^{+\infty}\int_{\boldsymbol{R}^d} G\big(V_T(\cdot, t)\big)\,dxdt = \frac{M}{2}\big(B_1(T)+B_2(T)+B_3(T)\big)+B_4(T),$$

where

$$B_1(T) = \int_0^{+\infty}\int_{\boldsymbol{R}^d} V_T(x, t)^2 - \big(\mathcal{T}_t v_T(x)\big)^2\,dxdt,$$

$$B_2(T) = \int_0^{+\infty}\int_{\boldsymbol{R}^d} \big(\mathcal{T}_t v_T(x)\big)^2 - \big(\mathcal{T}_t n_T(x)\big)^2\,dxdt,$$

$$B_3(T) = \int_0^{+\infty} \int_{\mathbf{R}^d} \left(\mathcal{T}_t n_T(x)\right)^2 dxdt,$$

$$B_4(T) = \int_0^{+\infty} \int_{\mathbf{R}^d} g\left(V_T(x, t)\right) V_T(x, t)^2 dxdt.$$

We will prove the following limits (let us recall that we assume (3.2) for simplicity):

$$B_1(T) \to 0, \qquad B_2(T) \to 0,$$

$$B_3(T) \to \frac{K}{2V} \langle \lambda, \varphi \rangle^2 \int_0^1 \int_0^1 \{-u_1^h - u_2^h + (u_1+u_2)^h\} \psi(u_1) \psi(u_2) du_1 du_2,$$

$$B_4(T) \to 0,$$

as $T \to +\infty$.

Limit of $B_1$. By (3.21) we obtain

$$\begin{aligned} 0 \leqslant -B_1(T) &= \int_0^{+\infty} \left(\int_{\mathbf{R}^d} \left(\mathcal{T}_t v_T(x)\right)^2 - V_T(x, t)^2 dx\right) dt \\ &= \int_0^{+\infty} \int_{\mathbf{R}^d} \left(\mathcal{T}_t v_T(x) - V_T(x, t)\right)\left(\mathcal{T}_t v_T(x) + V_T(x, t)\right) dxdt. \end{aligned}$$

Combining this with inequality (3.21) and equation (3.20), we see that the last equality is not greater than

$$\int_0^{+\infty} \int_{\mathbf{R}^d} \left(V \int_0^t \mathcal{T}_{t-t'} G\left(V_T(\cdot, t')\right)(x) dt'\right)\left(2\mathcal{T}_t v_T(x)\right) dxdt.$$

Taking into account the form of $G$ (Fact 3.1, property 4) we infer that this expression is equal to

$$B_{11}(T) + B_{12}(T),$$

where

$$B_{11}(T) = \int_0^{+\infty} \int_{\mathbf{R}^d} \left(V \frac{M}{2} \int_0^t \mathcal{T}_{t-t'} V_T(\cdot, t')^2 (x) dt'\right)\left(2\mathcal{T}_t v_T(x)\right) dxdt,$$

$$B_{12}(T) = \int_0^{+\infty} \int_{\mathbf{R}^d} \left(V \int_0^t \mathcal{T}_{t-t'} g\left(V_T(\cdot, t')\right) V_T(\cdot, t')^2 (x) dt'\right)\left(2\mathcal{T}_t v_T(x)\right) dxdt.$$

Once again we use inequality (3.21) and obtain

$$\begin{aligned} B_{11}(T) &\leqslant VM \int_0^{+\infty} \int_{\mathbf{R}^d} \left(\int_0^t \mathcal{T}_{t-t'} \left(\mathcal{T}_{t'} v_T(\cdot)\right)^2 (x) dt'\right)\left(\mathcal{T}_t v_T(x)\right) dxdt \\ &= VM \int_0^{+\infty} \int_0^t \int_{\mathbf{R}^d} \mathcal{T}_{t-t'} \left(\mathcal{T}_{t'} v_T(\cdot)\right)^2 (x) \mathcal{T}_t v_T(x) dxdt' dt. \end{aligned}$$

Applying (3.9) twice we see that the last expression is not greater than

$$VM \int_0^{+\infty} \int_0^t \int_{\mathbf{R}^d} \mathcal{T}_{t-t'} \left(\mathcal{T}_{t'} n_T(\cdot)\right)^2 (x)\, \mathcal{T}_t n_T(x)\, dx dt' dt$$

$$= MV \int_0^{+\infty} \int_0^t \int_{\mathbf{R}^d} \mathcal{T}_{t'} n_T(x)\, \mathcal{T}_{t'} n_T(x)\, \mathcal{T}_{2t-t'} n_T(x)\, dx dt' dt.$$

Using the Plancherel formula and (1.4) we infer that the last form is equal to

$$\frac{MV}{(2\pi)^{2d}} \int_0^{+\infty} \int_0^t \int_{\mathbf{R}^{2d}} \widehat{\mathcal{T}_{t'} n_T}(z_1)\, \widehat{\mathcal{T}_{t'} n_T}(z_2)\, \overline{\widehat{\mathcal{T}_{2t-t'} n_T}}(z_1+z_2)\, dz_1\, dz_2\, dt' dt$$

$$= \frac{MV}{(2\pi)^{2d}} \int_0^{+\infty} \int_0^t \int_{\mathbf{R}^{2d}} \exp\{-t'|z_1|^\alpha\}\, \hat{n}_T(z_1) \exp\{-t'|z_2|^\alpha\}\, \hat{n}_T(z_2)$$

$$\times \exp\{-(2t-t')|z_1+z_2|^\alpha\}\, \overline{\hat{n}_T}(z_1+z_2)\, dz_1\, dz_2\, dt' dt$$

$$= \frac{MV}{(2\pi)^{2d}} \int_{\mathbf{R}^{2d}} \hat{n}_T(z_1)\, \hat{n}_T(z_2)\, \overline{\hat{n}_T}(z_1+z_2) \int_0^{+\infty} \int_0^t \exp\{-t'|z_1|^\alpha\}$$

$$\times \exp\{-t'|z_2|^\alpha\} \exp\{-(2t-t')|z_1+z_2|^\alpha\}\, dt' dt dz_1\, dz_2$$

$$= \frac{MV}{(2\pi)^{2d}} \int_{\mathbf{R}^{2d}} \frac{1}{2|z_1+z_2|^\alpha \left(|z_1|^\alpha + |z_2|^\alpha + |z_1+z_2|^\alpha\right)}$$

$$\times \hat{n}_T(z_1)\, \hat{n}_T(z_2)\, \overline{\hat{n}_T}(z_1+z_2)\, dz_1\, dz_2 = (*).$$

Before proceeding further we will estimate $\hat{n}_T$:

$$|\hat{n}_T(z, r, t)| = \left| \int_0^t \mathcal{T}_{t-s} \widehat{\Psi_T(\cdot, r+t} - s)\, ds\, (z) \right|$$

$$= \left| \frac{1}{F_T} \int_0^t \exp\{-(t-s)|z|^\alpha\}\, \hat{\varphi}(z)\, \chi_T(r+t-s)\, ds \right|$$

$$\leqslant \frac{\sup \chi}{F_T} |\hat{\varphi}(z)| \int_0^t \exp\{-(t-s)|z|^\alpha\}\, ds.$$

Hence

$$|\hat{n}_T(z, r, t)| \leqslant \frac{C}{F_T} \frac{|\hat{\varphi}(z)|}{|z|^\alpha} [1 - \exp\{-t|z|^\alpha\}] \tag{3.23}$$

and this immediately implies (see (3.7))

$$|\hat{n}_T(z)| \leqslant \frac{C}{F_T} \frac{1}{|z|^\alpha} [1 - \exp\{-T|z|^\alpha\}]. \tag{3.24}$$

Here, and in what follows, $C$ denotes a generic constant. Coming back to $(*)$ and using the last inequality we obtain

$$|(*)| \leqslant \frac{C}{F_T^3} \int_{\mathbf{R}^{2d}} \frac{1}{2|z_1+z_2|^\alpha (|z_1|^\alpha + |z_2|^\alpha + |z_1+z_2|^\alpha)} \frac{1}{|z_1|^\alpha}[1-\exp\{-T|z_1|^\alpha\}]$$
$$\times \frac{1}{|z_2|^\alpha}[1-\exp\{-T|z_2|^\alpha\}] \frac{1}{|z_1+z_2|^\alpha}[1-\exp\{-T|z_1+z_2|^\alpha\}]\, dz_1\, dz_2,$$

which after substituting $T^{1/\alpha} z_1 = y_1$ and $T^{1/\alpha} z_2 = y_2$ yields

$$\frac{CT^5}{F_T^3\, T^{2d/\alpha}} \int_{\mathbf{R}^{2d}} \frac{1}{|y_1+y_2|^\alpha (|y_1|^\alpha + |y_2|^\alpha + |y_1+y_2|^\alpha)} \frac{1}{|y_1|^\alpha}[1-\exp\{-|y_1|^\alpha\}]$$
$$\times \frac{1}{|y_2|^\alpha}[1-\exp\{-|y_2|^\alpha\}] \frac{1}{|y_1+y_2|^\alpha}[1-\exp\{-|y_1+y_2|^\alpha\}]\, dy_1\, dy_2$$
$$\leqslant B'_{11}(T) \cdot B''_{11},$$

where

$$B'_{11}(T) = \frac{C' T^5}{F_T^3\, T^{2d/\alpha}},$$

$$B''_{11} = \int_{\mathbf{R}^{2d}} \frac{1}{|y_1+y_2|^\alpha (|y_1|^\alpha + |y_2|^\alpha + |y_1+y_2|^\alpha)} \frac{1}{|y_1|^\alpha}[1-\exp\{-|y_1|^\alpha\}]$$
$$\times \frac{1}{|y_2|^\alpha}[1-\exp\{-|y_2|^\alpha\}] \frac{1}{|y_1+y_2|^\alpha}[1-\exp\{-|y_1+y_2|^\alpha\}]\, dy_1\, dy_2.$$

The integral $B''_{11}$ is finite, which will be proved in the Appendix. The expression $B'_{11}(T)$ can be evaluated as follows:

$$B'_{11}(T) = T^{[10-3(3-d/\alpha)-4d/\alpha]/2} = T^{(1-d/\alpha)/2}.$$

As $1-d/\alpha < 0$, we get $B'_{11}(T) \to 0$; hence also $B_{11}(T) \to 0$.

From Fact 3.6 and inequalities (3.9) and (3.21) we know that $V_T(x, l) \to 0$ uniformly as $T \to 0$, and so $g(V_T(x, l)) \leqslant \varepsilon$ for $T$ sufficiently large. Hence

$$B_{12}(T) \leqslant \varepsilon \int_0^{+\infty} \int_{\mathbf{R}^d} \Big(V \int_0^t \mathcal{T}_{t-t'} V_T(\cdot, t')^2(x)\, dt'\Big)\big(2\mathcal{T}_t v_T(x)\big)\, dx dt \leqslant \frac{2\varepsilon}{M} B_{11}(T).$$

Thus $B_{12}(T) \to 0$ and also $B_1(T) \to 0$.

Limit of $B_2$. Let us first estimate the expression $n_T - v_T$ using (3.8) and (3.3):

$$n_T(x) - v_T(x) = \int_0^T \mathcal{T}_{T-u} \Psi_T(\cdot, T-u)(x)\, du$$
$$-\int_0^T \mathcal{T}_{T-u} \big[\Psi_T(\cdot, T-u)\big(1-v_T(\cdot, T-u, u)\big) - VG\big(v_T(\cdot, T-u, u)\big)\big](x)\, du,$$

$$n_T(x) - v_T(x)$$

$$= \int_0^T \mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, v_T(\cdot, T-u, u) + VG\left(v_T(\cdot, T-u, u)\right)\right](x)\, du.$$

Applying Fact 3.3 we see that the last expression is not greater than

$$\int_0^T \mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, v_T(\cdot, T-u, u) + Vc\left(v_T(\cdot, T-u, u)\right)^2\right](x)\, du,$$

where $c$ is a constant. By inequality (3.9) we get

(3.25) $$n_T(x) - v_T(x)$$

$$\leqslant \int_0^T \mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, n_T(\cdot, T-u, u) + Vc\left(n_T(\cdot, T-u, u)\right)^2\right](x)\, du.$$

We have

$$0 \leqslant -B_2(T) = \int_0^{+\infty} \Big( \int_{\mathbf{R}^d} \left(\mathcal{T}_t n_T(x)\right)^2 - \left(\mathcal{T}_t v_T(x)\right)^2 dx \Big)\, dt$$

$$= \int_0^{+\infty} \int_{\mathbf{R}^d} \Big(\mathcal{T}_t\big(n_T(\cdot) - v_T(\cdot)\big)(x)\Big)\Big(\mathcal{T}_t\big(v_T(\cdot) + n_T(\cdot)\big)(x)\Big)\, dx dt.$$

Applying (3.9) and (3.25) we infer that the last form is not greater than

$$2 \int_0^{+\infty} \int_{\mathbf{R}^d} \mathcal{T}_t b(x)\, \mathcal{T}_t n_T(x)\, dx dt,$$

where

$$b_T(x) = \int_0^T \mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, n_T(\cdot, T-u, u) + Vc\left(n_T(\cdot, T-u, u)\right)^2\right] du.$$

Now, applying the Plancherel formula, then interchanging the order of integration and integrating with respect to $t$, we get

$$\frac{2}{(2\pi)^d} \int_0^{+\infty} \int_{\mathbf{R}^d} \exp\{-2t|z|^\alpha\}\, \hat{b}_T(z)\, \hat{n}_T(z)\, dz dt$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{1}{|z|^\alpha} \hat{b}_T\, \hat{n}_T(z)\, dz = c'\left(B_{21}(T) + B_{22}(T)\right),$$

where

$$B_{21}(T) = \int_{\mathbf{R}^d} \frac{1}{|z|^\alpha} \Big\{ \int_0^T \widehat{\mathcal{T}_{T-u}\left[\Psi_T(\cdot, T-u)\, n_T(\cdot, T-u, u)\right]}(z)\, du \Big\}\, \hat{n}_T(z)\, dz,$$

$$B_{22}(T) = \int_{\mathbf{R}^d} \frac{1}{|z|^\alpha} \Big\{ \int_0^T \widehat{\mathcal{T}_{T-u}\left[Vc\left(n_T(\cdot, T-u, u)\right)^2(\cdot)\right]}(z)\, du \Big\}\, \hat{n}_T(z)\, dz.$$

First we shall compute $\lim_{T\to+\infty} B_{21}(T)$. We have

$$B_{21}(T) = \int_{\mathbf{R}^d} \frac{1}{|z|^\alpha}\{\int_0^T \exp\{-(T-u)|z|^\alpha\}\, \widehat{\Psi_T(\cdot, T-u) * n_T(\cdot, T-u, u)}(z)\}\, \hat{n}_T(z)\, dz.$$

The inner convolution can be estimated using the inequality (3.23) and simplification (3.2):

$$\begin{aligned}|\widehat{\Psi_T(\cdot, T-u) * n_T(\cdot, T-u, u)}(z)| &= |\chi_T(T-u)\, \widehat{\varphi_T(\cdot) * n_T(\cdot, T-u, u)}(z)| \\ &= \left|\chi_T(T-u) \int_{\mathbf{R}^d} \hat{\varphi}_T(z-x)\, \hat{n}_T(x, T-u, u)\, dx\right| \\ &\leqslant \frac{c(\chi)}{F_T^2}\chi_T(T-u) \int_{\mathbf{R}^d} |\hat{\varphi}(z-x)\, \hat{\varphi}(x)| \frac{1}{|x|^\alpha}\, dx \leqslant \frac{C}{F_T^2}.\end{aligned}$$

In the last inequality we use the fact that $\hat{\varphi}$ is bounded and $\hat{\varphi}(x)/|x|^\alpha$ is integrable. Hence we have

$$|\widehat{\Psi_T(\cdot, T-u) * n_T(\cdot, T-u, u)}(z)| \leqslant \frac{C}{F_T^2}. \tag{3.26}$$

Thus $B_{21}$ satisfies

$$|B_{21}(T)| \leqslant \frac{C}{F_T^2} \int_{\mathbf{R}^d} \frac{1}{|z|^\alpha} \int_0^T \exp\{-(T-u)|z|^\alpha\}\, du \cdot \hat{n}_T(z)\, dz.$$

Using (3.24) and integrating with respect to $u$ we see that the right-hand side of this inequality is not greater than

$$C' \frac{1}{F_T^3} \int_{\mathbf{R}^d} \frac{1}{|z|^\alpha} \frac{1}{|z|^\alpha} [1 - \exp\{-T|z|^\alpha\}] \frac{1}{|z|^\alpha} [1 - \exp\{-T|z|^\alpha\}]\, dz.$$

Substituting $zT^{1/\alpha} = y$ we infer that this expression equals

$$C' \frac{T^3}{F_T^3\, T^{d/\alpha}} \int_{\mathbf{R}^d} \frac{1}{|y|^\alpha} \frac{1}{|y|^\alpha} [1 - \exp\{-|y|^\alpha\}] \frac{1}{|y|^\alpha} [1 - \exp\{-|y|^\alpha\}]\, dy \leqslant B'_{21}(T) \cdot B''_{21},$$

where

$$B'_{21}(T) = C'' \frac{T^3}{F_T^3\, T^{d/\alpha}},$$

$$B''_{21} = \int_{\mathbf{R}^d} \frac{1}{|y|^\alpha} \frac{1}{|y|^\alpha} [1 - \exp\{-|y|^\alpha\}] \frac{1}{|y|^\alpha} [1 - \exp\{-|y|^\alpha\}]\, dy.$$

It is clear that the integral $B''_{21}$ in the last expression is finite since in a neighborhood of 0 the integrated expression is proportional to $1/|y|^\alpha$ and it is $O(1/|y|^{3\alpha})$ as $|y| \to +\infty$ (recall that $\alpha < d < 2\alpha$). Now only $B'_{21}$ needs to be evaluated

$$B'_{21}(T) = C''\,T^{[6-3(3-d/\alpha)-2d/\alpha]/2} = C''\,T^{(-3+d/\alpha)/2}.$$

Hence it is obvious that $B'_{21}(T) \to 0$ as $T \to 0$, and so $\lim_{T\to 0} B_{21}(T) = 0$.

Before proceeding to $B_{22}$ we will make the following estimation using the inequality (3.23):

$$\left|\widehat{(n_T(\cdot,\,T-u,\,u))^2}\right|(z) = \left|\int_{\boldsymbol{R}^d} \hat{n}_T(x,\,T-u,\,u)\,\hat{n}_T(z-x,\,T-u,\,u)\,dx\right|$$

$$\leqslant \frac{C}{F_T^2}\int_{\boldsymbol{R}^d} \frac{1}{|x|^\alpha}[1-\exp\{-u\,|x|^\alpha\}]\,\frac{1}{|z-x|^\alpha}[1-\exp\{-u\,|z-x|^\alpha\}]\,dx.$$

Substitution $xu^{1/\alpha} = y$ yields that the last expression is not greater than

$$u^{2-d/\alpha}\frac{C}{F_T^2}\int_{\boldsymbol{R}^d}\frac{1}{|y|^\alpha}[1-\exp\{-\,|y|^\alpha\}]\,\frac{1}{|zu^{1/\alpha}-y|^\alpha}[1-\exp\{-|zu^{1/\alpha}-y|^\alpha\}]\,dy \leqslant \frac{C'}{F_T^2}u^{2-d/\alpha},$$

since the integral can be regarded as a convolution of $\mathscr{L}^2$ functions, so it is bounded. This clearly implies

$$|B_{22}(T)| \leqslant \frac{C'}{F_T^2}\int_{\boldsymbol{R}^d}\frac{1}{|z|^\alpha}\int_0^T \exp\{-(T-u)\,|z|^\alpha\}\,u^{2-d/\alpha}\,du\cdot|\hat{n}_T(z)|\,dz$$

$$\leqslant C'\frac{T^{2-d/\alpha}}{F_T^2}\int_{\boldsymbol{R}^d}\frac{1}{|z|^\alpha}\int_0^T \exp\{-(T-u)\,|z|^\alpha\}\,du\cdot|\hat{n}_T(z)|\,dz.$$

By (3.24) the last expression is not greater than

$$C''\frac{T^{2-d/\alpha}}{F_T^3}\int_{\boldsymbol{R}^d}\frac{1}{|z|^\alpha}\frac{1}{|z|^\alpha}(1-\exp\{-T\,|z|^\alpha\})\frac{1}{|z|^\alpha}(1-\exp\{-T\,|z|^\alpha\})\,dz,$$

which after substituting $zT^{1/\alpha} = y$ can be rewritten in the form

$$C''\frac{T^{5-d/\alpha}}{F_T^3\,T^{d/\alpha}}\int_{\boldsymbol{R}^d}\frac{1}{|y|^\alpha}\frac{1}{|y|^\alpha}(1-\exp\{-\,|y|^\alpha\})\frac{1}{|y|^\alpha}(1-\exp\{-\,|y|^\alpha\})\,dy.$$

The integral is finite (the same proof as for $B''_{21}$) and

$$\frac{T^{5-d/\alpha}}{F_T^3\,T^{d/\alpha}} = T^{[10-2d/\alpha-3(3-d/\alpha)-2d/\alpha]/2} = T^{(1-d/\alpha)/2},$$

which yields $B_{22}(T) \to 0$ as $T \to +\infty$.

Limit of $B_3$. Applying the Plancherel formula to $B_3(T)$ we get

$$B_3(T) = \frac{1}{(2\pi)^d}\int_0^\infty\int_{\boldsymbol{R}^d}\exp\{-2t\,|z|^\alpha\}\,(\hat{n}_T(z))^2\,dz\,dt$$

$$= \frac{1}{(2\pi)^d}\int_{\boldsymbol{R}^d}(\hat{n}_T(z))^2\int_0^\infty\exp\{-2t\,|z|^\alpha\}\,dt\,dz = \frac{1}{2(2\pi)^d}\int_{\boldsymbol{R}^d}\frac{1}{|z|^\alpha}(\hat{n}_T(z))^2\,dz$$

$$= \frac{1}{2(2\pi)^d} \int_{R^d} \frac{1}{|z|^\alpha} \Big(\int_0^T \exp\{-(T-u)|z|^\alpha\} \hat{\varphi}_T(z) \chi_T(T-u)\,du\Big)^2 dz$$

$$= \frac{1}{2(2\pi)^d} \frac{1}{F_T^2} \int_{R^d} \frac{1}{|z|^\alpha} \Big(\int_0^T \exp\{-u|z|^\alpha\} \hat{\varphi}(z) \chi_T(u)\,du\Big)^2 dz.$$

Substituting $u' = u/T$ we obtain the last expression in the form

$$\frac{1}{2(2\pi)^d} \frac{T^2}{F_T^2} \int_{R^d} \frac{1}{|z|^\alpha} \Big(\int_0^1 \exp\{-Tu'|z|^\alpha\} \hat{\varphi}(z) \chi(u')\,du'\Big)^2 dz$$

$$= \frac{1}{2(2\pi)^d} \frac{T^2}{F_T^2} \int_0^1 \int_0^1 \int_{R^d} \frac{1}{|z|^\alpha} \exp\{-T(u_1+u_2)|z|^\alpha\} \big(\hat{\varphi}(z)\big)^2 \chi(u_1)\chi(u_2)\,du_1\,du_2\,dz.$$

Let $z = [T(u_1+u_2)]^{-1/\alpha} y$. Then the last expression takes the form

$$\frac{1}{2(2\pi)^d} \frac{T^{3-d/\alpha}}{F_T^2} \int_0^1 \int_0^1 \int_{R^d} (u_1+u_2) \frac{1}{|y|^\alpha} \exp\{-|y|^\alpha\} \big(\hat{\varphi}([T(u_1+u_2)]^{-1/\alpha} y)\big)^2$$
$$\times (u_1+u_2)^{-d/\alpha} \chi(u_1)\chi(u_2)\,du_1\,du_2\,dy.$$

Therefore, by the Lebesgue dominated convergence theorem and integration by parts, we obtain the limit of $B_3(T)$:

$$\lim_{T\to+\infty} B_3(T) = \frac{1}{2(2\pi)^d} \int_0^1 \int_0^1 \int_{R^d} (u_1+u_2)^{1-d/\alpha} \frac{1}{|y|^\alpha} \exp\{-|y|^\alpha\} \big(\hat{\varphi}(0)\big)^2 \chi(u_1)\chi(u_2)\,du_1\,du_2\,dy$$
$$= \frac{\Gamma(d/\alpha-1)}{2^d \alpha \Gamma(d/2)\pi^{d/2}} \langle\lambda, \varphi\rangle^2 \int_0^1 \int_0^1 (u_1+u_2)^{1-d/\alpha} \chi(u_1)\chi(u_2)\,du_1\,du_2$$
$$= \frac{K}{2V} \langle\lambda, \varphi\rangle^2 \int_0^1 \int_0^1 \{-u_1^h - u_2^h + (u_1+u_2)^h\} \psi(u_1)\psi(u_2)\,du_1\,du_2.$$

Limit of $B_4$. Firstly, let us notice that

$$B_1(T)+B_2(T)+B_3(T) = \int_0^{+\infty} \int_{R^d} V_T(x,t)^2\,dxdt,$$

and hence

$$\int_0^{+\infty} \int_{R^d} V_T(x,t)^2\,dxdt \to C \quad \text{as } T\to+\infty.$$

Secondly, by Fact 3.6 and the inequalities (3.21) and (3.9) we know that $V_T(x)\to 0$ uniformly as $T\to 0$. Hence $g(W_T(x)) \leqslant \varepsilon$ for $T$ sufficiently large, so

$$|B_4(T)| \leqslant \varepsilon \int_0^{+\infty} \int_{R^d} V_T(x,t)^2,$$

which clearly implies that $B_4(T)\to 0$ as $T\to+\infty$.

Putting the results together. Combining the previous results we conclude that

$$\lim_{T\to+\infty} B(T) = \exp\left\{\frac{MK}{4}\langle\lambda, \varphi\rangle^2 \int_0^1\int_0^1 \{-u_1^h - u_2^h + (u_1+u_2)^h\}\,\psi(u_1)\psi(u_2)\,du_1\,du_2\right\}$$

and finally, by (3.15),

$$\lim_{T\to+\infty} A(T)B(T) = \exp\left\{\frac{MK}{4}\langle\lambda, \varphi\rangle^2 \int_0^1\int_0^1 c_h(u_1, u_2)\psi(u_1)\psi(u_2)\,du_1\,du_2\right\},$$

where $c_h$ is the covariance function of the fractional Brownian motion defined by (1.9). This Laplace functional defines a process $\tilde{X}_T$ corresponding to the Gaussian process $X_T$ with the covariance (2.2), and hence Theorem 2.2 is proved.

## 4. APPENDIX

The appendix contains a technical fact used in the main proof.

FACT 4.1. *We have*

$$\int_{\mathbf{R}^{2d}} \frac{1}{|y_1+y_2|^\alpha(|y_1|^\alpha+|y_2|^\alpha+|y_1+y_2|^\alpha)}\frac{1}{|y_1|^\alpha}[1-\exp\{-|y_1|^\alpha\}]$$

$$\times\frac{1}{|y_2|^\alpha}[1-\exp\{-|y_2|^\alpha\}]\frac{1}{|y_1+y_2|^\alpha}[1-\exp\{-|y_1+y_2|^\alpha\}]\,dy_1\,dy_2 < +\infty.$$

Proof. Substituting $x = y_1+y_2$ and $z = y_2$ we get

$$\int_{\mathbf{R}^{2d}} \frac{1}{|x|^\alpha(|x|^\alpha+|z|^\alpha+|x-z|^\alpha)}\frac{1}{|x-z|^\alpha}[1-\exp\{-|x-z|^\alpha\}]$$

$$\times\frac{1}{|z|^\alpha}[1-\exp\{-|z|^\alpha\}]\frac{1}{|x|^\alpha}[1-\exp\{-|x|^\alpha\}]\,dxdz$$

$$= \int_{\mathbf{R}^{2d}} \frac{1}{|x|^\alpha}\frac{1}{|x|^\alpha}[1-\exp\{-|x|^\alpha\}]\frac{1}{|x|^\alpha+|z|^\alpha+|x-z|^\alpha}\frac{1}{|x-z|^\alpha}$$

$$\times[1-\exp\{-|x-z|^\alpha\}]\frac{1}{|z|^\alpha}[1-\exp\{-|z|^\alpha\}]\,dzdx = (*).$$

Let us investigate now

$$\int_{\mathbf{R}^{2d}} \frac{1}{|x|^\alpha+|z|^\alpha+|x-z|^\alpha}\frac{1}{|x-z|^\alpha}[1-\exp\{-|x-z|^\alpha\}]\frac{1}{|z|^\alpha}[1-\exp\{-|z|^\alpha\}]\,dz$$

$$\leqslant \int_{\mathbf{R}^d} \frac{1}{|z|^\alpha} \frac{1}{|z|^\alpha} [1-\exp\{-|z|^\alpha\}] \frac{1}{|x-z|^\alpha} [1-\exp\{-|x-z|^\alpha\}]\, dz$$

$$\leqslant c \int_{\mathbf{R}^d} \frac{1}{|z|^\alpha} \frac{1}{|z|^\alpha} [1-\exp\{-|z|^\alpha\}]\, dz.$$

The last integral is finite since in the neighborhood of 0 the integrated function is $O(1/|z|^\alpha)$ and for big $|z|$ is $O(1/|z|^{2\alpha})$. Going back to (*) we obtain

$$(*) \leqslant c_2 \int_{\mathbf{R}^d} \frac{1}{|x|^\alpha} \frac{1}{|x|^\alpha} [1-\exp\{-|x|^\alpha\}] < c_3,$$

by the same reason as above. ∎

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