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# CONVERGENCE RATE IN CLT FOR VECTOR-VALUED RANDOM FIELDS WITH SELF-NORMALIZATION

BY

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*To the memory of Professor Kazimierz Urbanik*

*Abstract.* Statistical version of the central limit theorem (CLT) with random matrix normalization is established for random fields with values in a space $\boldsymbol{R}^k$ ($k \geqslant 1$). Dependence structure of the field under consideration is described in terms of the covariance inequalities for the class of bounded Lipschitz "test functions" defined on finite disjoint collections of random vectors constituting the field. The main result provides an estimate of the convergence rate, over a family of convex bounded sets, in the CLT with random normalization.

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## 1. INTRODUCTION AND MAIN RESULTS

It is worthwhile to recall, as a motivation, the simple situation when $X, X_1, X_2, \ldots$ are i.i.d. random variables defined on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ with $\mathsf{E}X = a \in \boldsymbol{R}$ and $\operatorname{var} X = \sigma^2 \in \boldsymbol{R}_+$. Then the partial sums $S_n = X_1 + \ldots + X_n$, $n \in \boldsymbol{N}$, are asymptotically normal, i.e.

$$n^{-1/2}(S_n - na) \xrightarrow{D} Y \quad \text{as } n \to \infty, \tag{1.1}$$

where "$\xrightarrow{D}$" stands for weak convergence of distributions of random variables (vectors) and $Y \sim N(0, \sigma^2)$. Clearly, if $\sigma \neq 0$, (1.1) can be written in the usual form of the CLT:

$$\frac{S_n - na}{\sigma\sqrt{n}} \xrightarrow{D} Z \quad \text{as } n \to \infty,$$

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and $Z \sim N(0, 1)$. Hence, in this case, for any sequence of random variables $\hat{\sigma}_n \geqslant 0$ such that $\hat{\sigma}_n \xrightarrow{P} \sigma$ (i.e. converges in probability as $n \to \infty$), we conclude (stipulating that a fraction equals 0 if its denominator is 0) that

$$\frac{S_n - na}{\hat{\sigma}_n \sqrt{n}} \xrightarrow{D} Z \quad \text{as } n \to \infty. \tag{1.2}$$

The relation (1.2) provides a possibility to construct an approximate confidence interval for unknown mean value $a$ using consistent statistics $\hat{\sigma}_n = \hat{\sigma}_n(X_1, \ldots, X_n)$, $n \in N$. One usually employs the so-called *studentization*, namely, the empirical variance

$$\hat{\sigma}_n^2 = n^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2, \quad n \in N, \tag{1.3}$$

where the empirical mean $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$.

There are various generalizations of this approach. Note that for i.i.d. random vectors the situation is more involved. Let now $X, X_1, X_2, \ldots$ be i.i.d. random vectors with values in $\boldsymbol{R}^k$ such that the inner product $(v, X)$ is not a.s. constant for each $v$ belonging to the unit sphere $S^{k-1}$ in $\boldsymbol{R}^k$ (one says, respectively, that $X$ is *full*). Assume that there exist matrices $A_n$ and vectors $\boldsymbol{b}_n$ (both nonrandom) such that

$$A_n(S_n - \boldsymbol{b}_n) \xrightarrow{D} N(0, \boldsymbol{I}) \quad \text{as } n \to \infty,$$

for $S_n = \sum_{i=1}^{n} X_i$, where $\boldsymbol{I}$ is the unit $(k \times k)$-matrix. Then one writes $X \in$ GDOAN (generalized domain of attraction of the normal law). The analytic properties of GDOAN were studied in [20], [22] and [27]. In particular, if $X \in$ GDOAN, then $\mathsf{E}X$ exists and $A_n$ can be taken symmetric, non-singular and $\boldsymbol{b}_n = n\mathsf{E}X$. For the vector-valued case set also $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$ and

$$\bar{C}_n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)(X_i - \bar{X}_n)^*, \quad n \in N,$$

where "$*$" stands for the transposition of a vector. Thus $\bar{C}_n$ is the sample covariance matrix of $X_1, \ldots, X_n$, $n \in N$. If $X$ is full, then, due to [22], $\bar{C}_n$ is non-singular on event $D_n$, $n \in N$, with $\mathsf{P}(D_n) \to 1$ as $n \to \infty$. Therefore, one can introduce the statistics

$$\bar{T}_n = n^{1/2} \bar{C}_n^{-1/2} (S_n - n\mathsf{E}X), \quad n \in N, \tag{1.4}$$

where $\bar{C}_n^{-1/2}(\omega)$ means the zero matrix if $\omega \notin D_n$.

In [23] and [27] it was established that if $X \in$ GDOAN, then

$$\bar{T}_n \xrightarrow{D} N(0, \boldsymbol{I}) \quad \text{as } n \to \infty. \tag{1.5}$$

Giné et al. [19] proved the converse of this statement for real-valued random variables, i.e. for $k = 1$. In [18] the converse result was obtained for symmetric random vectors $X, X_1, X_2, \ldots$ with values in $\boldsymbol{R}^k$, $k \geqslant 1$.

For dependent random variables (or vectors in $\boldsymbol{R}^k$) new difficulties arise and statistics of other type than $\hat{\sigma}_n$ (or $\bar{C}_n$) appear. We are going to study a strictly stationary random field $\{X_j; j \in \boldsymbol{Z}^d\}$ with values in $\boldsymbol{R}^k$ ($k \geqslant 1$), having the dependence structure described by means of appropriate bounds for $|\mathrm{cov}(f(X_i, i \in I), g(X_j, j \in J))|$. Here $I$ and $J$ are disjoint finite subsets of $\boldsymbol{Z}^d$ and functions $f\colon \boldsymbol{R}^{k|I|} \to \boldsymbol{R}$ and $g\colon \boldsymbol{R}^{k|J|} \to \boldsymbol{R}$ belong to certain classes of "test functions", whereas $|I|$ stands for the cardinality of $I$. The aim of this paper is twofold, to provide for such fields an analogue of CLT with random matrix normalization and to estimate the convergence rate to the normal law as well. As far as we know, this is a first analysis of accuracy of normal approximation in such a setting, even for $k = 1$.

To clarify the dependence conditions we recall some basic concepts. In 1967 Esary, Proschan and Walkup [16] proposed the following

DEFINITION 1. A family $\{Y_t; t \in T\}$ of real-valued random variables is called *associated* if, for arbitrary finite sets $I, J \subset T$ and any coordinate-wise nondecreasing bounded functions $f\colon \boldsymbol{R}^{|I|} \to \boldsymbol{R}$ and $g\colon \boldsymbol{R}^{|J|} \to \boldsymbol{R}$, it follows that

$$\mathrm{cov}(f(X_s, s \in I), g(X_t, t \in J)) \geqslant 0. \tag{1.6}$$

The notation $f(X_s, s \in I)$ means that one can use $f(\vec{X}_I)$, where $\vec{X}_I$ is any vector obtained by ordering a collection of random variables $\{X_s; s \in I\}$.

Evidently, one can assume that $I = J$ in (1.6) (considering instead of $I$ and $J$ the set $I \cup J$). There are various modifications of this definition. For instance, Newman [25] introduced the notion of positive association (PA) imposing in (1.6) the complementary condition $I \cap J = \emptyset$. If, following Joag-Dev and Proschan [21], we suppose that, for any finite disjoint $I, J \subset T$ and any functions $f$, $g$ belonging to the class $BL$ ($BL$ stands for the collection of all bounded Lipschitz functions), the inequality (1.6) holds with opposite sign, then it leads to the so-called negative association (NA).

There are a number of important examples of PA or NA random systems. The main sources of interest here are percolation theory, statistical physics, mathematical statistics and reliability theory. One can refer to the pioneering papers by Harris, Lehmann, Fortuin, Kasteleyn and Ginibre; see also the book [10] and references therein.

Bulinski and Shabanovich [8] proved that if $\mathsf{E}X_t^2 < \infty$, $t \in T$, then either of PA and NA properties implies, for any bounded Lipschitz functions $f\colon \boldsymbol{R}^{|I|} \to \boldsymbol{R}$ and $g\colon \boldsymbol{R}^{|J|} \to \boldsymbol{R}$ (finite $I, J \subset T$, $I \cap J = \emptyset$), that

$$|\mathrm{cov}(f(X_s, s \in I), g(X_t, t \in J))| \leqslant \mathrm{Lip}(f)\,\mathrm{Lip}(g) \sum_{s \in I} \sum_{t \in J} |\mathrm{cov}(X_s, X_t)|, \tag{1.7}$$

where

$$\mathrm{Lip}(f) = \sup_{x \neq y} \{|f(x)-f(y)|/\|x-y\|_1\}, \quad \|x\|_1 = \sum_{m=1}^{|I|} |x_m|, \quad x \in \mathbf{R}^{|I|}.$$

Thus it is natural to introduce the next

DEFINITION 2. A random field $\{X_j; j \in \mathbf{Z}^d\}$ with values in $\mathbf{R}^k$ is called *$(BL, \theta)$-dependent* if there exists a sequence $\theta = (\theta_r)_{r \in \mathbf{N}}$ such that $\theta_r \searrow 0$, $r \to \infty$, and for arbitrary finite disjoint $I, J \subset \mathbf{Z}^d$ and any bounded Lipschitz functions $f: \mathbf{R}^{k|I|} \to \mathbf{R}$, $g: \mathbf{R}^{k|J|} \to \mathbf{R}$ it follows that

$$|\mathrm{cov}(f(X_i, i \in I), g(X_j, j \in J))| \leqslant \mathrm{Lip}(f)\,\mathrm{Lip}(g)(|I| \wedge |J|)\,\theta_{\mathrm{dist}(I,J)}, \tag{1.8}$$

where

$$\mathrm{dist}(I, J) = \inf\{\|i-j\|; i \in I, j \in J\} \quad \text{and} \quad \|t\| = \max_{1 \leqslant m \leqslant d} |t_m| \text{ for } t \in \mathbf{R}^d.$$

Obviously, any family of independent random vectors automatically satisfies (1.8) with any choice of a sequence $(\theta_r)_{r \in \mathbf{N}}$.

For stochastic processes ($d = 1$) this definition can be found in [15], for random fields in papers [5], [12] and [14]. Note that for PA or NA random field $\{X_j; j \in \mathbf{Z}^d\}$ inequality (1.7) enables us to use as $\theta_r$ the classical Cox–Grimmett coefficient

$$u_r = \sup_{s \in \mathbf{Z}^d} \sum_{t \in \mathbf{Z}^d, \|s-t\| \geqslant r} |\mathrm{cov}(X_s, X_t)|, \quad r \in N, \tag{1.9}$$

provided that $u_1 < \infty$. One can prove (see [10]) that, for a wide-sense stationary random field $\{X_j; j \in \mathbf{Z}^d\}$ with values in $\mathbf{R}^k$, the $(BL, \theta)$-dependence condition implies the following relations:

$$\sigma_{r,q} = \sum_{j \in \mathbf{Z}^d} |\mathrm{cov}(X_{0,r}, X_{j,q})| < \infty, \quad r, q = 1, \ldots, k. \tag{1.10}$$

We mention in passing that there exist other possibilities to choose the "test functions" $f$ and $g$ and other factors than $|I| \wedge |J|$ in (1.8); see, e.g., [1] and [15].

Now we return to the self-normalization problem. Peligrad and Shao [26] proposed two choices of statistics $\hat{\sigma}_n$ for mixing stochastic processes. Bulinski and Vronski [13] introduced for the associated random field a family of statistics comprising two above-mentioned choices. The vector-valued random fields and the corresponding (random) matrix normalization were studied in [6]. The problem of using self-normalization for the real-valued mixing random field is discussed in [7] in the context of radiobiology. In particular, one can employ the present stochastic model to describe the dependent "functional subunits" for irradiated organs or tissues.

For a finite set $U \subset \mathbf{Z}^d$ let us define

$$\partial U = \{s \in U: \inf_{t \in \mathbf{Z}^d \setminus U} \|s-t\| = 1\}.$$

DEFINITION 3. A sequence of finite sets $U_n \subset \mathbf{Z}^d$ ($n \in \mathbf{N}$) is called *regularly growing* (to infinity) if $|U_n| \to \infty$ and $|\partial U_n|/|U_n| \to 0$ as $n \to \infty$.

This notion is a discrete analogue of the concept of a family of sets in $\mathbf{R}^d$ *growing in the van Hove sense.*

Set $S(U_n) = \sum_{j \in U_n} X_j$, $U_n \subset \mathbf{Z}^d$, $|U_n| < \infty$, $n \in \mathbf{N}$. We shall use the following extension of the classical Newman's theorem.

THEOREM 1 (Bulinski and Shashkin [10]). *Let $X = \{X_j; j \in \mathbf{Z}^d\}$ be a strictly stationary $(BL, \theta)$-dependent random field with values in $\mathbf{R}^k$. Then, for any sequence of regularly growing sets $U_n \subset \mathbf{Z}^d$, $n \in \mathbf{N}$, it follows that*

$$(1.11) \qquad |U_n|^{-1/2}\left(S(U_n) - |U_n| \mathsf{E} X_0\right) \xrightarrow{D} N(0, C) \quad \text{as } n \to \infty,$$

*where $C$ is the matrix with elements*

$$(1.12) \qquad c_{r,q} = \sum_{j \in \mathbf{Z}^d} \operatorname{cov}(X_{0,r}, X_{j,q}), \quad r, q = 1, \ldots, k.$$

If $C$ is non-singular, then (1.11) amounts to

$$(1.13) \qquad (|U_n| C)^{-1/2}\left(S(U_n) - |U_n| \mathsf{E} X_0\right) \xrightarrow{D} N(0, I) \quad \text{as } n \to \infty.$$

Thus, if there is a sequence of statistical estimates $\hat{C}_n = \hat{C}(U_n) = \left(\hat{c}_{r,q}(U_n)\right)_{r,q=1}^k$ for $C$, such that for any $r, q = 1, \ldots, k$

$$(1.14) \qquad \hat{c}_{r,q}(U_n) \xrightarrow{P} c_{r,q} \quad \text{as } n \to \infty,$$

then by virtue of (1.13) and (1.14) we obtain

$$(1.15) \qquad \hat{T}_n := (|U_n| \hat{C}_n)^{-1/2}\left(S(U_n) - |U_n| \mathsf{E} X_0\right) \xrightarrow{D} N(0, I) \quad \text{as } n \to \infty.$$

We will consider the following statistics. For $j = (j_1, \ldots, j_d) \in U \subset \mathbf{Z}^d$ $(1 \leqslant |U| < \infty)$, $\mathbf{b} = \mathbf{b}(U) = (b_1, \ldots, b_d) \in \mathbf{N}^d$ and $r, q = 1, \ldots, k$, set

$$K_j(\mathbf{b}) = \{t \in \mathbf{Z}^d \colon |j_m - t_m| \leqslant b_m,\ m = 1, \ldots, d\}, \qquad Q_j = Q_j(U, \mathbf{b}) = U \cap K_j(\mathbf{b}),$$

and

$$(1.16) \qquad \hat{c}_{r,q}(U) = \frac{1}{|U|} \sum_{j \in U} |Q_j| \left(\frac{S_r(Q_j)}{|Q_j|} - \frac{S_r(U)}{|U|}\right)\left(\frac{S_q(Q_j)}{|Q_j|} - \frac{S_q(U)}{|U|}\right).$$

For real-valued associated random fields ($k = 1$) such estimates were introduced in [13] and for vector-valued ($k \geqslant 1$) quasi-associated random fields these estimates appeared in [6]. In the mentioned papers the condition $b_1 = b_2 = \ldots = b_d$ was used (i.e. $K_j(\mathbf{b})$ were the cubes). Now we consider a more general situation and set $\langle \mathbf{b} \rangle = b_1 b_2 \ldots b_d$.

Here is the *statistical version* of the CLT.

THEOREM 2. *Let the conditions of Theorem 1 be satisfied. Then for any sequence of nonrandom vectors $\mathbf{b}_n = (b_{n,1}, \ldots, b_{n,d}) \in \mathbf{N}^d$, $n \in \mathbf{N}$, such* *that*

$$(1.17)\qquad b_n := \min_i b_{n,i} \to \infty, \quad \frac{\langle b_n\rangle |\partial U_n|}{|U_n|} \to 0 \quad \text{as } n \to \infty,$$

*the relation* (1.14) *holds. If, moreover, the matrix C is non-singular, then* (1.15) *is valid.*

To estimate the convergence rate in the CLT with random normalization we impose the following additional conditions:

1° $U_n = \{(a_{n,1}, a_{n,1}+l_{n,1}] \times \ldots \times (a_{n,d}, a_{n,d}+l_{n,d}]\} \cap Z^d$, where $a_{n,m} \in Z$, $l_{n,m} \in N$ for $n \in N$, $m = 1, \ldots, d$.

2° $D_s := \sup_{j \in Z^d} \mathsf{E}\, \|X_j\|^s < \infty$ for some $s > 2$.

3° $X$ is a $(BL, \theta)$-dependent random field satisfying (1.8) with $\theta_r = O(r^{-\lambda})$, as $r \to \infty$, for some $\lambda$ such that

$$(1.18)\qquad \lambda > d\psi(s).$$

In (1.18), $s$ is taken the same as in 2°, and

$$(1.19)\qquad \psi(s) = \begin{cases} (s-1)/(s-2), & 2 < s \leqslant 4, \\ (3-\sqrt{s})(\sqrt{s}+1)/2, & 4 < s \leqslant t_0^2, \\ \dfrac{(s-1)\sqrt{(s-2)^2-3}-s^2+6s-11}{3s-12}, & s > t_0^2, \end{cases}$$

$t_0 \approx 2.1413$ being the maximal root of the equation $t^3+2t^2-7t-4=0$.

As shown in [11], under conditions 1°–3° we have

$$(1.20)\qquad \mathsf{E}\, |S(U_n)|^{2+\delta} \leqslant c_0 |U_n|^{1+\delta/2}$$

for some positive $\delta$ and $c_0$ depending only on $d$, $s$, $D_s$ and $\lambda$.

Remark 1. It is not difficult to provide an explicit formula for $\delta$. Namely, a closer inspection of the proof in [11] gives

$$(1.21)\qquad \delta = \begin{cases} \delta', & 2 < s \leqslant 4, \\ s-2-\sqrt{s}, & 4 < s \leqslant t_0^2, \\ \frac{2}{3}\left(s-2-\sqrt{(s-2)^2-3}\right), & s > t_0^2, \end{cases}$$

where $\delta' = \delta'(\lambda, d, s) = \min\{\delta_1, \delta_2\} \in (0, s-2]$ for $s \in (2, 4]$ and

$$\delta_1 = \frac{1}{2}\left(\left(\Delta_1^2+8(s-2)\lambda/d-8(s-1)\right)^{1/2}-\Delta_1\right), \quad \Delta_1 = 1-s+2\lambda/d,$$

$$\delta_2 = \Delta_2-\left(\Delta_2^2-4(s-2)(\lambda/d-1)\right)^{1/2}, \quad \Delta_2 = s-3+2\lambda/d.$$

Note that, for fixed $d$ and $s$, $\delta'(\lambda, d, s) \to s-2$ as $\lambda \to \infty$.

THEOREM 3. *Assume that the conditions of Theorem* 1 *are satisfied and, moreover,* 1°–3° *hold. Set* $a = 2+4/\delta$ *with* $\delta$ *taken from* (1.20). *Then*

$$\|\hat{c}_{r,q}(U_n)-c_{r,q}\|_{L^1} = O(l_n^{-\mu}), \tag{1.22}$$

*where* $l_n := \min_i l_{n,i}$ *and*

$$\mu = \begin{cases} d\lambda/(a(\lambda+d)) & \text{for } \lambda+1 < a \text{ and } d \leqslant a, \\ \lambda/(2\lambda+1) & \text{for } \lambda+1 \geqslant a \text{ and } d > a, \\ d\lambda/((a+d)\lambda+d) & \text{for } \lambda+1 \geqslant a \text{ and } d \leqslant a. \end{cases} \tag{1.23}$$

Remark 2. The case $\lambda+1 < a$ and $d > a$ is impossible. If both these inequalities take place, then $a-1 > \lambda > d\psi(s) > d > a$, which is wrong.

Next we can establish an estimate of the convergence rate in (1.15).

THEOREM 4. *Let the conditions of Theorem* 3 *be satisfied. Suppose that* $s \in (2, 3]$ *in* 2° *and also* $\lambda > ds/(s-2)$ *in* 3°. *Moreover, assume that* $|U_n| = O(l_n^M)$ *with* $d \leqslant M < 2d(s-1)/(s-2)$. *Then*

$$\sup_{B \in \mathscr{C}_k} |\mathsf{P}(\hat{T}_n \in B) - \mathsf{P}(Z \in B)| = O(l_n^{-\tau}), \tag{1.24}$$

*where* $\mathscr{C}_k$ *is the class of bounded convex sets in* $\boldsymbol{R}^k$, $Z$ *is a standard normal vector in* $\boldsymbol{R}^k$, $\mu$ *is defined in* (1.23),

$$\tau = \min\left\{\frac{2}{5}\mu, \frac{\nu}{3}\right\}, \quad \text{and} \quad \nu = \frac{M(\lambda(s-2)-ds)}{2\lambda+d(s+1+2\lambda(s-1))}.$$

COROLLARY 1. *Let* $U_n$, $n \in N$, *be finite subsets of* $\boldsymbol{Z}^d$ *growing like cubes, i.e.* $M = d$ (*the Fischer type growth condition*). *If* $s = 3$ *and* $d > 6$, *then, for the exponent* $\tau$ *in* (1.24), *it follows that* $\tau \to d/(6(1+2d))$ *as* $\lambda \to \infty$.

## 2. PROOFS

**Proof of Theorem 2.** We modify the proof of Theorem in [6]. First of all note that the estimates $(\hat{c}_{r,q}(U_n))_{r,q=1}^k$, introduced in (1.16), and elements of the matrix $C$ are invariant under the transformation $X_j \mapsto X_j - \mathsf{E}X_0$, $j \in U_n$. Consequently, without loss of generality we can assume that $\mathsf{E}X_0 = \boldsymbol{0} \in \boldsymbol{R}^k$. For a real-valued random variable $\xi$ let $\|\xi\|_{L^1}$ denote its norm in the space $L^1 = L^1(\Omega, \mathscr{F}, \mathsf{P})$. We shall show that, for any $r, q = 1, \ldots, k$,

$$\hat{c}_{r,q}(U_n) \xrightarrow{L^1} c_{r,q} \quad \text{as } n \to \infty. \tag{2.1}$$

Clearly, (2.1) yields (1.14). For each pair of $r, q = 1, \ldots, k$ we have the corresponding bound of the form

$$\|\hat{c}_{r,q}(U_n)-c_{r,q}\|_{L^1} \leqslant I_1(U_n)+I_2(U_n)+I_3(U_n),$$

where

$$I_1(U_n) := \left\| \hat{c}_{r,q}(U_n) - |U_n|^{-1} \sum_{j \in U_n} |Q_j|^{-1} S_r(Q_j) S_q(Q_j) \right\|_{L^1},$$

(2.2) $$I_2(U_n) := |U_n|^{-1} \left\| \sum_{j \in U_n} |Q_j|^{-1} \big(S_r(Q_j) S_q(Q_j) - \mathsf{E} S_r(Q_j) S_q(Q_j)\big) \right\|_{L^1},$$

$$I_3(U_n) := \left| |U_n|^{-1} \sum_{j \in U_n} |Q_j|^{-1} \mathsf{E} S_r(Q_j) S_q(Q_j) - c_{r,q} \right|.$$

I. To estimate $I_1$ observe that due to the wide-sense stationarity of a field $\{X_j; j \in \mathbf{Z}^d\}$ we have (see (1.10)), for a finite $Q \subset \mathbf{Z}^d$ and $r = 1, \ldots, k$,

(2.3) $$\mathsf{E} S_r^2(Q) \leqslant \sum_{i \in Q} \Big( \sum_{j \in \mathbf{Z}^d} |\mathrm{cov}(X_{0,r}, X_{j,r})| \Big) = |Q| \, \sigma_{r,r}.$$

Hence, by the Cauchy–Schwarz inequality,

(2.4) $$\mathsf{E} |S_r(Q_j) S_q(U_n)| \leqslant (\sigma_{r,r} \sigma_{q,q} |Q_j| \, |U_n|)^{1/2}, \qquad r, q = 1, \ldots, k.$$

Using (2.4) one can show that

(2.5) $$\begin{aligned} I_1(U_n) &\leqslant |U_n|^{-3} \mathsf{E} |S_r(U_n) S_q(U_n)| \sum_{j \in U_n} |Q_j| \\ &\quad + |U_n|^{-2} \sum_{j \in U_n} \big( \mathsf{E} |S_r(Q_j) S_q(U_n)| + \mathsf{E} |S_r(U_n) S_q(Q_j)| \big) \\ &\leqslant (\sigma_{r,r} \sigma_{q,q})^{1/2} \{ |K_0(\mathbf{b}_n)| \, |U_n|^{-1} + 2 |K_0(\mathbf{b}_n)|^{1/2} |U_n|^{-1/2} \} \end{aligned}$$

because $\sum_{j \in U_n} |Q_j| \leqslant |U_n| \, |K_0(\mathbf{b}_n)|$. Since

$$|K_0(\mathbf{b}_n)| = \prod_{i=1}^{d} (2b_{n,i} + 1) \leqslant 3^d \langle \mathbf{b}_n \rangle,$$

from (1.17) we conclude that $I_1(U_n) \to 0$ as $n \to \infty$.

II. To estimate $I_2$ let us introduce, for $c > 0$, two auxiliary functions

(2.6) $$h_1 = \mathrm{sign}(x) \min\{|x|, c\}, \qquad h_2(x) = x - h_1(x), \qquad x \in \mathbf{R}.$$

For a finite nonempty set $Q \subset \mathbf{Z}^d$ put $\bar{S}_r(Q) = S_r(Q)/\sqrt{|Q|}$, $r = 1, \ldots, k$. Note that $x = h_1(x) + h_2(x)$ for all $x \in \mathbf{R}$. Therefore,

(2.7) $$I_2(U_n) \leqslant \sum_{p,m=1}^{2} I_2^{(p,m)}(U_n),$$

where

(2.8) $$I_2^{(p,m)}(U_n) := |U_n|^{-1} \left\| \sum_{j \in U_n} h_p(\bar{S}_r(Q_j)) h_m(\bar{S}_q(Q_j)) - \mathsf{E} h_p(\bar{S}_r(Q_j)) h_m(\bar{S}_q(Q_j)) \right\|_{L^1}.$$

For $\mathbf{b} \in \mathbf{Z}^d$, $n \in \mathbf{N}$, let us introduce

$$T_n^{(\mathbf{b})} = \bigcup_{j \in \partial U_n} Q_j(U_n, \mathbf{b}).$$

Set $T_n := T_n^{(b_n)}$, $n \in N$, with $b_n$ satisfying (1.17). Then we have

$$(2.9) \quad I_2^{(1,2)}(U_n)+I_2^{(2,1)}(U_n)+I_2^{(2,2)}(U_n)$$
$$\leqslant 2|U_n|^{-1}\sum_{j\in U_n}\big(\mathsf{E}\,|h_1(\bar{S}_r(Q_j))\,h_2(\bar{S}_q(Q_j))|$$
$$+\mathsf{E}\,|h_2(\bar{S}_r(Q_j))\,h_1(\bar{S}_q(Q_j))|+\mathsf{E}\,|h_2(\bar{S}_r(Q_j))\,h_2(\bar{S}_q(Q_j))|\big)$$
$$\leqslant 2\big(\mathsf{E}\,|h_1(\bar{S}_r(K_0(b_n)))\,h_2(\bar{S}_q(K_0(b_n)))|+\mathsf{E}\,|h_2(\bar{S}_r(K_0(b_n)))\,h_1(\bar{S}_q(K_0(b_n)))|$$
$$+\mathsf{E}\,|h_2(\bar{S}_r(K_0(b_n)))\,h_2(\bar{S}_q(K_0(b_n)))|+3\,|T_n|\,|U_n|^{-1}(\sigma_{r,r}\sigma_{q,q})^{1/2}\big).$$

To obtain this bound we considered separately the summands with $j\in U_n\backslash T_n$ (then $|Q_j| = |K_0(b_n)|$ and there are no more than $|U_n|$ such terms) and $j\in T_n$. For the latter we used the property $|h_p(x)| \leqslant |x|$ for all $x\in \boldsymbol{R}$, $p = 1, 2$, and applied (2.4). Namely,

$$\mathsf{E}\,|h_p(\bar{S}_r(Q_j))\,h_m(\bar{S}_q(Q_j))| \leqslant \mathsf{E}\,|\bar{S}_r(Q_j)\,\bar{S}_q(Q_j)| \leqslant (\sigma_{r,r}\sigma_{q,q})^{1/2}.$$

To estimate the first, second and third terms on the right-hand side of (2.9) we use the Cauchy–Schwarz inequality and the property $h_2(x) \leqslant |x|$ for $|x| \geqslant c$, $h_2(x) = 0$ for $|x| < c$. For example, the first term is bounded in the following way:

$$\mathsf{E}\,|h_1(\bar{S}_r(K_0(b_n)))\,h_2(\bar{S}_q(K_0(b_n)))| \leqslant \big(\sigma_{r,r}\,\mathsf{E}\big(\bar{S}_q^2(K_0(b_n))\,I\{|\bar{S}_q(K_0(b_n))| \geqslant c\}\big)\big)^{1/2}.$$

Thus we establish that $I_2^{(1,2)}(U_n)+I_2^{(2,1)}(U_n)+I_2^{(2,2)}(U_n)$ is bounded by

$$(2.10) \quad 4\big(\sigma_{r,r}\,\mathsf{E}\big(\bar{S}_q^2(K_0(b_n))\,I\{|\bar{S}_q(K_0(b_n))| \geqslant c\}\big)\big)^{1/2}$$
$$+2\big(\sigma_{q,q}\,\mathsf{E}\big(\bar{S}_r^2(K_0(b_n))\,I\{|\bar{S}_r(K_0(b_n))| \geqslant c\}\big)\big)^{1/2}+6\,|T_n|\,|U_n|^{-1}(\sigma_{r,r}\sigma_{q,q})^{1/2}.$$

Now we use the following elementary result.

LEMMA 1 (see, e.g., [3], [6]). *Let* $X = \{X_j; j\in \boldsymbol{Z}^d\}$ *be a wide-sense stationary random field with values in* $\boldsymbol{R}^k$ *such that* (1.10) *holds. Then for any sequence of regularly growing sets* $U_n \subset \boldsymbol{Z}^d$, $n\in N$,

$$(2.11) \quad |U_n|^{-1}\,\mathsf{Var}\,S(U_n) \to C \quad \text{as } n\to\infty,$$

*where* $\mathsf{Var}\,S(U_n)$ *is the covariance matrix of* $S(U_n)$ *and the matrix* $C$ *was defined in* (1.12). *The relation* (2.11) *means the elementwise convergence of matrices.*

In view of (2.11) we have $\mathsf{var}\,\bar{S}_r^2(K_0(b_n)) \to c_{r,r}$ as $n\to\infty$, for each $r = 1, \ldots, k$.

Furthermore, by Theorem 1,

$$\bar{S}_r^2(K_0(b_n)) \xrightarrow{D} Z_r^2 \quad \text{as } n\to\infty,$$

where $Z_r \sim N(0, c_{r,r})$, so we may infer from Theorem 5.4 of [2] the uniform integrability of a family $\{\bar{S}_r^2(K_0(b_n))\}_{n=1}^{\infty}$ for each $r = 1, \ldots, k$. Using the inequality

$$|T_n| \leqslant \prod_{i=1}^{d} (2b_{n,i}+1)\,|\partial U_n|$$

and the condition (2.4) we see that, for any $\varepsilon > 0$, there exists $c = c(\varepsilon)$ (whence the functions $h_1$ and $h_2$) such that, for all $n$ large enough,

$$I_2^{(1,2)}(U_n)+I_2^{(2,1)}(U_n)+I_2^{(2,2)}(U_n) < \varepsilon.$$

Following [6] we can use the representation

$$\begin{aligned} U_n \times U_n &= \{(j, t)\colon j, t \in U_n\} \\ &= \{(j, t)\colon t \in K_j(3\boldsymbol{b}_n)\} \cup \{(j, t)\colon t \notin K_j(3\boldsymbol{b}_n)\} =: M_1 \cup M_2. \end{aligned}$$

Due to the Lyapunov inequality we see that

$$\left(I_2^{(1,1)}(U_n)\right)^2 \leqslant \frac{1}{|U_2|^2}(\Sigma_{1,n}+\Sigma_{2,n}),$$

where

$$\Sigma_{i,n} = \sum_{(j,t)\in M_i} \left|\operatorname{cov}\left(h_1(\bar{S}_r(Q_j))\,h_1(\bar{S}_q(Q_j)),\ h_1(\bar{S}_r(Q_t))\,h_1(\bar{S}_q(Q_t))\right)\right|, \qquad i = 1, 2.$$

Now note that if a random variable $\xi$ is such that $|\xi| \leqslant c^2$, then $|\operatorname{cov}(\xi, \eta)| \leqslant 2c^2\,\mathsf{E}\,|\eta|$. Obviously, $|h_1(x)\,h_1(y)| \leqslant c^2$ for all $x, y \in \boldsymbol{R}$. Thus, according to (2.4),

$$\begin{aligned} (2.12)\qquad \Sigma_{1,2} &\leqslant 2c^2 \sum_{(j,t)\in M_1} \mathsf{E}\left|h_1(\bar{S}_r(Q_j))\,h_1(\bar{S}_q(Q_j))\right| \\ &\leqslant 2c^2 \sum_{(j,t)\in M_1} (\sigma_{r,r}\,\sigma_{q,q})^{1/2} \leqslant 2\cdot 3^d c^2 (\sigma_{r,r}\,\sigma_{q,q})^{1/2}\,|K_0(\boldsymbol{b}_n)|\,|U_n|, \end{aligned}$$

because for a fixed $j$ (there are $|U_n|$ positions for $j$) there exist no more than $3^d|K_0(\boldsymbol{b}_n)|$ variants to choose $t$.

By the definition of $(BL, \theta)$-dependence we have

$$(2.13)\qquad \Sigma_{2,n} \leqslant \sum_{(j,t)\in M_2} \operatorname{Lip}(f_{|Q_j|})\operatorname{Lip}(f_{|Q_t|})(|Q_j| \wedge |Q_t|)\,\theta_{\operatorname{dist}(Q_j,Q_t)},$$

where

$$f_m(x_1, \ldots, x_{2m}) = h_1\left(m^{-1/2}(x_1+ \ldots +x_m)\right) h_1\left(m^{-1/2}(x_{m+1}+ \ldots +x_{2m})\right), \qquad m \in N$$

It is easy to verify that $\operatorname{Lip}(f_{|Q_j|}) = c/\sqrt{|Q_j|}$ (this is an immediate corollary to the Lemma in [4]). A sequence $(\theta_r)_{r\in N}$ is nonincreasing and $\theta_r \to 0$ as $r \to \infty$. Moreover, $b_n \leqslant \operatorname{dist}(Q_j, Q_t)$ for $(j, t) \in M_2$, which implies the inequality $\theta_{\operatorname{dist}(Q_j,Q_t)} \leqslant \theta_{b_n}$. Therefore,

$$(2.14)\qquad \frac{\Sigma_{2,n}}{|U_n|^2} \leqslant \frac{c^2\,\theta_{b_n}}{|U_n|^2} \sum_{(j,t)\in M_2} \left(\frac{|Q_j| \wedge |Q_t|}{\sqrt{|Q_j|}\,\sqrt{|Q_t|}}\right) \leqslant c^2\,\theta_{b_n} \to 0 \qquad \text{as } n \to \infty.$$

The last inequality is true as $(a \wedge b)/(\sqrt{a}\cdot\sqrt{b}) \leqslant 1$ for any positive $a$ and $b$.

In view of (2.12)–(2.14) and the condition (1.17), we have

$$(2.15)\quad I_2^{(1,1)}(U_n) \leqslant c\left(2\cdot 3^d(\sigma_{r,r}\sigma_{q,q})^{1/2}\left(\frac{|K_0(b_n)|}{|U_n|}\right)+\theta_{b_n}\right)^{1/2} \to 0 \quad \text{as } n\to\infty.$$

Thus, we come to the relation $I_2(U_n)\to 0$ as $n\to\infty$.

III. To estimate $I_3(U_n)$ let us note that

$$|U_n|^{-1}\sum_{j\in U_n}|Q_j|^{-1}\,\mathsf{E}\,S_r(Q_j)S_q(Q_j)$$
$$= |U_n|^{-1}\sum_{j\in U_n\setminus T_n}|Q_j|^{-1}\,\mathsf{E}\,S_r(Q_j)S_q(Q_j)+|U_n|^{-1}\sum_{j\in T_n}|Q_j|^{-1}\,\mathsf{E}\,S_r(Q_j)S_q(Q_j)$$
$$=\frac{|U_n\setminus T_n|}{|U_n|\,|K_0(b_n)|}\,\mathsf{E}\,S_r\big(K_0(b_n)\big)S_q\big(K_0(b_n)\big)+\frac{1}{|U_n|}\sum_{j\in T_n}|Q_j|^{-1}\,\mathsf{E}\,S_r(Q_j)S_q(Q_j).$$

By Lemma 1 we obtain

$$(2.16)\qquad |K_0(b_n)|^{-1}\,\mathsf{E}\,S_r\big(K_0(b_n)\big)S_q\big(K_0(b_n)\big)\to c_{r,q}\quad \text{as } n\to\infty.$$

According to (1.17) we have $|U_n|^{-1}|T_n|\to 0$ and $|U_n|^{-1}|U_n\setminus T_n|\to 1$ as $n\to\infty$.

The inequality $|Q_j|^{-1}\,\mathsf{E}\,|S_r(Q_j)S_q(Q_j)|\leqslant(\sigma_{r,r}\sigma_{q,q})^{1/2}$ (which is analogous to (2.4)) implies that $I_3(U_n)\to 0$ as $n\to\infty$. The proof is complete. ∎

**Proof of Theorem 3.** For $i=1,\dots,d$ and some $\alpha\in(0,1)$, introduce the sequences $(b_{n,i})_{n\in N}$ of positive integers $b_{n,i}:=[l_n^{\alpha-1}l_{n,i}]$, $n\in N$. Thus, for $b_n=(b_{n,1},\dots,b_{n,d})$, we have $|K_0(b_n)|=O(l_n^{d(\alpha-1)}|U_n|)$ as $n\to\infty$. Moreover, $|\partial U_n|=O(l_n^{-1}|U_n|)$ and $|T_n|=O(l_n^{\alpha-1}|U_n|)$. We shall estimate $I_r(U_n)$, $r=1,2,3$, defined in (2.2).

I. The inequality (2.5) yields

$$(2.17)\qquad I_1(U_n)\leqslant O\big(|K_0(b_n)|^{1/2}|U_n|^{-1/2}\big)=O\big(l_n^{d(\alpha-1)/2}\big).$$

II. To estimate $I_2(U_n)$ we use (2.10) and (2.15). For any random variable $\xi$ and positive numbers $A$, $\delta$ we have

$$A^\delta\,\mathsf{E}\big(\xi^2 I\{|\xi|>A\}\big)\leqslant \mathsf{E}\,|\xi|^{2+\delta}.$$

Therefore, taking into account (1.20) we obtain

$$\mathsf{E}\big(\bar S_q^2(K_0(b_n))I\{|\bar S_q(K_0(b_n))|\geqslant c\}\big)\leqslant c^{-\delta}\,\mathsf{E}\left(\frac{|S_q(K_0(b_n))|^{2+\delta}}{|K_0(b_n)|^{1+\delta/2}}\right)\leqslant c_0c^{-\delta}.$$

Choosing $c=c(l_n)=O(l_n^\beta)$ for some $\beta>0$, we derive the bound

$$(2.18)\qquad \Big(\mathsf{E}\big(\bar S_q^2(K_0(b_n))I\{|\bar S_q(K_0(b_n))|\geqslant c\}\big)\Big)^{1/2}\leqslant c_0^{1/2}c^{-\delta/2}=O\big(l_n^{-\beta\delta/2}\big).$$

The last summand in (2.10) can be estimated in the following way:

$$(2.19)\qquad 6\,|T_n|\,|U_n|^{-1}(\sigma_{r,r}\sigma_{q,q})^{1/2}=O\big(l_n^{\alpha-1}\big).$$

Applying (2.15) we conclude that

$$I_2^{(1,1)}(U_n) = O(l_n^{\beta+d(\alpha-1)/2}) + O(l_n^{\beta-\lambda\alpha/2}).$$

Consequently,

$$I_2(U_n) = O(l_n^{-\beta\delta/2} + l_n^{\alpha-1} + l_n^{\beta+d(\alpha-1)/2} + l_n^{\beta-\lambda\alpha/2}).$$

III. Now consider $I_3(U_n)$. Having fixed some $\gamma \in (0, 1)$, we can represent the set $\boldsymbol{Z}^d \setminus K_0(\boldsymbol{b}_n)$ as the union of the following disjoint sets:

$$V_1 = \{j \in \boldsymbol{Z}^d \setminus K_0(\boldsymbol{b}_n)\colon \operatorname{dist}(j, K_0(\boldsymbol{b}_n)) \leqslant b_n^\gamma\},$$

$$V_2 = \{j \in \boldsymbol{Z}^d \setminus K_0(\boldsymbol{b}_n)\colon \operatorname{dist}(j, K_0(\boldsymbol{b}_n)) > b_n^\gamma\}.$$

Thus one can write

$$\big||K_0(\boldsymbol{b}_n)|^{-1} \mathsf{E} S_r(K_0(\boldsymbol{b}_n)) S_q(K_0(\boldsymbol{b}_n)) - c_{r,q}\big| = |K_0(\boldsymbol{b}_n)|^{-1} \Big| \sum_{i \in K_0(\boldsymbol{b}_n)} \sum_{j \notin K_0(\boldsymbol{b}_n)} \mathsf{E} X_{i,r} X_{j,q}\Big|$$

$$=: J_{1,n} + J_{2,n},$$

where

$$J_{r,n} = |K_0(\boldsymbol{b}_n)|^{-1} \Big| \sum_{i \in K_0(\boldsymbol{b}_n)} \sum_{j \in V_r} \mathsf{E} X_{i,r} X_{j,q}\Big|, \qquad r = 1, 2. \tag{2.20}$$

For any $j_0 \in V_1$, $\big|\sum_{i \in K_0(\boldsymbol{b}_n)} \mathsf{E} X_{i,r} X_{j_0,q}\big| \leqslant \sigma_{r,q}$. Hence we have

$$J_{1,n} = O(|V_1|\,|K_0(\boldsymbol{b}_n)|^{-1}) = O(b_n^{\gamma-1}) = O(l_n^{\alpha(\gamma-1)}). \tag{2.21}$$

By the definition of $(BL, \theta)$-dependence there exists a constant $c_1 > 0$ such that, for any $R > 0$, the following inequality is valid:

$$\Big| \sum_{j\colon \|j\| \geqslant R} \operatorname{cov}(X_{0,r}, X_{j,q})\Big| \leqslant c_1 R^{-\lambda},$$

and therefore

$$J_{2,n} \leqslant |K_0(\boldsymbol{b}_n)|^{-1} |K_0(\boldsymbol{b}_n)|\, c_1 (b_n^\gamma)^{-\lambda} = O(l_n^{-\alpha\gamma\lambda}). \tag{2.22}$$

Clearly, the optimally chosen $\gamma$ satisfies the equation $\alpha(\gamma-1) = -\alpha\gamma\lambda$, i.e. $\gamma = 1/(\lambda+1)$. Due to (2.20)–(2.22) we obtain the bound

$$\big||K_0(\boldsymbol{b}_n)|^{-1} \mathsf{E} S_r(K_0(\boldsymbol{b}_n)) S_q(K_0(\boldsymbol{b}_n)) - c_{r,q}\big| = O(l_n^{-\alpha\lambda/(\lambda+1)}) \quad \text{as } n \to \infty. \tag{2.23}$$

Taking into account (2.19) we conclude that

$$I_3(U_n) = O(l_n^{\alpha-1} + l_n^{-\alpha\lambda/(\lambda+1)}).$$

Combining all bounds for $I_r(U_n)$, $r = 1, 2, 3$, $n \in N$, we see that (1.22) holds when

$$\mu = \max_{\alpha \in (0,1), \beta > 0} h(\alpha, \beta)$$

with

$$h(\alpha, \beta) = \min\left\{\beta\frac{\delta}{2}, 1-\alpha, \frac{d}{2}-\alpha\frac{d}{2}-\beta, \frac{\lambda}{2}\alpha-\beta, \alpha\frac{\lambda}{\lambda+1}\right\}. \tag{2.24}$$

Obviously, $\|\hat{c}_{r,q}(U_n)-c_{r,q}\|_{L^1} \to 0$ as $n \to \infty$, if $\mu > 0$. So we need $\alpha$ and $\beta$ satisfying the conditions

$$0 < \alpha < 1, \quad \beta > 0, \quad \frac{d}{2}-\alpha\frac{d}{2}-\beta > 0, \quad \frac{\lambda}{2}\alpha-\beta > 0.$$

Thus $\alpha$ and $\beta$ should belong to the triangle in $\mathbf{R}^2$ cut by the lines

$$\beta = 0, \quad \beta = \frac{d}{2}-\alpha\frac{d}{2}, \quad \beta = \frac{\lambda}{2}\alpha.$$

Consequently, $\mu = \max\{\mu_1, \mu_2\}$, where

$$\mu_1 = \max\{h(\alpha, \beta)\colon 0 < \alpha \leqslant d/(d+\lambda), 0 < \beta < \alpha\lambda/2\},$$

$$\mu_2 = \max\{h(\alpha, \beta)\colon d/(d+\lambda) < \alpha < 1, 0 < \beta < d(1-\alpha)/2\}.$$

Then elementary but tedious calculations lead to the final statement (1.23).

**Proof of Theorem 4.** We begin with an algebraic lemma for nonrandom matrices being of independent interest. For any $(k \times k)$-matrix $W$ and vector $x \in \mathbf{R}^k$ we use the operator norm

$$\|W\| = \|W\|_\infty = \max_{m=1,\ldots,k} \sum_{r=1}^{k} |w_{m,r}|$$

corresponding to the norm

$$\|x\| = \|x\|_\infty = \max_{m=1,\ldots,k} |x_m|,$$

and also the so-called spectral norm

$$\|W\|_2 = \max\{\sqrt{\lambda}\colon \lambda \text{ is an eigenvalue of } W^* W\}$$

corresponding to the norm $\|x\|_2 = \sqrt{\sum_{m=1}^{k} x_m^2}$.

LEMMA 2. *Let $T = (t_{i,j})_{i,j=1}^{k}$ be a symmetric matrix and $C = (c_{i,j})_{i,j=1}^{k}$ be a symmetric positive-definite matrix. Set*

$$\lambda := \frac{1}{\max_{i=1,\ldots,k}\{\sum_{j=1}^{k}(|c_{i,j}|+1)\}}, \qquad t := \frac{1-\|I-\lambda C\|_2}{2\lambda}. \tag{2.25}$$

*Let $\Delta$ satisfy $0 < \Delta < \min\{1, t, \lambda_{\min}\}$, where $\lambda_{\min}$ is the minimal eigenvalue of the matrix $C$. Then $T$ is non-singular and the inequality $\|T-C\| < \Delta$ implies*

$$\|T^{-1/2}-C^{-1/2}\| < \frac{\sqrt{k}}{2}t^{-3/2}\|T-C\|.$$

Proof. We divide the proof into several steps.

1) By the definition of $\lambda$, we have, for any $i = 1, \ldots, k$,

$$(2.26)\qquad 1 = \lambda \max_{i=1,\ldots,k} \{\sum_{j=1}^{k} (|c_{i,j}|+1)\} \geqslant \lambda \sum_{j=1}^{k} (|c_{i,j}|+1) > \lambda \sum_{j=1}^{k} |c_{i,j}|.$$

Therefore, for each $i$ we get $0 \leqslant \lambda \sum_{j\neq i} |c_{i,j}| < 1-\lambda |c_{i,i}|$. All the diagonal elements of symmetric positive-definite matrix $C$ are positive, so for any $i$ we obtain $\sum_{j\neq i} |\lambda c_{i,j}| < 1-\lambda c_{i,i}$. It means that $I-\lambda C$ is the matrix with diagonal domination, and due to the theorem by Gershgorin (see, e.g., [24], p. 192) $I-\lambda C > 0$. As $C > 0$, we have $I-(I-\lambda C) > 0$, i.e. $I > I-\lambda C$. Thus $0 < I-\lambda C < I$. Consequently, all the eigenvalues of $I-\lambda C$ belong to the interval (0, 1). Therefore, $\|I-\lambda C\|_2 < 1$ and $t > 0$ (see (2.25)).

2) By assumptions, for any $i = 1, \ldots, k$ we get

$$\sum_{j=1}^{k} |t_{i,j}-c_{i,j}| \leqslant \max_{i=1,\ldots,k} \sum_{j=1}^{k} |t_{i,j}-c_{i,j}| = \|T-C\| < \Delta.$$

Hence, for any $i$ and $j$ we have $|t_{i,j}|-|c_{i,j}| \leqslant |t_{i,j}-c_{i,j}| < \Delta$, and so

$$(2.27)\qquad t_{i,j} \leqslant |t_{i,j}| < |c_{i,j}|+\Delta.$$

As $\Delta < 1$ we conclude that, for every $i = 1, \ldots, k$,

$$1 = \lambda \max_{i=1,\ldots,k} \{\sum_{j=1}^{k} (|c_{i,j}|+1)\} > \lambda \sum_{j=1}^{k} (|c_{i,j}|+\Delta).$$

Thus, we get

$$(2.28)\qquad 1-\lambda(|c_{i,i}|+\Delta) > \lambda \sum_{j\neq i}^{k} (|c_{i,j}|+\Delta).$$

In view of (2.27) and (2.28) we obtain the following inequalities:

$$0 \leqslant \sum_{j\neq i}^{k} |\lambda t_{i,j}| < \lambda \sum_{j\neq i}^{k} (|c_{i,j}|+\Delta) < 1-\lambda(|c_{i,i}|+\Delta) < 1-\lambda t_{i,i}.$$

Therefore $I-\lambda T$ is a matrix with diagonal domination, and consequently $I-\lambda T > 0$.

3) A matrix $(T-C)$ is symmetric, so Gershgorin's theorem implies (see, e.g., [24], p. 192) that

$$(2.29)\qquad \|T-C\|_2 \leqslant \|T-C\|.$$

4) Let us show that $T > 0$. It suffices to verify that, for any vector $x$ such that $\|x\|_2 = 1$, the inequality $(Tx, x) > 0$ holds. Let $e_1, \ldots, e_k$ be an orthonormal system of eigenvectors for the matrix $C$, and $\lambda_1, \ldots, \lambda_k$ be the correspon-

ding eigenvalues. Then one can write $x = \alpha_1 e_1 + \ldots + \alpha_k e_k$ and

(2.30) $$(Cx, x) = \alpha_1^2 \lambda_1 + \ldots + \alpha_k^2 \lambda_k \geqslant \lambda_{\min} \|x\|_2 = \lambda_{\min}.$$

By the Cauchy–Schwarz inequality and (2.29) we see that

$$|((T-C)x, x)| \leqslant \|(T-C)x\|_2 \|x\|_2 \leqslant \|T-C\|_2 \|x\|_2^2$$
$$= \|T-C\|_2 \leqslant \|T-C\| < \Delta.$$

Thus $((T-C)x, x) > -\Delta$. Taking into account the last inequality, (2.30) and the condition $\Delta < \lambda_{\min}$ we have

$$(Tx, x) = ((T-C)x, x) + (Cx, x) > \lambda_{\min} - \Delta > 0.$$

Then $T > 0$ and $I - (I - \lambda T) > 0$, i.e. $I > I - \lambda T$. We obtain

(2.31) $$0 < I - \lambda T < I.$$

5) We have (see, e.g., [17], v. 2, p. 523), for $|x| < 1$,

$$(1-x)^{-1/2} = \sum_{m=0}^{\infty} a_m x^m, \qquad a_m = \frac{(2m-1)!!}{(2m)!!}.$$

Now the inequality $0 < I - \lambda C < I$ and (2.31) yield

$$T^{-1/2} - C^{-1/2} = \lambda^{1/2} \big(I - (I - \lambda T)\big)^{-1/2} - \lambda^{1/2} \big(I - (I - \lambda C)\big)^{-1/2}$$
$$= \lambda^{1/2} \Big( \sum_{m=0}^{\infty} a_m [(I - \lambda T)^m - (I - \lambda C)^m] \Big).$$

Note that, for any $(k \times k)$-matrices $W_1$ and $W_2$,

$$\|W_1^m - W_2^m\| \leqslant m \|W_1 - W_2\| (\|W_1\| \vee \|W_2\|)^{m-1},$$

where $a \vee b = \max\{a, b\}$, $m \in N$. Using this inequality we have

$$\|T^{-1/2} - C^{-1/2}\|_2 = \lambda^{1/2} \Big( \sum_{m=0}^{\infty} a_m \|(I - \lambda T)^m - (I - \lambda C)^m\|_2 \Big)$$
$$\leqslant \lambda^{1/2} \Big( \sum_{m=0}^{\infty} a_m m \lambda \|T - C\|_2 (\|I - \lambda T\|_2 \vee \|I - \lambda C\|_2)^{m-1} \Big)$$
$$< \lambda^{3/2} \|T - C\|_2 \Big( \sum_{m=0}^{\infty} a_m m (\|I - \lambda C\|_2 + \lambda \Delta)^{m-1} \Big),$$

since

$$\|I - \lambda T\|_2 = \|I - \lambda C + \lambda (C - T)\|_2 \leqslant \|I - \lambda C\|_2 + \lambda \|C - T\|_2$$
$$\leqslant \|I - \lambda C\|_2 + \lambda \|C - T\| < \|I - \lambda C\|_2 + \lambda \Delta.$$

Now $\Delta < t$ implies that

$$\|I - \lambda C\|_2 + \lambda \Delta < \|I - \lambda C\|_2 + \lambda t < 1.$$

Setting $y := \|I-\lambda C\|_2+\lambda t$, we obtain

$$\sum_{m=0}^{\infty} a_m m(\|I-\lambda C\|_2+\lambda\Delta)^{m-1} \leqslant \sum_{m=0}^{\infty} a_m m y^{m-1}$$

$$= \Big(\sum_{m=0}^{\infty} a_m y^m\Big)'_y = ((1-y)^{-1/2})'_y = \tfrac{1}{2}(1-y)^{-3/2} = \tfrac{1}{2}(\lambda t)^{-3/2}$$

because, by (2.25), $1-y = 1-\|I-\lambda C\|_2-\lambda t = 2\lambda t-\lambda t = \lambda t$. Hence, we get

$$\|T^{-1/2}-C^{-1/2}\|_2 < \tfrac{1}{2}t^{-3/2}\|T-C\|_2.$$

It is easily seen that, for any $(k\times k)$-matrix $W$, $k^{-1/2}\|W\| \leqslant \|W\|_2$. So, using (2.29) we conclude that the desired statement is established.

Now we shall apply Lemma 2 to a matrix $T = (\xi_{i,j})_{i,j=1}^{k}$ with random elements $\xi_{i,j}$.

COROLLARY 2. *If, for almost every $\omega\in\Omega$, $T$ satisfies the assumptions of Lemma 2, then an event $\{\|T-C\| < \Delta\}$ entails an event*

$$\left\{\|T^{-1/2}-C^{-1/2}\| < \frac{\sqrt{k}}{2}t^{-3/2}\Delta\right\}.$$

*Thus,*

$$\mathsf{P}\left(\|T^{-1/2}-C^{-1/2}\| \geqslant \frac{\sqrt{k}}{2}t^{-3/2}\Delta\right) \leqslant \mathsf{P}(\|T-C\| \geqslant \Delta).$$

Returning to the proof of Theorem 4, we shall employ the following elementary proposition (using the same notation as in Theorem 4).

LEMMA 3 (Bulinski and Shashkin [9]). *Let $\xi$ and $\eta$ be random vectors with values in $\boldsymbol{R}^k$. Then, for any $\varepsilon>0$,*

$$\sup_{B\in\mathscr{C}_k}|\mathsf{P}(\xi+\eta\in B)-\mathsf{P}(Z\in B)| \leqslant \sup_{B\in\mathscr{C}_k}|\mathsf{P}(\xi\in B)-\mathsf{P}(Z\in B)|+\mathsf{P}(\|\eta\|>\varepsilon)+A(k)\varepsilon,$$

*where $A(k)$ is a positive factor depending on $k$ only.*

Applying Lemma 3 for

$$\xi = (C|U_n|)^{-1/2}S(U_n), \qquad \eta = (\hat{C}_n^{-1/2}-C^{-1/2})|U_n|^{-1/2}S(U_n), \qquad \varepsilon=\varepsilon_n = l_n^{-2\mu/5}$$

we obtain

$$\sup_{B\in\mathscr{C}_k}\big|\mathsf{P}((\hat{C}_n|U_n|)^{-1/2}S(U_n)\in B)-\mathsf{P}(Z\in B)\big| \tag{2.32}$$

$$\leqslant \sup_{B\in\mathscr{C}_k}\big|\mathsf{P}((C|U_n|)^{-1/2}S(U_n)\in B)-\mathsf{P}(Z\in B)\big|$$

$$+\mathsf{P}\big(\|(\hat{C}_n^{-1/2}-C^{-1/2})|U_n|^{-1/2}S(U_n)\|>\varepsilon\big)+A(k)\varepsilon.$$

The second summand in (2.32) can be estimated in the following manner:

$$\text{(2.33)}\quad \mathsf{P}\big(\|(\hat{C}_n^{-1/2}-C^{-1/2})|U_n|^{-1/2}S_n\|>\varepsilon_n\big)$$
$$\leqslant \mathsf{P}(\|\hat{C}_n^{-1/2}-C^{-1/2}\|\,\||U_n|^{-1/2}S_n\|>\varepsilon_n,\ \|\hat{C}_n^{-1/2}-C^{-1/2}\|>\varepsilon_n^{3/2})$$
$$+\mathsf{P}(\|\hat{C}_n^{-1/2}-C^{-1/2}\|\,\||U_n|^{-1/2}S_n\|>\varepsilon_n,\ \|\hat{C}_n^{-1/2}-C^{-1/2}\|\leqslant\varepsilon_n^{3/2})$$
$$\leqslant \mathsf{P}(\|\hat{C}_n^{-1/2}-C^{-1/2}\|\geqslant\varepsilon_n^{3/2})+\mathsf{P}(\||U_n|^{-1/2}S_n\|>\varepsilon_n^{-1/2}).$$

Since $\varepsilon_n\to 0$ as $n\to\infty$, we deduce that, for all $n$ large enough,

$$0<\frac{2}{\sqrt{k}}t^{3/2}\varepsilon_n^{3/2}<\min\{1,\,t,\,\lambda_{\min}\},$$

where $t$ and $\lambda_{\min}$ were defined in (2.25).

Thus Corollary 3, the Chebyshev inequality and Theorem 3 imply that

$$\mathsf{P}(\|\hat{C}_n^{-1/2}-C^{-1/2}\|\geqslant\varepsilon_n^{3/2})\leqslant\mathsf{P}\Big(\|\hat{C}_n-C\|\geqslant\frac{2}{\sqrt{k}}t^{3/2}\varepsilon_n^{3/2}\Big)$$
$$\leqslant\Big(\frac{2}{\sqrt{k}}t^{3/2}\varepsilon_n^{3/2}\Big)^{-1}\mathsf{E}\|\hat{C}_n-C\|=\varepsilon_n^{-3/2}O(l_n^{-\mu})=O(l_n^{-2\mu/5}).$$

The second summand on the right-hand side of (2.33) has the following upper bound:

$$\sum_{i=1}^{k}\mathsf{P}\big(|U_n|^{-1/2}S_i(U_n)>\varepsilon_n^{-1/2}\big)\leqslant\varepsilon_n\sum_{i=1}^{k}\operatorname{var}\big(|U_n|^{-1/2}S_i(U_n)\big)=O(\varepsilon_n)=O(l_n^{-2\mu/5}),$$

where we used (2.3) and the choice of $\varepsilon_n$. Therefore,

$$\text{(2.34)}\quad \mathsf{P}\big(\|(\hat{C}_n^{-1/2}-C^{-1/2})|U_n|^{-1/2}S_n\|>\varepsilon_n\big)+A(k)\,\varepsilon_n=O(l_n^{-2\mu/5}).$$

To estimate $\sup_{B\in\mathscr{C}_k}\big|\mathsf{P}\big((C|U_n|)^{-1/2}S(U_n)\in B\big)-\mathsf{P}(Z\in B)\big|$ we apply the result of [9]. To this end consider a standard partition of $U_n$ (see, e.g., [7] and [9]) into "rooms" and "corridors". Set $p_n:=[l_n^\alpha]$ and $q_n:=[l_n^\beta]$, $0<\beta<\alpha<1$. Each edge $(a_{n,i},\,a_{n,i}+l_{n,i}]$ can be represented as a union of "large" and "small" intervals having the length $p_n,\,q_n,\,p_n,\,\ldots,\,q_n,\,\tilde{p}_n$, respectively, where $p_n\leqslant\tilde{p}_n\leqslant 3p_n$. We draw the hyperplanes, orthogonal to the corresponding $i$-th edge, through the end points of these intervals.

Let us enumerate the "big" blocks (with edges having the length $p_n$ or $\tilde{p}_n$): $U^{(i)}$, $i=1,\ldots,N$, $N=\prod_{i=1}^{d}[l_{n,i}/(p_n+q_n)]=O(l_n^{-\alpha d}|U_n|)=O(l_n^{M-\alpha d})$. Set $\tilde{U}_n=\bigcup_{i=1}^{N}U^{(i)}$, $U^{(0)}=U_n\setminus\tilde{U}_n$. Introduce $Y_i=\sum_{j\in U^{(i)}}X_j$ $(i=0,\ldots,N)$, $V_1^2=\sum_{i=1}^{N}\operatorname{Var}Y_i$, $V_0^2=\operatorname{Var}Y_0$, and recall

THEOREM 5 (Bulinski and Shashkin [9]). *Let the assumptions of Theorem 3 be satisfied for $s\in(2,3]$. Then, for a nonrandom $(k\times k)$-matrix $A$ and any*

$\gamma\in(0,\gamma_0(k))$, *we have*

(2.35) $$\sup_{B\in\mathscr{C}_k}|\mathsf{P}(AS_n\in B)-\mathsf{P}(Z\in B)|$$

$$\leqslant b\{\gamma+\gamma^{-2}(\|V_1^{-1}\|^2|\tilde{U}_n|\,\theta_q+|U^{(1)}|^s N\,\|V_1^{-1}\|^s+\|AV_1-I\|^2+\|A\|^2\,\|V_0^2\|)\},$$

*where b is a constant, $\mathscr{C}_k$ a class of bounded convex subsets of $\mathbf{R}^k$, and Z a standard Gaussian vector with values in $\mathbf{R}^k$.*

Let us take $A=|U_n|^{-1/2}C^{-1/2}$. Set $\tilde{S}_n=\sum_{j\in\tilde{U}_n}X_j$. Now we shall examine separately the summands on the right-hand side of (2.35).

1) The bound for $\|AV_1-I\|$. Note that

(2.36) $$\left\|C-\frac{V_1^2}{|U_n|}\right\|\leqslant\left\|C-\frac{\mathrm{Var}(\tilde{S}_n)}{|\tilde{U}_n|}\right\|+\left\|\frac{\mathrm{Var}(\tilde{S}_n)}{|\tilde{U}_n|}-\frac{V_1^2}{|\tilde{U}_n|}\right\|+\left\|\frac{V_1^2}{|\tilde{U}_n|}-\frac{V_1^2}{|U_n|}\right\|.$$

Analogously to (2.23) one can show that

$$\|C-|\tilde{U}_n|^{-1}\,\mathrm{Var}(\tilde{S}_n)\|=O(l_n^{\max\{\beta-\alpha,-\lambda\beta\}}).$$

Introduce the matrices $\mathrm{cov}(Y_i,Y_j):=(\mathrm{cov}(Y_{i,r},Y_{j,l}))_{r,l=1}^k$, $i,j=1,\ldots,N$. We have

$$\|\mathrm{cov}(Y_i,Y_j)\|\leqslant k\max_{r,l}|\mathrm{cov}(Y_{i,r},Y_{j,l})|=O(|U^{(1)}|\,\theta_{q_n})\quad\text{as } n\to\infty.$$

Hence the second summand in (2.36) can be estimated as follows:

$$\left\|\frac{\mathrm{Var}(\tilde{S}_n)}{|\tilde{U}_n|}-\frac{V_1^2}{|\tilde{U}_n|}\right\|=|\tilde{U}_n|^{-1}\left\|\mathrm{Var}\sum_{i=1}^N Y_i-\sum_{i=1}^N\mathrm{Var}\,Y_i\right\|$$

$$=|\tilde{U}_n|^{-1}\left\|\sum_{i,j=1,i\neq j}^N\mathrm{cov}(Y_i,Y_j)\right\|\leqslant|\tilde{U}_n|^{-1}\sum_{i,j=1,i\neq j}^N\|\mathrm{cov}(Y_i,Y_j)\|$$

$$=O(|\tilde{U}_n|^{-1}N^2|U^{(1)}|\,\theta_{q_n})=O(N\theta_{q_n})=O(l_n^{M-\alpha d-\lambda\beta}).$$

For the third summand in (2.36) we have

$$\left\|\frac{V_1^2}{|\tilde{U}_n|}-\frac{V_1^2}{|U_n|}\right\|\leqslant\left\|\frac{V_1^2}{|\tilde{U}_n|}\right\|\frac{|U_n\setminus\tilde{U}_n|}{|U_n|}\leqslant\left\|\frac{\mathrm{Var}\,Y_1}{|U^{(1)}|}\right\|\frac{q}{p+q}=O(l_n^{\beta-\alpha})$$

as $\|\mathrm{Var}\,Y_1\|/|U^{(1)}|<\infty$.

Thus, $\|C-|U_n|^{-1}V_1^2\|\leqslant O(l_n^{\max\{\beta-\alpha,M-\alpha d-\lambda\beta\}})$.

In view of Lemma 2 we conclude that

$$|U_n|^{1/2}\|A-V_1^{-1}\|=\left\|C^{-1/2}-\frac{V_1^{-1}}{|U_n|^{-1/2}}\right\|$$

$$\leqslant\frac{\sqrt{k}}{2}t^{-3/2}\left\|C-\frac{V_1^2}{|U_n|}\right\|\leqslant O(l_n^{\max\{\beta-\alpha,M-\alpha d-\lambda\beta\}}).$$

Consequently,

$$\|AV_1 - I\| \leqslant |U_n|^{1/2} \|A - V_1^{-1}\| \frac{\|V_1\|}{|\tilde{U}_n|^{1/2}} \frac{|\tilde{U}_n|^{1/2}}{|U_n|^{1/2}} \leqslant O\left(l_n^{\max\{\beta-\alpha, M-\alpha d-\lambda\beta\}}\right).$$

2) The bound for $\|A\|^2 \|V_0^2\|$. We have

$$\|A\|^2 \|V_0^2\| = \| |U_n|^{-1/2} C^{-1/2}\|^2 \|V_0^2\| = \|C^{-1/2}\|^2 \frac{\|V_0^2\|}{|U_n \backslash \tilde{U}_n|} \frac{|U_n \backslash \tilde{U}_n|}{|U_n|} = O\left(l_n^{\beta-\alpha}\right).$$

3) The bound for $|U^{(1)}|^s N \|V_1^{-1}\|^s$. Observe that $|\tilde{U}_n| = N|U^{(1)}|$, and so

$$\begin{aligned} |U^{(1)}|^s N \|V_1^{-1}\|^s &= \| |\tilde{U}_n|^{1/2} V_1^{-1}\|^s |\tilde{U}_n|^{-s/2} |U^{(1)}|^s N \\ &\leqslant \operatorname{const} N^{1-s/2} |U^{(1)}|^{s/2} \\ &= O\left(l_n^{(M-\alpha d)(1-s/2)+\alpha ds/2}\right) = O\left(l_n^{M(1-s/2)+\alpha d(s-1)}\right). \end{aligned}$$

4) Gathering the bounds obtained at steps 1), 2) and 3) we see that the coefficient of $\gamma^{-2}$ in (2.35) has the form

$$\begin{aligned} (2.37) \quad & O\left(l_n^{-\lambda\beta}\right) + O\left(l_n^{M(1-s/2)+\alpha d(s-1)}\right) + O\left(l_n^{2\max\{\beta-\alpha, M-\alpha d-\lambda\beta\}}\right) + O\left(l_n^{\beta-\alpha}\right) \\ &= O\left(l_n^{\max\{-\lambda\beta, M(1-s/2)+\alpha d(s-1), 2(M-\alpha d-\lambda\beta), \beta-\alpha\}}\right). \end{aligned}$$

It is not difficult to see that for

$$\alpha = \frac{M(s/2+1+\lambda s-2\lambda)}{2\lambda+d(s+1+2\lambda s-2\lambda)}, \quad \beta = \frac{M(s/2+1+ds)}{2\lambda+d(s+1+2\lambda s-2\lambda)},$$

due to the conditions concerning $\lambda$ and $M$ in Theorem 4 it follows that

$$0 < M(s/2-1) - \alpha d(s-1) = 2(\alpha d + \lambda\beta - M) = \alpha - \beta = \nu < \lambda\beta.$$

Taking $\gamma = l_n^{-\nu/3}$, in view of (2.35) and (2.37) we obtain

$$(2.38) \quad \sup_{B \in \mathscr{C}_k} \left| \mathsf{P}\left(C^{-1/2} |U_n|^{-1/2} S_n \in B\right) - \mathsf{P}(Z \in B) \right| = O\left(l_n^{-\nu/3}\right).$$

Thus (2.34) and (2.38) imply (1.24). The proof is complete. ∎

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