PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 25, Fasc. 2 (2005), pp. 241–255

ALMOST SURE LIMIT THEOREMS FOR SEMI-SELFSIMILAR PROCESSES

BY

ISTVÁN FAZEKAS (DEBRECEN) AND ZDZISŁAW RYCHLIK (LUBLIN)

Abstract. An integral analogue of the almost sure limit theorem is presented for semi-selfsimilar processes. In the theorem, instead of a sequence of random elements, a continuous time random process is involved; moreover, instead of the logarithmical average, the integral of delta-measures is considered. Then the theorem is applied to obtain almost sure limit theorems for semistable processes. Discrete versions of the above theorems are proved. In particular, the almost sure functional limit theorem is obtained for semistable random variables.

1991 AMS Mathematics Subject Classification: Primary: 60F17, 60F15; Secondary: 28D05.

Key words and phrases: Almost sure limit theorem, semi-selfsimilar process, semistable process, ergodic theorem, functional limit theorem.

1. INTRODUCTION AND NOTATION

Let ζ_n, $n \in N$, be a sequence of random elements defined on the probability space $(\Omega, \mathscr{A}, P)$. Almost sure limit theorems state that

$$\frac{1}{D_n}\sum_{k=1}^{n} d_k\, \delta_{\zeta_k(\omega)} \Rightarrow \mu, \quad \text{as } n \to \infty, \text{ for almost every } \omega \in \Omega, \tag{1.1}$$

where δ_x is the point mass at x and $\Rightarrow \mu$ denotes weak convergence to the probability measure μ.

In the simplest form of the almost sure central limit theorem (a.s. CLT)

$$\zeta_n = (X_1 + \ldots + X_n)/\sqrt{n},$$

where $X_1, X_2, \ldots$ are i.i.d. real vandom variables with mean 0 and variance 1, $d_k = 1/k$, $D_n = \log n$, and μ is the standard normal law $\mathscr{N}(0, 1)$; see [5], [18], [13]. Almost sure versions of several known usual limit theorems were proved, see e.g. [1], [2], [8], [9].

Major in [15] and [16] gave a unified approach to the almost sure limit theory. The starting point in [15] is an integral version of the a.s. limit theorem for selfsimilar processes. The final result in [16] is an a.s. functional limit theorem for variables being in the domain of attraction of a stable law. Here we shall follow [15]. We shall denote by $X(u) = X(u, \omega)$, $u \geq 0$, the underlying stochastic process.

Theorem 1 in [15] is an a.s. limit theorem for selfsimilar processes. Our first result (Theorem 2.1) is its extension to semi-selfsimilar processes. In [15] the proof is based on the continuous time ergodic theorem. Here we shall apply the discrete time ergodic theorem.

Our Theorem 3.1 is an almost sure limit theorem for semistable processes. We remark that the second part of Theorem 1 in [15] is an a.s. limit theorem for stable processes.

We prove discretized versions of Theorem 2.1. In Theorem 4.2 we consider random probability measures concentrated on some trajectories $X_{b_{j,n}}(\cdot, \omega)$. The result is a version of the first part of Theorem 3 in [15]. In Theorem 4.3 we consider random probability measures concentrated on step functions constructed from the trajectories $X_{b_{j,n}}(\cdot, \omega)$. The result is a generalization of the second part of Theorem 3 in [15].

Theorem 5.1 is an a.s. functional limit theorem for semistable distributions.

We mention an open problem. Can the approach of the present paper be used to prove an a.s. functional limit theorem for variables being in the domain of geometric partial attraction of a semistable distribution? We remark that Major in [16] was able to apply his original approach to obtain an a.s. functional limit theorem for variables being in the domain of attraction of a stable law. Moreover, (non-functional) a.s. limit theorems for variables being in the domain of geometric partial attraction of a semistable law were obtained in [3] (usual version) and in [7] (integral version).

We shall use the following notation. I_A is the indicator function of the set A. In the space $C[0, 1]$ we shall consider the supremum norm. Let ϱ denote the usual metric on $D[0, 1]$ (see [4]). I.e. let Λ denote the set of strictly increasing continuous λ: $[0, 1] \to [0, 1]$ functions satisfying $\lambda(0) = 0$ and $\lambda(1) = 1$. Then $\varrho(x, y) \leq \varepsilon$ if there exists a $\lambda \in \Lambda$ such that

$$\sup_{t \neq s} \log \left|(\lambda(t) - \lambda(s))/(t-s)\right| \leq \varepsilon \quad \text{and} \quad \left|x(t) - y(\lambda(t))\right| \leq \varepsilon$$

for any $t \in [0, 1]$. Both the above spaces endowed with the metrics mentioned are complete separable metric spaces.

2. A LIMIT THEOREM FOR SEMI-SELFSIMILAR PROCESSES

Let $(\Omega, \mathscr{A}, P)$ be a probability space. Let $X(u) = X(u, \omega)$, $u \geq 0$, be a real-valued stochastic process defined on Ω. We assume that $X(0) = 0$ a.s., $X(u)$ is non-trivial, and it is stochastically continuous. We can assume that $X(u, \omega)$, $(u, \omega) \in [0, \infty) \times \Omega$, is (Borel) measurable (see [10], Section 4.3).

$X(u)$, $u \geqslant 0$, is called a *semi-selfsimilar process* ([17], pp. 70–72) if there exists a $c > 1$ such that

$$\left\{\frac{X(cu)}{c^{1/\alpha}},\ u \geqslant 0\right\} \stackrel{d}{=} \{X(u),\ u \geqslant 0\}$$

for some $\alpha > 0$. The sign $\stackrel{d}{=}$ means that the finite-dimensional distributions are equal.

For $t > 0$ let $X_t(u)$, $u \in [0, 1]$, denote the following process:

$$X_t(u) = \frac{X(tu)}{t^{1/\alpha}}, \quad u \in [0, 1].$$

For any function x: $[0, \infty) \to \boldsymbol{R}$ and for $t > 0$ we shall use the notation

$$x_t(u) = \frac{x(tu)}{t^{1/\alpha}}, \quad u \geqslant 0.$$

We assume that the trajectories of $X(u)$, $u \geqslant 0$, are either continuous or càdlàg, i.e. they are continuous from the right and have limits from the left for all $u > 0$ (with probability 1). It means that $\{X_t(u), u \in [0, 1]\}$ can be considered as a random element of $C[0, 1]$ or $D[0, 1]$, respectively. Let μ_t denote the distribution of the process X_t on $C[0, 1]$ or $D[0, 1]$. We consider both $C[0, 1]$ and $D[0, 1]$ endowed with their usual topologies (i.e. they are complete separable metric spaces, see [4]).

We denote the tail σ-algebra of the process $X(u)$, $u \geqslant 0$, by $\mathscr{F}_\infty$. That is,

$$\mathscr{F}_\infty = \bigcap_{n=1}^{\infty} \sigma\{X(u)\colon\ u \geqslant n\}.$$

Theorem 1 in [15] is an a.s. limit theorem for selfsimilar processes. Our first theorem is its extension to semi-selfsimilar processes. In [15] the proof is based on the continuous time ergodic theorem. Here we shall apply the discrete time ergodic theorem. Our proof is shorter than the one in [15] because we apply a general lemma (Lemma 2.1) widely used in the a.s. limit theory.

THEOREM 2.1. *Let $X(u)$, $u \geqslant 0$, be a semi-selfsimilar process with càdlàg trajectories. Let $X(0) = 0$ a.s. Assume that the tail σ-algebra of the process $X(u)$ is trivial, i.e. for any $A \in \mathscr{F}_\infty$ $\boldsymbol{P}(A)$ is zero or one. Then for any bounded measurable functional $F\colon D[0, 1] \to \boldsymbol{R}$*

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \frac{1}{t} F[X_t(\cdot, \omega)]\, dt = \int_{D[0,1]} F[x]\, d\mu(x) \tag{2.1}$$

for almost all $\omega \in \Omega$, where

$$\mu = \frac{1}{\log c} \int_1^c \frac{1}{t} \mu_t\, dt \tag{2.2}$$

is a mixture of the distributions of the processes X_t.

Let $\mu_{T,\omega}$ be the following random measure on the space $D[0, 1]$:

(2.3) $$\mu_{T,\omega}(A) = \frac{1}{\log T}\int_1^T \frac{1}{t} I_A(X_t(\cdot, \omega))\,dt.$$

Then

(2.4) $$\lim_{T\to\infty} \mu_{T,\omega} = \mu \quad \textit{for almost all } \omega\in\Omega.$$

We remark that when $X(u)$, $u \geqslant 0$, has continuous trajectories, then in Theorem 2.1 $D[0, 1]$ can be substituted by $C[0, 1]$.

Proof. $F[X_t(\cdot, \omega)]$ is a measurable function of (t, ω). We can see that it is enough to prove (2.1) for $T = c^n$. Then

$$\frac{1}{\log T}\int_1^T \frac{1}{t} F[X_t(\cdot, \omega)]\,dt = \frac{1}{n}\int_0^n F[X_{c^s}(\cdot, \omega)]\,ds = \frac{1}{n}\sum_{i=1}^n \int_{i-1}^i F[X_{c^s}(\cdot, \omega)]\,ds.$$

Now consider $\boldsymbol{R}^{[0,\infty)}$ endowed with the distribution of the process $X(u)$, $u \geqslant 0$. Then S: $x \to x_c$ is a measure-preserving transformation of $\boldsymbol{R}^{[0,\infty)}$ into itself.

Let A be an invariant set of the transformation $X \to SX$. Then there is a set A' from the tail σ-algebra $\mathscr{F}_\infty$ such that the symmetric difference of A and A' has zero probability. The tail σ-algebra is trivial, therefore the transformation $X \to SX$ is ergodic.

For any function x: $[0, \infty) \to \boldsymbol{R}$ let $\tilde{F}(x) = 0$ if x is not a càdlàg function, and

$$\tilde{F}(x) = \int_0^1 F[x_{c^s}]\,ds$$

if x is a càdlàg function. (In $F[x_{c^s}]$, in the formula above, the function $x_{c^s}(u)$ is restricted to $u\in[0, 1]$.) Then $\tilde{F}$ is a bounded measurable functional on $\boldsymbol{R}^{[0,\infty)}$. Now, using the ergodic theorem, we obtain

$$\frac{1}{\log T}\int_1^T \frac{1}{t} F[X_t(\cdot, \omega)]\,dt = \frac{1}{n}\sum_{i=0}^{n-1}\int_0^1 F[S^i X_{c^s}(\cdot, \omega)]\,ds$$
$$= \frac{1}{n}\sum_{i=0}^{n-1} \tilde{F}[S^i X(\cdot, \omega)] \to E\tilde{F}[X] = \int_{D[0,1]} F[x]\,d\mu(x)$$

for almost every $\omega\in\Omega$. Therefore (2.1) is proved.

Now, applying Lemma 2.1 below, we can deduce (2.4) from (2.1). ■

The following lemma is widely used in the a.s. limit theory (see, e.g., [8]). Its proof follows from that of Theorem 11.3.3 in [7]. Let (M, ϱ) be a complete separable metric space. Let $BL(M)$ be the space of the Lipschitz continuous bounded functions g: $M \to \boldsymbol{R}$ with $\|g\|_{BL} = \|g\|_\infty + \|g\|_L < \infty$, where $\|g\|_\infty$ is the sup norm and

$$\|g\|_L = \sup_{x \neq y} \frac{|g(x)-g(y)|}{\varrho(x, y)}.$$

LEMMA 2.1. *Let μ be a finite Borel measure on M. Then there exists a countable set $\Theta \subset BL(M)$ (depending on μ) such that for any sequence of finite Borel measures μ_n, $n \in N$, on M we have: $\mu_n \Rightarrow \mu$, $n \to \infty$, if and only if for each $g \in \Theta$*

$$\int_M g(x)\, d\mu_n(x) \to \int_M g(x)\, d\mu(x), \qquad n \to \infty.$$

3. A LIMIT THEOREM FOR SEMISTABLE PROCESSES

The process $V(u)$, $u \geqslant 0$, is called a *Lévy process* ([17], p. 3) if $V(0) = 0$ a.s., $V(u)$ has independent and stationary increments, it is stochastically continuous, its trajectories are continuous from the right and have limits from the left for all $t > 0$ (with probability 1).

First we remark that the tail σ-algebra of the process $V(u)$ is trivial, i.e. for any $A \in \mathscr{F}_\infty$ $P(A)$ is zero or one. Its proof is similar to the proof of the Hewitt–Savage zero or one law.

For every infinitely divisible law μ on $\boldsymbol{R}$ there is a Lévy process $V(u)$ such that the distribution of $V(1)$ is μ ([17], p. 63). By the Lévy formula (see [11], Section 18) the characteristic function of the process $V(u)$ is

$$\varphi_{V(u)}(x) = E(e^{ixV(u)}) = \psi(u, x, b, \sigma^2, L(y), R(y)) \tag{3.1}$$

$$= \exp\left[u\left\{ibx - \frac{\sigma^2}{2}x^2 + \int_{-\infty}^{0}\left(e^{ixy} - 1 - \frac{ixy}{1+y^2}\right)dL(y) + \int_{0}^{\infty}\left(e^{ixy} - 1 - \frac{ixy}{1+y^2}\right)dR(y)\right\}\right],$$

$x \in \boldsymbol{R}$. Here $L(y)$ is (left-continuous and) non-decreasing on $(-\infty, 0)$ with $L(-\infty) = 0$ and $R(y)$ is (right-continuous and) non-decreasing on $(0, \infty)$ with $R(\infty) = 0$ and they satisfy

$$\int_{-\varepsilon}^{0} y^2\, dL(y) + \int_{0}^{\varepsilon} y^2\, dR(y) < \infty \qquad \text{for all } \varepsilon > 0.$$

To exclude the Gaussian case we assume that $\sigma = 0$ (and $0 < \alpha < 2$, see below).

If $V(u)$ is a non-Gaussian semistable process, then its Lévy's representation is the following (see [12] and [3]):

$$L(y) = M_L(y)/|y|^\alpha, \qquad y < 0, \tag{3.2}$$

is left continuous and non-decreasing, and

$$R(y) = -M_R(y)/y^{\alpha}, \quad y > 0, \tag{3.3}$$

is right-continuous and non-decreasing, where $0 < \alpha < 2$, $M_L(y)$ and $M_R(y)$ are non-negative bounded functions on $(-\infty, 0)$ and $(0, \infty)$, respectively, one of which has a strictly positive infimum and the other either has a strictly positive infimum or is identically zero; moreover, for the same period $c > 1$ $M_L(c^{1/\alpha} y) = M_L(y)$ for all $-\infty < y < 0$ and $M_R(c^{1/\alpha} y) = M_R(y)$ for all $0 < y < \infty$. For simplicity, we shall assume that $b = 0$.

If $V(u)$ is a non-Gaussian semistable process, then, using the Lévy representation, one can show that $X(u) = V(u) - b(u)$ is semi-selfsimilar if

$$b(u) = \begin{cases} (Ku)/(c^{1-1/\alpha} - 1) & \text{for } \alpha \neq 1, \\ (Ku \log u)/(\log c) & \text{for } \alpha = 1, \end{cases} \tag{3.4}$$

where

$$K = \int_{-\infty}^{0} \frac{y^3 (c^{2/\alpha} - 1)}{(1+y^2)(1+y^2 c^{2/\alpha})} dL(y) + \int_{0}^{\infty} \frac{y^3 (c^{2/\alpha} - 1)}{(1+y^2)(1+y^2 c^{2/\alpha})} dR(y). \tag{3.5}$$

Therefore we can apply Theorem 2.1 to the process $X(u)$ to obtain the following almost sure limit theorem for semistable processes. We remark that the second part of Theorem 1 in [15] is an a.s. limit theorem for stable processes.

THEOREM 3.1. *Let $V(u)$, $u > 0$, be a non-Gaussian semistable process, i.e. let the characteristic function of $V(u)$ be given by* (3.1)–(3.3) *with $\sigma = 0$, $b = 0$, and $0 < \alpha < 2$. Let $X(u) = V(u) - b(u)$ be semi-selfsimilar, i.e. let $b(u)$ be given by* (3.4) *and* (3.5). *Then* (2.1) *and* (2.4) *are satisfied for almost all $\omega \in \Omega$.*

4. DISCRETE VERSIONS OF THE LIMIT THEOREMS

We start with some preliminary results.

LEMMA 4.1. *Let $x \in D[0, 1]$. If $\lim_{u \to 1-0} |x(u) - x(1)| < \varepsilon$, then*

$$\limsup_{t \to 1-0} \varrho(x_t, x) < \varepsilon.$$

In particular, if $x(u)$ is continuous in 1, then $\lim_{t \to 1-0} \varrho(x_t, x) = 0$.

Proof. We follow the lines of [15], pp. 293–294. Let $t^*(t) = 1 - \sqrt{1-t}$, $t \in [0, 1]$. Let

$$\lambda_t(u) = \begin{cases} tu, & 0 \leqslant u \leqslant t^*(t), \\ (\sqrt{1-t} + t)(u-1) + 1, & t^*(t) \leqslant u \leqslant 1. \end{cases}$$

Then $\lambda_t\colon [0, 1] \to [0, 1]$, $\lambda_t(0) = 0$, $\lambda_t(1) = 1$, $\lambda_t(u)$ is strictly increasing and continuous. Moreover,

$$\lim_{t\to 1-0} \sup_{u\neq v} \left|\log\left(\frac{\lambda_t(u)-\lambda_t(v)}{u-v}\right)\right| = 0,$$

$$\lim_{t\to 1-0} \sup_{0\leqslant u\leqslant 1} |x_t(u)-x(\lambda_t(u))| = \lim_{u\to 1-0} |x(u)-x(1)| < \varepsilon.$$

These relations imply the result. ∎

LEMMA 4.2. *Let $X(t)$ be a process with trajectories in $D[0, 1]$, and let μ be a probability on $D[0, 1]$. Assume that for any bounded measurable functional $F\colon D[0, 1] \to \boldsymbol{R}$*

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \frac{1}{t} F[X_t(\cdot, \omega)]\, dt = \int_{D[0,1]} F[x]\, d\mu(x) \quad \text{for almost all } \omega\in\Omega. \tag{4.1}$$

Then μ almost all functions in $D[0, 1]$ are continuous at 1.

Proof. Let $\varepsilon > 0$ and let

$$F(x) = \boldsymbol{I}\{x\in D[0, 1]\colon |x(1)-x(1-0)| > \varepsilon\}.$$

Then F is a bounded measurable functional defined on $D[0, 1]$. Applying (4.1) to this functional, we obtain

$$0 = \mu\{x\in D[0, 1]\colon |x(1)-x(1-0)| > \varepsilon\}.$$

Now let $\varepsilon \downarrow 0$. Then the continuity of the measure μ gives the result. ∎

The following result is an extension of Theorem 2 in [15].

THEOREM 4.1. *Let us assume that for any bounded measurable functional $F\colon D[0, 1] \to \boldsymbol{R}$*

$$\lim_{T\to\infty} \int_{D[0,1]} F(x)\, d\mu_{T,\omega}(x) = \int_{D[0,1]} F(x)\, d\mu(x) \quad \text{for almost all } \omega\in\Omega, \tag{4.2}$$

where $\mu_{T,\omega}$ are probability measures depending on $\omega\in\Omega$ and $T\in\boldsymbol{R}$. Assume that μ almost all functions in $D[0, 1]$ are continuous in 1. *Then for almost all $\omega\in\Omega$*

$$\lim_{\eta\to 0} \lim_{T\to\infty} \mu_{T,\omega}\{x\colon \sup_{1-\eta\leqslant t\leqslant 1} \varrho(x_1, x_t) > \varepsilon\} = 0 \quad \text{for any } \varepsilon > 0. \tag{4.3}$$

Proof. Let $\eta > 0$ and $\varepsilon > 0$ be fixed. Let

$$F_{\eta,\varepsilon}(x) = \boldsymbol{I}\{x\in D[0, 1]\colon \sup_{1-\eta\leqslant t\leqslant 1} \varrho(x_1, x_t) > \varepsilon\}.$$

Then $F_{\eta,\varepsilon}$ is a bounded measurable functional defined on $D[0, 1]$. (To see the measurability of $F_{\eta,\varepsilon}$ one can apply the fact that $x \to x_t$ is a measurable mapping which can be checked by using projections; see [4], Theorem 14.5.)

By Lemma 4.1,

$$\lim_{\eta\downarrow 0} \int_{D[0,1]} F_{\eta,\varepsilon}(x)\, d\mu(x) = \lim_{\eta\downarrow 0} \mu\{x\colon \sup_{1-\eta\leqslant t\leqslant 1} \varrho(x_1, x_t) > \varepsilon\} = 0.$$

This implies the result because, by (4.2), for almost all $\omega\in\Omega$

$$\overline{\lim_{T\to\infty}}\, \mu_{T,\omega}\{x\colon \sup_{1-\eta\leqslant t\leqslant 1} \varrho(x_1, x_t) > \varepsilon\}$$
$$= \lim_{T\to\infty} \int_{D[0,1]} F_{\eta,\varepsilon}(x)\, d\mu_{t,\omega}(x) = \int_{D[0,1]} F_{\eta,\varepsilon}(x)\, d\mu(x).$$

The last quantity converges to 0 as $\eta \to 0$. ∎

COROLLARY 4.1. *Assume that the conditions of Theorem* 3.1 *are satisfied. Let* $\mu_{T,\omega}$ *be defined by* (2.3). *Then for almost all* $\omega\in\Omega$ *the relation* (4.3) *is satisfied.*

Proof. We check the conditions of Theorem 4.1. Let μ be defined by (2.2).

By Theorem 3.1, the relation (2.1) is satisfied. But in our case

$$\int_{D[0,1]} F[x]\, d\mu_{T,\omega}(x) = \frac{1}{\log T}\int_1^T \frac{1}{t} F[X_t(\cdot, \omega)]\, dt. \tag{4.4}$$

Consequently, the relation (4.2) is satisfied. Therefore (2.1) and Lemma 4.2 imply that μ almost all functions in $D[0, 1]$ are continuous at 1. So Theorem 4.1 implies the result. ∎

Now we prove discretized versions of Theorem 2.1. In the following theorem we consider random probability measures concentrated on some trajectories $X_{b_{j,n}}(\cdot, \omega)$. The result is a version of the first part of Theorem 3 in [15].

THEOREM 4.2. *Let* $X(t)$ *be a process with trajectories in* $D[0, 1]$, *and let* μ *be a probability on* $D[0, 1]$. *Assume that for any bounded measurable functional* $F\colon D[0, 1] \to \boldsymbol{R}$

$$\lim_{T\to\infty} \frac{1}{\log T}\int_1^T \frac{1}{t} F[X_t(\cdot, \omega)]\, dt \tag{4.5}$$
$$= \int_{D[0,1]} F[x]\, d\mu(x) \quad \textit{for almost all } \omega\in\Omega.$$

For any n let $1 = b_{1,n} < b_{2,n} < \ldots < b_{k_n,n} = b_n$ *be an increasing sequence of real numbers with the following properties*:

(4.6) $$\lim_{n\to\infty} b_n = \infty, \quad \lim_{n\to\infty} \frac{\log b_{j,n}}{\log b_n} = 0 \text{ for any fixed } j,$$

and

(4.7) $$\lim_{j\to\infty} \sup_{(k,n):j\leqslant k<k_n} \frac{b_{k+1,n}}{b_{k,n}} = 1.$$

For any fixed ω *and n define the random probability measure* $\hat{\mu}_{n,\omega}$ *on* $D[0, 1]$ *to be concentrated on the trajectories* $X_{b_{j,n}}(\cdot, \omega)$, $j = 1, \ldots, k_n - 1$, *and satisfying*

(4.8) $$\hat{\mu}_{n,\omega}\big(X_{b_{j,n}}(\cdot, \omega)\big) = \frac{1}{\log b_n} \int_{b_{j,n}}^{b_{j+1,n}} \frac{1}{u} du, \quad j = 1, \ldots, k_n - 1.$$

Then for almost all ω

(4.9) $$\lim_{n\to\infty} \hat{\mu}_{n,\omega} = \mu.$$

We need the following lemma.

LEMMA 4.3 (Lemma B in [15]). *Let* (M, ϱ) *be a complete separable metric space, let* $\mathscr{B}$ *denote the* σ*-algebra of its Borel sets. Let* μ, μ_n, *and* $\tilde{\mu}_n$, $n = 1, 2, \ldots,$ *be probability measures on* $(M, \mathscr{B})$. *Assume that* $\lim_{n\to\infty} \mu_n = \mu$. *Assume also that for any* $\varepsilon > 0$ *and for any compact set F*

(4.10) $$\liminf_{n\to\infty} \big(\tilde{\mu}_n(F^\varepsilon) - \mu_n(F)\big) \geqslant 0,$$

where $F^\varepsilon = \{x \in M: \varrho(x, F) < \varepsilon\}$. *Then* $\lim_{n\to\infty} \tilde{\mu}_n = \mu$.

Proof of Theorem 4.2. First we remark that, by (4.5), for the probability measures $\mu_{T,\omega}$ defined by (2.3) we have $\lim_{n\to\infty} \mu_{b_n,\omega} = \mu$ for almost all $\omega \in \Omega$. Therefore, by Lemma 4.3, it is enough to prove that for almost all ω, any $\varepsilon > 0$, and any compact set F

(4.11) $$\liminf_{n\to\infty} \big(\hat{\mu}_{n,\omega}(F^\varepsilon) - \mu_{b_n,\omega}(F)\big) \geqslant 0.$$

Define a probability measure on $[1, T]$ as follows:

$$\nu_T(C) = \frac{1}{\log T} \int_C \frac{1}{t} dt$$

for any Borel set $C \subseteq [1, T]$. Then for any Borel set $B \subseteq D[0, 1]$ we have

$$\mu_{b_n,\omega}(B) = \nu_{b_n}\{s:\ s \in [1, b_n),\ X_s(\cdot, \omega) \in B\},$$

$$\hat{\mu}_{n,\omega}(B) = \nu_{b_n}\{s:\ \exists j,\ 1 \leqslant j < k_n, \text{ such that } b_{j,n} \leqslant j < b_{j+1,n},\ X_{b_{j,n}}(\cdot, \omega) \in B\}.$$

For $\eta > 0$ and $\varepsilon > 0$ let

$$A(\varepsilon, \eta) = \{x \in D[0, 1]:\ \sup_{1-\eta \leqslant t \leqslant 1} \varrho(x_1, x_t) < \varepsilon\}.$$

By Lemma 4.2, μ almost all functions in $D[0, 1]$ are continuous at 1. Therefore we can apply Theorem 4.1. Consequently, for almost all ω we have: for fixed $\varepsilon > 0$ and $\delta > 0$ we can find $\eta > 0$ and n_0 such that

$$\mu_{b_n,\omega}\big(A(\varepsilon, \eta)\big) > 1 - \delta \quad \text{if } n > n_0. \tag{4.12}$$

For the above fixed $\varepsilon > 0$, $\delta > 0$, $\eta > 0$, and n_0 we choose $j_0 = j_0(\eta)$ and $n_1 > n_0$ such that

$$1 \leqslant \frac{b_{k+1,n}}{b_{k,n}} < 1 + \frac{\eta}{2} \quad \text{if } n \geqslant n_1 \text{ and } j_0 \leqslant k < k_n, \tag{4.13}$$

$$\frac{\log b_{j_0,n}}{\log b_n} < \delta \quad \text{if } n \geqslant n_1. \tag{4.14}$$

(This is possible because of (4.6) and (4.7).) Now we have

$$\begin{aligned}
\hat{\mu}_{n,\omega}(F^\varepsilon) &= \nu_{b_n}\{s:\ \exists j,\ 1 \leqslant j < k_n, \text{ such that } b_{j,n} \leqslant s < b_{j+1,n},\ X_{b_{j,n}}(\cdot, \omega) \in F^\varepsilon\} \\
&\geqslant \nu_{b_n}\{s:\ \exists j,\ j_0 \leqslant j < k_n, \text{ such that } b_{j,n} \leqslant s < b_{j+1,n},\ X_{b_{j,n}}(\cdot, \omega) \in F^\varepsilon\} \\
&= \sum_{j=j_0}^{k_n-1} \nu_{b_n}\{s:\ b_{j,n} \leqslant s < b_{j+1,n},\ X_{b_{j,n}}(\cdot, \omega) \in F^\varepsilon\} \\
&\geqslant \sum_{j=j_0}^{k_n-1} \nu_{b_n}\left\{s:\ b_{j,n} \leqslant s < b_{j+1,n},\ X_s(\cdot, \omega) \in F \cap A\left(\varepsilon, \frac{b_{j,n}}{b_{j+1,n}}\right)\right\} \\
&\geqslant \sum_{j=j_0}^{k_n-1} \nu_{b_n}\{s:\ b_{j,n} \leqslant s < b_{j+1,n},\ X_s(\cdot, \omega) \in F\} \\
&\quad - \sum_{j=j_0}^{k_n-1} \nu_{b_n}\left\{s:\ b_{j,n} \leqslant s < b_{j+1,n},\ X_s(\cdot, \omega) \notin A\left(\varepsilon, \frac{b_{j,n}}{b_{j+1,n}}\right)\right\} \\
&\geqslant \sum_{j=1}^{k_n-1} \nu_{b_n}\{s:\ b_{j,n} \leqslant s < b_{j+1,n},\ X_s(\cdot, \omega) \in F\} \\
&\quad - \sum_{j=1}^{j_0-1} \nu_{b_n}\{s:\ b_{j,n} \leqslant s < b_{j+1,n},\ X_s(\cdot, \omega) \in F\}
\end{aligned}$$

$$-\sum_{j=j_0}^{k_n-1} \nu_{b_n}\left\{s:\ b_{j,n} \leqslant s < b_{j+1,n},\ X_s(\cdot,\omega) \notin A\left(\varepsilon, \frac{b_{j,n}}{b_{j+1,n}}\right)\right\}$$

$$\geqslant \mu_{b_n,\omega}(F) - \nu_{b_n}([1, b_{j_0})) - \mu_{b_n,\omega}(\overline{A(\varepsilon,\eta)}) \geqslant \mu_{b_n,\omega}(F) - 2\delta.$$

In the last steps we have used (4.12), (4.13), and (4.14). ∎

In the following theorem we consider random probability measures concentrated on step functions constructed from the trajectories $X_{b_{j,n}}(\cdot,\omega)$. The result is a generalization of the second part of Theorem 3 in [15].

THEOREM 4.3. *Let $X(t)$ be a process with trajectories in $D[0,1]$, and let μ be a probability on $D[0,1]$. For any n let $0 = b_{0,n}$, $1 = b_{1,n} < b_{2,n} < \ldots < b_{k_n,n} = b_n$ be an increasing sequence of real numbers. For any fixed ω and n define the random probability measure $\hat{\mu}_{n,\omega}$ on $D[0,1]$ to be concentrated on the trajectories $X_{b_{j,n}}(\cdot,\omega)$, $j = 1, \ldots, k_n-1$. For any $\omega\in\Omega$ define the random step lines $\bar{X}_{b_{j,n}}(\cdot,\omega)$, $j = 1, \ldots, k_n-1$, as*

$$\bar{X}_{b_{j,n}}(s,\omega) = X_{b_{j,n}}\left(\frac{b_{l-1,n}}{b_{j,n}}, \omega\right) \quad \text{if } \frac{b_{l-1,n}}{b_{j,n}} \leqslant s < \frac{b_{l,n}}{b_{j,n}},\ l = 1, \ldots, j, \tag{4.15}$$

and

$$\bar{X}_{b_{j,n}}(1,\omega) = X_{b_{j,n}}(1,\omega).$$

Let the random probability measures $\bar{\mu}_{n,\omega}$ be concentrated on the above random step lines such that

$$\bar{\mu}_{n,\omega}(\bar{X}_{b_{j,n}}(\cdot,\omega)) = \hat{\mu}_{n,\omega}(X_{b_{j,n}}(\cdot,\omega)) = \frac{1}{\log b_n}\int_{b_{j,n}}^{b_{j+1,n}} \frac{1}{u}\,du, \quad j = 1, \ldots, k_n-1. \tag{4.16}$$

Suppose that for each fixed j the sequence $\{b_{j,n}:\ k_n \geqslant j\}$ is bounded,

$$\lim_{j\to\infty}\ \inf_{\{n:k_n\geqslant j\}} b_{j,n} = \infty, \tag{4.17}$$

and conditions (4.6) and (4.7) are satisfied. Assume that for almost all ω

$$\lim_{n\to\infty} \hat{\mu}_{n,\omega} = \mu.$$

Then for almost all ω

$$\lim_{n\to\infty} \bar{\mu}_{n,\omega} = \mu.$$

To prove this theorem we need the following preparation.

LEMMA 4.4 (Corollary of Lemma B in [15]). *Let (M,ϱ) be a complete separable metric space, let $\mathscr{B}$ denote the σ-algebra of its Borel sets. Let μ, μ_n, and*

$\tilde{\mu}_n$, $n = 1, 2, \ldots$, *be probability measures on* $(M, \mathscr{B})$. *Assume that for any* $\varepsilon > 0$ *and* $\delta > 0$ *there exist probability measures* $P_n^{\varepsilon,\delta}$, $n = 1, 2, \ldots$, *on the product space* $(M \times M, \mathscr{B} \times \mathscr{B})$ *such that*

(i) *the marginal distributions are* μ_n *and* $\tilde{\mu}_n$, *i.e.* $P_n^{\varepsilon,\delta}(A \times M) = \mu_n(A)$ *and* $P_n^{\varepsilon,\delta}(M \times A) = \tilde{\mu}_n(A)$ *for each* $A \in \mathscr{B}$, $n = 1, 2, \ldots$;

(ii) $\limsup_{n\to\infty} P_n^{\varepsilon,\delta}(\{(x, y)\colon \varrho(x, y) > \varepsilon\}) \leqslant \delta$.

If $\lim_{n\to\infty} \mu_n = \mu$, *then* $\lim_{n\to\infty} \tilde{\mu}_n = \mu$.

If (i) and (ii) are satisfied, then we say that μ_n and $\tilde{\mu}_n$ have a *good coupling*.

LEMMA 4.5 (Lemma C in [15]). *For* $x \in D[0, 1]$ *and* $\delta > 0$ *let*

$$g(x, \delta) = \sup\{\varrho(x, \bar{x}_{t_0,\ldots,t_s})\colon 0 = t_0 < t_1 < \ldots < t_s = 1,$$
$$t_j - t_{j-1} \leqslant \delta, j = 1, \ldots, s\},$$

where $\bar{x}_{t_0,\ldots,t_s}(t) = x(t_{j-1})$ *if* $t_{j-1} \leqslant t < t_j$, $j = 1, \ldots, s$, *and* $\bar{x}_{t_0,\ldots,t_s}(1) = x(1)$.

Then for any $x \in D[0, 1]$ *we have*

$$\lim_{\delta\to 0} g(x, \delta) = 0.$$

Moreover, for any compact set $K \subset D[0, 1]$ *we have*

$$\lim_{\delta\to 0} \sup_{x\in K} g(x, \delta) = 0.$$

Proof of Theorem 4.3. Apply Lemma 4.4 with $\mu_n = \hat{\mu}_{n,\omega}$ and $\tilde{\mu}_n = \bar{\mu}_{n,\omega}$. Define $P_n^{\varepsilon,\delta} = P_{n,\omega}$, $n = 1, 2, \ldots$, on the product space $(D[0, 1] \times D[0, 1])$ such that

$$(4.18) \qquad P_{n,\omega}\big(X_{b_{j,n}}(\cdot, \omega), \bar{X}_{b_{j,n}}(\cdot, \omega)\big) = \hat{\mu}_{n,\omega}\big(X_{b_{j,n}}(\cdot, \omega)\big) = \frac{1}{\log b_n} \int_{b_{j,n}}^{b_{j+1,n}} \frac{1}{u}\, du.$$

Then the marginal distributions of $P_{n,\omega}$ are $\hat{\mu}_{n,\omega}$ and $\bar{\mu}_{n,\omega}$. So it is enough to prove that for almost all ω

$$(4.19) \qquad \lim_{n\to\infty} P_{n,\omega}\big(A_n(\varepsilon, \omega)\big) = 0,$$

where

$$A_n(\varepsilon, \omega) = \big\{\big(X_{b_{j,n}}(\cdot, \omega), \bar{X}_{b_{j,n}}(\cdot, \omega)\big)\colon \varrho\big(X_{b_{j,n}}(\cdot, \omega), \bar{X}_{b_{j,n}}(\cdot, \omega)\big) > \varepsilon\big\},$$

where $\varepsilon > 0$ is arbitrary. As the sequence $\hat{\mu}_{n,\omega}$ is convergent, it is tight; therefore for any $\eta > 0$ there exists a compact set K such that $\hat{\mu}_{n,\omega}(K) > 1 - \eta$ for all $n = 1, 2, \ldots$ Therefore, to prove (4.19), it is enough to establish

$$(4.20) \qquad \lim_{n\to\infty} P_{n,\omega}\big(A_n(\varepsilon, \omega) \cap (K \times D[0, 1])\big) = 0$$

for any compact set K.

Now for any fixed $\eta > 0$ define $m(n) = m(n, \eta) = \max\{j: \log b_{j,n} \leqslant \eta \log b_n\}$. By (4.6), we have $\lim_{n\to\infty} m(n) = \infty$, and by (4.18)

$$\hat{\mu}_{n,\omega}\Big\{\bigcup_{j:j\leqslant m(n)} X_{b_{j,n}}(\cdot, \omega)\Big\} \leqslant \eta.$$

Let

$$A_n^\eta(\varepsilon, \omega) = \{(X_{b_{j,n}}(\cdot, \omega), \bar{X}_{b_{j,n}}(\cdot, \omega)): m(n, \eta) < j \leqslant k_n,$$
$$\varrho(X_{b_{j,n}}(\cdot, \omega), \bar{X}_{b_{j,n}}(\cdot, \omega)) > \varepsilon\}.$$

We have $P_{n,\omega}(A_n(\varepsilon, \omega)\setminus A_n^\eta(\varepsilon, \omega)) \leqslant \eta$. Consequently, instead of (4.20) we have to prove

(4.21) $$\lim_{n\to\infty} P_{n,\omega}(A_n^\eta(\varepsilon, \omega)\cap(K\times D[0, 1])) = 0.$$

Conditions (4.6), (4.7), (4.17) and the boundedness of the sequence $\{b_{j,n}: k_n \geqslant j\}$ imply that for any $\delta > 0$

(4.22) $$\sup_{m\leqslant j\leqslant k_n} \sup_{1\leqslant l\leqslant j} \left(\frac{b_{l,n}}{b_{j,n}} - \frac{b_{l-1,n}}{b_{j,n}}\right) < \delta$$

if n and m are large enough. This relation and Lemma 4.5 imply that

$$\lim_{n\to\infty} \sup_{\{j:j>m(n), X_{b_{j,n}}(\cdot,\omega)\in K\}} \varrho(X_{b_{j,n}}(\cdot, \omega), \bar{X}_{b_{j,n}}(\cdot, \omega)) = 0$$

for each compact set $K \in D[0, 1]$. Thus we obtain (4.21). ∎

5. A FUNCTIONAL LIMIT THEOREM FOR SEMISTABLE DISTRIBUTIONS

The following theorem is an a.s. functional limit theorem for semistable distributions.

THEOREM 5.1. *Let $\xi_1, \xi_2, \ldots$ be independent identically distributed random variables with non-Gaussian semistable distribution, i.e. the characteristic function of ξ_i is $\psi(1, x, 0, 0, L(y), R(y))$, where ψ is defined by* (3.1), *$L(y)$ and $R(y)$ satisfy* (3.2) *and* (3.3), *respectively. For the exponent assume $0 < \alpha < 2$ and for the period $c > 1$. Let $b(u)$ be defined by* (3.4)–(3.5). *Let $S_n = \xi_1 + \ldots + \xi_n - b(n)$, $n = 1, 2, \ldots, S_0 = 0$. Let*

$$Y_n(s, \omega) = \frac{S_{l-1}(\omega)}{n^{1/\alpha}}, \quad \frac{l-1}{n} \leqslant s < \frac{l}{n}, \ l = 1, \ldots, n,$$

and $Y_n(1, \omega) = S_n(\omega)/n^{1/\alpha}$. Let the measure $\mu_{n,\omega}$ on $D[0, 1]$ be defined by

$$\mu_{n,\omega}(A) = \frac{1}{\log n} \sum_{l=1}^{n-1} [\log(l+1) - \log l]\, I_A(Y_n(\cdot, \omega))$$

for any Borel set $A \subseteq D[0,1]$. Then for almost all ω

$$\lim_{n\to\infty} \mu_{n,\omega} = \mu,$$

where μ is defined by

$$\mu = \frac{1}{\log c} \int_1^c \frac{1}{t} \mu_t \, dt,$$

and μ_t is the distribution of $X_t(\cdot) = X(t\cdot)/t^{1/\alpha}$ while $X(u)$ is the semistable process defined in Theorem 2.1.

Proof: Let $b_{j,n} = j$ for $j = 0, 1, \ldots, n$ (in particular, $k_n = n$) and $n = 1, 2, \ldots$ It satisfies the conditions of Theorems 4.2 and 4.3.

Let $X(t)$ be the semistable process defined in Theorem 3.1. We can choose $S_n = X(n)$, $n = 0, 1, 2, \ldots$ Theorems 3.1, 4.2, and 4.3 give the result. ∎

Acknowledgement. The research was partially realized while the first-named author was visiting Institute of Mathematics, Maria Curie-Skłodowska University, Lublin, Poland. The first-named author was supported by the Hungarian Foundation of Scientific Researches under Grant No. OTKA T047067/2004 and Grant No. OTKA T048544/2005. He is grateful to Professor Péter Major for helpful conversations. Research of the second-named author was supported by DFG through the German-Polish project 436 POL 113/115/0-1.

REFERENCES

[1] P. Becker-Kern, *Almost sure limit theorems of mantissa type*, submitted to Acta Math. Hungar., 2005.

[2] I. Berkes and E. Csáki, *A universal result in almost sure central limit theory*, Stochastic Process. Appl. 94 (1) (2001), pp. 105–134.

[3] I. Berkes, E. Csáki, S. Csörgő and Z. Megyesi, *Almost sure limit theorems for sums and maxima from the domain of geometric partial attraction of semistable laws*, in: *Limit Theorems in Probability and Statistics*, Vol. I, I. Berkes, E. Csáki and M. Csörgő (Eds.), János Bolyai Math. Soc., Budapest 2002, pp. 133–157.

[4] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York 1968.

[5] G. A. Brosamler, *An almost everywhere central limit theorem*, Math. Proc. Cambridge Philos. Soc. 104 (1988), pp. 561–574.

[6] A. N. Chuprunov and I. Fazekas, *Integral analogues of almost sure limit theorems*, Period. Math. Hungar. 50 (1–2) (2005), pp. 61–78.

[7] R. M. Dudley, *Real Analysis and Probability*, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1989.

[8] I. Fazekas and Z. Rychlik, *Almost sure functional limit theorems*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 56 (2002), pp. 1–18.

[9] I. Fazekas and Z. Rychlik, *Almost sure central limit theorems for random fields*, Math. Nachr. 259 (2003), pp. 12–18.

[10] I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes*, W. B. Saunders Co., Philadelphia–London–Toronto 1969.

[11] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley, Reading, Massachusetts, 1954.

[12] V. M. Kruglov, *A certain extension of the class of stable distributions* (in Russian), Teor. Veroyatnost. i Primenen. 17 (4) (1972), pp. 723–732.

[13] M. T. Lacey and W. Philipp, *A note on the almost sure central limit theorem*, Statist. Probab. Lett. 9 (2) (1990), pp. 201–205.

[14] P. Lévy, *Théorie de l'addition des variables aléatoires*, Gauthier-Villars, Paris 1937.

[15] P. Major, *Almost sure functional limit theorems, Part I. The general case*, Studia Sci. Math. Hungar. 34 (1998), pp. 273–304.

[16] P. Major, *Almost sure functional limit theorems, Part II. The case of independent random variables*, Studia Sci. Math. Hungar. 36 (2000), pp. 231–273.

[17] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge 1999.

[18] P. Schatte, *On strong versions of the central limit theorem*, Math. Nachr. 137 (1988), pp. 249–256.

István Fazekas
Faculty of Informatics
University of Debrecen
P.O. Box 12
H-4010 Debrecen, Hungary
E-mail: fazekasi@inf.unideb.hu

Zdzisław Rychlik
Institute of Mathematics
Maria Curie-Skłodowska University
pl. M. Curie-Skłodowskiej 1
20-031 Lublin, Poland
E-mail: rychlik@golem.umcs.lublin.pl

Received on 13.9.2005