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CENTRAL LIMIT THEOREMS FOR RANDOM STAIN

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Abstract. The Central Limit Theorem in $\boldsymbol{R}$ and $\boldsymbol{R}^n$ is proved for two-parameter stochastic process proposed in [6] as a model of a stain of pollution.

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1. INTRODUCTION

We are concerned with the Central Limit Theorems for a certain class of two-parameter stochastic processes. This class has been proposed by Hensz-Chądzyńska et al. in [5] and [6] as a stochastic model of a stain of pollution. The stain is assumed to spread out radially from the center of its initial source (the origin) and to have a random shape at each moment. Moreover, many little "springs" of the further expansion of the stain appear irregularly on its edge. Therefore, the evolution in time of the edge of the stain may be treated as a realization of a two-parameter stochastic process $\xi(\cdot, \cdot)$ parametrized with time and direction. For fixed moment t and direction α, $\xi(t, \alpha)$ is a random distance from the origin of the boundary point of the stain at moment t, with radial coordinate α. The process $\xi(\cdot, \cdot)$ will be called a *random stain.* It is natural to assume that with probability one, for each direction α, the trajectory $\xi(\cdot, \alpha)$ is a non-decreasing continuous function of time and the velocity exists; moreover, the "springs" of expansion appear in Poissonian way. Therefore, the stain $\xi(\cdot, \cdot)$ is defined by formulas (2.1) and (2.2) in Section 2.

In [5] and [6] the asymptotic shape of the stain has been investigated and suitable Laws of Large Numbers are proved.

In this paper the asymptotic behavior in time of the distribution of the stain is studied.

In Sections 3 and 4 we consider one-dimensional distributions of the stain in a fixed direction. Under the natural assumption that the intensity function of increase has polynomial growth we prove the Central Limit Theorem: first for the stain with discrete time (Theorem 3.1) and then for the stain with continuous time (Theorem 4.1). Moreover, under the assumption of Theorem 3.1 we obtain a uniform estimate of the rate of convergence to the standard normal distribution function (Corollary 3.3). It is also shown that if the intensity function of increase has exponential growth, then the stain (even with discrete time) does not satisfy the Central Limit Theorem in any direction (Theorem 3.4).

In Section 5 we consider multidimensional distribution of the stain in fixed directions $\alpha_1, \ldots, \alpha_m$. Under certain additional assumptions we prove that the multidimensional Central Limit Theorem holds (Theorems 5.1 and 5.2) and we obtain explicit formulas for the limit covariance matrix (Theorem 5.2).

2. NOTATION AND PRELIMINARIES

We follow the notation used in [5] and [6]. By $S_1 = \{e^{i\alpha}: \alpha \in I\}$, where $I = [0, 2\pi)$, we denote the unit circle on the plane. We shall often identify points of S_1 with elements of I, in a natural way, with addition modulo 2π in I. Let f be a Borel nonnegative function defined on S_1 and ψ be a continuous nonnegative function defined on $[0, \infty)$. We denote by $(\pi_t, t \geqslant 0)$ the Poisson process with parameter $\lambda > 0$ and let $\tilde{t}_j$ be the time of occurrence of the j-th event for the process $(\pi_t, t \geqslant 0)$, $j = 1, 2, \ldots$ Moreover, let $(\alpha_j)_{j=1,2,\ldots}$ be a sequence of independent random variables uniformly distributed on S_1, independent of the process $(\pi_t, t \geqslant 0)$.

As in [6] we define the following two-parameter stochastic processes:

$$V(t, \alpha) = \sum_{\tilde{t}_i < t} \psi(\tilde{t}_i) f(\alpha - \alpha_i), \qquad t \geqslant 0, \ \alpha \in I, \tag{2.1}$$

and

$$\xi(T, \alpha) = \int_0^T V(t, \alpha)\, dt, \qquad T \geqslant 0, \ \alpha \in I. \tag{2.2}$$

Following [5] and [6] we call the process $\big(\xi(T, \alpha), T \geqslant 0, \alpha \in I\big)$ a ***stain with continuous time***. $V(t, \alpha)$ is called a *velocity of expansion* of the stain ξ at the moment t and in the direction α. The function f is understood as a 'spring' of further expansion of the stain. Finally, ψ is called the *intensity function* of ξ. Therefore, it is natural to assume that ψ is increasing in time. We consider two types of the growth of ψ: polynomial and exponential.

A ***stain with discrete time*** is defined in [6] as follows:

$$V(t, \alpha) = \sum_{\tau=0}^{t} \psi(\tau) \sum_{\{i: \tau \leqslant \tilde{t}_i < \tau+1\}} f(\alpha - \alpha_i) \quad \text{for } t = 0, 1, 2, \ldots, \tag{2.3}$$

where $\psi(\tau)$, $\tau = 0, 1, \ldots$, is a fixed sequence of positive numbers (describing the intensity of expansion of the stain). We set

$$\xi(T, \alpha) = \sum_{t=0}^{T-1} V(t, \alpha), \tag{2.4}$$

hence

$$\xi(T, \alpha) = \sum_{\tau=0}^{T-1} (T-\tau)\psi(\tau) X_\tau(\alpha), \tag{2.5}$$

where

$$X_\tau(\alpha) = \sum_{\{i:\, \tau \leqslant t_i < \tau+1\}} f(\alpha-\alpha_i). \tag{2.6}$$

The process $X_t(\alpha)$ can be represented equivalently (in the sense of probability distribution) in the following form (see [5] and [6]):

$$X_t(\alpha) = \sum_{i=1}^{N_t} f(\alpha-\alpha_{i,t}), \tag{2.7}$$

where $(N_\tau)_{\tau=0,1,\ldots}$ is a sequence of identically distributed independent random variables with Poisson distribution (with parameter $\lambda > 0$) and $(\alpha_{i,\tau})_{i=1,2,\ldots;\, \tau=0,1,\ldots}$ is a matrix of independent random variables uniformly distributed on I and independent of $(N_\tau)_{\tau=0,1,\ldots}$.

Remark 2.1. In both cases (2.6) and (2.7), $X_t(\alpha)$, $t = 0, 1, \ldots$, is a sequence of independent stochastic processes with the same finite-dimensional distributions stationary with respect to α.

3. CENTRAL LIMIT THEOREM IN A FIXED DIRECTION FOR A STAIN WITH DISCRETE TIME

THEOREM 3.1. *Let $\xi = \xi(T, \alpha)$, $T = 0, 1, \ldots$, be a stain with discrete time defined by* (2.5) *and* (2.6) *and let $\psi(t) = t^r$, $r \geqslant 0$. Then the stain satisfies the Central Limit Theorem in an arbitrary fixed direction α, that is*

$$\bar{\xi}_T = \frac{\xi(T, \alpha) - E[\xi(T, \alpha)]}{\sqrt{D^2 \xi(T, \alpha)}} \xrightarrow{d} \mathcal{N}(0, 1) \quad \text{as } T \to \infty. \tag{3.1}$$

In the proof of the theorem we use the following simple lemma.

LEMMA 3.2. *For $r \geqslant 0$ and $T = 1, 2, \ldots$ let*

$$c_T^2 = \sum_{t=0}^{T-1} (T-t)^2 t^{2r} \quad \text{and} \quad J_T = \int_0^T (T-t)^2 t^{2r}\, dt.$$

Then

$$(3.2)\qquad c_T^2/J_T \to 1 \quad \text{as } T \to \infty.$$

If (3.2) holds, then we say that the *asymptotic behavior of the sequences* $(c_T^2)_{T=1}^{\infty}$ *and* $(J_T)_{T=1}^{\infty}$ *is the same* and we write $c_T^2 \approx J_T$ as $T \to \infty$.

Proof of Theorem 3.1. Let us put, for a fixed α, $Y_t^T = (T-t)t^r X_t$, where X_t is defined in (2.7). Then

$$\xi(T, \alpha) = \sum_{t=0}^{T-1} Y_t^T.$$

Recall that X_t, $t = 1, 2, \ldots$, are mutually independent and identically distributed random variables and let us write $E[|X_t - EX_t|^3] := B$. Then $B < \infty$ and

$$E[|Y_t^T - EY_t^T|^3] = (T-t)^3 t^{3r} B.$$

Hence for any T the random variables $Y_0^T, \ldots, Y_{T-1}^T$ are independent.

To prove (3.1) we shall show that Liapunov's condition holds, i.e.

$$(3.3)\qquad \frac{1}{C_T^3} \sum_{t=0}^{T-1} E[|Y_t^T - EY_t^T|^3] \to 0 \quad \text{as } T \to \infty,$$

where $C_T^2 = \sum_{t=0}^{T-1} D^2 Y_t^T$. We first need to estimate

$$(3.4)\qquad \sum_{t=0}^{T-1} E[|Y_t^T - EY_t^T|^3] = \sum_{t=0}^{T-1} (T-t)^3 t^{3r} B$$

$$\leqslant B \int_0^T (T+1-t)^3 t^{3r}\, dt \leqslant B\left[(T+1)^3 \left\{\frac{T^{3r+1}}{3r+1}\right\} + 3(T+1)\left\{\frac{T^{3r+3}}{r+1}\right\}\right]$$

$$\leqslant BQ(r)(T+1)^3 T^{3r+1}, \quad \text{where } Q(r) = \frac{4r+2}{(3r+1)(r+1)}.$$

Now we want to find the order of C_T:

$$C_T^2 = \sum_{t=0}^{T-1} D^2 Y_t^T = \sum_{t=0}^{T-1} (T-t)^2 t^{2r} \sigma^2,$$

where we write $\sigma^2 = D^2 X_t$. Then, by Lemma 3.2,

$$C_T^2 \approx \sigma^2 \int_0^T (T-t)^2 t^{2r}\, dt = \sigma^2 \left[T^2 \int_0^T t^{2r}\, dt - 2T \int_0^T t^{2r+1}\, dt + \int_0^T t^{2r+2}\, dt\right]$$

$$= \sigma^2 [(2r+1)(r+1)(2r+3)]^{-1} T^{2r+3}.$$

Hence

$$(3.5)\qquad C_T^3 \approx H(r)\sigma^3 T^{3r+9/2}, \quad \text{where } H(r) = [(2r+1)(r+1)(2r+3)]^{-3/2}.$$

Now, from (3.3)–(3.5) we get

$$\frac{1}{C_T^3}\sum_{t=0}^{T-1} E[|Y_t^T - EY_t^T|^3] \approx \frac{1}{H(r)\,T^{3r+9/2}}\sum_{t=0}^{T-1} E[|Y_t^T - EY_t^T|^3]$$
$$\leqslant \frac{Q(r)}{H(r)} B\left(\frac{T+1}{T}\right)^3 T^{-1/2} \to 0 \quad \text{as } T\to\infty.$$

This means that the Liapunov condition is satisfied, which implies that the Central Limit Theorem holds. ∎

In Corollary 3.3 below we give a uniform estimate of the rate of convergence in Theorem 3.1. This is a consequence of the result by Berry and Esseen [2].

COROLLARY 3.3. *Let Φ be the standard normal distribution function and $\bar{F}_T(x)$ denote the distribution function of the random variable $\bar{\xi}_T$ defined in* (3.1). *Then we have the following asymptotic bound:*

$$\operatorname*{Sup}_{x\in \boldsymbol{R}} |\bar{F}_T(x) - \Phi(x)| \lessapprox c\frac{B}{\sigma^3} G(r)\, T^{-1/2},$$

where c is an absolute constant, $B = E[|X_t - EX_t|^3]$, $\sigma^2 = E[|X_t - EX_t|^2]$ and $G(r) = O(r^{7/2})$.

Proof. The rapidity of convergence can be evaluated by the Berry and Esseen theorem ([2], p. 7; [3], p. 515), where we put $\tilde{Y}_k = Y_k^T - EY_k^T$ and Y_k^T, $k = 0, \ldots, T-1$, is defined in the first line of the proof of Theorem 3.1. Then

$$\operatorname*{Sup}_{x\in \boldsymbol{R}} |\bar{F}_T(x) - \Phi(x)| \lessapprox CS_T^{-3}\sum_{k=1}^{T} \beta_{3k}, \qquad -\infty \leqslant x \leqslant \infty,$$

where

$$\beta_{3k} = E[|Y_k^T - EY_k^T|^3] < \infty, \qquad \sigma_k^2 = D^2 Y_k \quad \text{and} \quad S_T^2 = \sum_{k=0}^{T-1} \sigma_k^2 = C_T^2.$$

Therefore, by (3.4) and (3.5) we obtain

$$\operatorname*{Sup}_{x\in \boldsymbol{R}} |\bar{F}_T(x) - \Phi(x)| \lessapprox C\frac{BQ(r)(T+1)^3\, T^{3r+1}}{\sigma^3 H(r)\, T^{3r+9/2}} \lessapprox C\frac{B}{\sigma^3} G(r)\, T^{-1/2},$$

where $G(r) = Q(r)/H(r) = O(r^{-1})/O(r^{-9/2}) = O(r^{7/2})$. ∎

THEOREM 3.4. *Let $\xi = \xi(T, \alpha)$, $T = 0, 1, \ldots$, be a stain with discrete time, $f \equiv 1$, and $\psi(t) = e^t$. Then the stain does not satisfy the Central Limit Theorem in any direction.*

Proof. Let us put, for a fixed α, $Y_t^T = (T-t)\,e^t X_t$, where now $f \equiv 1$; then

$$X_t = \sum_{i=1}^{N_t} f(\alpha - \alpha_i) = N_t$$

and N_t has Poisson distribution with parameter λ. We assume for simplicity that $\lambda = 1$. Recall that

$$\xi(T, \alpha) = \sum_{t=0}^{T-1} Y_t^T.$$

We shall show that the characteristic function of

$$Z_T := \frac{\xi(T, \alpha) - E\xi(T, \alpha)}{\sqrt{D^2 \xi(T, \alpha)}}$$

does not converge to the characteristic function of $\mathcal{N}(0, 1)$.

We have

$$Z_T = \frac{1}{\beta_T} \sum_{t=0}^{T-1} (T-t) e^t (N_t - 1), \quad \text{where } \beta_T^2 = D^2 \xi(T, \alpha) = \sum_{t=0}^{T-1} (T-t)^2 e^{2t}.$$

Then

$$\frac{\beta_T^2}{e^{2T}} = \sum_{k=1}^{T} k^2 e^{-2k} \to \sum_{k=1}^{\infty} k^2 e^{-2k} = \frac{e^{-2} + e^{-4}}{(1-e^{-2})^3} \quad \text{as } T \to \infty$$

and $\beta_T^2 \approx \gamma^2 e^{2T}$, where $\gamma^2 = (e^{-2} + e^{-4})/(1-e^{-2})^3$.

We can write

$$Z_T := \frac{\gamma e^T}{\beta_T} \tilde{Z}_T, \quad \text{where } \frac{\gamma e^T}{\beta_T} \to 1,$$

and

$$\tilde{Z}_T := \frac{\xi(T, \alpha) - E\xi(T, \alpha)}{\gamma e^T} = \frac{1}{\gamma e^T} \sum_{t=0}^{T-1} (T-t) e^t (N_t - 1).$$

Then the sequence (Z_T) is convergent in distribution if and only if so is $(\tilde{Z}_T)$ (see e.g. [1], Ex. 25.7, or [7], p. 141), and in this case

$$\lim_{T \to \infty} \frac{\varphi_{Z_T}(s)}{\varphi_{\tilde{Z}_T}(s)} = 1 \quad \text{for each } s \in \mathbf{R}.$$

By properties of characteristic functions we have

$$\varphi_{\tilde{Z}_T}(s) = \varphi_{\gamma e^T \tilde{Z}_T}\left(\frac{s}{\gamma e^T}\right) = \prod_{t=0}^{T-1} \varphi_{(N_t - 1)}\left(\frac{s}{\gamma}(T-t) e^{-(T-t)}\right)$$

since

$$\gamma e^T \tilde{Z}_T = \sum_{t=0}^{T-1} (T-t) e^t (N_t - 1).$$

Recall that $\varphi_{N_t-1}(s) = \exp(e^{is}-is-1)$, and then

$$\varphi_{\tilde{Z}_T}(s) = \prod_{t=0}^{T-1} \exp\left[\exp\left(\frac{is}{\gamma}(T-t)e^{-(T-t)}\right)-\frac{is}{\gamma}(T-t)e^{-(T-t)}-1\right]$$

$$= \exp\left[\sum_{t=0}^{T-1} \exp\left(\frac{is}{\gamma}(T-t)e^{-(T-t)}\right)-i\sum_{t=1}^{T}\frac{s}{\gamma}(T-t)e^{-(T-t)}-T\right].$$

Consider the exponent in the above characteristic function

(3.6) $$\sum_{t=0}^{T-1} \exp\left[\frac{is}{\gamma}(T-t)e^{-(T-t)}\right]-i\sum_{t=0}^{T-1}\frac{s}{\gamma}(T-t)e^{-(T-t)}-T.$$

Since $e^{i\theta} = \cos\theta+i\sin\theta$, the imaginary part of (3.6) has the form

$$\sum_{k=1}^{T} \sin\left(\frac{s}{\gamma}ke^{-k}\right)-\sum_{k=1}^{T}\left(\frac{s}{\gamma}ke^{-k}\right) := S_{2,T}(s)-S_{1,T}(s).$$

Note that the series $\sum_{k=1}^{\infty} ke^{-k}$ is convergent (as the derivative of geometric series), and hence, as $T\to\infty$,

$$S_{1,T}(s)\to S_1(s) = \sum_{k=1}^{\infty}\left(\frac{s}{\gamma}ke^{-k}\right) = s\left[\gamma e\left(1-\frac{1}{e}\right)^2\right]^{-1} \quad \text{for } s\in\mathbf{R}.$$

Also, by the comparison test (the Weierstrass theorem),

$$S_{2,T}(s)\to S_2(s) := \sum_{k=1}^{\infty}\sin\left(\frac{s}{\gamma}ke^{-k}\right) \quad \text{as } T\to\infty$$

and the series is absolutely convergent almost uniformly in s (i.e. for any $M>0$, the convergence is uniform in $s\in[-M, M]$).

Consequently, S_1 and S_2 are continuous functions. Clearly, $S_1(0) = S_2(0) = 0$. But for $s>0$ we have

$$S_1(s)-S_2(s) = \sum_{k=1}^{\infty}\left(\frac{s}{\gamma}ke^{-k}\right)-\sum_{k=1}^{\infty}\sin\left(\frac{s}{\gamma}ke^{-k}\right) > \frac{s}{\gamma}e^{-1}-\sin\left(\frac{s}{\gamma}e^{-1}\right) > 0.$$

Therefore, the imaginary part of the exponent (3.6) converges, as $T\to\infty$, to the continuous function $g(s) = S_2(s)-S_1(s)$, which is different from 0 for s in a certain neighborhood of 0 and $g(0) = 0$. Hence, as $T\to\infty$, the characteristic functions $\varphi_{\tilde{Z}_T}(s)$ cannot converge to $\exp(-s^2/2)$, the characteristic function of standard normal distribution $\mathcal{N}(0, 1)$. This implies that the sequence $(\varphi_{Z_T}(s))_{T=1}^{\infty}$ does not converge to $\exp(-s^2/2)$. ∎

4. THE CENTRAL LIMIT THEOREM FOR A STAIN WITH CONTINUOUS TIME

In this section we investigate the limit theorems for a stain with continuous time, defined by formulas (2.1) and (2.2).

THEOREM 4.1. *Let $\xi = \xi(T, \alpha)$ be a stain with continuous time and define $\psi(t) = t^r$, $r \geqslant 0$. Then the stain satisfies the Central Limit Theorem, that is*

$$\frac{\xi(T,\alpha)-E\xi(T,\alpha)}{\sqrt{D^2\,\xi(T,\alpha)}} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } T\to\infty .$$

The simple analytic lemma below will be needed in the proof of Theorem 4.1.

LEMMA 4.2. *Fix $r \geqslant 0$. Then*

$$\text{(i)} \qquad \sum_{v=0}^{N-1} (N-v)\left[(v+1)^r - v^r\right] \leqslant O(N^{r+1}),$$

$$\text{(ii)} \qquad \sum_{v=0}^{N-1} (N-v)^2 \left[(v+1)^r - v^r\right]^2 \leqslant O(N^{2r+1}).$$

Proof of Theorem 4.1. The proof of the Central Limit Theorem for a stain with continuous time can be reduced to the Central Limit Theorem for a stain with discrete time. We have

$$\xi(T,\alpha) = \int_0^T V(t,\alpha)\,dt,\ T\geqslant 0, \quad \text{and} \quad V(t,\alpha) = \sum_{\tilde{t}_i<t} \psi(\tilde{t}_i)\, f(\alpha-\alpha_i),\ t\geqslant 0,$$

$$\begin{aligned}
\xi(T,\alpha) &= \int_{[0,N)} V(t,\alpha)\,dt + \int_{[N,T)} V(t,\alpha)\,dt \\
&= \sum_{k=0}^{N-1} \int_{[k,k+1)} \Big(\sum_{0\leqslant \tilde{t}_i<[t]+1} \psi(\tilde{t}_i)\, f(\alpha-\alpha_i) - \sum_{t\leqslant \tilde{t}_i<[t]+1} \psi(\tilde{t}_i)\, f(\alpha-\alpha_i)\Big)\, dt \\
&\quad + \int_{[N,T)} V(t,\alpha)\,dt \\
&= \sum_{k=0}^{N-1} \int_{[k,k+1)} \Big(\sum_{v=0}^{[t]} \sum_{v\leqslant \tilde{t}_i<v+1} \psi(\tilde{t}_i)\, f(\alpha-\alpha_i)\Big)\, dt \\
&\quad - \int_{[0,N)} \sum_{t\leqslant \tilde{t}_i<[t]+1} \psi(\tilde{t}_i)\, f(\alpha-\alpha_i) + \int_{[N,T)} V(t,\alpha)\,dt \\
&= \sum_{k=0}^{N-1} \sum_{v=0}^{k} \sum_{v\leqslant \tilde{t}_i<v+1} \psi(\tilde{t}_i)\, f(\alpha-\alpha_i) \\
&\quad - \int_{[0,N)} \sum_{t\leqslant \tilde{t}_i<[t]+1} \psi(\tilde{t}_i)\, f(\alpha-\alpha_i) + \int_{[N,T)} V(t,\alpha)\,dt .
\end{aligned}$$

Now we put

$$\bar{\xi}(N, \alpha) = \sum_{k=0}^{N-1} \sum_{v=0}^{k} \psi(v) X_v = \sum_{k=0}^{N-1} (N-v)\psi(v) X_v,$$

where

$$X_v = \sum_{v \leq \tilde{t}_i < v+1} f(\alpha - \alpha_i).$$

Then $\bar{\xi}(N, \alpha)$ is the stain with discrete time as in Theorem 3.1, and (X_v) is a sequence of independent identically distributed random variables. For simplicity we shall assume that $EX_v = 1$. We can write

$$\xi(T, \alpha) = \bar{\xi}(N, \alpha) + \sum_{k=0}^{N-1} \sum_{v=0}^{k} \sum_{v \leq \tilde{t}_i < v+1} \big(\psi(\tilde{t}_i) - \psi(v)\big) f(\alpha - \alpha_i)$$

$$- \int_{[0,N)} \sum_{t \leq \tilde{t}_i < [t]+1} \psi(\tilde{t}_i) f(\alpha - \alpha_i)\, dt + \int_{[N,T)} V(t, \alpha)\, dt$$

$$= \bar{\xi}(N, \alpha) + \zeta^{(1)}_{(N,\alpha)} - \zeta^{(2)}_{(N,\alpha)} + \zeta^{(3)}_{(T,\alpha)}.$$

Let us put $\xi(T, \alpha) = \bar{\xi}(N, \alpha) + A(T)$, where $A(T) = \zeta^{(1)}_{(N,\alpha)} - \zeta^{(2)}_{(N,\alpha)} + \zeta^{(3)}_{(T,\alpha)}$ and $N = [T]$ (the entire of T).

We want to prove that

$$\text{(4.1)} \qquad \frac{\xi(T, \alpha) - E\xi(T, \alpha)}{\sqrt{D^2 \xi(T, \alpha)}} \xrightarrow{d} \mathcal{N}(0, 1) \quad \text{as } T \to \infty.$$

From Theorem 3.1 we have

$$\text{(4.2)} \qquad \frac{\bar{\xi}(N, \alpha) - E\bar{\xi}(N, \alpha)}{\sqrt{D^2 \bar{\xi}(N, \alpha)}} \xrightarrow{d} \mathcal{N}(0, 1) \quad \text{as } N \to \infty.$$

Note that

$$\text{(4.3)} \qquad \frac{\xi(T, \alpha) - E\xi(T, \alpha)}{\sqrt{D^2 \xi(T, \alpha)}} = \frac{\big(\bar{\xi}(N, \alpha) - E\bar{\xi}(N, \alpha) + A(T) - EA(T)\big)}{\sqrt{D^2 \bar{\xi}(N, \alpha)}} \frac{\sqrt{D^2 \bar{\xi}(N, \alpha)}}{\sqrt{D^2 \xi(T, \alpha)}}.$$

By (4.2) and (4.3), the convergence (4.1) will be proved if we show the following:

$$\text{(4.4)} \qquad Z_T := \frac{A(T) - EA(T)}{\sqrt{D^2 \bar{\xi}(N, \alpha)}} \to 0 \text{ in } L^1 \quad \text{as } T \to \infty,$$

and

$$\text{(4.5)} \qquad \frac{\sqrt{D^2 \bar{\xi}(N, \alpha)}}{\sqrt{D^2 \xi(T, \alpha)}} \to 1 \quad \text{as } T \to \infty.$$

We first prove (4.4). Since f is nonnegative and ψ is non-decreasing, we have the estimate

$$0 \leqslant \zeta^{(1)}_{(N,\alpha)} = \sum_{k=0}^{N-1} \sum_{v=0}^{k} \sum_{v \leqslant \tilde{t}_i < v+1} (\psi(\tilde{t}_i) - \psi(v)) f(\alpha - \alpha_i)$$
$$\leqslant \sum_{k=0}^{N-1} \sum_{v=0}^{k} (\psi(v+1) - \psi(v)) \sum_{v \leqslant \tilde{t}_i < v+1} f(\alpha - \alpha_i)$$
$$= \sum_{v=0}^{N-1} \sum_{k=v}^{N-1} [\psi(v+1) - \psi(v)] X_v,$$

which gives

(4.6) $$0 \leqslant \zeta^{(1)}_{(N,\alpha)} \leqslant \sum_{v=0}^{N-1} (N-v)\big((v+1)^r - v^r\big) X_v.$$

Let us put $EX_v = 1$; then by Lemma 4.2 we obtain

(4.7) $$0 \leqslant E\zeta^{(1)}_{(N,\alpha)} \leqslant \sum_{v=0}^{N-1} (N-v)\big((v+1)^r - v^r\big) \leqslant O(N^{r+1}).$$

Similarly, we find

$$0 \leqslant \zeta^{(2)}_{(N,\alpha)} \leqslant \int_{[0,N)} \sum_{v \leqslant \tilde{t}_i < v+1}' \psi([t]+1) f(\alpha - \alpha_i)\, dt = \sum_{v=0}^{N-1} (v+1)^r X_v.$$

Consequently,

(4.8) $$0 \leqslant E\zeta^{(2)}_{(N,\alpha)} \leqslant O(N^{r+1}),$$

and

$$0 \leqslant \zeta^{(3)}_{(T,\alpha)} \leqslant \sum_{0 \leqslant \tilde{t}_i < N+1} \psi(\tilde{t}_i) f(\alpha - \alpha_i) = \sum_{v=0}^{N-1} \sum_{v \leqslant \tilde{t}_i < v+1} \psi(\tilde{t}_i) f(\alpha - \alpha_i) \leqslant \sum_{v=0}^{N-1} (v+1)^r X_v,$$

which yields

(4.9) $$0 \leqslant E\zeta^{(3)}_{(T,\alpha)} \leqslant O(N^{r+1}).$$

From (4.7)–(4.9) it follows that

(4.10) $$E\,|A(t)| \leqslant O(N^{r+1}).$$

Since $\bar{\xi}(N, \alpha) = \sum_{k=0}^{N-1} \sum_{v=0}^{k} \psi(v) X_v$ is a stain with discrete time, from the proof of Theorem 3.1 we have

(4.11) $$D^2\, \bar{\xi}(N, \alpha) = O(N^{2r+3}).$$

From (4.10) and (4.11) we obtain

$$\frac{E\,|A(t)|}{D\xi(N,\alpha)} \to 0 \quad \text{as } T \to \infty,$$

which implies (4.4).

Now, we want to prove the convergence (4.5). To this end, we investigate the asymptotic behavior of the variance $D^2 A(T)$ as $T \to \infty$. From (4.6) we have

$$0 \leqslant \zeta^{(1)}_{(N,\alpha)} \leqslant \sum_{k=0}^{N-1} (N-k)\,[(k+1)^r - k^r]\, X_k = \sum_{k=0}^{N-1} a_k X_k, \tag{4.12}$$

where $a_k = (N-k)\,[(k+1)^r - k^r]$, $k = 0, \ldots, N-1$, are positive numbers and X_k, $k = 0, \ldots, N-1$, are nonnegative independent identically distributed random variables. Hence, assuming $EX_k = 1$ and using the notation $b := E(X_k^2) = D^2 X_k + 1 = \sigma^2 + 1$, we have

$$E\Big(\big(\sum_{k=0}^{N-1} a_k X_v\big)^2\Big) = \sum_{k=0}^{N-1} a_k^2\, EX_k^2 + \sum_{k,j=0,k\neq j}^{N-1} a_k a_j EX_k EX_j = J_1 + J_2.$$

Note that from (4.7) we get the estimate

$$J_2 = \sum_{k,j=0,k\neq j}^{N-1} a_k a_j EX_k EX_j \leqslant \sum_{k=0}^{N-1} a_k \sum_{j=0}^{N-1} a_j \leqslant [O(N^{r+1})]^2 = O(N^{2r+2}).$$

We also have

$$J_1 = \sum_{k=0}^{N-1} a_k^2\, EX_k^2 = b \sum_{k=0}^{N-1} (N-k)^2\,[(k+1)^r - k^r]^2 \leqslant O(N^{2r+1}),$$

where the last inequality follows from Lemma 4.2. Hence, from (4.12) we obtain

$$E\,[(\zeta^{(1)}_{(N,\alpha)})^2] \leqslant O(N^{2r+2}). \tag{4.13}$$

Analogously, we have

$$E\,[(\zeta^{(2)}_{(N,\alpha)})^2] \leqslant O(N^{2r+2}) \tag{4.14}$$

and

$$E\,[(\zeta^{(3)}_{(N,\alpha)})^2] \leqslant O(N^{2r+2}). \tag{4.15}$$

The estimates (4.13)–(4.15) imply the inequality

$$\begin{aligned} E\,[A^2(T)] &= E\,[(\zeta^{(1)}_{(T,\alpha)} + \zeta^{(2)}_{(T,\alpha)} + \zeta^{(3)}_{(N,\alpha)})^2] \\ &\leqslant 3\,\{E\,[(\zeta^{(1)}_{(T,\alpha)})^2] + E\,[(\zeta^{(2)}_{(T,\alpha)})^2] + E\,[(\zeta^{(3)}_{(N,\alpha)})^2]\} \leqslant O(N^{2r+2}), \end{aligned}$$

and hence

$$D^2\,[A(T)] \leqslant E\,[A^2(T)] \leqslant O(N^{2r+2}). \tag{4.16}$$

We have

$$D^2 \xi(T, \alpha) = D^2 [\bar{\xi}(N, \alpha)] + D^2 [A(T)] + 2\mathrm{Cov}[\bar{\xi}(N, \alpha), A(T)],$$

and

$$-DXDY \leqslant \mathrm{Cov}(X, Y) \leqslant DXDY.$$

Therefore, using (4.11) and (4.16) we obtain the estimates:

$$(4.17) \quad \frac{D^2 \xi(T, \alpha)}{D^2 \bar{\xi}(N, \alpha)} \leqslant \frac{D^2 [\bar{\xi}(N, \alpha)] + D^2 [A(T)]}{D^2 \bar{\xi}(N, \alpha)} + \frac{2D[\bar{\xi}(N, \alpha)]\, D[A(T)]}{D^2 \bar{\xi}(N, \alpha)}$$

$$\leqslant 1 + \frac{O(N^{2r+2})}{O(N^{2r+3})} + \frac{O(N^{r+3/2})\, O(N^{r+1})}{O(N^{2r+3})}$$

$$\leqslant 1 + \frac{1}{O(N)} + \frac{1}{O(N^{1/2})} \to 1 \quad \text{as } T \to \infty \text{ and } N = [T],$$

$$(4.18) \quad \frac{D^2 \xi(T, \alpha)}{D^2 \bar{\xi}(N, \alpha)} \geqslant \frac{D^2 [\bar{\xi}(N, \alpha)] + D^2 [A(T)]}{D^2 \bar{\xi}(N, \alpha)} - \frac{2D[\bar{\xi}(N, \alpha)]\, D[A(T)]}{D^2 \bar{\xi}(N, \alpha)}$$

$$\geqslant 1 + \frac{1}{O(N)} - \frac{1}{O(N^{1/2})} \to 1 \quad \text{as } T \to \infty \text{ and } N = [T].$$

It follows from (4.17) and (4.18) that

$$\frac{D^2 \xi(T, \alpha)}{D^2 \bar{\xi}(N, \alpha)} \to 1 \quad \text{as } T \to \infty \text{ and } N = [T],$$

which implies (4.5). Thus, the proof is complete. ∎

5. MULTIDIMENSIONAL CENTRAL LIMIT THEOREM

In this section we investigate the convergence of finite-dimensional distributions of random stain to multidimensional Gaussian distribution.

THEOREM 5.1. *Let* $(\xi(N, \theta))$, $N = 1, 2, \ldots$, $\theta \in S_1$, *be a stain with discrete time, where* $\psi(t) = t^r$ $(r \geqslant 0)$ *and* f *is a nonnegative bounded Borel function on* S_1. *Let*

$$\varrho_N(\theta_1, \theta_2) = \frac{\mathrm{Cov}(\xi(N, \theta_1), \xi(N, \theta_2))}{D\xi(N, \theta_1) \cdot D\xi(N, \theta_2)}.$$

Assume that for any $\theta_1, \theta_2 \in S_1$ *there exists*

$$(5.1) \quad \lim_{N \to \infty} \varrho_N(\theta_1, \theta_2) = \varrho(\theta_1, \theta_2).$$

Then the sequence of stochastic processes $(\bar{\xi}(N,\cdot))_{N=1}^{\infty}$, *where*

$$\bar{\xi}(N,\theta)=\frac{\xi(N,\theta)-E\xi(N,\theta)}{D\xi(N,\theta)},$$

converges weakly in the sense of finite-dimensional distributions to a Gaussian process with zero mean value and covariance function $\varrho(\cdot,\cdot)\colon S_1\times S_1\to \mathbf{R}$ *which is defined in* (5.1). *This means that for each* $m=1,2,\ldots$ *and any fixed angles* $\theta_1,\theta_2,\ldots,\theta_m\in S_1$ *the distributions of the random vectors* $(\bar{\xi}(N,\theta_1),\ldots,\bar{\xi}(N,\theta_m))$, $N=1,2,\ldots$, *converge weakly as* $N\to\infty$ *to* $\mathcal{N}(0,\Sigma)$, *where* $\Sigma=[s_{j,l}]_{j,l=1,\ldots,m}$ *with* $s_{j,l}=\varrho(\theta_j,\theta_l)$ (*in particular,* $s_{j,j}=1$).

In the proof of Theorem 5.1 we use the Normal Correlation Theorem from [4] which we quote below.

Normal Correlation Theorem (Gikhman and Skorokhod [4]). *Let* $\{\eta_n\}$ *denote a sequence of random processes* $\eta_n(\theta)=\sum_{k=1}^{m_n}\alpha_{nk}(\theta)$, $\theta\in\Theta$, $n=1,2,\ldots$ *Suppose that the following three conditions are satisfied*:

(a) *For any fixed n, the random variables* $\alpha_{n1}(\theta_1),\alpha_{n2}(\theta_2),\ldots,\alpha_{nm_n}(\theta_{m_n})$ *are independent for arbitrary* $\theta_1,\theta_2,\ldots,\theta_{m_n}$, *have second order moments and*

$$E\alpha_{nk}(\theta)=0,\qquad E\alpha_{nk}^2(\theta)=b_{nk}^2(\theta),$$

where $\max_k b_{nk}^2(\theta)\to 0$ *as* $n\to\infty$.

(b) *The sequence of covariance functions* $R_n(\theta_1,\theta_2)=E[\eta_n(\theta_1)\eta_n(\theta_2)]$ *converges as* $n\to\infty$ *to some limit*

$$\lim_{n\to\infty}R_n(\theta_1,\theta_2)=R(\theta_1,\theta_2).$$

(c) *For every* θ, *the sums* $\eta_n(\theta)=\sum_{k=1}^{m_n}\alpha_{nk}(\theta)$ *satisfy Lindeberg's condition*: *For an arbitrary positive* ε,

$$\frac{1}{B_n^2}\sum_{k=1}^{m_n}\int_{|x|>\varepsilon B_n}x^2\,d\Pi_{nk}(\theta,x)\to 0,$$

where $\Pi_{nk}(\theta,x)$ *is the distribution function of the random variable* $\alpha_{nk}(\theta)$ *and*

$$B_n^2=\sum_{k=1}^{m_n}b_{nk}^2(\theta)=R_n(\theta,\theta).$$

Then the sequence $\{\eta_n(\theta)\}$ *converges weakly in the sense of finite-dimensional distributions as* $n\to\infty$ *to a Gaussian random function with expectation zero and covariance* $R(\theta_1,\theta_2)$.

Proof of Theorem 5.1. First, we verify that the assumptions of the Normal Correlation Theorem are satisfied.

Put

$$\eta_N(\theta)=\bar{\xi}(N,\theta)=\frac{\xi(N,\theta)-E\xi(N,\theta)}{D\xi(N,\theta)},\quad \text{where } \theta\in S_1.$$

Recall that

$$\xi(N,\theta)=\sum_{t=0}^{N-1} Y_t^N \quad \text{and} \quad D\xi(N,\theta)=C_N,$$

where C_N is defined in Theorem 3.1 and C_N does not depend on θ. Then under the notation of the Normal Correlation Theorem

$$\eta_N(\theta)=\sum_{k=0}^{N-1}\frac{Y_k^N(\theta)-EY_k^N(\theta)}{C_N},$$

that is

$$\alpha_{Nk}(\theta)=\frac{Y_k^N(\theta)-EY_k^N(\theta)}{C_N}, \qquad k=0,1,\ldots,N-1.$$

The first part of the condition (a) of the Normal Correlation Theorem is satisfied because, for fixed N, the processes $Y_0^N(\cdot),\ldots,Y_{N-1}^N(\cdot)$ are independent. Next, by Theorem 3.1 we have

$$b_{Nk}^2(\theta)=E\alpha_{Nk}^2(\theta)=\frac{D^2\left(Y_k^N(\theta)\right)}{C_N^2}=\frac{(N-k)^2k^{2r}\sigma^2}{C_N^2},$$

where

$$\sigma^2=D^2X_j(\theta) \quad \text{and} \quad C_N^2\approx\frac{\sigma^2}{(2r+1)(r+1)(2r+3)}N^{2r+3}.$$

Then

$$\max_k b_{Nk}^2(\theta)\leqslant\frac{O(N^{2r+2})}{O(N^{2r+3})}\to 0 \quad \text{as } N\to\infty.$$

Concerning the condition (b), note that

$$\varrho_N(\theta_1,\theta_2)=E[\eta_N(\theta_1)\eta_N(\theta_2)]=E[\bar{\xi}(N,\theta_1)\bar{\xi}(N,\theta_2)],$$

and hence (b) is satisfied by assumption (5.1).

Consider the condition (c). We verify that Liapunov's condition with $\delta=1$ holds for the random variables (α_{Nk}).

We have

$$B_N^2=\sum_{k=0}^{N-1}b_{Nk}^2(\theta)=\sum_{k=0}^{N-1}E\alpha_{Nk}^2=\frac{1}{C_N^2}\sum_{k=0}^{N-1}E[|Y_k^N-EY_k^N|^2]=1$$

and, by Theorem 3.1,

$$\frac{1}{B_N^3}\sum_{k=0}^{N-1}E[|\alpha_{Nk}-E\alpha_{Nk}|^3]=\sum_{k=0}^{N-1}E\left[\left|\frac{Y_k^N(\theta)-EY_k^N(\theta)}{C_N}\right|^3\right]$$

$$=\frac{1}{C_N^3}\sum_{k=0}^{N-1}E[|Y_k^N(\theta)-EY_k^N(\theta)|^3]\to 0 \quad \text{as } N\to\infty.$$

Hence Liapunov's condition holds for the random variables (α_{Nk}). Consequently, Lindeberg's condition holds for (α_{Nk}). This means that, for an arbitrary positive ε,

$$\frac{1}{B_N^2}\sum_{k=0}^{N-1}\int_{|x|>\varepsilon B_N} x^2\, d\Pi_{Nk}(\theta, x)\to 0,$$

where $\Pi_{Nk}(\theta, x)$ is the distribution function of the random variable α_{Nk}.

Therefore, all the assumptions of the Normal Correlation Theorem are satisfied and we conclude from this theorem that for each $m = 1, 2, \ldots$ and any fixed angles $\theta_1, \theta_2, \ldots, \theta_m \in S_1$ the distributions of the random vectors: $(\bar{\xi}(N, \theta_1), \ldots, \bar{\xi}(N, \theta_m))$, $N = 1, 2, \ldots$, converge weakly as $N\to\infty$ to normal distribution $\mathcal{N}(0, \Sigma)$ with zero mean and the covariance matrix $\Sigma = [s_{j,l}]_{j,l=1,\ldots,m}$ with $s_{j,l} = \varrho(\theta_j, \theta_l)$ (in particular, $s_{j,j} = 1$). ∎

Below we show under an additional assumption that condition (5.1) is satisfied and we obtain an explicit formula for the limit covariance matrix.

THEOREM 5.2. *Let $(\xi(N, \theta))$, $N = 1, 2, \ldots$, $\theta\in S_1$, be a stain with discrete time, where $\psi(t)\equiv 1$ and $f = 1_{[0,\pi]}$. Then for each $m = 1, 2, \ldots$ and any fixed angles $0\leqslant\theta_1\leqslant\theta_2\leqslant\ldots\leqslant\theta_m < 2\pi$ the distributions of the random vectors: $(\bar{\xi}(N, \theta_1), \ldots, \bar{\xi}(N, \theta_m))$, $N = 1, 2, \ldots$, converge weakly as $N\to\infty$ to $\mathcal{N}(0, \Sigma)$, where $\Sigma = [s_{j,l}]_{j,l=1,\ldots,m}$ with $s_{j,l} = s_{l,j}$ and for $j < l$*

$$s_{j,l} = \varrho(\theta_j, \theta_l) = \begin{cases} (\pi-(\theta_l-\theta_j))/\pi & \text{if } 0\leqslant\theta_l-\theta_j\leqslant\pi,\\ ((\theta_l-\theta_j)-\pi)/\pi & \text{if } \pi<\theta_l-\theta_j<2\pi,\end{cases}$$

and $s_{j,j} = 1$.

Before proving the theorem some preliminary calculations and two lemmas are needed.

Remark 5.3. Assume that a random variable $\mathfrak{R}$ has the uniform distribution on the unit circle S_1. For $f = 1_{[0,\pi]}$, we have

$$E[f(\mathfrak{R})] = 1/2 \tag{5.2}$$

and

$$E[f(\mathfrak{R})\cdot f(\mathfrak{R}+\beta)] = \frac{\pi-|\beta|}{2\pi}\quad \text{for } \beta\in[-\pi, \pi]. \tag{5.3}$$

LEMMA 5.4 (**Wald's identities**). *Assume that $(Z_j)_{j=1}^{\infty}$ are independent identically distributed random variables on $(\Omega, \mathcal{F}, P)$ and $\tau\colon \Omega\to\{0, 1, 2, \ldots\}$ is a random variable independent of $(Z_j)_{j=1}^{\infty}$.*

(i) *If Z_1 is nonnegative, $EZ_1 < \infty$ and $E\tau < \infty$, then*

$$E\Big(\sum_{j=1}^{\tau} Z_j\Big) = E(\tau)E(Z_1).$$

(ii) *Let h, g be Borel nonnegative functions on $\mathbf{R}$ such that $h(Z_1)$ and $g(Z_1)$ have the same distribution. If $E\big(h(Z_1)\big) < \infty$ and $E(\tau^2) < \infty$, then*

$$E\Big(\sum_{i,j=1,i\neq j}^{\tau} h(Z_i)g(Z_j)\Big) = [E(\tau^2)-E(\tau)]\big(E(h(Z_1))\big)^2.$$

LEMMA 5.5. *Let $\xi(N, \alpha)$ be a stain with discrete time, where $\psi(t) \equiv 1$ and $f = 1_{[0,\pi]}$. Then for fixed $\alpha \in S_1$, $\beta \in [-\pi, \pi]$ we have*

$$\mathrm{Cov}\big(\xi(N, \alpha), \xi(N, \alpha+\beta)\big) = \lambda\left(\frac{\pi-|\beta|}{2\pi}\right)\frac{N(N+1)(2N+1)}{6}.$$

Proof. From the form of $\xi(T, \alpha)$ in (2.4)–(2.7) we get

$$\begin{aligned}
&\mathrm{Cov}\big(\xi(N, \alpha), \xi(N, \alpha+\beta)\big)\\
&= E\Big[\sum_{t=0}^{N-1}(N-t)\big(X_t(\alpha)-EX_t(\alpha)\big)\Big]\Big[\sum_{t=0}^{N-1}(N-t)\big(X_t(\alpha+\beta)-EX_t(\alpha+\beta)\big)\Big]\\
&= \sum_{s,t=0,s\neq t}^{N-1}(N-t)(N-s)E\big(\{X_t(\alpha)-EX_t(\alpha)\}\{X_s(\alpha+\beta)-EX_s(\alpha+\beta)\}\big)\\
&\quad+\sum_{t=0}^{N-1}(N-t)^2 E\{X_t(\alpha)X_t(\alpha+\beta)-EX_t(\alpha)EX_t(\alpha+\beta)\} := J_1+J_2.
\end{aligned}$$

Since $\big(X_t(\cdot)\big)_{t=0}^{\infty}$ are independent processes, $J_1 = 0$ and

$$J_2 := \sum_{t=0}^{N-1}(N-t)^2[M_1-M_2],$$

where

$$M_1 = E[X_t(\alpha)X_t(\alpha+\beta)] \quad \text{and} \quad M_2 = EX_t(\alpha)EX_t(\alpha+\beta).$$

By (2.7) we have

$$\begin{aligned}
(5.4)\qquad M_1 &= E\Big[\Big\{\sum_{i=1}^{N_t} f(\alpha-\alpha_i)\Big\}\Big\{\sum_{j=1}^{N_t} f(\alpha+\beta-\alpha_j)\Big\}\Big]\\
&= E\Big[\sum_{j=0}^{N_t} f(\alpha-\alpha_j)f(\alpha+\beta-\alpha_j)\Big]+E\Big[\sum_{i,j=1,i\neq j}^{N_t} f(\alpha-\alpha_i)f(\alpha+\beta-\alpha_j)\Big].
\end{aligned}$$

Note that the random variables $Z_j = f(\alpha-\alpha_j)f(\alpha+\beta-\alpha_j)$, $j = 1, 2, \ldots$, and $\tau = N_t$ satisfy the assumptions of Lemma 5.4 (i) and, consequently,

$$E\Big[\sum_{j=0}^{N_t} f(\alpha-\alpha_j)f(\alpha+\beta-\alpha_j)\Big] = E[f(\alpha-\alpha_1)f(\alpha+\beta-\alpha_1)]E(N_t).$$

By (5.3) we have $E[f(\alpha-\alpha_1)f(\alpha+\beta-\alpha_1)] = (\pi-|\beta|)/2\pi$.

Consider the last term in (5.4). Put $Z_j = \alpha_j$, $j = 1, 2, \ldots$, $h(\cdot) = f(\alpha - \cdot)$, $g(\cdot) = f(\alpha+\beta-\cdot)$ and $\tau = N_t$. Then the assumptions of Lemma 5.4 (ii) are satisfied and from this lemma the equality follows:

$$E\Big[\sum_{i,j=1, i\neq j}^{N_t} f(\alpha-\alpha_i) f(\alpha+\beta-\alpha_j)\Big] = [E(N_t^2) - E(N_t)]\big(Ef(\alpha-\alpha_1)\big)^2.$$

Moreover, $Ef(\alpha-\alpha_1) = \frac{1}{2}$ by (5.2). Therefore,

$$M_1 = \frac{\pi-|\beta|}{2\pi} E(N_t) + \frac{1}{4}[E(N_t^2) - E(N_t)] = \frac{\pi-|\beta|}{2\pi}\lambda + \frac{\lambda^2}{4}$$

because N_t is a random variable with Poisson distribution (with parameter λ). Similarly, using Lemma 5.4 (i) and (5.2) we find

$$M_2 = E\Big[\sum_{i=1}^{N_t} f(\alpha-\alpha_i)\Big] E\Big[\sum_{j=1}^{N_t} f(\alpha+\beta-\alpha_j)\Big] = \frac{1}{4}(EN_t)^2 = \lambda^2/4.$$

Therefore,

$$J_2 = \sum_{t=0}^{N-1} (N-t)^2 \left(\frac{\pi-|\beta|}{2\pi}\lambda + \frac{\lambda^2}{4} - \frac{\lambda^2}{4}\right) = \lambda\left(\frac{\pi-|\beta|}{2\pi}\right) \sum_{k=1}^{N} k^2$$
$$= \lambda\left(\frac{\pi-|\beta|}{2\pi}\right)\left(\frac{N(N+1)(2N+1)}{6}\right),$$

and hence

$$\operatorname{Cov}\big(\xi(N,\alpha), \xi(N,\alpha+\beta)\big) = \lambda\left(\frac{\pi-|\beta|}{2\pi}\right)\left(\frac{N(N+1)(2N+1)}{6}\right). \ \blacksquare$$

Proof of Theorem 5.2. We verify that the assumptions of Theorem 5.1 are satisfied. We have

$$\varrho_N(\theta_j, \theta_l) = \frac{\operatorname{Cov}\big(\xi(N,\theta_j), \xi(N,\theta_l)\big)}{D\xi(N,\theta_j)\cdot D\xi(N,\theta_l)} \quad \text{and} \quad \varrho_N(\theta_j, \theta_l) = \varrho_N(\theta_l, \theta_j).$$

If $\theta_j = \theta_l$, then $\varrho_N(\theta_j, \theta_l) = 1$, $\lim_{N\to\infty} \varrho_N(\theta_j, \theta_l) = 1$.

If $\theta_j < \theta_l$ and $0 < \theta_l - \theta_j \leqslant \pi$, then we can use Lemma 5.5 with $\beta = \theta_l - \theta_j$, and then

$$s_{j,l} = \varrho_N(\theta_j, \theta_l) = \frac{\pi - (\theta_l - \theta_j)}{\pi}.$$

If $\pi < \theta_l - \theta_j < 2\pi$, then

$$\theta_l = \theta_j + (\theta_l - \theta_j) \overset{\mathrm{mod}\,2\pi}{=} \theta_j + \beta, \quad \text{where } \beta = -\big(2\pi - (\theta_l - \theta_j)\big)$$

and we also use Lemma 5.5 to obtain

$$s_{j,l} = \varrho_N(\theta_j, \theta_l) = \frac{(\theta_l - \theta_j) - \pi}{\pi}. \ \blacksquare$$

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