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ON THE FRACTIONAL RECORD VALUES

BY

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Abstract. We define the record-values process which may be considered as the collection of record values with non-integer or fractional indices. The alternative construction from the sample as well as the basic properties of the defined process are shown.

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1. INTRODUCTION

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent identically distributed random variables with a common distribution function (cdf) F and probability density function (pdf) f. Moreover, let $X_{1:n}, \ldots, X_{n:n}$ denote the order statistics of a sample $X_1, \ldots, X_n$.

For a fixed $k \geqslant 1$ we define the *k-th* (*upper*) *record times* $U_k(n)$, $n \geqslant 1$, of the sequence $\{X_n, n \geqslant 1\}$ as

$$U_k(1) = 1,$$

$$U_k(n+1) = \min\{j > U_k(n):\ X_{j:j+k-1} > X_{U_k(n):U_k(n)+k-1}\}, \quad n \geqslant 1,$$

and the *k-th* (*upper*) *record values* as

$$Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1} \quad \text{for } n \geqslant 1$$

(cf. [5]). Note that for $k = 1$ we have $Y_n^{(1)} = X_{U_1(n):U_1(n)} := R_n$ – the upper record values of the sequence $\{X_n, n \geqslant 1\}$, and that $Y_1^{(k)} = X_{1:k} = \min(X_1, \ldots, X_k)$.

Similarly, for a fixed $k \geqslant 1$ we define the *k-th lower record times* $L_k(n)$, $n \geqslant 1$, of the sequence $\{X_n, n \geqslant 1\}$ as

$$L_k(1) = 1,$$

$$L_k(n+1) = \min\{j > L_k(n):\ X_{k:j+k-1} < X_{k:L_k(n)+k-1}\}, \quad n \geqslant 1,$$

and the *k-th lower record values* as

$$Z_n^{(k)} = X_{k:L_k(n)+k-1} \qquad \text{for } n \geqslant 1$$

(cf. [11]). Note that for $k = 1$ we have $Z_n^{(1)} = X_{1:L_1(n)} := R'_n$ – the lower record values of the sequence $\{X_n, n \geqslant 1\}$, and $Z_1^{(k)} = X_{k:k} = \max(X_1, \ldots, X_k)$.

Stigler [13], by means of Dirichlet process, defined order statistics process, which may be considered as fractional order statistics, i.e. order statistics with non-integer index. A different approach to fractional order statistics is presented by Rohatgi and Saleh in [12]. Using Newton's binomial series expansion they defined a class of distribution functions $F_{r:\alpha}$ which may be interpreted as the distribution of the r-th order statistic with non-integral sample size $\alpha > 0$. Jones [8] gave an alternative construction of Stigler's uniform fractional order statistics. Namely, ordinary order statistics of a sample $U_1, \ldots, U_n$ from uniform distribution are used to construct random variables with the same joint distribution as Stigler's order statistics. Some applications of fractional order statistics are given in [7].

In this paper we define the record-values process, which can be considered as a family of k-th record values $Y_n^{(k)}$ with n replaced by a positive number t. In Section 2 we define the exponential record-values process by means of a gamma process. Next, we define the record-values process for an arbitrary distribution function F by a quantile transformation of the exponential record-values process. Then in Section 3 we establish that the record-values process is a Markov process. In Sections 4 and 5 we give an alternative construction of exponential fractional record values. Similar results for the k-th lower record-values process are summarized in Section 6. In Section 7 we give examples of evaluation of moments of fractional record values from special distributions. Finally, in Section 8 we give an application of fractional record values to the problem of point and interval estimation of the values of the inverse to hazard function of F.

2. RECORD-VALUES PROCESS

We start with a brief review of the distribution theory of k-th record values. It is known (cf. [5]) that if F is an absolutely continuous distribution function with pdf f, then the pdf of $Y_n^{(k)}$ is

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}\big(H(x)\big)^{n-1}\big(1-F(x)\big)^{k-1} f(x), \qquad x \in \boldsymbol{R},$$

where $H(x) := H_F(x) = -\log\big(1-F(x)\big)$ is the hazard function of F. The joint pdf of the random vector $(Y_1^{(k)}, \ldots, Y_n^{(k)})$ is

$$(2.1) \qquad f_{Y_1^{(k)},\ldots,Y_n^{(k)}}(x_1, \ldots, x_n) = k^n \prod_{i=1}^{n-1} \frac{f(x_i)}{1-F(x_i)}\big(1-F(x_n)\big)^{k-1} f(x_n)$$

for $-\infty < x_1 \leqslant \ldots \leqslant x_n < \infty$. Moreover, if $0 = j_0 < j_1 < \ldots < j_n$, then the vector $(Y_{j_1}^{(k)}, \ldots, Y_{j_n}^{(k)})$ has the joint pdf

$$(2.2)\quad f_{Y_{j_1}^{(k)},\ldots,Y_{j_n}^{(k)}}(x_1, \ldots, x_n) = k^{j_n} \prod_{i=1}^{n} \frac{(H(x_i)-H(x_{i-1}))^{j_i-j_{i-1}-1}h(x_i)}{(j_i-j_{i-1}-1)!}(1-F(x_n))^k$$

for $-\infty = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n < \infty$, where $h(x) = H'(x)$.

In this note $W_n^{(k)}$, $n \in N$, stands for the k-th record value from standard exponential distribution. It is known (see e.g. [2]) that for each $k \in N$ the sequence $\{W_n^{(k)}, n \geqslant 1\}$ of k-th record values from exponential distribution has the following property: for all $m, n \in N$ such that $n > m$, the random variables $W_m^{(k)}$ and $W_n^{(k)} - W_m^{(k)}$ are independent (and this property characterizes the exponential distribution). Moreover, we know that $W_m^{(k)}$ and $W_n^{(k)} - W_m^{(k)}$ are gamma $\Gamma(m, k)$ and $\Gamma(n-m, k)$ distributed, respectively, where $\Gamma(\alpha, \beta)$ denotes a gamma distribution with pdf

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \qquad x > 0,\ \alpha, \beta > 0.$$

The above facts motivate the following definition.

DEFINITION 1. Fix $k \in N$. Let $W^{(k)} = \{W^{(k)}(t), t \geqslant 0\}$ be a stochastic process such that:

(i) $W^{(k)}(0) = 0$ a.s.,

(ii) $W^{(k)}$ has independent increments,

(iii) if $t > s \geqslant 0$, then $W^{(k)}(t) - W^{(k)}(s)$ is gamma $\Gamma(t-s, k)$ distributed.

Then $\{W^{(k)}(t), t \geqslant 0\}$ is called the *exponential k-th record-values process*. The random variables $W^{(k)}(t)$, $t > 0$, are said to be *exponential fractional k-th record values*.

Note that $W^{(k)}(t)$, $t > 0$, is $\Gamma(t, k)$ distributed. Moreover, if $n \in N$ and $0 = t_0 < t_1 < \ldots < t_n$, then the joint pdf of the random vector

$$\hat{W} = (W^{(k)}(t_1), W^{(k)}(t_2) - W^{(k)}(t_1), \ldots, W^{(k)}(t_n) - W^{(k)}(t_{n-1}))$$

is

$$f_{\hat{W}}(x_1, \ldots, x_n) = k^{t_n} \prod_{i=1}^{n} \frac{x_i^{t_i-t_{i-1}-1}}{\Gamma(t_i-t_{i-1})} \exp\Big(-k \sum_{i=1}^{n} x_i\Big), \qquad x_1, \ldots, x_n \geqslant 0.$$

Therefore, the joint pdf of the random vector $W = (W^{(k)}(t_1), \ldots, W^{(k)}(t_n))$ is

$$(2.3)\qquad f_W(x_1, \ldots, x_n) = k^{t_n} \prod_{i=1}^{n} \frac{(x_i - x_{i-1})^{t_i-t_{i-1}-1}}{\Gamma(t_i-t_{i-1})} \exp(-kx_n)$$

for $0 = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n < \infty$.

Also, $(W^{(k)}(1), \ldots, W^{(k)}(n)) \stackrel{d}{=} (W_1^{(k)}, \ldots, W_n^{(k)})$, where $\stackrel{d}{=}$ means equality in distribution. More generally, if $t_m = j_m \in N$ and $1 \leqslant j_1 < \ldots < j_n$, then

$$(W^{(k)}(j_1), \ldots, W^{(k)}(j_n)) \stackrel{d}{=} (W_{j_1}^{(k)}, \ldots, W_{j_n}^{(k)}).$$

This can be stated by comparing (2.1) with (2.3) and (2.2) with (2.4) below. This explains the name for the process $W^{(k)}$, which has the same finite-dimensional marginal distributions as the sequence of k-th records from exponential distribution.

Let F be a distribution function and let $G(x) = 1-e^{-x}$, $x \geqslant 0$, be the standard exponential distribution function.

Definition 2. The stochastic process $Y^{(k)} = \{Y^{(k)}(t),\ t \geqslant 0\}$, where

$$Y^{(k)}(t) = F^{-1}\big(G(W^{(k)}(t))\big), \quad t \geqslant 0,$$

is called the *k-th record-values process* for distribution function F. The random variables $Y^{(k)}(t)$, $t > 0$, are said to be *fractional k-th record values* from F.

Suppose that F is absolutely continuous with the pdf f. Using the above definition one can easily show that $Y^{(k)}(t)$, $t > 0$, has the pdf

$$f_{Y^{(k)}(t)}(x) = \frac{k^t}{\Gamma(t)}\big(H(x)\big)^{t-1}\big(1-F(x)\big)^{k-1} f(x), \quad x \in \boldsymbol{R},$$

where H denotes the hazard function of F. Moreover, if $0 = t_0 < t_1 < \ldots < t_n$, then the random vector $Y := \big(Y^{(k)}(t_1), \ldots, Y^{(k)}(t_n)\big)$ has the joint pdf

$$\text{(2.4)} \quad f_Y(x_1, \ldots, x_n) = k^{t_n} \prod_{i=1}^{n} \frac{\big(H(x_i)-H(x_{i-1})\big)^{t_i-t_{i-1}-1} h(x_i)}{\Gamma(t_i-t_{i-1})}\big(1-F(x_n)\big)^k$$

for $-\infty = x_0 < x_1 \leqslant \ldots \leqslant x_n < \infty$, where $h(x) = H'(x)$.

Moreover, by (2.1) and (2.4) we have $\big(Y^{(k)}(1), \ldots, Y^{(k)}(n)\big) \stackrel{d}{=} (Y_1^{(k)}, \ldots, Y_n^{(k)})$, and using (2.2) and (2.4) we get $\big(Y^{(k)}(j_1), \ldots, Y^{(k)}(j_n)\big) \stackrel{d}{=} (Y_{j_1}^{(k)}, \ldots, Y_{j_n}^{(k)})$ for $1 \leqslant j_1 < \ldots < j_n$, $j_i \in N$. Therefore we can consider $Y^{(k)}(t)$ as $Y_n^{(k)}$ with index n replaced with arbitrary positive t.

3. THE MARKOV PROPERTY

Suppose that F is absolutely continuous with pdf f. Using (2.4) one can show that the conditional pdf of $Y^{(k)}(t+s)$, given $Y^{(k)}(t) = x$, $t, s > 0$, is

$$f_{Y^{(k)}(t+s)|Y^{(k)}(t)}(y \mid x) = \frac{k^s}{\Gamma(s)}\left(\frac{1-F(y)}{1-F(x)}\right)^k \big(H(y)-H(x)\big)^{s-1} h(y)$$

for $y \geqslant x$. Moreover, the conditional pdf of $Y^{(k)}(t)$, given $Y^{(k)}(t+s) = y$, is

$$f_{Y^{(k)}(t)|Y^{(k)}(t+s)}(x \mid y) = \frac{1}{B(t, s)}\left(\frac{H(x)}{H(y)}\right)^{t-1}\left(1-\frac{H(x)}{H(y)}\right)^{s-1}\frac{h(x)}{H(y)}$$

for $x \leqslant y$, where $B(t, s)$ denotes beta function determined by

$$B(t, s) = \int_0^1 x^{t-1}(1-x)^{s-1}\, dx, \qquad t, s > 0.$$

Also, by (2.4),

$$f_{Y^{(k)}(t_{n+1})|Y^{(k)}(t_1),\ldots,Y^{(k)}(t_n)}(x_{n+1} \mid x_1, \ldots, x_n) = f_{Y^{(k)}(t_{n+1})|Y^{(k)}(t_n)}(x_{n+1} \mid x_n),$$

which gives the following result.

PROPOSITION 1. *$\{Y^{(k)}(t), t \geqslant 0\}$ is a Markov process with the transition probabilities*

$$P\{Y^{(k)}(t+s) > y \mid Y^{(k)}(t) = x\} = 1 - \frac{1}{\Gamma(s)}\Gamma\big(s; k(H(y)-H(x))\big) \tag{3.1}$$

for $s > 0$, $y \geqslant x$, where

$$\Gamma(\alpha; x) = \int_0^x t^{\alpha-1} e^{-t}\, dt, \qquad \alpha > 0, x > 0, \tag{3.2}$$

denotes incomplete gamma function.

Note that if $t = n \in N$ and $s = 1$, the equation (3.1) reduces to

$$P\{Y^{(k)}(n+1) > y \mid Y^{(k)}(n) = x\} = \left(\frac{1-F(y)}{1-F(x)}\right)^k$$

for $y \geqslant x$, which agrees with the classical result (cf. [2], p. 97).

4. ALTERNATIVE CONSTRUCTION

In this section we show how to construct $W^{(k)}(t)$ using exponential k-th record values $\{W_n^{(k)}, n \geqslant 1\}$. For $t \geqslant 0$ we write $\{t\} = t - [t]$, where $[t]$ denotes the floor function and $\{t\}$ is called the *fractional part* of t.

THEOREM 1. *For $t \geqslant 0$*

$$W^{(k)}(t) \overset{d}{=} (1-B)\, W_{[t]}^{(k)} + B W_{[t]+1}^{(k)}, \tag{4.1}$$

where B is a beta $B(\{t\}, 1-\{t\})$ distributed random variable, independent of the sequence $\{W_n^{(k)}, n \geqslant 1\}$.

Remark 1. We put $B = 0$ a.s., if $\{t\} = 0$.

Proof. If $t = 0, 1, 2, \ldots$, then the right-hand side of (4.1) is simply $W_t^{(k)}$, which is $\Gamma(t, k)$ distributed. Now for $t \in (0, \infty)\backslash N$ let us denote the right-hand side of (4.1) by $W_t^{(k)}$. If $t \in (1, \infty)\backslash N$ and $n = [t]$, then the random vector

$(W_n^{(k)}, W_t^{(k)}, W_{n+1}^{(k)})$ has the pdf

$$f_{W_n^{(k)},W_t^{(k)},W_{n+1}^{(k)}}(u, v, w) = f_{W_t^{(k)}|W_n^{(k)},W_{n+1}^{(k)}}(u \mid v, w) f_{W_n^{(k)},W_{n+1}^{(k)}}(u, w)$$

$$= \frac{1}{B(\{t\}, 1-\{t\})}(v-u)^{\{t\}-1}(w-v)^{-\{t\}}\frac{k^{n+1}}{\Gamma(n)}u^{n-1}e^{-kw}$$

for $0 < u < v < w < \infty$. Therefore

$$f_{W_t^{(k)}}(v) = \frac{k^{n+1}}{\Gamma(n)B(\{t\}, 1-\{t\})}\int_0^v u^{n-1}(v-u)^{\{t\}-1}\,du\int_v^\infty (w-v)^{-\{t\}}e^{-kw}\,dw$$

$$= \frac{k^{n+1}}{\Gamma(n)B(\{t\}, 1-\{t\})}B(n, \{t\})v^{n+\{t\}-1}\frac{\Gamma(1-\{t\})}{k^{1-\{t\}}}e^{-kv}$$

$$= \frac{k^t}{\Gamma(t)}v^{t-1}e^{-kv}, \qquad v \geqslant 0,$$

which means that $W_t^{(k)} \sim \Gamma(t, k)$. Similar evaluations lead to (4.1) for $t \in (0, 1)$. ■

Remark 2. Other methods of the construction of gamma distributions and gamma processes can be found for instance in [4] and [6]. References [3] and [9] are also recommended.

5. MULTIDIMENSIONAL CASE

In the previous section we show how to construct the single random variable $W^{(k)}(t)$, $t > 0$, from the exponential k-th record values. Now we show how to construct the random vector $(W^{(k)}(t_1), W^{(k)}(t_2), \ldots, W^{(k)}(t_m))$, where $0 < t_1 < \ldots < t_m < \infty$. We start with the definition of m-dimensional generalized arc-sine distribution.

DEFINITION 3. The random variables $B_1, \ldots, B_m$ are said to have *m-dimensional generalized arc-sine distribution with parameters* $a_1, \ldots, a_m, a_{m+1} > 0$ *if* their joint pdf is of the form

$$f_{B_1,\ldots,B_m}(u_1, \ldots, u_m) = \Gamma\Big(\sum_{i=1}^{m+1} a_i\Big)\left\{\prod_{i=1}^{m+1}\frac{(u_i-u_{i-1})^{a_i-1}}{\Gamma(a_i)}\right\}$$

for $0 = u_0 < u_1 < \ldots < u_m < u_{m+1} = 1$.

Remark 3. Note that for $m = 1$ we obtain ordinary one-dimensional beta $B(a_1, a_2)$ distribution.

THEOREM 2. *Let* $n = t_0 < t_1 < \ldots < t_m < t_{m+1} = n+1$. *Define*

$$W_{t_i}^{(k)} = (1-B_i)W_n^{(k)} + B_i W_{n+1}^{(k)}, \qquad 1 \leqslant i \leqslant m,$$

where $(B_1, \ldots, B_m)$ *is a random vector with m-dimensional generalized arc-sine* *distribution with parameters* $a_i = t_i - t_{i-1}$, $1 \leqslant i \leqslant m+1$, *independent of the sequence* $\{W_n^{(k)}, n \geqslant 1\}$. *Then*

$$(W_{t_1}^{(k)}, \ldots, W_{t_m}^{(k)}) \stackrel{d}{=} (W^{(k)}(t_1), \ldots, W^{(k)}(t_m)).$$

Proof. The joint pdf of $W_n = (W_n^{(k)}, W_{t_1}^{(k)}, \ldots, W_{t_m}^{(k)}, W_{n+1}^{(k)})$ is of the form

$$(5.1)\quad f_{W_n}(u_0, \ldots, u_{m+1}) = f_{W_{t_1}^{(k)},\ldots,W_{t_m}^{(k)}|W_n^{(k)},W_{n+1}^{(k)}}(u_1, \ldots, u_m | u_0, u_{m+1}) f_{W_n^{(k)},W_{n+1}^{(k)}}(u_0, u_{m+1})$$

for $0 \leqslant u_0 < u_1 < \ldots < u_m < u_{m+1} < \infty$. Now, by (2.2), we get

$$(5.2)\quad f_{W_n^{(k)},W_{n+1}^{(k)}}(u_0, u_{m+1}) = \frac{k^{n+1}}{\Gamma(n)} u_0^{n-1} \exp(-ku_{m+1}), \qquad 0 < u_0 < u_{m+1}.$$

Moreover, the conditional pdf of $W_{t_1}^{(k)}, \ldots, W_{t_m}^{(k)}$, given $W_n^{(k)} = u_0$, $W_{n+1}^{(k)} = u_{m+1}$, is the same as the joint pdf of the vector $\boldsymbol{B}' = (u_{m+1} - u_0)\boldsymbol{B} + \boldsymbol{u}_0$, where $\boldsymbol{B} = (B_1, \ldots, B_m)$ and $\boldsymbol{u}_0 = (u_0, \ldots, u_0) \in \boldsymbol{R}^m$. Therefore

$$(5.3)\quad f_{W_{t_1}^{(k)},\ldots,W_{t_m}^{(k)}|W_n^{(k)},W_{n+1}^{(k)}}(u_1, \ldots, u_m | u_0, u_{m+1}) = \prod_{i=1}^{m+1} \frac{(u_i - u_{i-1})^{t_i - t_{i-1} - 1}}{\Gamma(t_i - t_{i-1})}$$

for $u_0 < u_1 < \ldots < u_m < u_{m+1}$. Combining (5.1), (5.2) and (5.3) we obtain

$$f_{W_{t_1}^{(k)},\ldots,W_{t_m}^{(k)}}(u_1, \ldots, u_m) = \frac{k^{n+1}}{\Gamma(n)} \int_0^{u_1} u_0^{n-1} \frac{(u_1 - u_0)^{t_1 - t_0 - 1}}{\Gamma(t_1 - t_0)} du_0$$

$$\times \prod_{i=2}^{m} \frac{(u_i - u_{i-1})^{t_i - t_{i-1} - 1}}{\Gamma(t_i - t_{i-1})} \int_{u_m}^{\infty} \frac{(u_{m+1} - u_m)^{t_{m+1} - t_m - 1}}{\Gamma(t_{m+1} - t_m)} e^{-ku_{m+1}} du_{m+1}$$

$$= k^{t_m} \prod_{i=1}^{m} \frac{(u_i - u_{i-1})^{t_i - t_{i-1} - 1}}{\Gamma(t_i - t_{i-1})} e^{-ku_m},$$

which is the same as (2.3) with $n = m$. ∎

Theorem 2 allows us to construct $(W^{(k)}(t_1), \ldots, W^{(k)}(t_m))$ in the case when $[t_1] = [t_m]$. Now we consider the general case. Let $i \equiv t_{i,0} < t_{i,1} < \ldots < t_{i,m_i} < t_{i,m_i+1} \equiv i+1$, $i = 0, 1, \ldots, n$, where $n = [t_m]+1$ and m_i denotes the number of $W^{(k)}(t)$ in W with $i < t < i+1$. Our aim is to construct the vector of the k-th fractional record values

$$W = (W^{(k)}(t_{i,j}), 1 \leqslant j \leqslant m_i, 0 \leqslant i \leqslant n-1)$$

using the sequence $\{W_n^{(k)}, n \geqslant 1\}$. This is done in the following theorem.

THEOREM 3. *Under the above assumptions we define*

$$W_{t_{i,j}}^{(k)} = (1 - B_j^{(i)}) W_i^{(k)} + B_j^{(i)} W_{i+1}^{(k)}, \qquad 1 \leqslant j \leqslant m_i, 0 \leqslant i \leqslant n-1,$$

where $\boldsymbol{B}^{(i)} = (B_1^{(i)}, \ldots, B_{m_i}^{(i)})$, $i = 0, 1, \ldots, n-1$, *is a random vector with* m_i-*dimensional generalized arc-sine distribution with parameters* $a_j^{(i)} = t_{i,j} - t_{i,j-1}$,

$j = 0, 1, \ldots, m_i+1$. *Suppose that* $\boldsymbol{B}^{(0)}, \boldsymbol{B}^{(1)}, \ldots, \boldsymbol{B}^{(n-1)}$ *and* $\{W_n^{(k)}, n \geqslant 1\}$ *are mutually independent. Then*

$$W \stackrel{d}{=} (W_{t_{0,1}}^{(k)}, \ldots, W_{t_{0,m_0}}^{(k)}, W_{t_{1,1}}^{(k)}, \ldots, W_{t_{1,m_1}}^{(k)}, \ldots, W_{t_{n-1,1}}^{(k)}, \ldots, W_{t_{n-1,m_{n-1}}}^{(k)}).$$

Proof. This easily follows from Theorem 2 and the independence of increments of k-th record values $\{W_n^{(k)}, n \geqslant 1\}$. ∎

6. LOWER RECORD-VALUES PROCESS

We start with a brief review of the distribution theory of k-th lower record values. It is known (cf. [11]) that if F is an absolutely continuous distribution function with pdf f, then the pdf of $Z_n^{(k)}$ is

$$f_{Z_n^{(k)}}(x) = \frac{k^n}{(n-1)!} (\bar{H}(x))^{n-1} (F(x))^{k-1} f(x), \quad x \in \boldsymbol{R}, \tag{6.1}$$

where $\bar{H}(x) := \bar{H}_F(x) = -\log F(x)$. The random vector $(Z_1^{(k)}, \ldots, Z_n^{(k)})$ has the joint pdf

$$f_{Z_1^{(k)}, \ldots, Z_n^{(k)}}(x_1, \ldots, x_n) = k^n \prod_{i=1}^{n-1} \frac{f(x_i)}{F(x_i)} (F(x_n))^{k-1} f(x_n) \tag{6.2}$$

for $x_1 \geqslant \ldots \geqslant x_n$. Moreover, if $0 = j_0 < j_1 < \ldots < j_n$, then the vector $(Z_{j_1}^{(k)}, \ldots, Z_{j_n}^{(k)})$ has the joint pdf

$$f_{Z_{j_1}^{(k)}, \ldots, Z_{j_n}^{(k)}}(x_1, \ldots, x_n) = k^{j_n} \prod_{i=1}^{n} \frac{(\bar{H}(x_{i-1}) - \bar{H}(x_i))^{j_i - j_{i-1} - 1} \bar{h}(x_i)}{(j_i - j_{i-1} - 1)!} (F(x_n))^k \tag{6.3}$$

for $\infty = x_0 > x_1 \geqslant \ldots \geqslant x_n > -\infty$, where $\bar{h}(x) = \bar{H}'(x)$.

Let $V_n^{(k)}$, $n \in \boldsymbol{N}$, stand for the k-th record value from standard negative exponential distribution with the cdf $G^*(x) = e^x$, $x \leqslant 0$. Using (6.1) and (6.3) one can show that for each $k \in \boldsymbol{N}$ the sequence $\{V_n^{(k)}, n \geqslant 1\}$ of k-th lower record values from negative exponential distribution has the following property: for all $m, n \in \boldsymbol{N}$ such that $n > m$ the random variables $V_m^{(k)}$ and $V_n^{(k)} - V_m^{(k)}$ are independent. Moreover, $V_m^{(k)}$ and $V_n^{(k)} - V_m^{(k)}$ are negative gamma $N\Gamma(m, k)$ and $N\Gamma(n-m, k)$ distributed, respectively, where $N\Gamma(\alpha, \beta)$ denotes a negative gamma distribution with pdf

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} |x|^{\alpha-1} e^{\beta x}, \quad x < 0, \ \alpha, \beta > 0.$$

The above facts motivate the following definition.

DEFINITION 4. Fix $k \in \boldsymbol{N}$. Let $V^{(k)} = \{V^{(k)}(t), t \geqslant 0\}$ be a stochastic process such that:

(i) $V^{(k)}(0) = 0$ a.s.,

(ii) $V^{(k)}$ has independent increments,

(iii) if $t > s \geqslant 0$, then $V^{(k)}(t) - V^{(k)}(s)$ is negative gamma $N\Gamma(t-s, k)$ distributed.

Then $\{V^{(k)}(t), t \geqslant 0\}$ is called the *negative exponential k-th lower record-values process.* The random variables $V^{(k)}(t)$, $t > 0$, are said to be *negative exponential fractional k-th lower record values.*

Note that $V^{(k)}(t)$, $t > 0$, is $N\Gamma(t, k)$ distributed. Moreover, if $n \in N$ and $0 = t_0 < t_1 < \ldots < t_n$, and

$$\hat{V} = \big(V^{(k)}(t_1),\ V^{(k)}(t_2) - V^{(k)}(t_1),\ \ldots,\ V^{(k)}(t_n) - V^{(k)}(t_{n-1})\big),$$

then the joint pdf of $\hat{V}$ is

$$f_{\hat{V}}(x_1, \ldots, x_n) = k^{t_n} \prod_{i=1}^{n} \frac{|x_i|^{t_i - t_{i-1} - 1}}{\Gamma(t_i - t_{i-1})} \exp\big(k \sum_{i=1}^{n} x_i\big), \qquad x_1, \ldots, x_n \leqslant 0.$$

Therefore, the joint pdf of $V = \big(V^{(k)}(t_1), V^{(k)}(t_2), \ldots, V^{(k)}(t_n)\big)$ is of the form

$$f_V(x_1, \ldots, x_n) = k^{t_n} \prod_{i=1}^{n} \frac{(x_{i-1} - x_i)^{t_i - t_{i-1} - 1}}{\Gamma(t_i - t_{i-1})} \exp(kx_n) \tag{6.4}$$

for $0 = x_0 \geqslant x_1 \geqslant \ldots \geqslant x_n > -\infty$.

Note that by (6.2) and (6.4) we have $\big(V^{(k)}(1), \ldots, V^{(k)}(n)\big) \stackrel{d}{=} (V_1^{(k)}, \ldots, V_n^{(k)})$ and, more generally, if $t_m = j_m \in N$ and $1 \leqslant j_1 < \ldots < j_n$, then using (6.3) and (6.4) we get

$$\big(V^{(k)}(j_1), \ldots, V^{(k)}(j_n)\big) \stackrel{d}{=} (V_{j_1}^{(k)}, \ldots, V_{j_n}^{(k)}).$$

This explains the name for the process $V^{(k)}$, which has the same finite-dimensional marginal distributions as the sequence of k-th lower record from negative exponential distribution.

Let F be a distribution function and let $G^*(x) = e^x$, $x \leqslant 0$, be the standard negative exponential distribution function.

DEFINITION 5. The stochastic process $Z^{(k)} = \{Z^{(k)}(t), t \geqslant 0\}$, where

$$Z^{(k)}(t) = F^{-1}\big(G^*\big(V^{(k)}(t)\big)\big), \qquad t \geqslant 0,$$

is called the *k-th lower record-values process for distribution function F.* The random variables $Z^{(k)}(t)$, $t > 0$, are said to be *fractional k-th lower record values from F.*

Suppose that F is absolutely continuous with the pdf f. Using the above definition one can easily show that $Z^{(k)}(t)$, $t > 0$, has the pdf

$$f_{Z^{(k)}(t)}(x) = \frac{k^t}{\Gamma(t)} \big(\bar{H}(x)\big)^{t-1} \big(F(x)\big)^{k-1} f(x), \qquad x \in \boldsymbol{R}.$$

Moreover, if $0 = t_0 < t_1 < \ldots < t_n$, then the joint pdf of the random vector $\boldsymbol{Z} := \left(Z^{(k)}(t_1), \ldots, Z^{(k)}(t_n)\right)$ is

$$(6.5) \qquad f_{\boldsymbol{Z}}(x_1, \ldots, x_n) = k^{t_n} \prod_{i=1}^{n} \frac{\left(\bar{H}(x_i) - \bar{H}(x_{i-1})\right)^{t_i - t_{i-1} - 1} \bar{h}(x_i)}{\Gamma(t_i - t_{i-1})} (F(x_n))^k$$

for $\infty = x_0 > x_1 \geqslant \ldots \geqslant x_n > -\infty$.

Note that by (6.2) and (6.5) we get $\left(Z^{(k)}(1), \ldots, Z^{(k)}(n)\right) \stackrel{d}{=} (Z_1^{(k)}, \ldots, Z_n^{(k)})$, and, more generally, for $1 \leqslant j_1 < \ldots < j_n$, $j_i \in N$, by (6.3) and (6.5) we have

$$\left(Z^{(k)}(j_1), \ldots, Z^{(k)}(j_n)\right) \stackrel{d}{=} (Z_{j_1}^{(k)}, \ldots, Z_{j_n}^{(k)}).$$

Therefore we can consider $Z^{(k)}(t)$ as $Z_n^{(k)}$ with n replaced with arbitrary positive t.

Now the following results hold true.

PROPOSITION 2. *$\{Z^{(k)}(t), t \geqslant 0\}$ is a Markov process with the transition probabilities*

$$P\{Z^{(k)}(t+s) < y \mid Z^{(k)}(t) = x\} = 1 - \frac{1}{\Gamma(s)} \Gamma\left(s; k\left(\bar{H}(y) - \bar{H}(x)\right)\right)$$

for $s > 0$, $y \leqslant x$.

THEOREM 4. *For $t \geqslant 0$*

$$V^{(k)}(t) \stackrel{d}{=} (1-B) V_{[t]}^{(k)} + B V_{[t]+1}^{(k)},$$

where B is a beta $B(\{t\}, 1-\{t\})$ distributed random variable, independent of the sequence $\{V_n^{(k)}, n \geqslant 1\}$.

THEOREM 5. *Let $n = t_0 < t_1 < \ldots < t_m < t_{m+1} = n+1$. Define*

$$V_{t_i}^{(k)} = (1-B_i) V_n^{(k)} + B_i V_{n+1}^{(k)}, \qquad 1 \leqslant i \leqslant m,$$

where $(B_1, \ldots, B_m)$ is a random vector with m-dimensional generalized arc-sine distribution with parameters $a_i = t_i - t_{i-1}$, $1 \leqslant i \leqslant m+1$, independent of the sequence $\{V_n^{(k)}, n \geqslant 1\}$. Then

$$(V_{t_1}^{(k)}, \ldots, V_{t_m}^{(k)}) \stackrel{d}{=} \left(V^{(k)}(t_1), \ldots, V^{(k)}(t_m)\right).$$

Let $i \equiv t_{i,0} < t_{i,1} < \ldots < t_{i,m_i} < t_{i,m_i+1} \equiv i+1$, for $i = 0, 1, \ldots, n$, and

$$V = \left(V^{(k)}(t_{i,j}), 1 \leqslant j \leqslant m_i, 0 \leqslant i \leqslant n-1\right).$$

THEOREM 6. *Under the above assumptions we define*

$$V_{t_{i,j}}^{(k)} = (1-B_j^{(i)}) V_i^{(k)} + B_j^{(i)} V_{i+1}^{(k)}, \qquad 1 \leqslant j \leqslant m_i, \; 0 \leqslant i \leqslant n-1,$$

where $\boldsymbol{B}^{(i)} = (B_1^{(i)}, \ldots, B_{m_i}^{(i)})$, $i = 0, 1, \ldots, n-1$, is a random vector with m_i-dimensional generalized arc-sine distribution with parameters $a_j^{(i)} = t_{i,j} - t_{i,j-1}$, $j = 0, 1, \ldots, m_i+1$. Suppose that $\boldsymbol{B}^{(0)}, \boldsymbol{B}^{(1)}, \ldots, \boldsymbol{B}^{(n-1)}$ and $\{V_n^{(k)}, n \geqslant 1\}$ are mutually independent. Then

$$V \stackrel{d}{=} (V_{t_{0,1}}^{(k)}, \ldots, V_{t_{0,m_0}}^{(k)}, V_{t_{1,1}}^{(k)}, \ldots, V_{t_{1,m_1}}^{(k)}, \ldots, V_{t_{n-1,1}}^{(k)}, \ldots, V_{t_{n-1,m_{n-1}}}^{(k)}).$$

The proofs of Proposition 2 and Theorems 4, 5 and 6 are similar to the proofs of Proposition 1 and Theorems 1, 2 and 3, respectively, with obvious modifications.

7. MOMENTS OF FRACTIONAL RECORD VALUES

In this section we present some examples of evaluations of moments of fractional record values.

EXAMPLE 1. *Uniform distribution.*
Let

$$F(x) = \begin{cases} 0, & x \leqslant 0, \\ x, & x \in (0, 1), \\ 1, & x \geqslant 1. \end{cases}$$

Then for $x \in (0, 1)$ we have $f(x) = 1$, $H(x) = -\log(1-x)$ and $h(x) = (1-x)^{-1}$. Therefore the pdf of $Y^{(k)}(t)$, $t > 0$, is

$$f_{Y^{(k)}(t)}(x) = \frac{k^t}{\Gamma(t)}\left(-\log(1-x)\right)^{t-1}(1-x)^{k-1}, \qquad x \in (0, 1),$$

and for $n \in N$

$$E\left(Y^{(k)}(t)\right)^n = \frac{k^t}{\Gamma(t)}\int_0^1 x^n\left(-\log(1-x)\right)^{t-1}(1-x)^{k-1}\,dx$$

$$= \frac{k^t}{\Gamma(t)}\int_0^\infty (1-e^{-z})^n e^{-kz} z^{t-1}\,dz.$$

Using Newton's binomial formula we get

$$E\left(Y^{(k)}(t)\right)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\left(\frac{k}{k+j}\right)^t.$$

For instance,

$$EY^{(k)}(t) = 1-\left(\frac{k}{k+1}\right)^t$$

and

$$E\left(Y^{(k)}(t)\right)^2 = 1-2\left(\frac{k}{k+1}\right)^t+\left(\frac{k}{k+2}\right)^t.$$

Therefore

$$\operatorname{Var} Y^{(k)}(t) = \frac{k^t(k+1)^{2t}-k^{2t}(k+2)^t}{(k+1)^{2t}(k+2)^t}.$$

Similarly, for $0<t<s$

$$\operatorname{Cov}\big(Y^{(k)}(t),\, Y^{(k)}(s)\big)=\left(\frac{k}{k+1}\right)^{s}\left[\left(\frac{k+1}{k+2}\right)^{t}-\left(\frac{k}{k+1}\right)^{t}\right].$$

EXAMPLE 2. *Weibull distribution.*
Let

$$F(x)=\begin{cases}1-\exp(-\lambda x^{\alpha}), & x\geqslant 0,\\ 0, & x<0.\end{cases}$$

Then $f(x)=\alpha\lambda x^{\alpha-1}\exp(-\lambda x^{\alpha})$, $H(x)=\lambda x^{\alpha}$ and $h(x)=\alpha\lambda x^{\alpha-1}$. Therefore

$$f_{Y^{(k)}(t)}(x)=\frac{\alpha(k\lambda)^{t}}{\Gamma(t)}x^{\alpha t-1}\exp(-k\lambda x^{\alpha}),\qquad x\geqslant 0,$$

which for $\beta>0$ gives

$$E\big(Y^{(k)}(t)\big)^{\beta}=\frac{\Gamma(t+\beta/\alpha)}{(k\lambda)^{\beta/\alpha}\,\Gamma(t)}.$$

For instance,

$$\operatorname{Var}Y^{(k)}(t)=\frac{1}{(k\lambda)^{2/\alpha}\,\Gamma^{2}(t)}\left(\Gamma\left(t+\frac{2}{\alpha}\right)\Gamma(t)-\Gamma^{2}\left(t+\frac{1}{\alpha}\right)\right).$$

Moreover, for $0<t<s$

$$\operatorname{Cov}\big(Y^{(k)}(t),\, Y^{(k)}(s)\big)=\frac{\Gamma(t+1/\alpha)}{(k\alpha)^{2/\alpha}}\left\{\frac{\Gamma(s+2/\alpha)}{\Gamma(s+1/\alpha)}-\frac{\Gamma(s+1/\alpha)}{\Gamma(s)}\right\}.$$

EXAMPLE 3. *Single-parameter Pareto distribution.*
Consider the single-parameter Pareto distribution function

$$F(x)=\begin{cases}0, & x<1,\\ 1-1/x^{\alpha}, & x\geqslant 1,\end{cases}$$

where $\alpha>0$. Then for $x\geqslant 1$ we have $f(x)=\alpha/x^{\alpha+1}$, $H(x)=\alpha\log x$ and $h(x)=\alpha/x$. Therefore the pdf of $Y^{(k)}(t)$ is

$$f_{Y^{(k)}(t)}(x)=\frac{(k\alpha)^{t}}{\Gamma(t)}\frac{(\log x)^{t-1}}{x^{k\alpha+1}},\qquad x\geqslant 1.$$

Therefore for $\beta>0$

$$E\big(Y^{(k)}(t)\big)^{\beta}=\left(\frac{k\alpha}{k\alpha-\beta}\right)^{t},$$

provided that $\beta<k\alpha$. If $\alpha>2/k$, this easily gives

$$\operatorname{Var}Y^{(k)}(t)=\left(\frac{k\alpha}{k\alpha-2}\right)^{t}-\left(\frac{k\alpha}{k\alpha-1}\right)^{2t}.$$

Similarly, for $0<t<s$

$$\operatorname{Cov}\left(Y^{(k)}(t),\, Y^{(k)}(s)\right)=\left(\frac{k\alpha}{k\alpha-1}\right)^{s}\left[\left(\frac{k\alpha-1}{k\alpha-2}\right)^{t}-\left(\frac{k\alpha}{k\alpha-1}\right)^{t}\right].$$

EXAMPLE 4. *Two-parameter Pareto distribution (Lomax distribution).* For the two-parameter Pareto distribution

$$F(x)=\begin{cases}1-(\lambda/(\lambda+x))^{\alpha}, & x>0,\\ 0, & x\leqslant 0,\end{cases}\qquad \lambda>0,\ \alpha>0,$$

we have

$$E\left(Y^{(k)}(t)\right)^{n}=\lambda^{n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\left(\frac{k\alpha}{k\alpha-j}\right)^{t},\qquad n<k\alpha.$$

Therefore, if $k\alpha>2$, then

$$\operatorname{Var} Y^{(k)}(t)=\lambda^{2}\left\{\left(\frac{k\alpha}{k\alpha-2}\right)^{t}-\left(\frac{k\alpha}{k\alpha-1}\right)^{2t}\right\}.$$

Also

$$\operatorname{Cov}\left(Y^{(k)}(t),\, Y^{(k)}(s)\right)=\lambda^{2}\left(\frac{k\alpha}{k\alpha-1}\right)^{s}\left[\left(\frac{k\alpha-1}{k\alpha-2}\right)^{t}-\left(\frac{k\alpha}{k\alpha-1}\right)^{t}\right].$$

EXAMPLE 5. *Generalized Pareto distribution.*
For the generalized Pareto distribution with pdf

$$f(x)=\begin{cases}(1+\alpha x)^{-1-1/\alpha}, & x\geqslant 0, & \text{if } \alpha>0,\\ (1+\alpha x)^{-1-1/\alpha}, & 0\leqslant x\leqslant -1/\alpha, & \text{if } \alpha<0,\\ e^{-x}, & x\geqslant 0, & \text{if } \alpha=0,\\ 0, & \text{otherwise,} & \end{cases}$$

we have for $n\in N$, $\alpha\neq 0$,

$$E\left(Y^{(k)}(t)\right)^{n}=\frac{1}{\alpha^{n}}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\left(\frac{k}{k-j\alpha}\right)^{t},$$

where

$$\begin{array}{ll} n<k/\alpha & \text{if } \alpha>0,\\ n\in N & \text{if } \alpha<0.\end{array}$$

For instance, if $2\alpha<k$, then

$$\operatorname{Var} Y^{(k)}(t)=\frac{1}{\alpha^{2}}\left\{\left(\frac{k}{k-2\alpha}\right)^{t}-\left(\frac{k}{k-\alpha}\right)^{2t}\right\}.$$

Moreover, for $0 < t < s$

$$\mathrm{Cov}\big(Y^{(k)}(t),\, Y^{(k)}(s)\big) = \frac{1}{\alpha^2}\left(\frac{k}{k-\alpha}\right)^s \left[\left(\frac{k-\alpha}{k-2\alpha}\right)^t - \left(\frac{k}{k-\alpha}\right)^t\right].$$

EXAMPLE 6. *Inverse exponential distribution.*
Let

$$F(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x < 0. \end{cases}$$

Then for $x > 0$ we have $f(x) = x^{-2} e^{-1/x}$ and $\bar{H}(x) = x^{-1}$, and $\bar{h}(x) = x^{-2}$. Therefore

$$f_{Z^{(k)}(t)}(x) = \frac{k^t}{\Gamma(t)} \frac{e^{-k/x}}{x^{t+1}}, \qquad x > 0,$$

and for $\alpha > 0$

$$E\big(Z^{(k)}(t)\big)^\alpha = k^\alpha \frac{\Gamma(t-\alpha)}{\Gamma(t)},$$

provided that $t > \alpha$. For instance, for $t > 1$

$$EZ^{(k)}(t) = \frac{k}{t-1},$$

and for $t > 2$

$$E\big(Z^{(k)}(t)\big)^2 = \frac{k^2}{(t-1)(t-2)},$$

which implies

$$\mathrm{Var}\, Z^{(k)}(t) = \frac{k^2}{(t-1)^2(t-2)}, \qquad t > 2.$$

EXAMPLE 7. *Gumbel distribution.*
Let

$$F(x) = \exp(-e^{-x}), \qquad x \in \boldsymbol{R}.$$

First we consider the case $\gamma = 0$ which corresponds to Gumbel distribution. Then

$$f_{Z^{(k)}(t)}(x) = \frac{k^t}{\Gamma(t)} \exp(-ke^{-x}) e^{-tx}, \qquad x \in \boldsymbol{R},$$

and for $n \in \boldsymbol{N}$

$$E\big(Z^{(k)}(t)\big)^n = \frac{k^t}{\Gamma(t)} \int_{-\infty}^{\infty} x^n \exp(-ke^{-x}) e^{-tx}\, dx = \frac{k^t}{\Gamma(t)} \int_0^\infty (-\log u)^n e^{-ku} u^{t-1}\, du$$

$$= \frac{1}{\Gamma(t)} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (\log k)^{n-j} \Gamma^{(j)}(t),$$

where $\Gamma^{(j)}$, $j \geqslant 1$, denotes the j-th derivative of gamma function and $\Gamma^{(0)} = \Gamma$. Therefore

$$EZ^{(k)}(t) = \log k - \frac{\Gamma'(t)}{\Gamma(t)}$$

and

$$E\left(Z^{(k)}(t)\right)^2 = \frac{1}{\Gamma(t)}\{(\log k)^2 \Gamma(t) - 2\Gamma'(t)\log k + \Gamma''(t)\}.$$

This gives

$$\operatorname{Var} Z^{(k)}(t) = \frac{\Gamma(t)\Gamma''(t) - \left(\Gamma'(t)\right)^2}{\left(\Gamma(t)\right)^2},$$

which is positive since Γ is log-convex function on $(0, \infty)$.

Moreover, for $0 < t < s$

$$EZ^{(k)}(t)Z^{(k)}(s) = \frac{k^s}{\Gamma(t)\Gamma(s-t)} \int_{-\infty}^{\infty} y\exp(-ke^{-y})e^{-y} \int_{y}^{\infty} xe^{-tx}(e^{-y}-e^{-x})^{s-t-1}\,dx\,dy$$

$$= \frac{k^s}{\Gamma(t)\Gamma(s-t)} \int_{-\infty}^{\infty} y\exp(-ke^{-y})e^{-sy}\Big\{\int_0^{\infty} ze^{-tz}(1-e^{-z})^{s-t-1}\,dz$$

$$+ y\int_0^{\infty} e^{-tz}(1-e^{-z})^{s-t-1}\,dz\Big\}\,dy.$$

We have

$$\int_0^{\infty} e^{-tz}(1-e^{-z})^{s-t-1}\,dz = B(t, s-t) = \frac{\Gamma(t)\Gamma(s-t)}{\Gamma(s)}$$

and

$$\int_0^{\infty} e^{-tz}(1-e^{-z})^{s-t-1}\,dz = B(t, s-t)\left(\frac{\Gamma'(s)}{\Gamma(s)} - \frac{\Gamma'(t)}{\Gamma(t)}\right).$$

Hence

$$EZ^{(k)}(t)Z^{(k)}(s) = E\left(Z^{(k)}(s)\right)^2 + \left(\frac{\Gamma'(s)}{\Gamma(s)} - \frac{\Gamma'(t)}{\Gamma(t)}\right)EZ^{(k)}(s),$$

and

$$\operatorname{Cov}\left(Z^{(k)}(t), Z^{(k)}(s)\right) = \operatorname{Var} Z^{(k)}(s) = \frac{\Gamma(s)\Gamma''(s) - \left(\Gamma'(s)\right)^2}{\left(\Gamma(s)\right)^2}.$$

EXAMPLE 8. *Generalized extreme value distributions.*
Let

$$F(x)=\begin{cases}\exp\big(-(1-\gamma x)^{1/\gamma}\big), & x<1/\gamma,\ \gamma>0,\\ \exp\big(-(1-\gamma x)^{1/\gamma}\big), & x>1/\gamma,\ \gamma<0,\\ \exp(-e^{-x}), & x\in\mathbf{R},\ \gamma=0.\end{cases}$$

The case $\gamma=0$ corresponds to Gumbel distribution which has been considered in Example 7. For $\gamma\neq 0$ we obtain

$$f_{Z^{(k)}(t)}(x)=\frac{k^t}{\Gamma(t)}(1-\gamma x)^{t/\gamma-1}\exp\big(-k(1-\gamma x)^{1/\gamma}\big),$$

and for $n\in\mathbf{N}$

$$E\big(Z^{(k)}(t)\big)^n=\frac{1}{\gamma^n\Gamma(t)}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{\Gamma(\gamma i+t)}{k^{\gamma i}},\qquad t>\max(0,\,-n\gamma).$$

Therefore for $t>\max(0,\,-\gamma)$

$$EZ^{(k)}(t)=\frac{1}{\gamma}-\frac{\Gamma(\gamma+t)}{\gamma k^{\gamma}\Gamma(t)},$$

and for $t>\max(0,\,-2\gamma)$

$$E\big(Z^{(k)}(t)\big)^2=\frac{1}{\gamma^2}-\frac{2\Gamma(t+\gamma)}{\gamma^2k^{\gamma}\Gamma(t)}+\frac{\Gamma(t+2\gamma)}{\gamma^2k^{2\gamma}\Gamma(t)}.$$

Hence

$$\operatorname{Var}Z^{(k)}(t)=\frac{\Gamma(t+2\gamma)\Gamma(t)-\Gamma^2(\gamma+t)}{\gamma^2k^{2\gamma}\Gamma^2(t)}.$$

Moreover (cf. [1]), for $0<t<s$

$$\operatorname{Cov}\big(Z^{(k)}(t),\,Z^{(k)}(s)\big)=\frac{\Gamma(t+\gamma)}{\gamma^2k^{2\gamma}\Gamma(t)}\left\{\frac{\Gamma(s+2\gamma)}{\Gamma(s+\gamma)}-\frac{\Gamma(s+\gamma)}{\Gamma(s)}\right\}.$$

8. AN APPLICATION

Let $\{Y^{(k)}(t),\,t\geqslant 0\}$ be the k-th record-values process for an absolutely continuous distribution function F with pdf f and the hazard function $H(x)=-\log\big(1-F(x)\big)$. Let ψ_F stand for the inverse function of H, i.e.

$$\psi_F(u)=H^{-1}(u)=F^{-1}(1-e^{-u}),\qquad u\geqslant 0.$$

As an application of fractional record values we consider the problem of estimation of $\psi_F(u)$ for $u>0$, which is equivalent to the estimation of x_p, the p-th quantile of F, by putting $u = -\log(1-p)$, $p \in (0, 1)$. The problem of the estimation of x_p by fractional order statistics is considered in [7] and [10].

Using Taylor's formula to ψ_F in a neighbourhood of u we get

$$\psi_F(x)-\psi_F(u) = \psi'_F(u)(x-u)+\tfrac{1}{2}\psi''_F(u)(x-u)^2+\tfrac{1}{6}\psi'''_F(u)(x-u)^3+\ldots$$

Using $Y^{(k)}(t) \stackrel{d}{=} \psi_F(W^{(k)}(t))$, putting $x = W^{(k)}(t)$ and taking expectations, we obtain

$$(8.1)\quad EY^{(k)}(t) = \psi_F(u)+\psi'_F(u)E\left(W^{(k)}(t)-u\right) + \tfrac{1}{2}\psi''_F(u)E\left(W^{(k)}(t)-u\right)^2+\tfrac{1}{6}\psi'''_F(u)E\left(W^{(k)}(t)-u\right)^3+\ldots$$

Taking into account that $W^{(k)}(ku)$ is $\Gamma(ku, k)$ distributed, we see that if $t = ku$, then $E\left(W^{(k)}(t)-u\right) = 0$ and $E\left(W^{(k)}(t)-u\right)^2 = u/k$. Putting these quantities into (8.1) we get

$$EY^{(k)}(ku) = \psi_F(u)+\frac{u\psi''_F(u)}{2k}+\frac{1}{6}\psi'''_F(u)E\left(W^{(k)}(ku)-u\right)^3+\ldots$$

Therefore $Y^{(k)}(ku)$ can be considered as an estimator of the value $\psi_F(u)$.

DEFINITION 6. The *estimator $\hat{\psi}_F(u)$ of the inverse to hazard function at the point u based on the k-th fractional record values* is defined as

$$\hat{\psi}_F(u) = Y^{(k)}(ku), \qquad u > 0.$$

Note that using the fractional record values instead of the ordinary record values allows us to reduce the bias of $\hat{\psi}_F(u)$.

We consider also the estimator of $\psi_F(u)$ based on the sequence $\{Y_n^{(k)}, n \geqslant 1\}$ of k-th record values from F.

DEFINITION 7. The *estimator $\tilde{\psi}_F(u)$ of $\psi_F(u)$ based on the k-th record values from F* is defined as

$$\tilde{\psi}_F(u) = (1-\{ku\})\, Y^{(k)}_{[ku]}+\{ku\}\, Y^{(k)}_{[ku]+1},$$

where $[x]$ and $\{x\}$ stand for the integral and fractional part of a real number x.

Note that the values of $\tilde{\psi}_F(u)$ may be obtained from empirical data, on the contrary to $\hat{\psi}_F(u)$. The values of $\hat{\psi}_F(u)$ can be approximated by the values of $\tilde{\psi}_F(u)$, as stated in the following theorem.

THEOREM 7. *Let $\varepsilon = \{ku\}$. Then*

$$(8.2)\qquad E\left(\tilde{\psi}_F(u)-\hat{\psi}_F(u)\right) = \frac{\varepsilon(1-\varepsilon)}{2k^2}\left(\psi''_F(u)+u\psi^{(3)}_F(u)\right)+O(k^{-3}).$$

Proof. Let $\mu'_j = E\left(W^{(k)}(t)-t/k\right)^j$, $j \in N$, stand for the j-th central moment of $W^{(k)}(t)$ and let $c = t/k-u$. Then for $j \geqslant 2$

$$\mu'_j = \frac{1}{k^j}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}t^{j-i}\frac{\Gamma(t+j)}{\Gamma(j)} = O\left(\frac{1}{k^j}\right),$$

which implies for $r \geqslant 2$

$$E\left(W^{(k)}(t)-u\right)^r = \sum_{j=0}^{r}\binom{r}{j}c^{r-j}\mu_j' = O\left(\frac{1}{k^r}\right).$$

Moreover, by (8.1) the left-hand side of (8.2) may be written as

$$\psi_F(u)+\psi_F'(u)M_1+\tfrac{1}{2}\psi_F''(u)M_2+\tfrac{1}{6}\psi_F'''(u)M_3+\ldots, \tag{8.3}$$

where

$$\begin{aligned}M_r &= (1-\varepsilon)E\left(W_{[ku]}^{(k)}-u\right)^r+\varepsilon E\left(W_{[ku]+1}^{(k)}-u\right)^r-E\left(W^{(k)}(t)-u\right)^r\\ &= \sum_{j=0}^{r-1}\binom{r}{j}\frac{\mu_j'}{k^{r-j}}\{(1-\varepsilon)(-\varepsilon)^{r-j}+\varepsilon(1-\varepsilon)^{r-j}\}.\end{aligned}$$

Therefore

$$M_1 = 0, \quad M_2 = \frac{\varepsilon(1-\varepsilon)}{k^2}, \quad M_3 = \frac{\varepsilon(1-\varepsilon)}{k^2}\left(3u-\frac{2\varepsilon^2-2\varepsilon+1}{k}\right).$$

Putting these expressions into (8.3) we get (8.2). ■

Now we show how to construct the confidence intervals for $\psi_F(u)$ using $\hat{\psi}_F(u)$ and $\tilde{\psi}_F(u)$. As $W^{(k)}(t) \sim \Gamma(t, k)$, we obtain

$$P\left(Y^{(k)}(t) \leqslant \psi_F(u)\right) = P\left(W^{(k)}(t) \leqslant u\right) = \frac{\Gamma(t;\, ku)}{\Gamma(t)},$$

where $\Gamma(\alpha;\, x)$ is incomplete gamma function given by (3.2). Therefore, for $0 < t < s$

$$P\left(Y^{(k)}(t) \leqslant \psi_F(u) \leqslant Y^{(k)}(s)\right) = \frac{\Gamma(t;\, ku)}{\Gamma(t)}-\frac{\Gamma(s;\, ku)}{\Gamma(s)}. \tag{8.4}$$

If $t, s \in N$ and $t = n$, $s = n+r$, then (8.4) takes the form

$$P\left(Y_n^{(k)} \leqslant \psi_F(u) \leqslant Y_{n+r}^{(k)}\right) = e^{-ku}\sum_{i=n}^{n+r-1}\frac{(ku)^i}{i!}.$$

Therefore, to construct the $100(1-\alpha)\%$ confidence interval of the form

$$\left(Y^{(k)}(t),\, Y^{(k)}(s)\right),$$

we choose as t and s the solutions to the equations

$$\frac{\Gamma(t;\, ku)}{\Gamma(t)} = 1-\frac{\alpha}{2}, \tag{8.5}$$

$$\frac{\Gamma(s;\, ku)}{\Gamma(s)} = \frac{\alpha}{2}. \tag{8.6}$$

Alternatively, t and s can be approximated as follows:

(8.7) $$t \approx \Gamma_{ku,1}^{-1}(\alpha/2),$$

(8.8) $$s \approx \Gamma_{ku,1}^{-1}(1-\alpha/2),$$

where $\Gamma_{a,b}^{-1}(p)$, $p \in (0, 1)$, denotes the quantile of order p of gamma $\Gamma(a, b)$ distribution.

Note that in general the values given in (8.7) and (8.8) are easier to find. However, for the values of t and s determined by (8.5) and (8.6) the coverage probability is exactly $1-\alpha$, while for t and s determined by (8.7) and (8.8) the coverage probability is only approximately equal to $1-\alpha$.

To summarize the above consideration, we define the exact $100(1-\alpha)\%$ confidence interval for $\psi_F(u)$ as

$$\left(\hat{\psi}_F(t/k),\ \hat{\psi}_F(s/k)\right),$$

where t and s are given by (8.5) and (8.6), respectively. But in practice we propose using the approximate $100(1-\alpha)\%$ confidence interval for $\psi_F(u)$ defined by

$$\left(\tilde{\psi}_F(t/k),\ \tilde{\psi}_F(s/k)\right),$$

where t and s are given by (8.5) and (8.6), respectively.

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