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A LOGARITHMIC SOBOLEV INEQUALITY FOR ONE-DIMENSIONAL MULTIVALUED STOCHASTIC DIFFERENTIAL EQUATIONS

BY

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Abstract. We establish a logarithmic Sobolev inequality for a one-dimensional multivalued stochastic differential equation associated with the subdifferential of a convex lower semicontinuous function, using an explicit expression for the Malliavin derivative of the considered process. This result is given under some mild conditions on the coefficients.

1. INTRODUCTION AND MAIN RESULT

In this note we prove a logarithmic Sobolev inequality for a class of one-dimensional multivalued stochastic differential equations associated with the subdifferential of a convex lower semicontinuous function h. This includes important examples including the one-dimensional diffusion processes with zero or two reflecting barriers, the reflected Bessel process and the Bang–Bang process. The inequality generalizes the one established in Capitaine [1] for one-dimensional diffusion processes. The main tool is the explicit expression of the stochastic derivative of reflected diffusion processes as given in Lépingle et al. [3], which turns out to be positive and dominated by a stochastic process as is the case of a standard diffusion process.

Let $W = \{W_t\colon t \geqslant 0\}$ be a one-dimensional standard Brownian motion defined on the probability space $(\Omega, \mathscr{F}, \mathscr{F}_t\colon t \geqslant 0, \boldsymbol{P})$, where $\Omega = \mathscr{C}_0([0, T], \boldsymbol{R})$ is the space of real-valued continuous functions on $[0, T]$ vanishing at the origin, $\boldsymbol{P}$ is the Wiener measure and $\mathscr{F}_t$ in the natural filtration of W completed with the $\boldsymbol{P}$-null sets of $\mathscr{F}$.

Let $h\colon \boldsymbol{R} \to]-\infty, +\infty]$ be a proper convex and lower semicontinuous function, that is the interior I of its domain

$$\mathrm{Dom}(h) = \{x \in \boldsymbol{R}\colon h(x) < +\infty\}$$

is nonempty. The multivalued maximal monotone operator ∂h is defined by its graph

$$\mathrm{Gr}(\partial h) = \{(x, y) \in \boldsymbol{R}^2\colon \text{ for all } z \in \boldsymbol{R},\ h(z) \geqslant h(x) + y(z-x)\}.$$

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Let us now introduce our multivalued stochastic differential equation (MSDE). Let the real-valued functions σ and b be Lipschitz, let h be a proper convex and lower semicontinuous function and suppose that η is an $\mathscr{F}_0$ random variable taking its values in $\bar{I}$, the closure of the interior of Dom(h).

Given σ, b, η, h and the Brownian motion W, Lépingle and Marois [2] proved that there exists a unique pair (Y, K) of $\mathscr{F}_t$-adapted and continuous processes such that:

- For each $0 \leqslant t \leqslant T$, Y_t takes its values in $\bar{I}$ and $Y_0 = \eta$.
- dK_t is a σ-finite random measure on $[0, T]$, $K_0 = 0$ and is adapted in the sense that, for any measurable mapping φ: $[0, T] \to \boldsymbol{R}_+$, the process $\int_0^t \varphi(s)\, dK_s$ is $\mathscr{F}_t$-measurable.
- (Y, K) solves the SDE:

$$dY_t = \sigma(Y_t)\, dW_t + b(Y_t)\, dt - dK_t, \qquad Y_0 = \eta.$$

- For every pair of optional processes (α, β) such that $(\alpha, \beta) \in \mathrm{Gr}(\partial h)$ the measure

$$(Y_t - \alpha_t)(dK_t - \beta_t\, dt)$$

is a.s. positive over $[0, T]$.

In short, the pair (Y, K) is called a *solution of the problem* Eq(η, σ, b, h).

The following important examples are special cases of the problem Eq(η, σ, b, h):

- **Reflecting SDE in zero.** This corresponds to the convex function

$$h(x) = \begin{cases} +\infty & \text{for } x < 0, \\ 0 & \text{for } x \geqslant 0. \end{cases}$$

- **Reflection with two obstacles** 0 **and** 1 corresponds to the convex function

$$h(x) = \begin{cases} +\infty & \text{for } x \notin [0, 1], \\ 0 & \text{for } x \in [0, 1]. \end{cases}$$

- **Reflected Bessel process of order** $\alpha > 1$ corresponds to the convex function

$$h(x) = \begin{cases} +\infty & \text{for } x \leqslant 0, \\ [(1-\alpha)/2] \log x & \text{for } x > 0. \end{cases}$$

- **Bang–Bang process** corresponds to the convex function $h(x) = |x|$.

Capitaine [1] has proved a logarithmic Sobolev inequality when there is no reflection $(K \equiv 0)$ and the diffusion coefficient σ is assumed to be Lipschitz non-degenerate and $\mathscr{C}^2$ whereas the drift coefficient is assumed Lipschitz and $\mathscr{C}^1$. The main result in this note, Proposition 1.1, is a logarithmic Sobolev inequality for the reflecting diffusion described above. As a by-product, we obtain the logarithmic Sobolev inequality for the standard SDE considered in Capitaine [1], under fairly weak conditions on the coefficients.

Let μ denote the positive σ-finite measure associated with h and given by the formula $\mu(a, b) = h'(b) - h'(a)$ for $a < b$ and a and b in $\boldsymbol{I}$, $\mu(\boldsymbol{I}^c) = 0$, where, here and in the sequel, g' denotes the right derivative of the function g.

PROPOSITION 1.1. *Let the pair* (Y, K) *be the solution of the problem* $\mathrm{Eq}(\eta, \sigma, b, h)$. *Furthermore, assume that* η *is in* $L^p(\Omega, \mathscr{F}_0)$ *with* $p \geqslant 4$ *and that* σ *and* b *satisfy the following conditions*:

(1) σ *is the difference of two convex functions,* $\sigma(x) > 0$ *for all* x *in* $\boldsymbol{I}$, *and* σ *and* b *are Lipschitz functions.*

(2) *The measure* $-\sigma'(Y_t)\,dK_t$ *is a.s. positive on* $\boldsymbol{R}_+$.

(3) *There exists a positive constant* c *such that the measure*

$$\sigma\left(\frac{1}{2}\sigma' - \frac{b}{\sigma}\right)'(dx) + \mu(dx) - c\,dx \text{ is positive.} \tag{1.1}$$

Then, for any function f *in* $\mathscr{C}_b^1(\boldsymbol{R}, \boldsymbol{R})$,

$$\boldsymbol{E}[f^2(Y_t)\log f^2(Y_t)] - \boldsymbol{E}[f^2(Y_t)]\log \boldsymbol{E}[f^2(Y_t)] \leqslant \frac{1-\exp(-2ct)}{c}\boldsymbol{E}\left[(f'(Y_t)\,\sigma(Y_t))^2\right]. \tag{1.2}$$

Remark.

- The logarithmic Sobolev inequality (1.2) coincides with the one corresponding to the case $K \equiv 0$, established in Capitaine [1] under stronger conditions on the coefficients. If furthermore σ is $\mathscr{C}^2$ and b is $\mathscr{C}^1$ in Proposition 1.1, the condition (1.1) coincides with the main condition leading to the logarithmic Sobolev inequality in Capitaine [1], namely

$$\sigma^{-1}(L\sigma - b'\sigma) \geqslant -c,$$

where $c \geqslant 0$ and L is the infinitesimal generator of the diffusion with coefficients σ and b.

- Using the method of proof of Proposition 2 in Capitaine [1], we can easily extend the logarithmic Sobolev inequality (1.2) to every cylindric function of the process Y. For ease of exposition, we omit the details.

- The condition (2) in Proposition 1.1 is satisfied if e.g. h is a decreasing function and $\sigma'(x) \geqslant 0$ dx a.e.

2. PROOF OF THE MAIN RESULT

It is proved in Lépingle et al. [3], Proposition 2.7, that if η is in $L^p(\Omega, \mathscr{F}_0)$ with $p \geqslant 2$, then we have the explicit formula for the Malliavin derivative of the process Y:

$$D_r Y_t = \mathbf{1}_{B_{r,T}}(t)\left\{\sigma(Y_r) + \int_r^t \sigma'(Y_s) D_r Y_s \, dW_s + \int_r^t b'(Y_s) D_r Y_s \, ds - D_r K_t\right\},$$

where $B_{r,T}$ in a random set, in the case when h is decreasing and is affine on $[a, b]^c$, $\mathbf{1}_{B_{r,T}}(t) = [r, T]$. Moreover, it follows from Theorem 3.2 in [3] that if η is in $L^p(\Omega, \mathscr{F}_0)$ with $p \geqslant 4$, then a.s. in $[0, t] \times \Omega$

$$0 \leqslant D_r Y_t \leqslant V_t(r) := U_t(r) \exp\left(-\int_{\mathbf{R}} \frac{L_t^x - L_r^x}{\sigma^2(x)} \mu(dx)\right), \tag{2.1}$$

where L_s^x denotes the local time of the process Y at x at time s and

$$U_t(r) = \sigma(Y_r) \exp\left(\int_r^t \sigma'(Y_s) \, dW_s - \int_r^t \frac{1}{2}(\sigma')^2(Y_s) \, ds + \int_r^t b'(Y_s) \, ds\right).$$

As in [1], we consider the reflected diffusion Y as a functional of W. Hence the logarithmic Sobolev inequality for functionals of the Brownian motion yields

$$E[f^2(Y_t) \log f^2(Y_t)] - E[f^2(Y_t)] \log E[f^2(Y_t)] \leqslant 2E \int_0^t (f'(Y_t) D_r Y_t)^2 \, dr.$$

Now, by (2.1), the above inequality becomes

$$E[f^2(Y_t) \log f^2(Y_t)] - E[f^2(Y_t)] \log E[f^2(Y_t)] \leqslant 2E \int_0^t (f'(Y_t) V_t(r))^2 \, dr.$$

In order to get the inequality, we make use of the following upper bound of the process $U_t(r)$.

LEMMA 2.1. *Under the conditions of Proposition* 1.1 *we have*

$$U_t(r) \leqslant \sigma(Y_t) \exp\left(-\int_{\mathbf{R}} \frac{L_t^x - L_r^x}{\sigma(x)} \left(\frac{1}{2}\sigma' - \frac{b}{\sigma}\right)'(dx)\right).$$

Proof. For t in $[0, T]$, we set

$$M_t = \exp\left(-\int_0^t \sigma'(Y_s) \, dW_s - \int_0^t \left(b' - \frac{1}{2}[\sigma']^2\right)(Y_s) \, ds\right),$$

$$A_t = \exp\left(-\int_0^t \int_{\mathbf{R}} \frac{d_u L_u^x}{\sigma(x)} \left(\frac{1}{2}\sigma' - \frac{b}{\sigma}\right)'(dx)\right).$$

Now, integrating by parts we get

$$A_t \sigma(Y_t) M_t = A_r \sigma(Y_r) M_r + \int_r^t A_u M_u \, d\sigma(Y_u) + \int_r^t \sigma(Y_u) M_u \, dA_u + \int_r^t A_u \sigma(Y_u) \, dM_u + \int_r^t A_u \, d\langle \sigma(Y_\cdot), M_\cdot \rangle_u.$$

Using the Meyer–Tanaka formula we obtain

$$d\sigma(Y_u) = (\sigma' \sigma)(Y_u) \, dW_u + (\sigma' b)(Y_u) \, du + \frac{1}{2} d_u \int_{\mathbf{R}} L_u^x \sigma''(dx) - \sigma'(Y_u) \, dK_u,$$

$$dM_u = -M_u \sigma'(Y_u) \, dW_u + M_u ([\sigma']^2 - b')(Y_u) \, du,$$

$$d\langle \sigma(Y_\cdot), M_\cdot \rangle_u = -M_u ([\sigma']^2 \sigma)(Y_u) \, du.$$

Substituting in the above formula we obtain

$$\begin{aligned} A_t \sigma(Y_t) M_t = {} & A_r \sigma(Y_r) M_r + \int_r^t A_u M_u (\sigma' \sigma)(Y_u) \, dW_u \\ & + \int_r^t A_u M_u (\sigma' b)(Y_u) \, du + \frac{1}{2} \int_r^t A_u M_u \int_{\mathbf{R}} d_u L_u^x \sigma''(dx) \\ & - \int_r^t A_u M_u \sigma'(Y_u) \, dK_u - \int_r^t A_u \sigma(Y_u) M_u \int_{\mathbf{R}} \frac{d_u L_u^x}{\sigma(x)} \left(\frac{1}{2} \sigma' - \frac{b}{\sigma} \right)' (dx) \\ & - \int_r^t A_u M_u (\sigma' \sigma)(Y_u) \, dW_u + \int_r^t A_u M_u \big(\sigma ([\sigma']^2 - b') \big)(Y_u) \, du \\ & - \int_r^t A_u M_u (\sigma [\sigma']^2)(Y_u) \, du. \end{aligned}$$

Therefore,

$$\begin{aligned} A_t \sigma(Y_t) M_t = {} & A_r \sigma(Y_r) M_r + \frac{1}{2} \int_r^t A_u M_u \int_{\mathbf{R}} d_u L_u^x \sigma''(dx) \\ & - \frac{1}{2} \int_r^t A_u \sigma(Y_u) M_u \int_{\mathbf{R}} \frac{d_u L_u^x}{\sigma(x)} \sigma''(dx) - \int_r^t A_u M_u \sigma'(Y_u) \, dK_u. \end{aligned}$$

Consequently, since for a.a. ω the measure in s, $d_s L_s^x$, is carried by the set $\{u\colon\ Y_u(\omega) = x\}$, it follows that

$$A_t \sigma(Y_t) M_t = A_r \sigma(Y_r) M_r - \int_r^t A_u M_u \sigma'(Y_u) dK_u.$$

Now, since by the condition (2) the integral $-\int_r^t A_u M_u \sigma'(Y_u) \, dK_u$ is non-negative, it follows that

$$U_t(r) = \sigma(Y_r) \frac{M_r}{M_t} \leqslant \sigma(Y_t) \frac{A_t}{A_r}. \ \blacksquare$$

Proof of Proposition 1.1. Applying Lemma 2.1 we get

$$E[f^2(Y_t)\log f^2(Y_t)] - E[f^2(Y_t)]\log E[f^2(Y_t)]$$
$$\leqslant E\left[(f'(Y_t)\sigma(Y_t))^2 \int_0^t \exp\left(-2\int_{\mathbf{R}} \frac{L_t^x - L_r^x}{\sigma^2(x)}\left(\sigma\left(\frac{1}{2}\sigma' - \frac{b}{\sigma}\right)'(dx) + \mu(dx)\right)\right) dr\right].$$

But, the condition (1.1) yields

$$\int_{\mathbf{R}} \frac{L_t^x - L_r^x}{\sigma^2(x)}\left(\sigma\left(\frac{1}{2}\sigma' - \frac{b}{\sigma}\right)'(dx) + \mu(dx)\right) \geqslant c\int_{\mathbf{R}} \frac{L_t^x - L_r^x}{\sigma^2(x)} dx = c(t-r),$$

by the occupation density formula. Therefore,

$$\int_0^t \exp\left(-2\int_{\mathbf{R}} \frac{L_t^x - L_r^x}{\sigma^2(x)}\left(\sigma\left(\frac{1}{2}\sigma' - \frac{b}{\sigma}\right)'(dx) + \mu(dx)\right)\right) dr$$
$$\leqslant \int_0^t \exp(2c(r-t))\,dr = \frac{1-\exp(-2ct)}{2c}. \quad \blacksquare$$

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