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STEREOLOGICAL FORMULAE FOR SIZE DISTRIBUTIONS VIA MARKED POINT PROCESSES

BY

DIETRICH STOYAN (FREIBERG)

Abstract. This paper deals with the intersection of a non-random k-dimensional flat with a subset of R^d which is the union of infinitely many particles. It is described by a stationary marked point process Φ, where the *points* are the positions and the *marks* determine the forms of particles. Then the intersection set can be described similarly by a stationary marked point process Ψ. The intensity and the Palm mark distribution of Ψ can be expressed in terms of the corresponding characteristics of Φ. The formulae generalize well-known formulae for the Boolean model (i.e. independently marked Poisson process) to a quite general case. Furthermore, they are similar or equivalent to those for cross-sections of compact sets by Poisson flat processes. Since the formulae connect k-dimensional characteristics with d-dimensional ones, they are of stereological interest.

1. Introduction. A classical problem of stochastic geometry is the study of random cross-sections of bodies with planes or lines. There are two approaches which in some cases yield identical results. One may fix a body and assume that the intersecting planes (lines) are members of a stationary Poisson plane (line) process (see [4] and [2], ch. 4) or fix a plane (line) and consider infinitely many copies of a body scattered in the space whose positions (for example, midpoints) form a stationary Poisson process (see [2], ch. 5).

In this paper we study a situation in the spirit of the second approach. We consider an infinite set of random particles in random positions in R^d and assume that this structure can be described by a stationary marked point process. This is a very general assumption, since there may be dependences between point distances and shape and size of particles. Consequently, the following model may be a good approximation of many real structures.

Let $\hat{\Phi} = \{x_n\}$ be a point process in R^d and let $\{K_n\}$ be a sequence of non-empty compact subsets of R^d. Moreover, let C_n stand for the n-th member of the considered families of bodies. Assume that $C_n = x_n + K_n$, i.e., x_n is the position of C_n for $n = 1, 2, \dots$ Assume that $\Phi = \{[x_n; K_n]\}$ is a stationary marked point process in R^d and that $[\mathscr{K}', \mathfrak{K}']$ is its space of marks. Here $\mathscr{K}'$ is the set of all non-empty compact subsets of R^d, and $\mathfrak{K}'$ is the Borel σ-algebra of subsets of $\mathscr{K}'$ generated by the Hausdorff metric [2]. Finally, assume that $\hat{\Phi}$ has a finite intensity, say λ. Then the intensity measure Λ of Φ is of the form $\Lambda = \lambda\mu \times D$, where μ is the d-dimensional Lebesgue measure and D is a distribution on $\mathfrak{K}$ called the *Palm mark distribution* (see [1]).

Let $\mathfrak{F}$ be the intersecting k-dimensional flat ($1 \leqslant k < d$), i.e., a k-dimensional subset of R^d parallel to a k-dimensional subspace. The assumption of stationarity ensures that results established for a particular k-dimensional flat hold for all k-dimensional flats of R^d parallel to $\mathfrak{F}$. Thus, we may assume without loss of generality that

$$\mathfrak{F} = \{(z_1, \dots, z_d)\colon z_{k+1} = \dots = z_d = 0\}.$$

Consider now the following marked point process $\Psi = \{[y_m; L_m]\}$ with points in $\mathfrak{F}$ and marks which are subsets of $\mathfrak{F}$. Every marked point $[y_m; L_m]$ results from a marked point $[x_{n(m)}; K_{n(m)}]$ of Φ which produces an intersection with $\mathfrak{F}$, i.e.

$$x_n + K_n \cap \mathfrak{F} \neq \emptyset.$$

If $x_{n(m)} = (t_1, \dots, t_d)$, then $y_m = (t_1, \dots, t_k, 0, \dots, 0)$ (with $d-k$ zeros) and $L_m = \big((x_{n(m)} + K_{n(m)}) \cap \mathfrak{F}\big) - y_m$, i.e., y_m is the projection of $x_{n(m)}$ onto $\mathfrak{F}$ and L_m is the intersection of $K_{n(m)} + x_{n(m)} - y_m$ with $\mathfrak{F}$.

Note that Ψ is a stationary marked point process on $\mathfrak{F} \times \mathscr{K}'_k$, where $\mathscr{K}'_k$ is the set of all non-empty compact subsets of $\mathfrak{F}$. The intensity measure of Ψ, say Λ_k, takes the form $\Lambda_k = \lambda_k \mu_k \times D_k$, where λ_k is the intensity of $\{y_m\}$, μ_k is the Lebesgue measure on $\mathfrak{F}$, and D_k is the Palm mark distribution defined on the σ-algebra $\mathfrak{K}'_k$ of elements of $\mathfrak{K}'$ in $\mathfrak{F}$. We interpret D_k as the distribution of a typical body in $\mathfrak{F}$.

2. The general formulae for λ_k and D_k. To establish general formulae for the intensity λ_k and the distribution D_k we introduce the following notation.

Let $\varphi\colon R^{d-k} \times \mathscr{K}' \to \mathscr{K}'_k$ be defined for $(t_{k+1}, \dots, t_d) \in R^{d-k}$ and $C \in \mathscr{K}'$ by

$$\varphi(t_{k+1}, \dots, t_d, C) = \big((0, \dots, 0, t_{k+1}, \dots, t_d) + C\big) \cap \mathfrak{F}.$$

Moreover, for every $\mathscr{A} \in \mathscr{K}'_k$ let

$$M_k(\mathscr{A}) = \int_{\mathscr{K}'} \int_{R^{d-k}} 1_{\mathscr{A}}\big(\varphi(t_{k+1}, \dots, t_d, C)\big)\, \mu_{d-k}(dt_{k+1} \times \dots \times dt_d)\, D(dC). \tag{2.1}$$

THEOREM. *We have*

$$\lambda_k = \lambda M_k(\mathscr{K}'_k), \tag{2.2}$$

$$D_k(\mathscr{A}) = M_k(\mathscr{A})/M_k(\mathscr{K}'_k), \quad \mathscr{A} \in \mathscr{K}'_k. \tag{2.3}$$

Proof. Let $A = [0, 1)^k \times \{0\}^{d-k} \times \mathscr{A}$. Then

$$\Lambda_k(A) = \lambda_k D_k(\mathscr{A}). \tag{2.4}$$

Moreover, with $x = (t_1, \ldots, t_d)$ we have

$$\Lambda_k(A) = \mathrm{E} \sum_{[x,C]\in\Phi} 1_A\big(t_1, \ldots, t_k, 0, \ldots, 0, ((x+C)\cap\mathfrak{F})-(t_1, \ldots, t_k, 0, \ldots, 0)\big)$$

$$= \int_{R^d \times \mathscr{K}'} 1_A\big(t_1, \ldots, t_k, 0, \ldots, 0, \varphi(t_{k+1}, \ldots, t_d, C)\big)\Lambda(dx \times dC)$$

$$= \lambda \int_{R^d \times \mathscr{K}'} 1_{[0,1)}(t_1) \ldots 1_{[0,1)}(t_k) 1_{\mathscr{A}}\big(\varphi(t_{k+1}, \ldots, t_d, C)\big)\mu_d(dt_1 \times \ldots \times dt_d)D(dC)$$

$$= \lambda \int_{\mathscr{K}'} \int_{R^{d-k}} 1_{\mathscr{A}}\big(\varphi(t_{k+1}, \ldots, t_d, C)\big)\mu_{d-k}(dt_{k+1} \times \ldots \times dt_d)D(dC)$$

$$= \lambda M_k(\mathscr{A}).$$

This combined with (2.4) yields $\lambda_k D_k(\mathscr{A}) = \lambda M_k(\mathscr{A})$. Finally, letting $\mathscr{A} = \mathscr{K}'_k$ gives (2.2), and this implies (2.3).

Note that the distribution D_k depends only on the Palm mark distribution D of Φ and it is independent of the other characteristics of Φ. Thus the formulae in [2], p. 146, for intersections of the Boolean model (i.e. independently marked Poisson process) with flats remain true for the general model considered here, where the point process $\{x_n\}$ needs not be a Poisson process and where dependences between positions and particles are possible.

M_k defined by (2.1) corresponds to the measure of flats (in the sense of integral geometry) which intersect a random set with distribution D giving an intersection figure in A. Thus, for the study of random cross-sections of a compact body with distribution D by flats parallel to $\mathfrak{F}$, the usual methods of integral geometry and geometrical probability would also yield (2.3). Consequently, it is possible to replace the study of random cross-sections (in the sense of integral geometry) of a random set with distribution D by that of the intersection of a fixed flat with a stationary marked point process having the mark distribution D. This may simplify the solution in some cases, as R. V. Ambartzumian suggested to me. An example is the study of random cross-sections of the "typical" polyhedron of a Poisson hyperplane process.

Clearly, formulae (2.2) and (2.3) may help to solve stereological problems in which λ_k and D_k are given and λ and D are unknown. In [3] and [5] the particular cases of spherical and convex particles have been studied by the use of methods similar to those in this paper.

3. Two particular cases. In this section we find formulae for M_k for two examples which are closely related to two problems studied by Matheron ([2], p. 70-71 and 87-88). Whereas Matheron considered the cross-section of a fixed set, say C, with Poisson flat processes, we treat here the intersection of structures as above with a fixed flat, where the mark distribution is related to C. For convenience of the reader we preserve the notation introduced in [2]. Thus T denotes the functional of a random closed set, $F(C)$ the surface of C, b_k the volume of the unit ball in R^k, $\mathcal{S}_k$ the family of all k-dimensional subspaces of R^d, $\bar{\omega}_k$ the unique probability measure on $\mathcal{S}_k$ which is invariant under rotations, $S^{\perp}$ the subspace orthogonal to S, $\Pi_{S\perp}$ the projection onto $S^{\perp}$ and, finally, $\gamma(r, u)$ the $(d-1)$-dimensional volume of the projection of the set $B \cap (ru+B)$ onto the hyperplane orthogonal to the direction u of the translation vector ru.

(i) *Random cross-sections of a compact set* B. Assume that Φ has the following particular property:

The marks K_n are generated from a (non-random) compact set B by rotating it independently according to a distribution G on $\mathcal{S}_k$, which means that the functional T of the marks,

$$T(K) = D(\mathcal{K}_K), \qquad \mathcal{K}_K = \{C \in \mathcal{K}' \colon C \cap K \neq \emptyset\}, K \in \mathcal{K}',$$

is given by

$$T(K) = \int_{\mathcal{S}_k} \chi_{B,K}(S)\, G^*(dS).$$

Here we identify $S \in \mathcal{S}_k$ with the rotation ω_S which transforms $\mathfrak{F}$ in S and write

$$\chi_{B,K}(S) = \begin{cases} 1 & \text{if } \omega_S B \cap K \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

G^* is the distribution on $\mathcal{S}_k$ with $G^*(\mathcal{R}) = G(\mathcal{R}^{-1})$, where $\mathcal{R} \subset \mathcal{S}_k$ and $\mathcal{R}^{-1}$ is the set of rotations inverse to those of $\mathcal{R}$.

In our case, the formula for $M_k(\mathcal{A})$ takes a simple form, since the distribution D is completely determined by G, and $\varphi(t_{k+1}, \ldots, t_d, C)$ depends only on ω_S or S,

$$\varphi(t_{k+1}, \ldots, t_d, C) = \varphi(t_{k+1}, \ldots, t_d, S).$$

Hence

$$M_k(\mathcal{A}) = \int_{\mathcal{S}_k} \int_{R^{d-k}} 1_{\mathcal{A}}\big(\varphi(t_{k+1}, \ldots, t_d, S)\big)\, \mu_{d-k}(dt_{k+1} \times \ldots \times dt_d)\, G^*(dS).$$

Let $K \in \mathcal{K}'_k$ and let $\mathcal{K}_{k,K}$ be the set of all elements of $\mathcal{K}'_k$ having a common element with K. For $\mathcal{A} = \mathcal{K}_{k,K}$ we have

$$\text{(3.1)} \quad \int_{R^{d-k}} 1_{\mathcal{K}_{k,K}}\big(\varphi(t_{k+1}, \ldots, t_d, S)\big)\, \mu_{d-k}(dt_{k+1} \times \ldots \times dt_d)$$
$$= \mu_{d-k}\Big(\big\{y \colon y = (t_{k+1}, \ldots, t_d),\ \big((0, \ldots, 0, t_{k+1}, \ldots, t_d) + \omega_S B\big) \cap K \neq \emptyset\big\}\Big).$$

Using the notation $\{\omega_S B, K\}$ for the set $\{...\}$ in (3.1), we obtain

$$M_k(\mathscr{K}_{k,K}) = \int_{\mathscr{S}_k} \mu_{d-k}(\{\omega_S B, K\})\, G^*(dS).$$

Furthermore,

$$M_k(\mathscr{K}'_k) = \int_{\mathscr{S}_k} \mu_{d-k}(\Pi_{S\perp} B)\, G(dS).$$

The functional of the "typical" intersection set corresponding to D_k is then given by

$$T_B(K) = \int_{\mathscr{S}_k} \mu_{d-k}(\{\omega_S B, K\})\, G^*(dS) / \int_{\mathscr{S}_k} \mu_{d-k}(\Pi_{S\perp} B)\, G(dS).$$

This formula is quite similar to (3-5-4) in [2]. The difference results from the fact that our intersections are contained in the (fixed) $\mathfrak{F}$ whereas in [2] the intersection sets are contained in the intersecting random flats.

(ii) *Random intercepts*. Let Φ be defined as in (i). Assume additionally that B is a compact convex set with non-empty interior and that $G = \bar{\omega}_1$. To study intersections with lines, we assume that

$$\mathfrak{F} = \{(z_1, \ldots, z_d)\colon z_2 = \ldots = z_d = 0\}.$$

We are interested in obtaining the probability $P(L \geqslant l)$ that the length L of a "typical" intersection segment in $\mathfrak{F}$ is greater than or equal to l, i.e.,

$$P(L \geqslant l) = D_1(\mathscr{L}_l),$$

where $\mathscr{L}_l$ is the set of all intervals in $\mathfrak{F}$ with length greater than or equal to l.

In the considered case we have

$$M_1(\mathscr{L}_l) = \int_{\mathscr{K}'} \int_{R^{d-1}} 1_{\mathscr{L}_l}\big(\varphi(t_2, \ldots, t_d, C)\big)\, \mu_{d-1}(dt_2 \times \ldots \times dt_d)\, D(dC).$$

Since the distribution D is given by $\bar{\omega}_1$ and since $\varphi(t_2, \ldots, t_d, C)$ is for fixed $t_2, \ldots, t_d$ a function only of the direction u of $S \in \mathscr{S}_1$, i.e.,

$$\varphi(t_2, \ldots, t_d, C) = \varphi(t_2, \ldots, t_d, u),$$

we obtain

$$M_1(\mathscr{L}_l) = \int_{\mathscr{S}_1} \int_{R^{d-1}} 1_{\mathscr{L}_l}\big(\varphi(t_2, \ldots, t_d, u)\big)\, \mu_{d-1}(dt_2 \times \ldots \times dt_d)\, \bar{\omega}_1(du)$$

$$= \int_{\mathscr{S}_1} \gamma(l, u)\, \bar{\omega}_1(du).$$

Since B is convex and compact, we have $M_1(\mathscr{K}'_1) = M_1(\mathscr{L}_0)$. From the integral geometry it is known ([2], p. 78) that

$$\int_{\mathscr{S}_1} \gamma(0, u)\, \bar{\omega}_1(du) = d b_d / F(B)\, b_{d-1} = c,$$

which finally gives

$$P(L \geqslant l) = c \int_{\mathscr{S}_1} \gamma(l, u)\bar{\omega}_1(du).$$

This is Matheron's [2] formula (4-3-3).

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Sektion Mathematik
Bergakademie Freiberg
DDR 9200 Freiberg

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