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CENTRAL LIMIT THEOREM IN HÖLDER SPACES

BY

ALFREDAS RAČKAUSKAS (VILNIUS) AND CHARLES SUQUET (LILLE)

Abstract. Stochastic processes are considered within the framework of Hölder spaces H_α^0 as paths spaces. Using Ciesielski's isomorphisms between H_α^0 and sequences spaces via the Faber Schauder triangular functions allows us to express our basic assumptions in terms of second differences of the processes, giving more flexibility. We obtain general conditions for the existence of a version with paths in H_α^0 and the tightness of sequences of random elements in these spaces. Central limit theorems in H_α^0 are established and convergence rates are given with respect to Prohorov and bounded Lipschitz metrics. As an application, we study the weak Hölder convergence of the characteristic empirical process.

0. Introduction. The space $\mathscr{C}[0, 1]$ of continuous functions is a classical framework for many limit theorems in the theory of stochastic processes. The $\mathscr{C}[0, 1]$-weak convergence of a sequence of stochastic processes ξ_n gives useful results about the convergence in distribution of continuous functionals of the paths. In many situations, the processes ξ_n are known to have almost surely paths with at least some Hölder regularity and the same happens for the limiting process ξ (see, for instance, invariance principles for random polygonal lines, perturbed empirical processes, empirical characteristic functions). Looking for limit theorems with respect to some weak Hölder convergence instead of the $\mathscr{C}[0, 1]$ convergence is then a natural question. One important interest of this new functional framework is to provide more continuous functionals of the paths (see Hamadouche [8], [9] for a list of examples). Moreover, when a process ξ is known by its finite-dimensional distributions, the question of the existence of a version with almost all paths of some given Hölder regularity has its own interest.

The first result in this direction goes back to the Kolmogorov sufficient condition for the existence of a version of ξ with continuous paths, namely:

$$P\{|\xi(t+h)-\xi(t)| > \lambda\} \leqslant c\lambda^{-\gamma} h^{1+\delta}, \tag{1}$$

where $c, \delta > 0$ and $\gamma > 1$ are constants. In fact, the same condition gives a version of ξ with α-Hölder regularity for any $0 < \alpha < \delta/\gamma$. Ciesielski [4] gave sufficient conditions for a Gaussian process to have a version with Hölderian paths. Lamperti [12] proved for the polygonal line interpolating the Donsker Prohorov partial sums process an α-Hölder invariance principle for any $\alpha < 1/2$ (in the most favorable case). This result was derived again by Kerkyacharian and Roynette [11] by another method using Ciesielski's analysis [3] of Hölder spaces by triangular functions. The problem of the existence of a version with Hölderian paths was studied by Ibragimov [10] and by Nobelis [15]. Recently, Hamadouche [9] extended Lamperti's invariance principle to dependent variables. Hamadouche [8] proved the weak α-Hölder convergence of the polygonal uniform empirical process to the Brownian bridge for any $0 < \alpha < 1/4$ and the optimality of this bound.

All these results rely on some control of the first differences

$$\Delta_h^1 \xi(t) = \xi(t+h) - \xi(t).$$

In this paper, we propose to use instead the second differences:

$$\Delta_h^2 \xi(t) = \xi(t+h) + \xi(t-h) - 2\xi(t).$$

The usefulness of $\Delta_h^2 \xi(t)$ in the problem of sample paths differentiability is known (see Cramér and Leadbetter [5]). From an analytical point of view, there is no loss in working with $\Delta_h^2 f$ to study the Hölder regularity of a function f. This observation goes back to Zygmund [22] who investigating smooth functions noticed that a necessary and sufficient condition that a continuous and periodic function $f(x)$ should satisfy a Hölder condition of order α, $0 < \alpha < 1$, is that

$$\Delta_h^2 f(x) = O(h^\alpha) \quad \text{as } h \to +0,$$

uniformly in x. The role of Δ_h^2 is now well understood in the more general context of Besov spaces (see Peetre [18]).

This fact stimulated us to study the central limit theorem for processes in Hölder spaces in terms of second differences. It turns out that the second difference allows us to recognize more processes, which admits a version with almost all paths in Hölder space. In particular, as a special case of our Theorem 1.2, we obtain the following version of the Kolmogorov theorem: if there exists $\delta > 0$, $\gamma > 1$ and a constant $c > 0$ such that for all $t, t \pm h \in [0, 1]$

$$P\{|\Delta_h^2 \xi(t)| > \lambda\} \leqslant c\lambda^{-\gamma} h^{1+\delta}, \tag{2}$$

then there exists a version of ξ with sample paths in $H_\alpha^0[0, 1]$, where $0 < \alpha < \min(1, \delta/\gamma)$. To get a feeling of the advantage of condition (2) over condition (1), let us test both of them on the following crude bench-mark (more interesting examples will be given in Section 1). Let g be a deterministic function with first derivative of Hölder regularity τ $(0 < \tau < 1)$, $g(1) = 1$ and consider the process

$$\xi(t) = Xg(t), \quad t \in [0, 1],$$

where $X = \xi(1)$ is a random variable with finite variance (as we are interested in processes satisfying the central limit theorem, this is a natural assumption). We have then the obvious estimates:

$$\boldsymbol{P}\{|\Delta_h^i \xi(t)| > \lambda\} \leqslant \boldsymbol{E}X^2 |\Delta_h^i g(t)|^2 \lambda^{-2}, \qquad i = 1, 2.$$

Now $|\Delta_h^1 g(t)| \leqslant \|g'\|_\infty |h|$ and $|\Delta_h^2 g(t)| \leqslant \|g'\|_\tau |h|^{1+\tau}$, so condition (1) corresponds only to a Hölder regularity $\alpha < 1/2$ for ξ, whereas condition (2) corresponds to $\alpha < \min(\tau + 1/2, 1)$.

The paper is organized as follows. In the preliminary section we collected necessary facts about Hölder spaces with the main references to Ciesielski [3]. The main auxiliary result which gives an estimate for the Hölder modulus of continuity of a process in terms of a second difference is also stated in Section 1. This estimate yields several sufficient conditions that a process $\xi = (\xi(t),\ t \in [0, 1])$ should have almost all paths in Hölder space. In Section 2 we study the embedding of Hölder spaces. We prove in particular that if $1 \geqslant \beta > \alpha + 1/p$, then the embedding $H_\beta[0, 1] \to H_\alpha[0, 1]$ is p-absolutely summing. Section 3 is devoted to the central limit theorem for random processes with paths in Hölder spaces. In particular we prove that the random process $\xi = (\xi(t),\ t \in [0, 1])$ satisfies the central limit theorem in the space $H_\alpha^0[0, 1]$, $0 < \alpha < 1$, provided the following conditions are satisfied:

(i) for each $t \in [0, 1]$, $\boldsymbol{E}\xi(t) = 0$, $\boldsymbol{E}\xi^2(t) < \infty$;

(ii) there exists a positive random variable M such that $\boldsymbol{E}M^2 < \infty$ and, for all $t, h \in [0, 1]$ such that $t \pm h \in [0, 1]$,

$$|\Delta_h^2 \xi(t)| \leqslant M h^\alpha \ln^{-\beta}(1/h), \qquad \text{where } \beta > 3/2.$$

The central limit theorem is supplied with convergence rate estimates with respect to Prohorov and bounded Lipschitz metrics.

The last section is devoted to the central limit theorem for the empirical characteristic process (e.ch.p.) in Hölder spaces. For the relevance of this process in many statistical problems, we refer to the introduction in Feuerverger and Mureika [7]. The study of the e.ch.p. in the functional framework of $\mathscr{C}[-1/2, 1/2]$ was achieved in Marcus [14] and Csörgő [6]. Our Hölderian CLT for the e.ch.p. is a first step in the investigation of the convergence of e.ch.p. with a stronger topology than $\mathscr{C}[-1/2, 1/2]$ one.

1. Preliminaries. Throughout $\boldsymbol{T} = [0, 1]$ and $0 < \alpha < 1$. Denote by H_α the set of real-valued continuous functions $x\colon \boldsymbol{T} \to \boldsymbol{R}$ such that $w_\alpha(x, 1) < \infty$, where

$$w_\alpha(x, \delta) = \sup_{t,s\in \boldsymbol{T}, 0 < |t-s| < \delta} \frac{|x(t) - x(s)|}{|s-t|^\alpha}.$$

The set H_α is a Banach space when endowed with the norm $\|x\|_\alpha = |x(0)| + w_\alpha(x, 1)$. Set

$$H_\alpha^0 = \{x \in H_\alpha\colon \lim_{\delta \to 0} w_\alpha(x, \delta) = 0\}.$$

Then H_α^0 is a closed subspace of H_α.

Let

$$\Lambda(t) = \begin{cases} 2t & \text{if } 0 \leqslant t \leqslant 1/2, \\ 2(1-t) & \text{if } 1/2 < t \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 2^j + k$, $0 \leqslant k < 2^j$, $j = 0, 1, \ldots$, set

$$\Lambda_n(t) = \Lambda_{jk}(t) = \Lambda(2^j t - k), \qquad t \in [0, 1].$$

Define moreover $\Lambda_{-1}(t) = \chi_{[0,1]}(t)$ and $\Lambda_0(t) = \Lambda_{-1,0}(t) = t$. Then the family $\{\Lambda_n, n \geqslant -1\}$ is a Schauder basis on the space H_α^0: each function $x \in H_\alpha^0$ admits the series representation

$$x(t) = x(0) + \sum_{n=0}^{\infty} \lambda_n(x) \Lambda_n(t), \qquad t \in [0, 1], \tag{3}$$

with $\lambda_{-1}(x) = \lambda_{-2,0}(x) = x(0)$, $\lambda_0(x) = \lambda_{-1,0}(x) = x(1) - x(0)$, and for $n = 2^j + k$, $j \geqslant 0$, $0 \leqslant k < 2^j$:

$$\lambda_n(x) = \lambda_{jk}(x) = x\big((k+1/2)2^{-j}\big) - \tfrac{1}{2}\big\{x(k2^{-j}) + x\big((k+1)2^{-j}\big)\big\}.$$

By Ciesielski [3], the norm $\|x\|_\alpha$ is equivalent to the sequence norm

$$\|x\|_\alpha^{\text{seq}} = \sup_{j \geqslant -2} 2^{(j+1)\alpha} \max_{0 \leqslant k < 2^j} |\lambda_{jk}(x)|.$$

For $j \geqslant 0$, let us write $E_j x$ for the polygonal line interpolating x between the dyadic points $k2^{-j}$ $(0 \leqslant k \leqslant 2^j)$. We clearly have

$$E_j x = \sum_{i<j} \sum_{0 \leqslant k < 2^i} \lambda_{ik}(x) \Lambda_{ik},$$

whence

$$\|x - E_j x\|_\alpha^{\text{seq}} = \sup_{i \geqslant j} \max_{0 \leqslant k < 2^j} |\lambda_{jk}(x)|. \tag{4}$$

The following estimates are easily obtained:

$$\|E_j x\|_\alpha \leqslant 3\|x\|_\alpha, \tag{5}$$

$$\|x - E_j x\|_\alpha \leqslant 4 w_\alpha(x, 2^{-j}), \tag{6}$$

$$w_\alpha(x, 1) \leqslant 2^j w_\alpha(x, 2^{-j}), \tag{7}$$

$$\|x\|_\alpha \leqslant |x(0)| + 2^j w_\alpha(x, 2^{-j}). \tag{8}$$

For our purpose, the norm $\|x\|_\alpha^{\text{seq}}$ is more tractable than $\|x\|_\alpha$. Ciesielski proved that a *continuous* function x belongs to H_α^0 if and only if $\lim_{n\to\infty} n^\alpha \lambda_n(x) = 0$. We give here a more quantitative version of this result for further use.

PROPOSITION 1.1. *We have the following estimate: for* $x \in H_\alpha^0$ *and* $j \geqslant 0$

$$(9)\qquad w_\alpha(x, 2^{-j}) \leqslant |\lambda_0(x)|\, 2^{-j(1-\alpha)} + \frac{8}{2^\alpha(2-2^\alpha)} \|x - E_0 x\|_\alpha^{\mathrm{seq}}\, 2^{(f(j)-j)(1-\alpha)} + \left(\frac{8}{2^\alpha(2-2^\alpha)} + \frac{2}{2^\alpha - 1}\right) \|x - E_{f(j)} x\|_\alpha^{\mathrm{seq}},$$

where we can choose for f *any function* $N \to N$ *such that* $f(n) \leqslant n$ *and the sequences* $f(n)$ *and* $n - f(n)$ *are non-decreasing with limit infinity.*

Proof. Set

$$c_{jk} = 2^{(j+1)\alpha} \lambda_{jk}(x) \quad \text{and} \quad c_j = \max_{0 \leqslant k < 2^j} |c_{jk}|.$$

Fix s and t in $\boldsymbol{T}$. From (3) we get

$$(10)\qquad x(t) - x(s) = \lambda_0(x)(t-s) + \sum_{j \geqslant 0} \sum_{0 \leqslant k < 2^j} \lambda_{jk}(x)\big(\Lambda_{jk}(t) - \Lambda_{jk}(s)\big).$$

Now observe that for fixed s and t there are at most two non-vanishing terms in the block indexed by j in the series above. Moreover, as the maximal slope of Λ_{jk} is 2^{j+1}, we have

$$|\Lambda_{jk}(t) - \Lambda_{jk}(s)| \leqslant \min(1;\, 2^{j+1}|t-s|).$$

Let j_0 be the integer defined by $2^{-j_0-1} \leqslant |t-s| < 2^{-j_0}$ and $j_1 = f(j_0)$. The splitting

$$|x(t) - x(s)| \leqslant |\lambda_0(x)(t-s)| + 2\sum_{j=0}^{j_1} c_j 2^{-(j+1)\alpha} 2^{j+1} |t-s| + 2\sum_{j=j_1+1}^{j_0} c_j 2^{-(j+1)\alpha} 2^{j+1} |t-s| + 2\sum_{j > j_0} c_j 2^{-(j+1)\alpha},$$

after some elementary computations leads to

$$\frac{|x(t)-x(s)|}{|t-s|^\alpha} \leqslant |\lambda_0(x)|\, 2^{-j_0(1-\alpha)} + \frac{8}{2^\alpha(2-2^\alpha)} 2^{(j_1-j_0)(1-\alpha)} \sup_{j \leqslant j_1} c_j + \frac{8}{2^\alpha(2-2^\alpha)} \sup_{j_1 < j \leqslant j_0} c_j + \frac{2}{2^\alpha - 1} \sup_{j > j_0} c_j$$

and finally to

$$(11)\qquad \frac{|x(t)-x(s)|}{|t-s|^\alpha} \leqslant |\lambda_0(x)|\, 2^{-j_0(1-\alpha)} + \frac{8}{2^\alpha(2-2^\alpha)} \sup_{j \geqslant 0} c_j 2^{(j_1-j_0)(1-\alpha)} + \left(\frac{8}{2^\alpha(2-2^\alpha)} + \frac{2}{2^\alpha - 1}\right) \sup_{j > j_1} c_j.$$

Let $\delta = 2^{-J}$. By our assumptions on f, taking the supremum over $0 < |t-s| < \delta$ on the left-hand side of (11) and over $j_0 \geqslant J$ on the right-hand side gives the desired bound for $w_\alpha(x, \delta)$.

As a special case of the previous computation we obtain for $J = 0$:

$$w_\alpha(x, 1) \leqslant |\lambda_0(x)| + \left(\frac{16}{2^\alpha(2-2^\alpha)} + \frac{2}{2^\alpha - 1}\right) \sup_{j \geqslant 0} \max_{0 \leqslant k < 2^j} 2^{(j+1)\alpha} |\lambda_{jk}(x)|. \tag{12}$$

Recall that a *Young function* ϕ is a convex increasing function on R^+ such that $\phi(0) = 0$ and $\lim_{t \to \infty} \phi(t) = \infty$. If Z is a random variable such that $E\phi(|Z|/c) < \infty$ for some $c > 0$, then we set

$$\|Z\|_\phi = \inf\{c > 0\colon\ E\phi(|Z|/c) \leqslant 1\}.$$

Throughout σ will denote an increasing function on $\boldsymbol{T}$ such that $\sigma(0) = 0$. For the sake of convenience, we recall here the basic inequalities used throughout the paper to handle the maxima of random variables. The first of them is Lemma 11.3 (p. 303) in Ledoux and Talagrand [13].

LEMMA 1.1. *Let (X_i) be positive random variables on some probability space $(\Omega, \mathscr{F}, P)$ such that for all $1 \leqslant i \leqslant n$ and all $A \in \mathscr{F}$*

$$\int_A X_i \, dP \leqslant c_i P(A) \phi^{-1}\left(\frac{1}{P(A)}\right), \tag{13}$$

where ϕ is some Young function and c_i is a constant. Then, for every set $A \in \mathscr{F}$,

$$\int_A \max_{1 \leqslant i \leqslant n} X_i \, dP \leqslant cP(A) \phi^{-1}\left(\frac{n}{P(A)}\right) \quad \textit{with } c = \max_{1 \leqslant i \leqslant n} c_i.$$

The following two lemmas provide practical conditions to satisfy (13).

LEMMA 1.2. *Let (X_i) be positive random variables on some probability space $(\Omega, \mathscr{F}, P)$ and ϕ some Young function. Then X_i satisfies* (13) *with $c_i = \|X_i\|_\phi$. In particular,*

$$E \max_{1 \leqslant i \leqslant n} X_i \leqslant \phi^{-1}(n) \max_{1 \leqslant i \leqslant n} \|X_i\|_\phi.$$

Proof. By convexity of ϕ and Jensen's inequality, for any $b > \|X_i\|_\phi$ we have

$$\begin{aligned}\int_A X_i \, dP &= bP(A) \int_A \phi^{-1} \phi(X_i/b) P^{-1}(A) \, dP \\ &\leqslant bP(A) \phi^{-1}\big(P^{-1}(A) E\phi(X_i/b)\big) \leqslant bP(A) \phi^{-1}\big(P^{-1}(A)\big).\end{aligned}$$

Taking the infimum over $b > \|X_i\|_\phi$, we see that (13) is satisfied and we can apply Lemma 1.1.

LEMMA 1.3. *Let (X_i) be positive random variables on some probability space $(\Omega, \mathscr{F}, P)$ satisfying for some constant $1 < p < \infty$ and each $i = 1, \ldots, n$*

$$a_i = \sup_{t > 0} t P^{1/p}(X_i > t) < \infty.$$

Then X_i *satisfies* (13) *with* $c_i = qa_i$, *where* $q = p/(p-1)$. *In particular, putting* $a = \max_{1 \leqslant i \leqslant n} a_i$, *we get*

$$E \max_{1 \leqslant i \leqslant n} X_i \leqslant q a n^{1/p}.$$

Proof. Integrating by parts we have

$$\int_A X_i \, dP = P(A) \int_0^\infty P\{X_i > t \mid A\} \, dt \leqslant P(A) \frac{p}{p-1} \sup_{t>0} t P^{1/p}\{X_i > t \mid A\}$$

$$\leqslant P^{1-1/p}(A) \frac{p}{p-1} \sup_{t>0} t P^{1/p}\{X_i > t\}.$$

So we can apply Lemma 1.1 with $\phi(t) = t^p$.

THEOREM 1.1. *Assume the process* $\xi = \{\xi(t), t \in T\}$ *is defined on the probability space* $(\Omega, \mathscr{F}, P)$ *and satisfies: for each set* $A \in \mathscr{F}$ *and all* $t, t \pm h \in T$

$$\int_A |\Delta_h^2 \xi(t)| \, dP \leqslant \sigma(h) P(A) \phi^{-1}\left(\frac{1}{P(A)}\right). \tag{14}$$

Then for each $A \in \mathscr{F}$, *and integers* $K > J \geqslant 0$,

$$\int_A \|E_K \xi - E_J \xi\|_\alpha^{\mathrm{seq}} \, dP \leqslant 2^{1+\alpha} P(A) \int_{2^{-K}}^{2^{-J}} \frac{\sigma(u)}{u^{\alpha+1}} \phi^{-1}\left(\frac{1}{P(A) u}\right) du. \tag{15}$$

Proof. Set

$$I_j = \int_A \max_{0 \leqslant k < 2^j} |\lambda_{jk}(\xi)| \, dP, \quad j \geqslant 0.$$

By the condition (14), for each measurable set A we obtain

$$\int_A |\lambda_{jk}(\xi)| \, dP \leqslant \sigma(2^{-j-1}) P(A) \phi^{-1}\left(\frac{1}{P(A)}\right).$$

Now Lemma 1.1 yields

$$I_j \leqslant P(A) \sigma(2^{-j-1}) \phi^{-1}\left(\frac{2^j}{P(A)}\right). \tag{16}$$

Putting together (4) and (16) we get

$$\int_A \|E_K \xi - E_J \xi\|_\alpha^{\mathrm{seq}} \, dP \leqslant P(A) \sum_{J \leqslant j \leqslant K} 2^{\alpha(j+1)} \sigma(2^{-j-1}) \phi^{-1}\left(\frac{2^j}{P(A)}\right). \tag{17}$$

The result follows by comparing series and integral.

Theorem 1.1 yields several sufficient conditions that the process $\xi = (\xi(t), t \in T)$ should have a version with paths in Hölder space H_α. The following result is a version of the Kolmogorov theorem, whereas Theorem 1.3 gives a version of the corresponding Ibragimov result.

THEOREM 1.2. *Let $p > 1$. Assume that the process $\xi = (\xi_t,\ t \in \boldsymbol{T})$ is continuous in probability and satisfies the condition: for each $t \in \boldsymbol{T}$, $h > 0$ such that $t \pm h \in \boldsymbol{T}$,*

$$\boldsymbol{P}\{|\Delta_h^2 \xi(t)| > \lambda\} \leqslant \frac{c}{\lambda^p} \sigma^p(h). \tag{18}$$

If

$$\int_0^1 \frac{\sigma(u)}{u^{\alpha+1+1/p}}\, du < \infty, \tag{19}$$

then ξ admits a version with almost all paths in the space H_α^0.

Proof. By Lemma 1.3, the inequality (18) yields

$$\int_A |\Delta_h^2 \xi(t)|\, d\boldsymbol{P} \leqslant \frac{pc^{1/p}}{p-1} \boldsymbol{P}(A) \boldsymbol{P}^{-1/p}(A) \sigma(h)$$

and from Theorem 1.1 it is easily seen that $(E_J \xi)$ is a.s. a Cauchy sequence in H_α^0, and therefore $\lim_{J\to\infty} E_J \xi$ exists a.s. If this limit is denoted by $\tilde{\xi}$, then it is easy to see that

$$\tilde{\xi}(t) = \xi(0) + (\xi(1) - \xi(0))t + \sum_{j=0}^{\infty} \lambda_{jk}(\xi) \Lambda_{jk}(t),$$

which is the version of ξ with paths in H_α^0.

Combining the bound (17) with Proposition 1.1 we get

COROLLARY 1.1. *If the random element ζ in $\mathscr{C}[0, 1]$ satisfies (18) and (19), then for each $j \geqslant 0$:*

$$\boldsymbol{E} w_\alpha(\zeta, 2^{-j}) \leqslant 2^{-j(1-\alpha)} \boldsymbol{E}|\zeta(1) - \zeta(0)| + C_\alpha 2^{(f(j)-j)(1-\alpha)} I(1) + C_\alpha I(2^{-f(j)}),$$

where C_α is a constant depending only on α, f is as in Proposition 1.1, and

$$I(\delta) = \int_0^\delta \frac{\sigma(u)}{u^{\alpha+1+1/p}}\, du.$$

THEOREM 1.3. *Let ϕ be a Young function. Assume that the process $\xi = (\xi_t,\ t \in \boldsymbol{T})$ is continuous in probability and satisfies the condition: for each $t \in \boldsymbol{T}$, $h > 0$ such that $t \pm h \in \boldsymbol{T}$,*

$$\|\Delta_h^2 \xi(t)\|_\phi \leqslant c\sigma(h). \tag{20}$$

If

$$\int_0^1 \frac{\sigma(u)}{u^{\alpha+1}} \phi^{-1}\left(\frac{1}{u}\right) du < \infty, \tag{21}$$

then the process ξ admits a version with almost all paths in the space H_α^0.

Proof. One has to note (see Lemma 1.2) that the condition (20) yields

$$\int_A |\Delta_h^2 \xi(t)|\, d\boldsymbol{P} \leqslant c\boldsymbol{P}(A) \phi^{-1}\left(\frac{1}{\boldsymbol{P}(A)}\right) \sigma(h)$$

for all measurable sets A. The proof can be completed by the same arguments as above.

COROLLARY 1.2. *If the random element ζ in $\mathscr{C}[0, 1]$ satisfies* (20) *and* (21), *then for each $j \geqslant 0$:*

$$Ew_\alpha(\zeta, 2^{-j}) \leqslant 2^{-j(1-\alpha)} E|\zeta(1)-\zeta(0)| + C_\alpha 2^{(f(j)-j)(1-\alpha)} I'(1) + C_\alpha I'(2^{-f(j)}),$$

where C_α is a constant depending only on α, f is as in Proposition 1.1, *and*

$$I'(\delta) = \int_0^\delta \frac{\sigma(u)}{u^{\alpha+1}} \phi^{-1}\left(\frac{1}{u}\right) du.$$

EXAMPLE 1. Consider a continuous in probability symmetric p-stable process $(X(t), t \in \boldsymbol{T})$, $1 \leqslant p < 2$, with the stochastic integral representation

$$(X(t), t \in \boldsymbol{T}) \stackrel{\mathscr{D}}{=} \left(\int_E f(t, x) M(dx), t \in \boldsymbol{T}\right),$$

where $(E, \mathscr{E})$ is a measurable space, M is a symmetric p-stable random measure with σ-finite spectral measure m, and $\stackrel{\mathscr{D}}{=}$ denotes equality in distribution (see Samorodnitsky and Taqqu [20]). Define

$$\sigma^p(h) = \sup_{t \in \boldsymbol{T}} \int_E |f(t+h, x) + f(t-h, x) - 2f(t, x)|^p m(dx).$$

COROLLARY 1.3. *Let $\alpha \in (0, 1)$. If σ is an increasing function such that*

$$\int_0^1 \frac{\sigma(u)}{u^{\alpha+1+1/p}} du < \infty, \tag{22}$$

then $(X(t), t \in \boldsymbol{T})$ admits a version with almost all paths in the space H_α^0.

Proof. It is well known (see Samorodnitsky and Taqqu [20]) that

$$\boldsymbol{P}\{|\Delta_h^2 X(t)| > \lambda\} \leqslant C\lambda^{-p} \sigma^p(h).$$

The result now follows by Theorem 1.2.

The necessary and sufficient conditions for the stable process with index $p \in (0, 1)$ to have a version with a.s. paths in H_α are found by Nolan [16]. Let us also remark that by Theorem 1.3 the condition (22) yields that the function $f(t)$, $t \in \boldsymbol{T}$, considered as a process defined on $(E, \mathscr{E}, m)$ has a version with a.s. paths in H_α^0.

EXAMPLE 2. Let $(X(t), t \in \boldsymbol{T})$ be a continuous in probability centered Gaussian process. Under which conditions on its covariance structure is the process Hölderian?

COROLLARY 1.4. *Let*

$$\sigma^2(h) = \sup_{t \in \boldsymbol{T}} \boldsymbol{E}|\Delta_h^2 X(t)|^2.$$

If σ is an increasing function such that

$$\int_0^1 \frac{\sigma(u)}{u^{\alpha+1}} |\ln u|^{1/2} du < \infty,$$

then $(X(t), t \in \boldsymbol{T})$ admits a version with almost all paths in the space H_α^0.

Proof. Apply Theorem 1.3 with the function $\phi_2 = \exp(t^2)-1$, $t\in R$, noting that

$$\|\Delta_h^2 X(t)\|_{\phi_2} \leqslant \sqrt{2}\sigma(h).$$

We end this section with some tightness conditions related to the above results.

THEOREM 1.4. *The sequence* (ζ_N) *of random elements in the Hölder space* H_α^0 *is tight if and only if the following two conditions are satisfied:*

(i) $\lim_{b\to\infty} \sup_N \boldsymbol{P}\{|\zeta_N(0)| > b\} = 0$;

(ii) *for each* $\varepsilon > 0$, $\lim_{\delta\to\infty} \sup_N \boldsymbol{P}\{w_\alpha(\zeta_N, \delta) > \varepsilon\} = 0$.

Proof. The estimates (6) and (8) enable us to use the flat concentration criterion (see Lemma 2.2, p. 40, in Ledoux and Talagrand [13]) to derive the tightness of (ζ_N) from (i) and (ii).

The necessity of (i) for the tightness of (ζ_N) is obvious. For the necessity of (ii), we use the following lemma, the proof of which can be found in Suquet [21].

LEMMA 1.4. *Let* $\mathscr{K}$ *be a compact family (for the topology of weak convergence) of probability measures on the separable metric space* H. *Let* $(F_j, j \geqslant 1)$ *be a sequence of closed subsets of* H *decreasing to* $\emptyset$. *Define the functions* $u_j\colon \mathscr{K} \to [0, 1]$, $\mu \mapsto u_j(\mu) = \mu(F_j)$. *Then the sequence* (u_j) *uniformly converges to zero on* $\mathscr{K}$.

Since the functionals $w_\alpha(\cdot, \delta)$ are continuous and decreasing to zero in δ on H_α^0, the choice of the closed sets

$$F_j = \{x\in H_\alpha^0\colon\ w_\alpha(x, \delta_j) \geqslant \varepsilon\}, \qquad \delta_j \downarrow 0,$$

shows the necessity of (ii) for the tightness of (ζ_N).

Combining Theorem 1.4 with Corollaries 1.1 and 1.2, we obtain the following sufficient conditions:

COROLLARY 1.5. *The sequence* (ζ_N) *of random elements in the Hölder space* H_α^0 *is tight if:*

(i) $\lim_{b\to\infty} \sup_N \boldsymbol{P}\{|\zeta_N(k)| > b\} = 0$, $k = 0, 1$;

(ii) *for each* $t\in \boldsymbol{T}$, $h > 0$ *such that* $t\pm h\in \boldsymbol{T}$,

$$\sup_{N\geqslant 1} \boldsymbol{P}\{|\Delta_h^2 \zeta_N(t)| > \lambda\} \leqslant \frac{c}{\lambda^p} \sigma^p(h),$$

where $p > 1$ *and the function* σ *satisfies* (19).

COROLLARY 1.6. *The sequence* (ζ_N) *of random elements in the Hölder space* H_α^0 *is tight if:*

(i) $\lim_{b\to\infty} \sup_N \boldsymbol{P}\{|\zeta_N(k)| > b\} = 0$, $k = 0, 1$;

(ii) *for some Young function* ϕ *and each* $t\in \boldsymbol{T}$, $h > 0$ *such that* $t\pm h\in \boldsymbol{T}$,

$$\sup_{N\geqslant 1} \|\Delta_h^2 \zeta_N(t)\|_\phi \leqslant c\sigma(h),$$

where the functions σ *and* ϕ *satisfy* (21).

Remark. In all the results of this section σ was supposed increasing. In fact, this hypothesis was used only in the comparison between series and integrals starting from (17). If we drop the assumption of monotonicity, all the results remain valid provided the conditions involving integrals of σ (like (19), (21), (22)) are replaced by the corresponding ones involving series.

2. Embedding maps between Hölder spaces. In this section we prove some useful properties of embedding operator between Hölder spaces. Set

$$H_\sigma = \left\{x\colon T \to R\colon \|x\|_\sigma = |x(0)| + \sup_{s \neq t} \frac{|x(t)-x(s)|}{\sigma(|t-s|)} < \infty\right\}$$

and

$$H_\sigma^0 = \left\{x \in H_\sigma\colon \lim_{\delta \to 0} \sup_{|s-t|<\delta} \frac{|x(t)-x(s)|}{\sigma(|t-s|)} = 0\right\}.$$

Recall that a continuous linear operator $u\colon E \to F$ between Banach spaces E and F is of type p, $1 < p \leqslant 2$, if there exists a constant $c > 0$ such that, for any finite sequence $(x_i) \subset E$,

$$E\Big\|\sum_i \varepsilon_i u x_i\Big\| \leqslant c\big(\sum_i \|x_i\|^p\big)^{1/p},$$

where (ε_i) means a Rademacher sequence.

LEMMA 2.1. *Let* $1 < p \leqslant 2$ *and* $q = p/(p-1)$. *If*

$$\int_0^1 \frac{\sigma(s)}{s^{1+\alpha}} \ln^{1/q} s^{-1}\, ds < \infty, \tag{23}$$

then the embedding map $H_\sigma \to H_\alpha^0$ *is of type* p.

Proof. It is more convenient to use here the sequential norm. Fix the finite sequence $(x_i) \in H_\sigma$ and let (ε_i) be a Rademacher sequence. Let $\phi_q(x) = \exp(x^q) - 1$. It is known (see Lemma 4.3, p. 93, in Ledoux and Talagrand [13]) that there exists a constant $c_p > 0$ such that for each finite sequence (a_i) of real numbers

$$\Big\|\sum_i \varepsilon_i a_i\Big\|_{\phi_q}^p \leqslant c_p \sum_i |a_i|^p. \tag{24}$$

Observe that $|\lambda_{-2,0}(x)| = |x(0)| \leqslant \|x\|_\sigma$ and, for $j \geqslant -1$, we have $|\lambda_{jk}(x)| \leqslant \sigma(2^{-j-1})\|x\|_\sigma$. Let us start with the special case $j = -2$. Using the elementary inequality $E|X| \leqslant \phi^{-1}(1)\|X\|_\phi$ between L^1 and Orlicz norms and (24) we get

$$E\Big|\sum_i \varepsilon_i x_i(0)\Big| \leqslant (\ln 2)^{1/q} c_p^{1/p}\big(\sum_i |x_i(0)|^p\big)^{1/p} \leqslant (\ln 2)^{1/q} c_p^{1/p}\big(\sum_i \|x\|_\sigma^p\big)^{1/p}. \tag{25}$$

Now for $j \geqslant -1$ and $0 \leqslant k < 2^j$ we have

$$\Big\|\lambda_{jk}\big(\sum_i \varepsilon_i x_i\big)\Big\|_{\phi_q} \leqslant c_p^{1/p}\big(\sum_i |\lambda_{jk}(x_i)|^p\big)^{1/p} \leqslant c_p^{1/p}\sigma(2^{-j-1})\big(\sum_i \|x_i\|_\sigma^p\big)^{1/p}. \tag{26}$$

Lemma 1.2 gives then

$$E \max_{1 \leqslant i \leqslant n} \left|\lambda_{jk}\big(\sum_i \varepsilon_i x_i\big)\right| \leqslant c_p^{1/p} \sigma(2^{-j-1}) \ln^{1/q}(2^j) \big(\sum_i \|x_i\|_\sigma^p\big)^{1/p}. \tag{27}$$

Finally, since

$$E\Big\|\sum_i \varepsilon_i x_i\Big\|_\alpha^{\mathrm{seq}} \leqslant E\Big|\sum_i \varepsilon_i x_i(0)\Big| + \sum_{j \geqslant -1} E\Big|\lambda_{jk}\big(\sum_i \varepsilon_i x_i\big)\Big|,$$

the conclusion follows from (23), (25) and (27).

Recall that an operator $u\colon E \to F$ is said to be *p-absolutely summing* if there exists a constant $c > 0$ such that for any finite sequence $(x_i) \subset E$

$$\sum_i \|ux_i\|^p \leqslant c \sup\big\{\sum_i |f(x_i)|^p\colon f \in E^*, \|f\| \leqslant 1\big\}.$$

As usual, E^* denotes the topological dual of E.

LEMMA 2.2. *Let $p > 1$. If $1 \geqslant \beta > \alpha + 1/p$, then the embedding operator $H_\beta \to H_\alpha$ is p-absolutely summing.*

Proof. Working with the sequential norms in H_β and H_α, we shall use the well-known Pietsch theorem [19]. Consider for $j \geqslant -2$, $0 \leqslant j < 2^j$ the linear functionals $\lambda_{jk}^*\colon H_\beta \to R$ defined by

$$\lambda_{jk}^*(x) = 2^{\beta(j+1)} \lambda_{jk}(x).$$

Obviously, $K_\beta = \{\lambda_{jk}^*;\ j \geqslant -2,\ 0 \leqslant j < 2^j\}$ is a norming set, that is

$$\|x\|_\beta^{\mathrm{seq}} = \sup_{f \in K_\beta} |f(x)|.$$

Define the measure

$$\mu = \sum_{j=-2}^{\infty} 2^{p(\alpha-\beta)j} \sum_{0 \leqslant k < 2^j} \delta_{\lambda_{jk}^*},$$

where δ_α is Dirac measure. It is a bounded measure supported by K_β since the series $\sum_j 2^{(p(\alpha-\beta)+1)j}$ converges. Now we have

$$\begin{aligned}
\int_{K_\beta} |f(x)|^p \mu(df) &= \sum_{j=-2}^{\infty} 2^{p(\alpha-\beta)j} \sum_{0 \leqslant k < 2^j} |\lambda_{jk}^*(x)|^p \\
&= \sum_{j=-2}^{\infty} 2^{p\alpha j} \sum_{0 \leqslant k < 2^j} |\lambda_{jk}(x)|^p \geqslant (\|x\|_\alpha^{\mathrm{seq}})^p.
\end{aligned}$$

Hence the embedding map is p-absolutely summing by the Pietsch theorem.

Remark. From the last step of the proof of Lemma 2.2 with $p = 2$ it is easily seen that the embedding $I_{\beta\alpha}\colon H_\beta \to H_\alpha$ factorizes through the Hilbert space

$$H = \Big\{u = (u_n)\colon \|u\|^2 = \sum_{j=0}^{\infty} 2^{2\alpha j} \sum_{k=0}^{2^j-1} u_{2^j+k}^2 < \infty\Big\}.$$

If we denote the corresponding factorization by $I_\lambda I_\beta$, where I_β: $H_\beta \to H$ and I_α: $H \to H_\alpha$, then the operator I_α is γ-radonifying in the sense that if ν denotes the canonical Gaussian measure on H, then νI_α^{-1} is a Gaussian Radon measure on H_α.

3. CLT in Hölder spaces. For a random element $\xi \in H_\alpha^0$ we denote by $\xi_1, \ldots, \xi_N$ independent copies of ξ and

$$\zeta_N = N^{-1/2} \sum_{k=1}^{N} \xi_k.$$

Recall that a random element ξ satisfies the central limit theorem (we write $\xi \in \mathrm{CLT}(H_\alpha^0)$) if the sequence (ζ_N) converges in distribution in H_α^0.

THEOREM 3.1. *Assume that the random element ξ in H_α^0 satisfies the following conditions:*

(i) $E\xi(t) = 0$ *and* $E\xi^2(t) < \infty$ *for all* $t \in \boldsymbol{T}$;

(ii) *there exists a positive random variable M and an increasing function σ: $R_+ \to R$, $\sigma(0) = 0$ such that $EM^2 < \infty$ and*

$$|\Delta_h^2 \xi(t)| \leqslant M\sigma(h) \quad \text{for all } t, t \pm h \in \boldsymbol{T},$$

where

$$\int_0^1 \frac{\sigma(s)}{s^{1+\alpha}} \sqrt{\ln s^{-1}}\, ds < \infty.$$

Then $\xi \in \mathrm{CLT}(H_\alpha^0)$ and $E\|\xi\|_\alpha^2 < \infty$.

Proof. Consider the Rademacher sequence (ε_l) which is independent of (ξ_k). It may be thought as constructed on another probability space Ω'. By Theorem 10.14 in Ledoux and Talagrand [13] it suffices to prove that, for almost every ω of the probability space Ω supporting the ξ_l, the sequence of random elements $\tilde{\zeta}_N^\omega$ of H_α^0 defined on Ω' by

$$\tilde{\zeta}_N^\omega(t) = N^{-1/2} \sum_{l=1}^{N} \varepsilon_l \xi_l(\omega, t), \qquad t \in \boldsymbol{T},$$

converges in distribution.

For the convergence of finite-dimensional distributions, fix $t_1 < \ldots < t_m$ in $\boldsymbol{T}$ and note that for any scalars $a_1, \ldots, a_m$

$$E_\varepsilon\Big(\sum_{i=1}^{m} a_i \tilde{\zeta}_N^\omega(t_i)\Big)^2 = \sum_{i,j=1}^{m} a_i a_j \frac{1}{N} \sum_{l=1}^{N} \xi_l(\omega, t_i)\, \xi_l(\omega, t_j).$$

Using (i) and the strong law of large numbers we see that the factor of $a_i a_j$ in the above formula converges to $E_\xi(\xi(t_i)\,\xi(t_j))$ for almost every $\omega \in \Omega$. Hence the convergence of finite-dimensional distributions of $\tilde{\zeta}_N^\omega$ holds for almost every ω by the finite-dimensional CLT. To check the tightness, we use Corollary 1.6 whose condition (i) is a simple by-product of the case $m = 1$ above. Using (24) with $p = q = 2$, we get

$$\|\Delta_h^2 \tilde{\zeta}_N^\omega(t)\|_{\Phi_2}^2 = \frac{1}{N}\Big\|\sum_{l=1}^N \varepsilon_l \Delta_h^2 \xi_l(\omega, t)\Big\|_{\Phi_2}^2$$
$$\leqslant \frac{C}{N}\sum_{l=1}^N |\Delta_h^2 \xi_l(\omega, t)|^2 \leqslant \frac{C}{N}\sum_{l=1}^N M_l^2(\omega)\sigma^2(h),$$

where $M_1, \ldots, M_N$ are independent copies of the random variable M. By the strong law of large numbers, for almost every $\omega \in \Omega$, $\sup_{N \geqslant 1} N^{-1}\sum_{l=1}^N M_l^2(\omega)$ is finite, and so $\tilde{\zeta}_N^\omega$ satisfies condition (ii) of Corollary 1.6, which completes the proof. ▪

THEOREM 3.2. *Let $\sigma\colon R^+ \to R$ be an increasing function with $\sigma(0) = 0$. Let $p \geqslant 2$. Assume that the process $\xi = (\xi_t, t \in T)$ satisfies the following conditions:*
(i) *$E\xi(t) = 0$ and $E\xi^2(t) < \infty$ for all $t \in T$;*
(ii) *for each $t \in T$, $h > 0$ such that $t \pm h \in T$*

$$E|\Delta_h^2 \xi(t)|^p \leqslant \sigma^p(h) \quad \text{and} \quad \int_0^1 \frac{\sigma(u)}{u^{\alpha+1+1/p}}\,du < \infty.$$

Then $\xi \in \mathrm{CLT}(H_\alpha^0)$.

Proof. From condition (i) we obtain the convergence of finite-dimensional distributions of (ζ_N) and condition (i) of Corollary 1.5. To complete the proof of the tightness, observe that, by the Rosenthal L_p-inequality,

$$E|\Delta_h^2 \zeta_N(t)|^p \leqslant cE|\Delta_h^2 \xi(t)|^p \leqslant c\sigma^p(h),$$

from which condition (ii) of Corollary 1.5 is obtained.

We supplement the central limit theorem in Hölder spaces with results on a convergence rate. Recall that the Prohorov metric on the space of probability measures on the Banach space B is defined by

$$\pi(\mu, \nu) = \inf\{\varepsilon > 0\colon\ \mu(F) \leqslant \nu(F^\varepsilon) + \varepsilon \text{ for all closed sets } F \subset B\},$$

where $F^\varepsilon = \{x \in B\colon \inf_{y \in F}\|x - y\| < \varepsilon\}$. The bounded Lipschitz metric is defined by

$$d_{\mathrm{BL}}(\mu, \nu) = \sup\{|\textstyle\int f\,d\mu - \int f\,d\nu|\colon f \in \mathscr{L}_1(B)\},$$

where $\mathscr{L}_1(B)$ denotes the set of functions $f\colon B \to R$ that satisfy the condition

$$\sup_x |f(x)| + \sup_{x \neq y}\frac{|f(x) - f(y)|}{\|x - y\|} \leqslant 1.$$

Let π^α and d_{BL}^α denote the Prohorov and bounded Lipschitz metrics in the space of probability measures on H_α, respectively.

THEOREM 3.3. *Assume that the process $\xi = (\xi(t), t \in T)$ satisfies the following conditions:*

(i) $E\xi(t) = 0$ *and* $E\xi^2(t) < \infty$ *for all* $t \in T$;

(ii) *there exist a positive random variable* M *and an increasing function* $\sigma\colon R_+ \to R$, $\sigma(0) = 0$, *such that* $EM^3 < \infty$ *and*

$$|\Delta_h^2 \xi(t)| \leqslant M\sigma(h) \quad \textit{for all } t,\ t \pm h \in T.$$

If for $1 \geqslant \beta > 1/2$

$$\int_0^1 \frac{\sigma(s)}{s^{1+\beta}} \sqrt{\ln s^{-1}}\, ds < \infty,$$

then for each $0 \leqslant \alpha < \beta - 1/2$

$$\pi^\alpha(\mu_N, \mu) \leqslant C(EM^3)^{1/4} N^{-1/8}, \qquad d_{\mathrm{BL}}^\alpha(\mu_N, \mu) \leqslant C(EM^3)^{1/3} N^{-1/6},$$

where μ_N *is the distribution of* ζ_N *whereas* μ *is the distribution of the limiting Gaussian process* ζ.

Proof. The Remark after the proof of Lemma 2.2 and results in Bentkus and Račkauskas [2] (see also Paulauskas and Račkauskas [17]) yield

$$\pi^\alpha(\mu_N, \mu) \leqslant C(E\|\xi\|_\beta^3 + E\|\zeta\|_\beta^3)^{1/4} N^{-1/8},$$

$$d_{\mathrm{BL}}^\alpha(\mu_N, \mu) \leqslant C(E\|\xi\|_\beta^3 + E\|\zeta\|_\beta^3)^{1/3} N^{-1/6},$$

where the constant C depends only on the covariance operator of the Gaussian random element ζ. Evidently, condition (ii) and Ciesielski [3] yield

$$E\|\xi\|_\beta^3 \leqslant CE \sup_j 2^{3\beta j} \max_{0 \leqslant k < 2^j} |\lambda_{kj}(\xi)|^3 \leqslant CEM^3.$$

By Theorem 1.1 and tail behavior of Gaussian random elements,

$$E\|\zeta\|_\beta \leqslant \sum_j 2^{\beta j} \sqrt{j} \max_{0 \leqslant k < 2^j} (E|\lambda_{jk}(\zeta)|^2)^{1/2}.$$

According to condition (i), $E\lambda_{jk}^2(\zeta) = E\lambda_{jk}^2(\xi)$. Finally, since $E\|\zeta\|_\beta^3 \leqslant C(E\|\zeta\|_\beta)^3$, using (ii) we obtain $E\|\zeta\|_\beta^3 \leqslant CEM^3$.

Let L denote the Lévy metric in the space of distribution functions. We shall write $L(\xi, \eta)$ for $L(F_\xi, F_\eta)$, where F_ξ denotes the distribution function of ξ. Recall that

$$L(\xi, \eta) = \inf\{\varepsilon > 0\colon F_\xi(x) \leqslant F_\eta(x+\varepsilon) + \varepsilon \text{ and } F_\eta(x) \leqslant F_\xi(x+\varepsilon) + \varepsilon \text{ for all } x \in R\}.$$

THEOREM 3.4. *Assume that the conditions of Theorem* 3.3 *are satisfied and* $0 < \alpha < \beta - 1/2$. *Let* $F\colon H_\alpha^0 \to R$ *satisfy*

$$\sup_{x \neq y} \frac{|F(x) - F(y)|}{\|x-y\|_\alpha} = C < \infty.$$

Then

$$L(F(\zeta_N), F(\zeta)) \leqslant C(EM^3)^{1/4} N^{-1/8}.$$

Proof. This is a direct corollary to Lemma 2.2 and Theorem 5.3.6 and Proposition 5.3.3 in Paulauskas and Račkauskas [17] (see pp. 124 and 123, respectively).

The function $F(x)$ equal to the q-variation norm, where $q \geqslant 1/\alpha$, is of a particular interest.

4. CLT for an empirical characteristic process. Let $X, X_1, X_2, \ldots$ be independent identically distributed random variables with probability distribution function $F(x)$ and characteristic function

$$c(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x).$$

Let $F_N(x)$ denote the empirical distribution function based on the first N observations $X_1, \ldots, X_N$, i.e.

$$F_N(x) = \frac{1}{N} \sum_{k=1}^{N} \mathbf{1}_{[X_k, \infty)}(x),$$

where $\mathbf{1}_A$ is the indicator function of the set A. The *empirical characteristic function* corresponding to F_N is

$$c_N(t) = \int_{-\infty}^{\infty} e^{itx}\, dF_N(x) = \frac{1}{N} \sum_{k=1}^{N} \exp(it X_k)$$

and an *empirical characteristic process* is

$$Y_N(t) = \sqrt{N}\left(c_N(t) - c(t)\right), \qquad t \in R.$$

It is well known that $c_N(t)$ converges to $c(t)$ uniformly on each finite interval (more strong convergence is proved in Csörgő [6]). Marcus [14] has found necessary and sufficient conditions that the process $\left(Y_N(t),\ t \in [-1/2, 1/2]\right)$ converges in distribution in the space $\mathscr{C}[-1/2, 1/2]$ of continuous complex functions on $[-1/2, 1/2]$ endowed with usual supremum norm. Given the experience of the Hölderian central limit theorem it seems natural to assert that $Y_N \xrightarrow{\mathscr{D}} Y$ in a complex space H_α^0.

From now on, H_α and H_α^0 are the spaces of complex-valued continuous functions x: $\boldsymbol{T} \to C$ such that $\omega_\alpha(x, 1) < \infty$ and $\lim_{\delta \to 0} \omega_\alpha(x, \delta) = 0$, respectively. Both spaces are endowed with the norm $\|\cdot\|_\alpha$.

Evidently, $Y_N(t) = N^{-1/2} \sum_{k=1}^{N} \xi_k(t)$, where

$$\xi_k(t) = \exp(it X_k) - c(t), \qquad k = 1, \ldots, N.$$

Hence Y_N is a random element in H_α^0 if and only if the deterministic function $c(t)$ is itself in H_α^0.

An elementary computation gives

$$\Delta_h^2 \xi(t) = -4e^{itX} \sin^2(hX/2) + 4E\left(e^{itX} \sin^2(hX/2)\right). \tag{28}$$

This leads us to express our hypotheses in terms of the moments of $\sin^2(hX/2)$, introducing the functions

$$\tau_k(u) = \int_{-\infty}^{\infty} \sin^{2k}(ux)\,dF(x), \qquad k \geqslant 1.$$

This choice avoids putting the conditions on the moments of the X_i.

By the multidimensional central limit theorem, the finite-dimensional distributions of (Y_N) converge to those of a complex Gaussian process Y with $EY(t) = 0$ and covariance $EY(t)\,\overline{Y(s)} = c(t-s) - c(t)c(-s)$, $s, t \in \boldsymbol{T}$. The process Y can be represented in the form

$$Y(t) = \int_{-\infty}^{+\infty} e^{itx}\,dW(F(x)) - W(1)\,c(t),$$

where $W(t)$ is a standard Wiener process.

THEOREM 4.1. *Let* $0 < \alpha < 1$. *If*

$$\sum_{j=1}^{\infty} 2^{\alpha j} \sqrt{j}\,\tau_2^{1/2}(2^{-j}) < \infty, \tag{29}$$

then (Y_N) *converges in distribution in the space* H_α^0.

Proof. Let us observe first that

$$|\Delta_h^2 c(t)| \leqslant 4\tau_1(h/2) \leqslant 4\tau_2^{1/2}(h/2),$$

so using the sequential norm, it is clear from (29) that c belongs to H_α^0 and (Y_N) can be considered as a sequence of random elements in H_α^0.

We use the same randomization method (and the same notation) as in the proof of Theorem 3.1, the only difference being in the fulfilment of condition (ii) of Corollary 1.6. Write

$$\tilde{Y}_N^\omega(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \varepsilon_k \xi_k(t), \qquad t \in \boldsymbol{T}.$$

By the series version of Corollary 1.6 (see the Remark at the end of Section 1), it suffices to check that, for almost every ω,

$$\sup_N \sum_{j\geqslant 1} 2^{(j+1)\alpha} \phi_2^{-1}(2^j) \sup_t \|\Delta_{2^{-j-1}}^2 \tilde{Y}_N^\omega(t)\|_{\phi_2} < \infty.$$

Using (28), we obtain with absolute constants a and b

$$\begin{aligned}\|\Delta_{2^{-j-1}}^2 \tilde{Y}_N^\omega(t)\|_{\phi_2} &\leqslant \frac{a}{N} \sum_{l=1}^{N} |\Delta_{2^{-j-1}}^2 \xi_l(\omega, t)|^2 \\ &\leqslant \frac{b}{N} \sum_{l=1}^{N} \sin^4\left(\frac{X_l(\omega)}{2^{j+2}}\right) + bE^2 \sin^2\left(\frac{X_1}{2^{j+2}}\right).\end{aligned}$$

Using (29) and $\tau_1 \leqslant \tau_2^{1/2}$, to treat the contribution of $E^2 \sin^2(X_1 2^{-j-2})$ reduces the problem to prove that, for almost every ω,

$$(30) \qquad \sup_N \frac{1}{N} \sum_{l=1}^{N} \sum_{j=1}^{\infty} 2^{j\alpha} \sqrt{j} \sin^4\left(\frac{X_l(\omega)}{2^{j+2}}\right) < \infty .$$

Write $Z_l(\omega)$ for the sum of the series indexed by j in (30) and observe that, by (29),

$$EZ_l = \sum_{j=1}^{\infty} 2^{j\alpha} \sqrt{j} \tau_2 (2^{-j-2}) \leqslant \sum_{j=1}^{\infty} 2^{j\alpha} \sqrt{j} \tau_2^{1/2} (2^{-j-2}) < \infty .$$

Hence (30) follows from the strong law of large numbers for the i.i.d. sequence (Z_l) and the proof is complete.

The following result supplements Corollary 1 in Csörgő [6].

COROLLARY 4.1. *Assume that the distribution function F has a non-empty monotonicity interval. Let*

$$F_1(x) = \int_0^1 |x(t)|^2 \, dt, \qquad F_2(x) = \int_0^1 (\Re x(t))^2 \, dt, \qquad F_3(x) = \int_0^1 (\Im x(t))^2 \, dt.$$

Then there exists a constant $C > 0$ such that

$$\sup_{r \geqslant 0} |P\{F_i(Y_N) < r\} - P\{F_i(Y) < r\}| \leqslant CN^{-1}$$

holds for each $i = 1, 2, 3$.

Proof. The result follows easily from Bentkus and Götze [1]. We have to check that the limiting random element Y considered in the space $L_2(0, 1)$ is at least of dimension $d \geqslant 9$. By the representation (20) it is enough to show that the random element $Y_0 = \int_{-\infty}^{+\infty} e^{itx} \, dW(F(x))$ is at least of dimension $d \geqslant 9$. This is ensured by monotonicity of the distribution function F in a certain non-empty interval. Indeed, assume that F is monotone on the interval (a, b) and consider functions $f_1, \ldots, f_M \in L_2(0, 1)$ such that $g_1, \ldots, g_M$ have non-intersecting supports on (a, b), where

$$g_k(x) = \int_0^1 e^{itx} f_k(t) \, dt, \qquad x \in R, \; k = 1, \ldots, M.$$

Then the random variables

$$\int_0^1 Y_0(t) f_k(t) \, dt = \int_0^1 g_k(x) \, dW(F(x)), \qquad k = 1, \ldots, M,$$

are independent. This yields that Y is at least M-dimensional in $L_2(0, 1)$.

Consider the following functionals on the space H_α^0:

$$F_4(x) = \int_0^1 \int_0^1 \frac{|x(t)-x(s)|^2}{|t-s|^{2\alpha}} dt\, ds, \quad F_5(x) = \int_0^1 \int_0^1 \frac{|\Re(x(t)-x(s))|^2}{|t-s|^{2\alpha}} dt\, ds,$$

$$F_6(x) = \int_0^1 \int_0^1 \frac{|\Im(x(t)-x(s))|^2}{|t-s|^{2\alpha}} dt\, ds.$$

COROLLARY 4.2. *Assume that the distribution function F has a non-empty monotonicity interval and*

$$\int_0^1 \int_0^1 |t-s|^{-4\alpha} \tau_2(t-s)\, dt\, ds < \infty.$$

Then there exists a constant $C > 0$ such that

$$\sup_{r \geqslant 0} |P\{F_i(Y_N) < r\} - P\{F_i(Y) < r\}| \leqslant CN^{-1}$$

holds for each $i = 4, 5, 6$.

The proof is similar to that to Corollary 4.1.

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Department of Mathematics, Vilnius University
Naugarduko 24, Lt-2006 Vilnius, Lithuania
E-mail address: alfredas@ieva.maf.vu.lt

Laboratoire de Statistique et Probabilités
Bât. M2
Université des Sciences et Technologies de Lille
F-59655 Villeneuve d'Ascq Cedex France
E-mail address: suquet@alea.univ-lille1.fr

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