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LEVEL CROSSINGS AND LOCAL TIME FOR REGULARIZED GAUSSIAN PROCESSES

BY

CORINNE BERZIN (PARIS), JOSÉ R. LEÓN (CARACAS) AND JOAQUÍN ORTEGA (CARACAS)

Abstract. Let $\{X_t, t \in [0, 1]\}$ be a centred stationary Gaussian process defined on (Ω, A, P) with covariance function satisfying

$$r(t) \sim 1 - C|t|^{2\alpha}, \quad 0 < \alpha < 1, \quad \text{as } t \to 0.$$

Define the regularized process

$$X^\varepsilon = \varphi_\varepsilon * X \quad \text{and} \quad Y^\varepsilon = X^\varepsilon/\sigma_\varepsilon, \quad \text{where } \sigma_\varepsilon^2 = \operatorname{var} X_t^\varepsilon,$$

φ_ε is a kernel which approaches the Dirac delta function as $\varepsilon \to 0$ and $*$ denotes the convolution. We study the convergence of

$$Z_\varepsilon(f) = \varepsilon^{-a(\alpha)} \int_{-\infty}^{\infty} \left[\frac{N^{Y^\varepsilon}(x)}{c(\varepsilon)} - L_X(x)\right] f(x)\, dx \quad \text{as } \varepsilon \to 0,$$

where $N^V(x)$ and $L_V(x)$ denote, respectively, the number of crossings and the local time at level x for the process V in $[0, 1]$ and

$$c(\varepsilon) = \left(2 \operatorname{var}(\dot{X}_t^\varepsilon)/\pi \operatorname{var}(X_t^\varepsilon)\right)^{1/2}.$$

The limit depends on the value of α.

1. INTRODUCTION

A natural way to approximate the local time L_X of an irregular process X_t is to consider regularizations by convolution: $X_t^\varepsilon = \varphi_\varepsilon * X_t$, where $\varphi_\varepsilon(\cdot) = \varepsilon^{-1}\varphi(\cdot/\varepsilon)$, φ being a continuous function, and to study the asymptotic behaviour of the number of crossings $N^{X^\varepsilon}(u)$ of the level u by the process X_t^ε on the interval [0, 1]. Wschebor [16] showed that for Brownian motion $N^{X^\varepsilon}(0)$ with an adequate normalization tends to $L_X(0)$, the local time at the origin, as $\varepsilon \to 0$, in L^p for any $p \geqslant 1$, and a similar result holds for multiparametric Brownian motion. Azaïs and Florens [1] extended this result to a class of stationary Gaussian processes, and Berzin and Wschebor [6] considered the multiparametric case. In view of these results it is natural to consider the speed at which the convergence takes place, and then to study the convergence as ε goes to 0 of the difference between $L_X(u)$ and $N^{X^\varepsilon}(u)$, adequately normalized, divided by the square root of the speed.

Let us look more closely at this problem. Let $Y_t^\varepsilon = X_t^\varepsilon/\sigma_\varepsilon$, $\sigma_\varepsilon^2 = \operatorname{var} X_t^\varepsilon$. By the results in [3], Theorem 2, and [13], we have

$$N^{Y^\varepsilon}(u) = \lim_{\delta\to 0}\frac{1}{2\delta}\int_0^1 \mathbf{1}_{(u-\delta,u+\delta)}(Y_s^\varepsilon)\left|\frac{\dot{X}_s^\varepsilon}{\sigma_\varepsilon}\right| ds$$

and there is a similar expression for the local time:

$$L_X(u) = \lim_{\delta\to 0}\frac{1}{2\delta}\int_0^1 \mathbf{1}_{(u-\delta,u+\delta)}(X(s))\,ds.$$

One could now try to study the L^2-convergence, say, by fixing δ, looking at the second order moment of the difference as $\varepsilon\to 0$, dividing then by the speed of convergence of the difference between these two random variables and looking at the limit as $\delta\to 0$. However, the expression obtained after making $\varepsilon\to 0$ goes to infinity as $\delta\to 0$, so that this approach does not work.

We consider a related problem, substituting the indicator function of the interval $(u-\delta, u+\delta)$ by a function which does not depend on δ, thus avoiding the problem of the divergence of the moments. We look at Gaussian processes such that, near the origin, the covariance is of the form $1-L(|t|)|t|^{2\alpha}$ for $0<\alpha<1$, where $L(t)\to C>0$ as $t\to 0^+$ and satisfies certain additional conditions set in Section 2. By [4] these processes have continuous local times. We study the asymptotic behaviour of $Z_\varepsilon(f)$, where

$$(1)\qquad Z_\varepsilon(f) = \frac{1}{\varepsilon^{a(\alpha)}}\zeta_\varepsilon(f),\qquad \zeta_\varepsilon(f) = \int_{-\infty}^{\infty} f(x)\left[\frac{N^{Y^\varepsilon}(x)}{c(\varepsilon)} - L_X(x)\right]dx,$$

with $c(\varepsilon) = \left(2\operatorname{var}(\dot{X}_t^\varepsilon)/\pi\operatorname{var}(X_t^\varepsilon)\right)^{1/2}$ and f a continuous function satisfying certain regularity conditions and in $L^4(\phi(x)\,dx)$, where $\phi(x)$ is the standard Gaussian density. We obtain three different types of limit depending on the value of α. For $0<\alpha<1/4$, which corresponds to the class of processes with more irregular paths, $a(\alpha)=2\alpha$ and the L^2-limit can be written as

$$K_\alpha \int_{-\infty}^{\infty} Hf(x)\,L_X(x)\,dx,$$

where H is Hermite's differential operator for the standard Gaussian measure: $Hf(x) = xf'(x) - f''(x)$. If $1/4<\alpha<3/4$, $a(\alpha)=1/2$, we have weak convergence and the limit variable has a conditional Gaussian distribution, given the sample path of the process. Finally, for $3/4<\alpha<1$ we have L^2-convergence with $a(\alpha) = 2(1-\alpha)$ and the limit can be written as a multiple stochastic integral in the infinite chaos. To get the appropriate normalization for each case we obtain in Theorem 1 the variance of $\zeta_\varepsilon(f)$.

The techniques employed vary according to the value of α. We can split $Z_\varepsilon(f)$ in two terms:

$$(2)\qquad Z_\varepsilon(f) = \frac{1}{\varepsilon^{a(\alpha)}}\int_0^1 f(Y_s^\varepsilon)\,g(\dot{Y}_s^\varepsilon)\,ds + \frac{1}{\varepsilon^{a(\alpha)}}\int_0^1 [f(Y_s^\varepsilon) - f(X_s)]\,ds \equiv T_1 + T_2,$$

where

$$g(x) = \sqrt{\pi/2}\,|x| - 1, \qquad \dot{Y}_s^\varepsilon = \frac{\dot{X}_s}{\dot{\sigma}_\varepsilon}, \qquad \text{and} \qquad \dot{\sigma}_\varepsilon^2 = \operatorname{var}\dot{X}_t^\varepsilon,$$

and $Z_\varepsilon(f)$ is shown to be equivalent to T_1 except when $0 < \alpha < 1/4$. In the latter case we show that it is enough to consider $f = H_n$ for $n \in N$, where $\{H_n,\ n \geqslant 0\}$ are Hermite's polynomials, orthogonal with respect to the standard Gaussian measure and with leading coefficient equal to 1. The term T_2 measures intuitively the distance between $f(Y^\varepsilon)$ and its limit $f(X)$.

If $1/4 < \alpha < 3/4$, we show that the finite-dimensional distributions of

$$S_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}}\int_0^t g(\dot{Y}_s^\varepsilon)\,ds$$

converge to those of Brownian motion and then use a construction similar to that of the stochastic integral in L^2 to obtain the result. Finally, in the last case the limit is given by the first term of the Hermite expansion of the function g. The Itô–Wiener formula for multiple stochastic integrals is used to obtain the result.

Estimation for the local time of diffusions when the regularization is done by using continuous piecewise linear functions obtained from a sequence of partitions has been used by Florens-Zmirou [11] to estimate the variance of a diffusion. We think that the techniques used in this work can be employed to consider similar problems for Gaussian processes.

Recently, Azaïs and Wschebor [2] have shown, for any continuous function f, the a.s. convergence of $\int_{-\infty}^{\infty} f(x)\,N^{X^\varepsilon}(x)\,dx$ for X in a class of Gaussian processes which includes ours.

The results obtained in this paper have been announced under slightly different hypotheses in [14]. Some minor mistakes in Theorem 1 are corrected herein.

2. HYPOTHESIS AND NOTATION

(H1) *For the process* X: $\{X_t,\ t \in [0, 1]\}$ is a standard stationary Gaussian process defined on (Ω, A, P) with covariance function

$$r(t) = E(X_0 X_t) = 1 - |t|^{2\alpha} L(t),\ 0 < \alpha < 1, \qquad \lim_{t \to 0^+} L(t) = C > 0,$$

where $L \geqslant 0$ is even and has two continuous derivatives except at the origin, which satisfy $|t| L'(|t|) = O(1)$ and $t^2 L''(|t|) = O(1)$ as $t \to 0$. Furthermore, when $\alpha > 1/4$ we suppose that the process has a spectral density $h(u) = M(u)/u^{\gamma}$, $\gamma = 2\alpha + 1$, and $M(u)$ has a limit when u goes to infinity.

(H2) *For the kernel* φ: φ is even, $\varphi \geqslant 0$, and

$$\operatorname{supp}\varphi \subseteq [-1, 1], \qquad \varphi \in C^1, \qquad \int_{-1}^{+1} \varphi(t)\,dt = 1.$$

We shall use the Hermite polynomials, which can be defined by

$$\exp\left(tx - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

They are an orthogonal system for the standard Gaussian measure ϕ and, if $h \in L^2(\phi(x)\,dx)$,

$$h(x) = \sum_{n=0}^{\infty} \hat{h}_n H_n(x) \quad \text{and} \quad \|h\|_{2,\phi}^2 = \sum_{n=0}^{\infty} n!\, \hat{h}_n^2.$$

Mehler's formula [8] gives a simple form to compute the covariance between two L^2-functions of Gaussian r.v.'s: If (X, Y) is a Gaussian random vector having correlation ϱ, then

$$E[h(X)k(Y)] = \sum_{n=0}^{\infty} \hat{h}_n \hat{k}_n n!\, \varrho^n. \tag{3}$$

(H3) *For the function* f: $f \in L^4(\phi(x)\,dx)$ and is continuous. We assume that f' and f'' belong to $L^2(\phi(x)\,dx)$.

Define

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(t/\varepsilon), \qquad X_\varepsilon(t) = \varphi_\varepsilon * X(t), \qquad \psi = \varphi * \varphi,$$

$$g(x) = \sqrt{\frac{\pi}{2}}|x| - 1 = \sum_{n=1}^{\infty} a_{2n} H_{2n}(x), \qquad \sigma_\varepsilon^2 = \operatorname{var}(X_t^\varepsilon),$$

$$\chi^2 = \int_{-\infty}^{\infty} \ddot{\psi}(z)\,|z|^{2\alpha}\,dz, \qquad \dot{Y}_t^\varepsilon = \frac{\dot{X}_t^\varepsilon}{\dot{\sigma}_\varepsilon}, \qquad \dot{\sigma}_\varepsilon^2 = \operatorname{var}(\dot{X}_t^\varepsilon),$$

$$c(\varepsilon) = \sqrt{\frac{2}{\pi}\frac{\dot{\sigma}_\varepsilon}{\sigma_\varepsilon}}, \qquad K_\alpha = \frac{C}{2}\int_{-\infty}^{\infty} \psi(u)\,|u|^{2\alpha}\,du,$$

$\Gamma(X, Y)$ is the covariance between X and Y, $\phi(x, y; r)$ is the bivariate Gaussian density with correlation r, and Const denotes a constant whose value may change during a proof.

The exponent $a(\alpha)$ is defined as $a(\alpha) = 2\alpha$ for $0 < \alpha < 1/4$, $a(\alpha) = 1/2$ for $1/4 < \alpha < 3/4$ and $a(\alpha) = 2(1-\alpha)$ for $3/4 < \alpha < 1$.

Note that if f' and f'' are in $L^2(\phi(x)\,dx)$ and we denote by c_n the Hermite coefficients of f, then

$$Hf(x) = xf'(x) - f''(x) = \sum_{n=1}^{\infty} nc_n H_n(x).$$

Remark. The results we obtain are also valid under the hypothesis that, for some $\beta \in [-1, 1)$, $c_n^2 n!\, n^2 \sim n^\beta$ when n goes to infinity. Examples of functions satisfying this condition but not (H3) are the indicators of intervals. The proof can be seen in [5].

3. RESULTS

THEOREM 1. *Under the hypotheses* H1, H2 *and* H3 *we have the following*: $\zeta_\varepsilon(f)$ *defined in* (1) *satisfies*

$$E(\zeta_\varepsilon^2(f)) = O(\varepsilon^{4(1-\alpha)} + \varepsilon + \varepsilon^{4\alpha}).$$

Moreover:

(i) *If* $3/4 < \alpha < 1$, *the limit of the variance divided by* $\varepsilon^{4(1-\alpha)}$ *is*

$$(4) \qquad \frac{1}{C^2\chi^4} \sum_{m=0}^{\infty} \sum_{j=(m-2)\vee 0}^{m} m!\, c_m^2 \binom{m}{j} \binom{2}{m-j} \times \int_0^1 (1-v)\, r^j(v) [\dot r(v)]^{2(m-j)} [-\ddot r(v)]^{j-m+2}\, dv.$$

(ii) *If* $1/4 < \alpha < 3/4$, *the variance divided by* ε *converges to*

$$(5) \qquad 2 \sum_{m=0}^{\infty} m!\, c_m^2 \sum_{l=1}^{\infty} (2l)!\, a_{2l}^2 \left(\frac{1}{\chi^2}\right)^{2l} \int_0^{\infty} \Big[\int_{-\infty}^{\infty} \ddot\psi(u) |w-u|^{2\alpha}\, du\Big]^{2l} dw.$$

(iii) *If* $0 < \alpha < 1/4$, *the variance divided by* $\varepsilon^{4\alpha}$ *converges to*

$$(6) \qquad 2K_\alpha^2 \sum_{m=0}^{\infty} m^2\, m!\, c_m^2 \int_0^1 (1-v)\, r^m(v)\, dv = E\Big[K_\alpha \int_{-\infty}^{\infty} Hf(x) L_X(x)\, dx\Big]^2.$$

THEOREM 2. *Under the hypotheses* H1, H2 *and* H3 *we have*

(i) *If* $0 < \alpha < 1/4$, $a(\alpha) = 2\alpha$, $Z_\varepsilon(f)$ *defined in* (1) *converges in* $L^2(\Omega)$ *when* $\varepsilon \to 0$ *to*

$$K_\alpha \sum_{n=1}^{\infty} c_n n \int_0^1 H_n(X_s)\, ds = K_\alpha \int_{-\infty}^{\infty} Hf(x) L_X(x)\, dx.$$

(ii) *If* $1/4 < \alpha < 3/4$, $a(\alpha) = 1/2$, $Z_\varepsilon(f)$ *converges weakly when* $\varepsilon \to 0$ *to an r.v.* Y *in* $L^2(\Omega)$ *and the conditional distribution* $(Y/X_s,\ 0 \leqslant s \leqslant 1)$ *is Gaussian with* *zero mean and random variance equal to*

$$\sigma^2 \int_0^1 f^2(X_s)\, ds,$$

where

$$\sigma^2 = 2 \sum_{l=1}^{\infty} (2l)!\, a_{2l}^2 \int_0^{\infty} \sigma_{2l}^2(v)\, dv \quad \text{and} \quad \sigma_{2l}^2(v) = \left[\chi^{-2} \int_{-\infty}^{\infty} \ddot{\psi}(z)\, |v-z|^{2\alpha}\, dz\right]^{2l}.$$

(iii) *If* $3/4 < \alpha < 1$, $a(\alpha) = 2(1-\alpha)$, $Z_\varepsilon(f)$ *converges in* $L^2(\Omega)$ *when* $\varepsilon \to 0$ *to*

$$-[C\chi^2]^{-1} a_2 \Bigg[\sum_{k=2}^{\infty} \frac{c_{k-2}}{k!} \int_{\mathbf{R}^k} K(\lambda_1 + \ldots + \lambda_k) \times \sum_{\pi \in \Pi_k} \lambda_{\pi^{-1}(k-1)} \lambda_{\pi^{-1}(k)}\, dZ_X(\lambda_1) \ldots dZ_X(\lambda_k) \Bigg],$$

where $K(\lambda) = [\exp(i\lambda) - 1]/i\lambda$, Π_k *is the set of permutations of* $\{1, 2, \ldots, k\}$, dZ_X *is the random spectral measure associated with* X *and the integral is an Itô–Wiener integral* [10] *(remember that* (a_n) *are the Hermite coefficients for* $g(x) = \sqrt{\pi/2}\,|x| - 1$*).*

Comments. (a) One way to try to prove Theorem 1 is the following: since $f(x) = \sum_{m=0}^{\infty} c_m H_m(x)$, by using a generalization of the Banach–Kac formula shown in [15], $\zeta_\varepsilon(f)$ can be written as

$$(7) \quad \zeta_\varepsilon(f) = \int_0^1 \big(f(Y_s^\varepsilon)(\pi/2)^{1/2}\, |\dot{Y}_s^\varepsilon| - f(X_s)\big)\, ds$$

$$= \int_0^1 f(Y_s^\varepsilon)\big((\pi/2)^{1/2}\, |\dot{Y}_s^\varepsilon| - 1\big)\, ds + \int_0^1 \big(f(Y_s^\varepsilon) - f(X_s)\big)\, ds$$

$$\equiv S_1 + S_2$$

$$= \sum_{m=0}^{\infty} c_m \int_0^1 H_m(Y_s^\varepsilon)\big((\pi/2)^{1/2}\, |\dot{Y}_s^\varepsilon| - 1\big)\, ds + \sum_{m=0}^{\infty} c_m \int_0^1 \big(H_m(Y_s^\varepsilon) - H_m(X_s)\big)\, ds,$$

but since $g(x) = (\pi/2)^{1/2}\, |x| - 1 = \sum_{l=1}^{\infty} a_{2l} H_{2l}(x)$, S_1 can be written as

$$S_1 = \sum_{m=0}^{\infty} \sum_{l=1}^{\infty} c_m a_{2l} \int_0^1 H_m(Y_s^\varepsilon)\, H_{2l}(\dot{Y}_s^\varepsilon)\, ds$$

$$= \sum_{k=2}^{\infty} \sum_{l=1}^{[k/2]} c_{k-2l}\, a_{2l} \int_0^1 H_{k-2l}(Y_s^\varepsilon)\, H_{2l}(\dot{Y}_s^\varepsilon)\, ds.$$

To prove the L^2-convergence, one can calculate $E(S_1^2)$ and $E(S_2^2)$. Under assumptions allowing the interchange of expectations, sums and integrals, one can write the second order moment of S_1 as

$$E(S_1^2) = \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \sum_{l_1=1}^{[k_1/2]} \sum_{l_2=1}^{[k_2/2]} c_{k_1-2l_1} a_{2l_1} c_{k_2-2l_2} a_{2l_2}$$

$$\times \int_0^1 \int_0^1 E\{H_{k_1-2l_1}(Y_{s'}^{\varepsilon}) H_{2l_1}(\dot{Y}_{s'}^{\varepsilon}) H_{k_2-2l_2}(Y_s^{\varepsilon}) H_{2l_2}(\dot{Y}_s^{\varepsilon})\} \, ds\, ds'.$$

Now the diagram formula for the expected value of the product of Hermite polynomials evaluated at Gaussian r.v.'s with known covariance matrix gives an expression in terms of powers of the covariances between the variables. From this expression it can be seen that $E(S_1^2)\varepsilon^{-4+4\alpha}$ converges to (4) when $\varepsilon \to 0$ if $\alpha > 3/4$, and that $E(S_1^2)\varepsilon^{-1}$ converges to (5) as $\varepsilon \to 0$ if $\alpha < 3/4$. On the other hand, S_2 can be expressed directly in terms of the covariances and we infer that $E(S_2^2) = O(\varepsilon^{1+2\alpha}) + O(\varepsilon^{4\alpha}) + O(\varepsilon^2)$ and for $\alpha < 1/4$ that $E(S_2^2)\varepsilon^{-4\alpha}$ converges to (6) as $\varepsilon \to 0$.

In this paper we will use a different approach, making direct calculations with Gaussian densities instead of using Hermite expansions and the diagram formula, since the application of the diagram formula requires more restrictive conditions on the Hermite coefficients for the function f, and the interchange between limits, sums and integrals is difficult.

(b) In (ii) of Theorem 2 one could say that, given the σ-algebra generated by $\{X_s, 0 \leqslant s \leqslant 1\}$, the limit random variable Y is the stochastic integral of $f(X_s)$ with respect to the Brownian motion W limit of S_t^{ε}, i.e. $\sigma \int_0^1 f(X_s)\, dW(s)$.

4. PROOFS

LEMMA 1. *We have*

(i)
$$\frac{\partial}{\partial r}\phi(x_1, x_2; r) = \phi(x_1, x_2; r)\left[\frac{(x_2 - r x_1)(x_1 - r x_2)}{(1-r^2)^2} + \frac{r}{1-r^2}\right],$$

(ii)
$$\frac{\partial^2}{\partial^2 r}\phi(x_1, x_2; r) = \phi(x_1, x_2; r)\left\{\left[\frac{(x_2 - r x_1)(x_1 - r x_2)}{(1-r^2)^2}\right]^2 + \frac{6r(x_2 - r x_1)(x_1 - r x_2)}{(1-r^2)^3} - \frac{x_1^2 - 2r x_1 x_2 + x_2^2}{(1-r^2)^2} + \frac{1+r^2}{(1-r^2)^2}\right\}.$$

The lemma follows from the properties of the multivariate Gaussian density. ∎

Before proving Theorem 1 we shall give an alternative expression for the limit variance in the case $\alpha > 3/4$.

LEMMA 2. *Define* $s(v) = (1-r^2(v))^{1/2}$. *If* $\alpha > 3/4$, *then the expression* (4) *is equal to*

$$\int_0^1 (1-v) \int_{R^2} f(x_1) f(r(v)x_1 + s(v)x_2) \left[\frac{(\dot{r}(v))^4 (x_2^2-1)}{2C^2 \chi^4 s^2(v)} \left[\left(x_1 - \frac{r(v)}{s(v)} x_2 \right)^2 - \frac{1}{s^2(v)} \right] \right.$$
$$+ \frac{2(\dot{r}(v))^2 x_2}{C^2 \chi^4 s(v)} \left[\ddot{r}(v) + \frac{r(v)(\dot{r}(v))^2}{s^2(v)} \right] \left(x_1 - \frac{r(v)}{s(v)} x_2 \right)$$
$$\left. + \frac{1}{C^2 \chi^4} \left[\ddot{r}(v) + \frac{r(v)(\dot{r}(v))^2}{s^2(v)} \right]^2 \right] \phi(x_1) \phi(x_2) \, dx_1 \, dx_2 \, dv.$$

Proof. We consider separately the terms corresponding to the three possible values of j in the second sum of (4). Using Mehler's formula (3) and the relation

$$\sum_{m=0}^{\infty} m! \, c_m^2 r^m(v) = E(f(X_u) f(X_{u+v})),$$

for $j = m$ we obtain

$$\frac{1}{C^2 \chi^4} \sum_{m=0}^{\infty} m! \, c_m^2 \int_0^1 (1-v) r^m(v) (\ddot{r}(v))^2 \, dv$$
$$= \frac{1}{C^2 \chi^4} \int_0^1 (1-v) (\ddot{r}(v))^2 \int_{R^2} f(x_1) f(r(v)x_1 + s(v)x_2) \phi(x_1) \phi(x_2) \, dx_1 \, dx_2 \, dv.$$

Using (i) of Lemma 1 and the equality

$$\sum_{m=1}^{\infty} m! \, c_m^2 m r^{m-1}(v) = \frac{\partial}{\partial r} E(f(X_u) f(X_{u+v}))\big|_{r=r(v)},$$

for $j = m-1$ we get

$$\frac{2}{C^2 \chi^4} \sum_{m=1}^{\infty} m! \, c_m^2 m \int_0^1 (1-v) r^{m-1}(v) \dot{r}^2(v) \ddot{r}(v) \, dv$$
$$= \frac{2}{C^2 \chi^4} \int_0^1 (1-v) \int_{R^2} f(x_1) f(r(v)x_1 + s(v)x_2) \left[\frac{\dot{r}^2(v) \ddot{r}(v)}{s(v)} x_2 \left(x_1 - \frac{r(v) x_2}{s(v)} \right) \right.$$
$$\left. + \frac{\dot{r}^2(v) \ddot{r}(v) r(v)}{s^2(v)} \right] \phi(x_1) \phi(x_2) \, dx_1 \, dx_2 \, dv.$$

Finally, since

$$\sum_{m=2}^{\infty} m! \, c_m^2 m(m-1) r^{m-2}(v) = \frac{\partial^2}{\partial r^2} E(f(X_u) f(X_{u+v}))\big|_{r=r(v)}$$

and using (ii) of Lemma 1, we get for $j = m-2$

$$\frac{1}{2C^2\chi^4}\sum_{m=1}^{\infty} m!\,c_m^2\,m(m-1)\int_0^1 (1-v)\,r^{m-2}(v)\big(\dot r(v)\big)^4\,dv$$

$$= \frac{1}{2C^2\chi^4}\int_0^1 (1-v)\big(\dot r(v)\big)^4 \int_{R^2} f(x_1)\,f\big(r(v)x_1+s(v)x_2\big)\Big\{\frac{x_2^2\big(x_1-r(v)x_2\big)^2}{s^2(v)}$$

$$+\frac{6r(v)\,x_2\big(x_1-r(v)x_2\big)}{s^3(v)}-\frac{x_1^2+x_2^2}{s^2(v)}+\frac{1+2r^2(v)}{s^4(v)}\Big\}\,\phi(x_1)\,\phi(x_2)\,dx_1\,dx_2\,dv.$$

Adding up the three expressions we obtain the result. ■

LEMMA 3. *Let*

$$\varrho_\varepsilon(v) = r_\varepsilon(v)(\sigma_\varepsilon)^{-2} = \Gamma(Y_0^\varepsilon,\, Y_v^\varepsilon), \qquad \dot\varrho_\varepsilon(v) = \Gamma(Y_0^\varepsilon,\, \dot Y_v^\varepsilon), \qquad -\ddot\varrho_\varepsilon(v) = \Gamma(\dot Y_0^\varepsilon,\, \dot Y_v^\varepsilon);$$

then, for any $\eta > 0$, *uniformly for* $v \in [\eta, 1]$, *as* $\varepsilon \to 0$,

$$\varrho_\varepsilon(v) \to r(v), \qquad \dot\varrho_\varepsilon(v)\,\varepsilon^{\alpha-1} \to \frac{\dot r(v)}{\sqrt{C}\,\chi}, \qquad \ddot\varrho_\varepsilon(v)\,\varepsilon^{2(\alpha-1)} \to \frac{\ddot r(v)}{C\chi^2}.$$

The proof is based on the relation $\dot\sigma_\varepsilon^2 \sim C\chi^2\,\varepsilon^{2\alpha-2}$ which can be obtained from the equality

$$\dot\sigma_\varepsilon^2\,\varepsilon^{2-2\alpha} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dot\varphi(u)\,\dot\varphi(v)\,|u-v|^{2\alpha} L(\varepsilon|u-v|)\,du\,dv$$

since this integral converges to $C\chi^2$ as $\varepsilon \to 0$. ■

Proof of Theorem 1. Let us start with (7):

$$\zeta_\varepsilon(f) = \int_0^1 f(Y_s^\varepsilon)\,g(\dot Y_s^\varepsilon)\,ds + \int_0^1 \big(f(Y_s^\varepsilon)-f(X_s)\big)\,ds = S_1+S_2,$$

and use the decomposition (2) of $Z_\varepsilon(f) = \zeta_\varepsilon(f)/\varepsilon^{a(\alpha)}$, where the exponent $a(\alpha)$ is defined in Section 2,

$$Z_\varepsilon(f) = \frac{1}{\varepsilon^{a(\alpha)}}\int_0^1 f(Y_s^\varepsilon)\,g(\dot Y_s^\varepsilon)\,ds + \frac{1}{\varepsilon^{a(\alpha)}}\int_0^1 [f(Y_s^\varepsilon)-f(X_s)]\,ds = T_1+T_2.$$

The proof will proceed as follows: in Part 1 we obtain an expression for $E(S_1^2)$ for all values of α. In Part 1A we consider the case $\alpha > 3/4$ and show, using Lemmas 5 and 7, that $E(T_1^2)$ converges to (4) as ε goes to zero. In Part 1B we consider $\alpha < 3/4$ and prove in Lemma 8 that $\varepsilon^{-1}E(S_1^2)$ converges to (5). In Part 2 we prove that $E(S_2^2) = O(\varepsilon^{4\alpha})+o(\varepsilon)$; hence this term only matters when $\alpha < 1/4$ and we show in this case that $E(T_2^2)$ converges to (6).

Part 1. *The second order moment of* S_1:

(8) $$E(S_1^2) = 2\int_0^1 (1-v)\, E[f(Y_0^\varepsilon)\, g(\dot Y_0^\varepsilon)\, f(Y_u^\varepsilon)\, g(\dot Y_u^\varepsilon)]\, du$$
$$= 2\int_0^1 (1-v) \int_{R^4} f(x) f(y)\, g(\dot x)\, g(\dot y)\, \phi_\varepsilon(x, \dot x, y, \dot y; v)\, dx\, d\dot x\, dy\, d\dot y\, dv,$$

where $\phi_\varepsilon(x, \dot x, y, \dot y; s-s')$ is the joint Gaussian density for the variables Y_s^ε, $\dot Y_s^\varepsilon$, $Y_{s'}^\varepsilon$, $\dot Y_{s'}^\varepsilon$. Integrals and expectation can be interchanged since f is in $L^2(\mathbf{R}, \phi(x)\, dx)$. We now fix v and make a change of variables to transform this into a standard Gaussian density (remember that ϱ_ε, $\dot\varrho_\varepsilon$ and $\ddot\varrho_\varepsilon$ are defined in Lemma 3):

$$x = x_1, \qquad y = \varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2,$$
$$\dot x = \delta_1 x_2 + \delta_2 x_3 \qquad \text{and} \qquad \dot y = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4,$$

where

$$\delta_1(v) = \dot\varrho_\varepsilon(v)\big(1-\varrho_\varepsilon^2(v)\big)^{-1/2}, \qquad \delta_2(v) = \big(1-\delta_1^2(v)\big)^{1/2}, \qquad c_1(v) = -\dot\varrho_\varepsilon(v),$$

(9) $$c_2(v) = \varrho_\varepsilon(v)\,\delta_1(v), \qquad c_3(v) = \delta_2^{-1}(v)\big(\ddot\varrho_\varepsilon(v) - \varrho_\varepsilon(v)\,\delta_1^2(v)\big),$$
$$c_4(v) = (1-c_1^2-c_2^2-c_3^2)^{1/2}.$$

We can write the inner integral in (8) as

(10) $$\int_{R^2} f(x_1)\, f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big)$$
$$\times \int_{R^2} g(\delta_1 x_2 + \delta_2 x_3)\, g(c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4)$$
$$\times\, \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\, dx_1\, dx_2\, dx_3\, dx_4.$$

LEMMA 4. *Define*

$$m_0 = c_3\,\delta_2^{-1}, \qquad m_1 = \delta_1\,\delta_2^{-1} x_2, \qquad m_2 = \delta_2^{-1}(c_1 x_1 + c_2 x_2),$$
$$q(z) = \exp(-z^2/2) \qquad \textit{and} \qquad p(z) = z\int_0^z q(y)\, dy.$$

Then, for x_1, x_2 *and* v *fixed,*

(11) $$\int_{R^2} g(\delta_1 x_2 + \delta_2 x_3)\, g(c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4)\,\phi(x_3)\,\phi(x_4)\, dx_3\, dx_4$$
$$= K_2 + K_3 + K_4,$$

where

$$K_2 = \delta_2^2\, m_0 \Big(\int_0^{m_1} \xi^2 \exp(-\xi^2/2)\, d\xi - m_1\, q(m_1)\Big)\Big(\int_0^{m_2} \xi^2 \exp(-\xi^2/2)\, d\xi - m_2\, q(m_2)\Big),$$

$$K_3 = \frac{\pi}{2}\delta_2^2 \sum_{n=2}^{\infty} a_n(m_1)\, a_n(m_2)\, n!\, m_0^n,$$

$$K_4 = [\delta_2(p(m_1)+q(m_1))-1]\,[\delta_2(p(m_2)+q(m_2))-1],$$

and $a_n(m_i)$, $n \geqslant 0$, *are the Hermite coefficients of the functions* $G_i(x) = |m_i+x|$, $i = 1, 2$.

Proof. Since $\delta_2^2 = c_3^2+c_4^2$, the integral in (11) is

$$\int_{R^2} [(\pi/2)^{1/2}\delta_2 |m_1+x_3|-1]$$

$$\times [(\pi/2)^{1/2}\delta_2 |m_2+m_0 x_3+(1-m_0^2)^{1/2} x_4|-1]\, \phi(x_3)\, \phi(x_4)\, dx_3\, dx_4$$

and this can be written as $J_1+J_2+J_3+1$, where

$$J_1 = \frac{\pi}{2}\delta_2^2 \int_{R^2} |m_1+x_3|\, |m_2+m_0 x_3+(1-m_0^2)^{1/2} x_4|\, \phi(x_3)\, \phi(x_4)\, dx_3\, dx_4$$

$$= \frac{\pi}{2}\delta_2^2 E(G_1(X)\, G_2(Y)),$$

$$J_2 = -\delta_2(p(m_1)+q(m_1)), \qquad J_3 = -\delta_2(p(m_2)+q(m_2))$$

with (X, Y) standard Gaussian r.v.'s with correlation m_0. Thus, by Mehler's formula (3), we obtain

$$J_1 = \frac{\pi}{2}\delta_2^2 \sum_{n=0}^{\infty} a_n(m_1)\, a_n(m_2)\, n!\, m_0^n.$$

Let us look closely at the terms in this sum. For $n = 0$ we have

$$K_1 \equiv \frac{\pi}{2}\delta_2^2\, a_0(m_1)\, a_0(m_2) = \delta_2^2(p(m_1)+q(m_1))(p(m_2)+q(m_2)),$$

for $n = 1$ we get

$$K_2 \equiv \frac{\pi}{2}\delta_2^2 m_0\, a_1(m_1)\, a_1(m_2)$$

$$= \delta_2^2 m_0 \Big(\int_0^{m_1} \xi^2 \exp(-\xi^2/2)\, d\xi - m_1 q(m_1)\Big)\Big(\int_0^{m_2} \xi^2 \exp(-\xi^2/2)\, d\xi - m_2 q(m_2)\Big),$$

and for $n \geqslant 2$ we obtain

$$(12)\qquad K_3 \equiv \frac{\pi}{2}\delta_2^2 \sum_{n=2}^{\infty} a_n(m_1)\, a_n(m_2)\, n!\, m_0^n = \pi\delta_2^2\, a_2(m_1)\, a_2(m_2)\, m_0^2 + O(m_0^3).$$

Thus $J_1 = K_1 + K_2 + K_3$, and defining

$$K_4 \equiv K_1 + J_2 + J_3 + 1 = \left[\delta_2\big(p(m_1)+q(m_1)\big)-1\right]\left[\delta_2\big(p(m_2)+q(m_2)\big)-1\right],$$

we complete the proof. ■

Using (10) and Lemma 4 we can now write (8) as

$$(13)\qquad E(S_1^2) = 2\int_0^1 (1-v) \int_{R^2} f(x_1)\, f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big) \times [K_2+K_3+K_4]\, \phi(x_1)\, \phi(x_2)\, dx_1\, dx_2\, dv.$$

For $\eta \in (0, 1)$ split (13) into $l_1(\eta)+l_2(\eta)$, where $l_1(\eta)$ corresponds to the integral over $v < \eta$ and $l_2(\eta)$ to the integral over the rest.

Part 1A. $\alpha > 3/4$.

In Lemma 5 we show that $l_2(\eta)/\varepsilon^{4(1-\alpha)}$ converges to (1) as $\varepsilon \to 0$ and $\eta \to 0$ in this order. In Lemma 7 we prove that $|l_1(\eta)|/\varepsilon^{4(1-\alpha)} = O(\eta^{4\alpha-3})$ as $\eta \to 0$. Making $\eta \to 0$ we infer that $E(T_1^2)$ converges to (1). We will prove in Part 2 that $E(T_2^2) \to 0$ if $\alpha > 1/4$. This shows (i) of Theorem 1.

Lemma 5. *If $\alpha > 3/4$, then $l_2(\eta)/\varepsilon^{4(1-\alpha)}$ converges to* (1) *as $\varepsilon \to 0$ and $\eta \to 0$ in this order.*

Proof. As before, $s^2(v) = 1 - r^2(v)$ and we use the same notation as in Lemma 4 and the results of Lemma 3. For $\eta > 0$, uniformly for $v \in [\eta, 1]$ and for any value of α, as $\varepsilon \to 0$, we obtain

$$\frac{m_1}{\varepsilon^{1-\alpha}} \to \frac{\dot r(v)\, x_2}{\sqrt{C}\,\chi s(v)},\qquad \frac{m_2}{\varepsilon^{1-\alpha}} \to \frac{-\dot r(v)}{\sqrt{C}\,\chi}\left(x_1 - \frac{r(v)\, x_2}{s(v)}\right),$$

$$\frac{\delta_2 - 1}{\varepsilon^{2(1-\alpha)}} \to \frac{-(\dot r(v))^2}{2C\chi^2 s^2(v)},\qquad \frac{m_0}{\varepsilon^{2(1-\alpha)}} \to \frac{-1}{C\chi^2}\left[\ddot r(v) + \frac{r(v)(\dot r(v))^2}{s^2(v)}\right].$$

Thus we have

$$\frac{K_4}{\varepsilon^{4(1-\alpha)}} \to \frac{(\dot r(v))^4 (x_2^2-1)}{4C^2\chi^4 s^2(v)}\left[\left(x_1 - \frac{r(v)\, x_2}{s(v)}\right)^2 - \frac{1}{s^2(v)}\right],$$

$$\frac{K_2}{\varepsilon^{4(1-\alpha)}} \to \frac{(\dot r(v))^2 x_2}{C^2\chi^4 s(v)}\left[\ddot r(v) + \frac{r(v)(\dot r(v))^2}{s^2(v)}\right]\left(x_1 - \frac{r(v)\, x_2}{s(v)}\right),$$

$$\frac{K_3}{\varepsilon^{4(1-\alpha)}} \to \frac{1}{2C^2\chi^4}\left[\ddot r(v) + \frac{r(v)(\dot r(v))^2}{s^2(v)}\right]^2.$$

Hence, using the Dominated Convergence Theorem, we see that $l_2(\eta)/\varepsilon^{4(1-\alpha)}$ converges, as $\varepsilon \to 0$, to

$$\int_\eta^1 (1-v) \int_{R^2} f(x_1) f(r(v)x_1+s(v)x_2)\Bigg[\frac{(\dot{r}(v))^4 (x_2^2-1)}{2C^2\chi^4 s^2(v)}\Bigg[\Big(x_1-\frac{r(v)x_2}{s(v)}\Big)^2-\frac{1}{s^2(v)}\Bigg] + \frac{2(\dot{r}(v))^2 x_2}{C^2\chi^4 s(v)}\Bigg[\ddot{r}(v)+\frac{r(v)(\dot{r}(v))^2}{s^2(v)}\Bigg]\Big(x_1-\frac{r(v)x_2}{s(v)}\Big) + \frac{1}{C^2\chi^4}\Bigg[\ddot{r}(v)+\frac{r(v)(\dot{r}(v))^2}{s^2(v)}\Bigg]^2\Bigg]\phi(x_1)\phi(x_2)\,dx_1\,dx_2\,dv,$$

and taking into account Lemma 2, this goes to (1) as $\eta \to 0$. ∎

We give now a technical result needed for the proof of Lemma 7.

LEMMA 6. *With the notation set in* (9), *if* $\alpha > 3/4$ *for* M *large enough and* ε *small such that* $M\varepsilon < \eta$, *then*

$$(14)\qquad \varepsilon^{-4(1-\alpha)} \int_{M\varepsilon}^{\eta} \left(|\ddot{\varrho}_\varepsilon(v)|\,\delta_1^2(v)\,\delta_2^{-2}(v)+\delta_1^4(v)\,\delta_2^{-2}(v)+\delta_1^4(v)+\dot{\varrho}_\varepsilon^2(v)\,\delta_2^{-2}(v)\right)dv = O(\eta^{4\alpha-3}).$$

Proof. Let M be large and ε be small so that $M\varepsilon < \eta$. For any $v \in (M\varepsilon, \eta)$ we have

$$1-\varrho_\varepsilon(v) = \frac{v^{2\alpha}}{\sigma_\varepsilon^2}\int_{-2}^{2}\psi(u)\left[\left|1-\frac{\varepsilon u}{v}\right|^{2\alpha} L(v-\varepsilon u)-\left|\frac{\varepsilon u}{v}\right|^{2\alpha} L(|\varepsilon u|)\right]du.$$

Using (H1) we have $L(|\varepsilon u|) \leqslant 2C$, $L(v-\varepsilon u) \geqslant C/2$ for ε small, and also $\varepsilon u/v \leqslant 2/M$. Hence

$$1-\varrho_\varepsilon(v) \geqslant \frac{v^{2\alpha}}{\sigma_\varepsilon^2}\int_{-2}^{2}\psi(u)\,du\left[\left(1-\frac{2}{M}\right)^{2\alpha}\frac{C}{2}-\left(\frac{2}{M}\right)^{2\alpha}2C\right] \geqslant Q_\alpha v^{2\alpha},$$

where Q_α is a constant dependent on α.

We also need a lower bound for $1-\varrho_\varepsilon^2(v)-\dot{\varrho}_\varepsilon^2(v)$. Since $L(v-\varepsilon u)$ and $(v-\varepsilon u)L'(v-\varepsilon u)$ are bounded, using

$$|\dot{\varrho}_\varepsilon(v)| \leqslant \mathrm{Const}\,\varepsilon^{1-\alpha}v^{2\alpha-1}\left[\int_{-2}^{2}\psi(u)\left(1-\frac{\varepsilon u}{v}\right)^{2\alpha-1}du\right] \leqslant \mathrm{Const}\,\varepsilon^{1-\alpha}v^{2\alpha-1}\left(1+\frac{2}{M}\right)^{2\alpha-1} \leqslant \mathrm{Const}\,\varepsilon^{1-\alpha}v^{2\alpha-1},$$

for M large we obtain $1-\varrho_\varepsilon^2(v)-\dot{\varrho}_\varepsilon^2(v) \geqslant \mathrm{Const}\,v^{2\alpha}$. Consequently, the results follow for the two middle terms of the left-hand side of (14). For the first one, using the Schwarz inequality we have

$$\varepsilon^{-4(1-\alpha)} \int_{M\varepsilon}^{\eta} |\ddot{\varrho}_\varepsilon(v)| \, \delta_1^2(v) \, \delta_2^{-2}(v) \, dv \leqslant \mathrm{Const}\,\varepsilon^{-2(1-\alpha)} \Big[\int_{M\varepsilon}^{\eta} \ddot{\varrho}_\varepsilon^2(v) \, dv \Big]^{1/2} \eta^{(4\alpha-3)/2}.$$

By Jensen's inequality the integral above is bounded by $\mathrm{Const}\,\eta^{(4\alpha-3)}$. For the last term in (14) it is enough to prove that $1-\varrho_\varepsilon^2(v) \leqslant \mathrm{Const}\, v^{2\alpha}$ but this is a consequence of behaviour of the spectral density at infinity. ∎

The next lemma shows that for $\alpha > 3/4$, the terms near the diagonal are negligible. For the proof we use again Lemma 4.

LEMMA 7. *If* $\alpha > 3/4$, *then* $|l_1(\eta)| \, \varepsilon^{4\alpha-4} = O(\eta^{4\alpha-3})$ *as* $\eta \to 0$.

Proof. We use the same notation as before. Remember that

$$l_1(\eta) = 2\int_0^{\eta} (1-v) \int_{R^2} f(x_1) \, f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big) \times [K_2 + K_3 + K_4] \, \phi(x_1) \, \phi(x_2) \, dx_1 \, dx_2 \, dv,$$

where ϱ_ε is defined in Lemma 3, and K_2, K_3 and K_4 in Lemma 4.

We split $[0, \eta]$ into two intervals: $[0, M\varepsilon]$ and $[M\varepsilon, \eta]$, where M and ε will be chosen later. It is easy to show that the integral over $[0, M\varepsilon]$ is $O(\varepsilon)$. The other integral is

$$(15) \qquad 2 \int_{M\varepsilon}^{\eta} (1-v) \int_{R^2} f(x_1) \, f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big) \times [K_2 + K_3 + K_4] \, \phi(x_1) \, \phi(x_2) \, dx_1 \, dx_2 \, dv \equiv L_1 + L_2 + L_3.$$

The first term in (15) is

$$|L_1| = \Big|2 \int_{M\varepsilon}^{\eta} (1-v) \int_{R^2} f(x_1) \, f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big) K_2 \, \phi(x_1) \, \phi(x_2) \, dx_1 \, dx_2 \, dv\Big|$$

$$\leqslant 2 \int_{M\varepsilon}^{\eta} (1-v) \int_{R^2} \big|f(x_1) \, f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big)\big| \times |\delta_2 \, c_3 \, m_1 \, m_2| \, \phi(x_1) \, \phi(x_2) \, dx_1 \, dx_2 \, dv$$

$$= \mathrm{Const} \int_{M\varepsilon}^{\eta} (1-v) \int_{R^2} \big|f(x_1) \, f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big)\big| \left| x_1 - \frac{\varrho_\varepsilon x_2}{(1-\varrho_\varepsilon^2)^{1/2}} \right| \times |\dot{\varrho}_\varepsilon \, \delta_1 \, \delta_2^{-1} \, x_2 \, c_3| \, \phi(x_1) \, \phi(x_2) \, dx_1 \, dx_2 \, dv.$$

Consider the expression

$$|\dot{\varrho}_\varepsilon| \int_{R^2} \big|f(x_1) \, f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big)\big| \, |x_2| \left| x_1 - \frac{\varrho_\varepsilon x_2}{(1-\varrho_\varepsilon^2)^{1/2}} \right| \phi(x_1) \, \phi(x_2) \, dx_1 \, dx_2,$$

which by changing the variables $z = \varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2$ takes the form

$$|\delta_1| \int_{R^2} |f(x_1) f(z)| \left|\frac{z-\varrho_\varepsilon x_1}{(1-\varrho_\varepsilon^2)^{1/2}}\right| \left|\frac{x_1-\varrho_\varepsilon z}{(1-\varrho_\varepsilon^2)^{1/2}}\right| \phi(x_1, z; \varrho_\varepsilon)\, dx_1\, dz$$

$$\leqslant |\delta_1| E\big(f^2(N)\big) E(|N|^2),$$

where N is a standard Gaussian r.v. Hence, using (9) and Lemma 6, we obtain

$$(16) \quad |L_1| \leqslant \text{Const} \int_{M\varepsilon}^{\eta} (1-v)\,[|\ddot{\varrho}_\varepsilon \delta_1^2 \delta_2^{-2}| + |\varrho_\varepsilon \delta_1^4 \delta_2^{-2}|]\, dv \leqslant \text{Const}\, \eta^{4\alpha-3} \varepsilon^{4-4\alpha}.$$

For the second term in (15) use (12) to get

$$|L_2| = |2 \int_{M\varepsilon}^{\eta} (1-v) \int_{R^2} f(x_1) f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big) K_3\, \phi(x_1)\, \phi(x_2)\, dx_1\, dx_2\, dv|$$

$$\leqslant \text{Const} \int_{M\varepsilon}^{\eta} (1-v) \int_{R^2} |f(x_1) f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big)| \left[1+\frac{m_1^2}{2}\right]\left[1+\frac{m_2^2}{2}\right]$$

$$\times\, \delta_2^2 m_0^2\, \phi(x_1)\, \phi(x_2)\, dx_1\, dx_2\, dv.$$

Writing out the product in the integrand we get four integrals. We shall consider only the first one as the others can be treated similarly. Using Lemma 6 we obtain

$$(17) \qquad |L_2| \leqslant \text{Const} \int_{M\varepsilon}^{\eta} (1-v) \int_{R^2} |f(x_1) f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big)|$$

$$\times\, \delta_2^2 m_0^2\, \phi(x_1)\, \phi(x_2)\, dx_1\, dx_2\, dv$$

$$\leqslant \text{Const}\, \|f\|_2^2 \int_{M\varepsilon}^{\eta} (1-v)\, c_3^2\, dv \leqslant \text{Const}\, \|f\|_2^2 \Big[\int_{M\varepsilon}^{\eta} \delta_2^{-2} (\ddot{\varrho}_\varepsilon^2 + \varrho_\varepsilon^2 \delta_1^4)\, dv\Big]$$

$$\leqslant \text{Const}\, \eta^{4\alpha-3} \varepsilon^{4-4\alpha}.$$

Finally, the last term in (15) is

$$|L_3| = |\int_{M\varepsilon}^{\eta} (1-v) \int_{R^2} f(x_1) f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big) K_4\, \phi(x_1)\, \phi(x_2)\, dx_1\, dx_2\, dv|$$

$$\leqslant \text{Const} \int_{M\varepsilon}^{\eta} (1-v) \int_{R^2} |f(x_1) f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big)|\, [\delta_2 \big(p(m_1) + q(m_1)\big) - 1]$$

$$\times\, [\delta_2 \big(p(m_2) + q(m_2)\big) - 1]\, \phi(x_1)\, \phi(x_2)\, dx_1\, dx_2\, dv$$

$$\leqslant \text{Const} \int_{M\varepsilon}^{\eta} (1-v) \int_{R^2} |f(x_1) f\big(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2\big)| \frac{\varepsilon^{4-4\alpha}}{\sigma_\varepsilon^4 C^2 \chi^4}$$

$$\times \Big[\int_{-\infty}^{\infty} \psi(u)\dot{r}(v-\varepsilon u)\,du\Big]^4 \left[\left(x_1 - \frac{\varrho_\varepsilon x_2}{(1-\varrho_\varepsilon^2)^{1/2}}\right)^2 + \frac{1}{1-\varrho_\varepsilon^2}\right]$$

$$\times (x_2^2+1)\frac{1}{1-\varrho_\varepsilon^2}\phi(x_1)\phi(x_2)\,dx_1\,dx_2\,dv.$$

Using the same argument as before we get

$$\int_{R^2} |f(x_1) f(\varrho_\varepsilon x_1 + (1-\varrho_\varepsilon^2)^{1/2} x_2)| \big((x_1(1-\varrho_\varepsilon^2)^{1/2} - \varrho_\varepsilon x_2)^2 + 1\big)$$

$$\times (x_2^2+1)\phi(x_1)\phi(x_2)\,dx_1\,dx_2 = E\big(f^2(N)\big)E(N^2+1)^2.$$

Therefore

$$(18) \qquad |L_3| \leqslant \mathrm{Const}\frac{\varepsilon^{4-4\alpha}}{\sigma_\varepsilon^4 C^2 \chi^4}\int_{M\varepsilon}^{\eta}\frac{1-v}{(1-\varrho_\varepsilon^2)^2}\Big[\int_{-\infty}^{\infty}\psi(u)\dot{r}(v-\varepsilon u)\,du\Big]^4 dv$$

$$\leqslant \mathrm{Const}\,\eta^{4\alpha-3}\varepsilon^{4-4\alpha}.$$

The expressions (16), (17) and (18) show the result. ■

This finishes the consideration of Part 1A and, together with Part 2 below, completes the proof of (i) in Theorem 1.

Part 1B. $\alpha < 3/4$.

We prove now that $\varepsilon^{-1}E(S_1^2)$ converges to (5).

LEMMA 8. *If* $\alpha < 3/4$, *then* $\varepsilon^{-1}E(S_1^2)$ *converges to* (5) *as* $\varepsilon \to 0$.

Proof. Remember that $E(S_1^2)$ is given by (8) and consider

$$\frac{2}{\varepsilon}\int_0^\eta (1-u)E\{f(Y_0^\varepsilon)g(\dot{Y}_0^\varepsilon)f(Y_u^\varepsilon)g(\dot{Y}_u^\varepsilon)\}\,du.$$

For ε small enough, divide the domain of integration into $[0, M\varepsilon]$ and $[M\varepsilon, \eta]$. Using the same type of arguments as in the proofs of Lemmas 5 and 6 we get for the second integral

$$\Big|\int_{M\varepsilon}^{\eta}(1-u)E\{f(Y_0^\varepsilon)g(\dot{Y}_0^\varepsilon)f(Y_u^\varepsilon)g(\dot{Y}_u^\varepsilon)\}\,du\Big| \leqslant \mathrm{Const}\,\varepsilon M^{4\alpha-3}$$

for M large enough. Note that in this case the bound is obtained by using the lower limit of the integral. For the first integral, making $u = \varepsilon w$, we have

$$2\int_0^M (1-\varepsilon w)E\{f(Y_0^\varepsilon)g(\dot{Y}_0^\varepsilon)f(Y_{\varepsilon w}^\varepsilon)g(\dot{Y}_{\varepsilon w}^\varepsilon)\}\,dw.$$

Let $\Sigma_\varepsilon(w)$ be the covariance matrix of the Gaussian vector $(Y_0^\varepsilon, \dot{Y}_{\varepsilon w}^\varepsilon, Y_0^\varepsilon, \dot{Y}_{\varepsilon w}^\varepsilon)$; then

$$\Sigma_\varepsilon(w) \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta(w) \\ 0 & 0 & \theta(w) & 1 \end{pmatrix} \equiv \Sigma(w),$$

where $\theta(w) = \chi^{-2} \int_{-\infty}^{\infty} \ddot{\psi}(u) |w-u|^{2\alpha} du$ and $\varepsilon \to 0$. Hence the measure

$$\mu_\varepsilon = \mathscr{L}(Y_0^\varepsilon, Y_{\varepsilon w}^\varepsilon, \dot{Y}_0^\varepsilon, \dot{Y}_{\varepsilon w}^\varepsilon)$$

tends weakly to $\mu = \mathscr{L}(X_0, X_0, Z_1(w), Z_2(w))$ which is Gaussian with covariance $\Sigma(w)$. On the other hand, we have

$$E\{f(Y_0^\varepsilon) g(\dot{Y}_0^\varepsilon) f(Y_{\varepsilon w}^\varepsilon) g(\dot{Y}_{\varepsilon w}^\varepsilon)\}^2 \leqslant \|f\|_4^4 \|g\|_4^4 < \infty.$$

These two facts together with (3) imply that $E\{f(Y_0^\varepsilon) g(\dot{Y}_0^\varepsilon) f(Y_{\varepsilon w}^\varepsilon) g(\dot{Y}_{\varepsilon w}^\varepsilon)\}$ converges to

$$E[f^2(X_0)] E[g(Z_1(w)) g(Z_2(w))] = \|f\|^2 \sum_{k=1}^{\infty} a_{2k}^2 (2k)! [\theta(w)]^{2k}.$$

Then using the Dominated Convergence Theorem we obtain

$$2\int_0^M (1-\varepsilon w) E\{f(Y_0^\varepsilon) g(\dot{Y}_0^\varepsilon) f(Y_{\varepsilon w}^\varepsilon) g(\dot{Y}_{\varepsilon w}^\varepsilon)\} dw \to 2\|f\|^2 \sum_{k=1}^{\infty} a_{2k}^2 (2k)! \int_0^M [\theta(w)]^{2k} dw.$$

Letting $\varepsilon \to 0$, and then $M \to \infty$, we obtain the result since the integral over $[\eta, 1]$ goes to 0. ∎

Part 2. *The second order moment of* S_2.

We shall prove now that $E(S_2^2) = O(\varepsilon^{4\alpha}) + o(\varepsilon)$ and for $\alpha < 1/4$ that $E(S_2^2)\varepsilon^{-4\alpha}$ converges to (6) as $\varepsilon \to 0$. Remember that from (7) we have

$$S_2 = \sum_{m=0}^{\infty} c_m \int_0^1 (H_m(Y_s^\varepsilon) - H_m(X_s)) ds$$

and using again Mehler's formula (3) we get

$$\text{(19)} \quad E(S_2^2) = 2\sum_{m=1}^{\infty} m! c_m^2 \int_0^1 (1-v) \left\{ \left[\frac{1}{\sigma_\varepsilon^2} \int_{-2}^{2} \psi(u) r(|v-\varepsilon u|) du \right]^m - 2\left[\frac{1}{\sigma_\varepsilon} \int_{-1}^{1} \varphi(u) r(|v-\varepsilon u|) du \right]^m + r^m(v) \right\} dv.$$

We divide the domain of integration for v into three intervals: $[0, M\varepsilon]$, $[M\varepsilon, \eta]$, and $[\eta, 1]$, where ε, M, and η will be chosen later and satisfy $M\varepsilon < \eta$. Making $v = \varepsilon w$, we obtain the first expression in the form

$$(20)\quad 2\varepsilon \sum_{m=1}^{\infty} m!\, c_m^2 \int_0^M (1-\varepsilon w)\left\{\left[\frac{1}{\sigma_\varepsilon^2}\left(1 - C\varepsilon^{2\alpha} \int_{-2}^{2} \psi(u)\, |w-u|^{2\alpha} L(\varepsilon(w-u))\, du\right)\right]^m \right.$$
$$\left. -\frac{2}{\sigma_\varepsilon^m}\left[1 - C\varepsilon^{2\alpha} \int_{-1}^{1} \varphi(u)\, |w-u|^{2\alpha} L(\varepsilon(w-u))\right]^m + \left[1 - C\varepsilon^{2\alpha} |w|^{2\alpha} L(\varepsilon w)\right]^m\right\} dw$$
$$= o(\varepsilon).$$

LEMMA 9. *We have*

$$(21)\quad \sum_{m=1}^{\infty} m!\, c_m^2 \int_\eta^1 (1-v)\left\{\left[\frac{1}{\sigma_\varepsilon^2} \int_{-\infty}^{\infty} \psi(u)\, r(|v-\varepsilon u|)\, du\right]^m \right.$$
$$\left. -2\left[\frac{1}{\sigma_\varepsilon} \int_{-1}^{1} \varphi(u)\, r(|v-\varepsilon u|)\, du\right]^m + r^m(v)\right\} dv = O(\varepsilon^2) + O(\varepsilon^{4\alpha}).$$

Proof. For $v \in [\eta, 1]$, since $\varepsilon < \eta$ and $u \in [-1, 1]$, $\varepsilon u < \eta \leqslant v$ and

$$r(|v-\varepsilon u|) = r(v) - \varepsilon u \dot{r}(v) + \frac{\varepsilon^2 u^2}{2} \ddot{r}(\theta_\varepsilon) \quad \text{with } v - \varepsilon u < \theta_\varepsilon < v.$$

Also

$$\int_{-1}^{1} u\varphi(u)\, du = \int_{-2}^{2} u\psi(u)\, du = 0,$$

and the left-hand side of (21) is

$$(22)\quad \sum_{m=1}^{\infty} m!\, c_m^2 \int_\eta^1 (1-v)\left\{\left[\frac{1}{\sigma_\varepsilon^2}\left(r(v) + \frac{\varepsilon^2}{2} \int_{-2}^{2} \psi(u)\, u^2 \ddot{r}(\theta_\varepsilon)\, du\right)\right]^m \right.$$
$$\left. -2\left[\frac{1}{\sigma_\varepsilon}\left(r(v) + \frac{\varepsilon^2}{2} \int_{-1}^{1} \varphi(u)\, u^2 \ddot{r}(\theta_\varepsilon)\, du\right)\right]^m + r^m(v)\right\} dv.$$

We develop this expression using the Binomial Theorem. The first term is:

$$(23)\quad \sum_{m=1}^{\infty} m!\, c_m^2 \int_\eta^1 (1-v)\left(\frac{r(v)}{\sigma_\varepsilon^2}\right)^m (1-\sigma_\varepsilon^m)^2\, dv.$$

Since our domain of integration is $[\eta, 1]$, there exists $\hat{\varrho} < 1$ such that, for ε small enough, $|r(v)/\sigma_\varepsilon^2| \leqslant \hat{\varrho} < 1$ for any $v \in [\eta, 1]$ and $\sum_{m=1}^{\infty} m!\, m^2 c_m^2 \hat{\varrho}^m < \infty$ because $f \in L^2(\phi(x)\, dx)$. Moreover,

$$(24)\quad \frac{(1-\sigma_\varepsilon^m)^2}{\varepsilon^{4\alpha}} \to C^2 \frac{m^2}{4}\left[\int_{-2}^{2} \psi(u)\, |u|^{2\alpha}\, du\right]^2 \quad \text{and} \quad \frac{(1-\sigma_\varepsilon^m)^2}{\varepsilon^{4\alpha}} \leqslant \text{Const}\, m^2$$

and (23) is equal to $O(\varepsilon^{4\alpha})$. The rest of the terms in the binomial expansion of (22) are

$$\sum_{m=1}^{\infty} m!\,c_m^2 \int_{\eta}^{1} (1-v) \sum_{k=1}^{m} \binom{m}{k} \frac{1}{2^k} \frac{\varepsilon^{2k}}{\sigma_\varepsilon^{2m}} \Big[\int_{-2}^{2} \psi(u) u^2 \ddot{r}(\theta_\varepsilon)\, du\Big]^k (r(v))^{m-k}\, dv$$

$$-2\sum_{m=1}^{\infty} m!\,c_m^2 \int_{\eta}^{1} (1-v) \sum_{k=1}^{m} \binom{m}{k} \frac{1}{2^k} \frac{\varepsilon^{2k}}{\sigma_\varepsilon^{m}} \Big[\int_{-1}^{1} \varphi(u) u^2 \ddot{r}(\theta_\varepsilon)\, du\Big]^k (r(v))^{m-k}\, dv$$

$$= \sum_{m=1}^{\infty} m!\,c_m^2 \frac{1}{\sigma_\varepsilon^{2m}} \int_{\eta}^{1} (1-v) \sum_{k=1}^{m} \binom{m}{k} \frac{\varepsilon^{2k}}{2^k} (r(v))^{m-k}$$

$$\times \Big\{\Big[\int_{-2}^{2} \psi(u) u^2 \ddot{r}(\theta_\varepsilon)\, du\Big]^k - 2\sigma_\varepsilon^m \Big[\int_{-1}^{1} \varphi(u) u^2 \ddot{r}(\theta_\varepsilon)\, du\Big]^k\Big\}\, dv.$$

Defining $\|\ddot{r}\|_{\eta,1} = \sup_{\eta \leqslant v \leqslant 1} |\ddot{r}(v)|$, we see that the absolute value of this expression is bounded by

$$\text{Const} \sum_{m=1}^{\infty} m!\,c_m^2 \int_{\eta}^{1} (1-v) \sum_{k=1}^{m} \binom{m}{k} \frac{\varepsilon^{2k}}{\sigma_\varepsilon^{2m}} |r(v)|^{m-k} \|\ddot{r}\|_{\eta,1}^k\, dv.$$

For $k \geqslant 2$ and using

$$\binom{m}{l+1} \leqslant m \binom{m-1}{l}$$

we get

$$\sum_{m=1}^{\infty} m!\,c_m^2 \int_{\eta}^{1} (1-v) \sum_{k=2}^{m} \binom{m}{k} \frac{\varepsilon^{2k}}{\sigma_\varepsilon^{2m}} |r(v)|^{m-k} \|\ddot{r}\|_{\eta,1}^k\, dv$$

$$= \varepsilon^2 \sum_{m=1}^{\infty} m!\,c_m^2 \int_{\eta}^{1} (1-v) \sum_{l=1}^{m-1} \binom{m}{l+1} \frac{\varepsilon^{2l}}{\sigma_\varepsilon^{2m}} |r(v)|^{m-l-1} \|\ddot{r}\|_{\eta,1}^{l+1}\, dv$$

$$\leqslant \left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^2 \|\ddot{r}\|_{\eta,1} \sum_{m=1}^{\infty} c_m^2 m!\, m \int_{\eta}^{1} (1-v) \sum_{l=1}^{m-1} \binom{m-1}{l} \left|\frac{r(v)}{\sigma_\varepsilon^2}\right|^{m-l-1} \left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^{2l} \|\ddot{r}\|_{\eta,1}^{l}\, dv$$

$$\leqslant \left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^2 \|\ddot{r}\|_{\eta,1} \sum_{m=1}^{\infty} c_m^2 m!\, m \int_{\eta}^{1} (1-v) \left[\left|\frac{r(v)}{\sigma_\varepsilon^2}\right| + \left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^2 \|\ddot{r}\|_{\eta,1}\right]^{m-1} dv = O(\varepsilon^2)$$

since we have seen that $|r(v)/\sigma_\varepsilon^2| \leqslant \hat{\varrho} < 1$, and for ε small enough the term inside the square brackets is less than ϱ^*, where $0 \leqslant \varrho^* < 1$ and $\sum_{m=1}^{\infty} c_m^2 mm!\, (\varrho^*)^{m-1} < \infty$. A similar argument shows that the term for $k = 1$ is also $O(\varepsilon^2)$. ∎

We still have to consider the part of (19) corresponding to the integral over $[M\varepsilon, \eta]$.

LEMMA 10. *We have*

$$(25)\quad \sum_{m=1}^{\infty} m!\,c_m^2 \int_{M\varepsilon}^{\eta} (1-v)\left\{\left[\frac{1}{\sigma_\varepsilon^2}\int_{-2}^{2} \psi(u)\,r(|v-\varepsilon u|)\,du\right]^m - 2\left[\frac{1}{\sigma_\varepsilon}\int_{-1}^{1} \varphi(u)\,r(|v-\varepsilon u|)\,du\right]^m + r^m(v)\right\} dv = O(\varepsilon^{4\alpha}) + o(\varepsilon).$$

Proof. Using again the binomial expansion and (24) we infer that the first term is

$$(26)\quad \sum_{m=1}^{\infty} m!\,c_m^2 (1-\sigma_\varepsilon^m)^2 \int_{M\varepsilon}^{\eta} (1-v)\left(\frac{r(v)}{\sigma_\varepsilon^2}\right)^m dv \leqslant \text{Const} \sum_{m=1}^{\infty} m!\,c_m^2\, m^2\, \varepsilon^{4\alpha} \int_{M\varepsilon}^{\eta} \left(\frac{r(v)}{\sigma_\varepsilon^2}\right)^m dv.$$

Let us prove that for η fixed and ε small enough, $r(v) < \sigma_\varepsilon^2$ for $v \in [M\varepsilon, \eta]$. By (H1) this is the same as

$$(27)\quad L(v)\,v^{2\alpha} > \varepsilon^{2\alpha} \int_{-\infty}^{\infty} \psi(z)\,|z|^{2\alpha} L(\varepsilon z)\,dz.$$

Since L is continuous, for $v \in [M\varepsilon, \eta]$ and some constant C_1, $L(v)\,v^{2\alpha} \geqslant C_1 M^{2\alpha} \varepsilon^{2\alpha}$ and (27) is satisfied as long as

$$M^{2\alpha} > \frac{4}{CC_1} \sup_{[0,1]} L(v)\,K_\alpha.$$

By (H3), given the relation $f'' \in L^2(\phi(x)\,dx)$, the expression (26) is equal to $O(\varepsilon^{4\alpha})$. The terms corresponding to $1 \leqslant k \leqslant m$ in the binomial expansion are bounded in absolute value by

$$\sum_{m=1}^{\infty} c_m^2\, m!\, \frac{1}{\sigma_\varepsilon^{2m}} \int_{M\varepsilon}^{\eta} \sum_{k=1}^{m} \left(\frac{\varepsilon^2}{2}\right)^k \binom{m}{k} (r(v))^{m-k} \times \{[\int_{-2}^{2} \psi(u)\,u^2\,|\ddot{r}(\theta_\varepsilon)|\,du]^k - 2\sigma_\varepsilon^m [\int_{-1}^{1} \varphi(u)\,u^2\,|\ddot{r}(\theta_\varepsilon)|\,du]^k\}\,dv$$

$$\leqslant \text{Const} \sum_{m=1}^{\infty} c_m^2\, m!\, \frac{1}{\sigma_\varepsilon^{2m}} \int_{M\varepsilon}^{\eta} \sum_{k=1}^{m} \left(\frac{\varepsilon^2}{2}\right)^k \binom{m}{k} (r(v))^{m-k} (v^{2\alpha-2})^k\,dv.$$

The term for $k = 1$ is bounded by

$$(28)\quad \text{Const}\left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^2 \sum_{m=1}^{\infty} c_m^2\, m!\, m \int_{M\varepsilon}^{\eta} \left(\frac{r(v)}{\sigma_\varepsilon^2}\right)^{m-1} \frac{1}{v^{2-2\alpha}}\,dv.$$

The expression corresponding to $k \geqslant 2$ can be bounded by

$$(29)\quad \mathrm{Const}\left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^2 \sum_{m=1}^{\infty} c_m^2 m!\, m \int_{M\varepsilon}^{\eta} \sum_{l=1}^{m-1} \binom{m-1}{l} \left(\frac{r(v)}{\sigma_\varepsilon^2}\right)^{m-l-1} \left(\frac{\varepsilon^2}{\sigma_\varepsilon^2 v^{2-2\alpha}}\right)^{l} \frac{1}{v^{2-2\alpha}}\, dv$$

$$\leqslant \mathrm{Const}\left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^2 \sum_{m=1}^{\infty} c_m^2 m!\, m \int_{M\varepsilon}^{\eta} \left(\frac{r(v)}{\sigma_\varepsilon^2} + \frac{\varepsilon^2}{\sigma_\varepsilon^2 v^{2-2\alpha}}\right)^{m-1} \frac{1}{v^{2-2\alpha}}\, dv.$$

Taking M large enough, for $v \in [M\varepsilon, \eta]$ we have

$$\frac{r(v)}{\sigma_\varepsilon^2} + \frac{\varepsilon^2}{\sigma_\varepsilon^2 v^{2-2\alpha}} < 1,$$

and since by (H3) the series $\sum c_m^2 m!\, m$ is convergent, it is easy to see that (28) and (29) are $o(\varepsilon)$. Using (20), (21) and (25) we finally obtain $E(S_2^2) = O(\varepsilon^{4\alpha}) + o(\varepsilon)$. ∎

Finally, if $\alpha < 1/4$, we can now see using (24) that

$$\lim_{\varepsilon\to 0} E[\varepsilon^{-4\alpha} S_2^2] = \lim_{\varepsilon\to 0} E(T_2^2) = \lim_{\varepsilon\to 0} \frac{2}{\varepsilon^{4\alpha}} \sum_{m=1}^{\infty} m!\, c_m^2 (1-\sigma_\varepsilon^m)^2 \int_{M\varepsilon}^{1} (1-v) \frac{r^m(v)}{\sigma_\varepsilon^{2m}}\, dv$$

$$= 2K_\alpha^2 \sum_{m=1}^{\infty} m^2 m!\, c_m^2 \int_0^1 (1-v)\, r^m(v)\, dv.$$

This series is convergent by (H3) and this concludes the proof of Theorem 1. ∎

Proof of Theorem 2. Recall the decomposition of $Z_\varepsilon(f)$ given by (2):

$$Z_\varepsilon(f) = \frac{1}{\varepsilon^{a(\alpha)}} \int_0^1 f(Y_s^\varepsilon)\, g(\dot{Y}_s^\varepsilon)\, ds + \frac{1}{\varepsilon^{a(\alpha)}} \int_0^1 [f(Y_s^\varepsilon) - f(X_s)]\, ds \equiv T_1 + T_2.$$

We divide the proof into three cases according to the value of α.

Case 1. $0 < \alpha < 1/4$.

In this case $a(\alpha) = 2\alpha$ and the important term in the development of $Z_\varepsilon(f)$ is T_2. By Theorem 1 we see that if f satisfies (H3), $f(x) = \sum_{n=0}^{\infty} c_n H_n(x)$, and then

$$\sum_{n=1}^{\infty} c_n^2 n!\, n^2 \int_0^1 (1-v)\, r^n(v)\, dv < \infty,$$

and

$$E(T_2^2) \to 2K_\alpha^2 \sum_{n=1}^{\infty} c_n^2 n!\, n^2 \int_0^1 (1-v)\, r^n(v)\, dv \quad \text{as } \varepsilon \to 0.$$

This result shows that it is enough to prove the convergence for a finite linear combination of Hermite polynomials. Furthermore, the limits for polynomials with different degrees are orthogonal, and this implies that it is sufficient to consider the limit for only one polynomial, i.e. $f = H_n$. Thus, we want to show that

$$U_2 \equiv \varepsilon^{-2\alpha} \int_0^1 [H_n(Y_s^\varepsilon) - H_n(X_s)]\, ds \to K_\alpha n \int_0^1 H_n(X_s)\, ds$$

in $L^2(\Omega)$ as $\varepsilon \to 0$. We know that

$$E(U_2^2) \to 2K_\alpha^2 n^2 n! \int_0^1 (1-v) r^n(v)\, dv \qquad \text{as } \varepsilon \to 0,$$

and the covariance term is

$$2K_\alpha E\Big(U_2 \int_0^1 n H_n(X_s)\, ds\Big)$$
$$= (2n) 2K_\alpha \int_0^1 (1-v) \varepsilon^{-2\alpha} E\big[\big(H_n(Y_0^\varepsilon) - H_n(X_0)\big) H_n(X_v)\big]\, dv.$$

Following the same argument as in the proof of Theorem 1, it is easy to show that the last term converges, as $\varepsilon \to 0$, to

$$2K_\alpha^2 (2n^2) n! \int_0^1 (1-v) r^n(v)\, dv.$$

Hence, if f is a function satisfying (H3), we have

$$T_2 \to K_\alpha \sum_{n=1}^{\infty} c_n n \int_0^1 H_n(X_s)\, ds = K_\alpha \int_{-\infty}^{\infty} Hf(x) L_X(x)\, dx$$

in $L^2(\Omega)$ as $\varepsilon \to 0$.

Case 2. $1/4 < \alpha < 3/4$.

The proof will proceed in several steps. In (a) we prove that

$$S_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \int_0^t g(\dot{Y}_s^\varepsilon)\, ds$$

converges, in the sense of weak convergence of the finite-dimensional distributions, to a Brownian motion W. In (b) we show that $X(t)$ and $W(t)$ are asymptotically independent. Finally, in (c) we prove the convergence of T_1.

(a) *Convergence to a Brownian motion.*

We will study the convergence of S_t^ε. We consider first its second moment. Using Mehler's formula (3) we obtain

$$E(S_t^\varepsilon)^2 = \frac{1}{\varepsilon} \int_0^t \int_0^t \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} a_{2l} a_{2k} E\big(H_{2l}(\dot{Y}_s^\varepsilon) H_{2k}(\dot{Y}_{s'}^\varepsilon)\big)\, ds\, ds'$$
$$= \frac{2}{\varepsilon} \int_0^t (t-w) \sum_{l=1}^{\infty} a_{2l}^2 (2l)!\, \sigma_{2l,\varepsilon}^2(w)\, dw$$

with

$$\sigma_{2l,\varepsilon}^2(w) = \left[-\frac{\varepsilon^{-2}}{\dot{\sigma}_\varepsilon^2} \int_{-2}^{2} \ddot{\psi}(z)\, r(w-\varepsilon z)\, dz \right]^{2l}.$$

We split the domain of integration into two parts: $[0, \eta]$ and $[\eta, t]$. In the first integral, making $w = \varepsilon v$, we have

$$2 \int_0^{\eta/\varepsilon} (t-\varepsilon v) \sum_{l=1}^{\infty} a_{2l}^2 (2l)!\, \sigma_{2l,\varepsilon}^2(\varepsilon v)\, dv.$$

For the second integral, since $\sum_{l=1}^{\infty} a_{2l}^2(2l)! < \infty$, we infer that it is $O(\varepsilon^{3-4\alpha})$. Furthermore, the convergence of the last series, and the fact that $\sigma_{2l,\varepsilon}^2(\varepsilon v) < \mathrm{Const}\,(v-1)^{2(2\alpha-2)}$ for v large enough and $|\sigma_{2l,\varepsilon}^2(\varepsilon v)| < 1$ otherwise allow us to prove that

$$E(S_t^\varepsilon)^2 \to 2t \int_0^{\infty} \sum_{l=1}^{\infty} a_{2l}^2 (2l)!\, \sigma_{2l}^2(v)\, dv, \tag{30}$$

where σ_{2l}^2 is defined in (ii) of Theorem 2.

Next, we will study the convergence of the moments of $\sum_{j=1}^{m} d_j (S_{t_j}^\varepsilon - S_{t_{j-1}}^\varepsilon)$, where the d_j are real constants and $0 = t_0 < t_1 < \ldots < t_m$, $m \in N$. We will use the method of Breuer–Major [8]. Define

$$g_M(x) = \sum_{l=1}^{M} a_{2l} H_{2l}(x) \quad \text{and} \quad g_M^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t g_M(\dot{Y}_s^\varepsilon)\, ds.$$

With the same argument used to prove (30) we see that

$$E\big(S_t^\varepsilon - g_M^\varepsilon(t)\big)^2 \leqslant 2t \sum_{l=M+1}^{\infty} \int_0^{\infty} a_{2l}^2 (2l)!\, \sigma_{2l}^2(v)\, dv + O(\varepsilon^{(M+1)(2-2\alpha)-1}),$$

and then

$$\lim_{M\to\infty} \lim_{\varepsilon\to 0} E\big(S_t^\varepsilon - g_M^\varepsilon(t)\big)^2 = 0. \tag{31}$$

Hence to study the weak convergence of S_t^ε, it is enough to consider that of $g_M^\varepsilon(t)$. Let us look at

$$E\Big(\sum_{j=1}^{m} d_j \big(g_M^\varepsilon(t_j) - g_M^\varepsilon(t_{j-1})\big)\Big)^p$$

$$= (1/\varepsilon)^{p/2} \sum_{l_1,\ldots,l_p=1,M} a_{2l_1} \ldots a_{2l_p} \sum_{j_1,\ldots,j_p=1,m} d_{j_1} \ldots d_{j_p}$$

$$\times \int_{t_{j_1-1}}^{t_{j_1}} \ldots \int_{t_{j_p-1}}^{t_{j_p}} E\{H_{2l_1}(\dot{Y}_{s_1}^\varepsilon) \ldots H_{2l_p}(\dot{Y}_{s_p}^\varepsilon)\}\, ds_1 \ldots ds_p.$$

Let $(l_1, \ldots, l_p)$ be fixed; we are interested in

$$(1/\varepsilon)^{p/2} \sum_{j_1,\ldots,j_p=1,m} d_{j_1} \ldots d_{j_p} \int_{t_{j_1-1}}^{t_{j_1}} \int_{t_{j_p-1}}^{t_{j_p}} E\{H_{2l_1}(\dot{Y}^{\varepsilon}_{s_1}) \ldots H_{2l_p}(\dot{Y}^{\varepsilon}_{s_p})\}\, ds_1 \ldots ds_p.$$

To calculate the expectation we use the diagram formula (see [8], pp. 431 and 432):

$$E\Big(\prod_{i=1}^{p} H_{2l_i}(\dot{Y}^{\varepsilon}_{s_i})\Big) = \sum_{G\in\Gamma} I_G,$$

where G is an undirected graph with $2l_1 + \ldots + 2l_p$ edges and p levels (see [8], p. 431, for definitions), $\Gamma = \Gamma(l_1, l_2, \ldots, l_p)$ denotes the set of diagrams having these properties, and

$$I_G = \prod_{w\in G(V)} [-\ddot{\varrho}_{\varepsilon}(s_{d_1(w)} - s_{d_2(w)})],$$

where $G(V)$ denotes the set of edges of G; the edges w are oriented, beginning in $d_1(w)$ and finishing in $d_2(w)$.

The diagrams are called *regular* [8] (p. 432) if their levels can be matched in a way such that no edge passes between levels in different pairs, otherwise they are called *irregular*. Consider a regular diagram G^* and let i be the permutation such that

$$\big(i(1), i(2)\big), \ldots, \big(i(p-1), i(p)\big)$$

defines the diagram with $p = 2q$. The contribution of this diagram is

$$(1/\varepsilon)^{q} \sum_{j_1,\ldots,j_p=1,m} d_{j_1} \ldots d_{j_p} \prod_{k=1}^{q} \Big[\int_{t_{j_{i(2k-1)}-1}}^{t_{j_{i(2k-1)}}} \int_{t_{j_{i(2k)}-1}}^{t_{j_{i(2k)}}} [E(\dot{Y}^{\varepsilon}_u \dot{Y}^{\varepsilon}_v)]^{e(k)}\, du\, dv \Big],$$

where $e(k)$ is the number of edges linking $i(2k-1)$ with $i(2k)$. But

$$(1/\varepsilon) \int_{t_{j-1}}^{t_j} \int_{t_{j'-1}}^{t_{j'}} [E(\dot{Y}^{\varepsilon}_u \dot{Y}^{\varepsilon}_v)]^{2l}\, du\, dv \to 0$$

as $\varepsilon \to 0$ except when $j = j'$, and in this case

$$(1/\varepsilon) \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} [E(\dot{Y}^{\varepsilon}_u \dot{Y}^{\varepsilon}_v)]^{2l}\, du\, dv \to 2(t_j - t_{j-1}) \int_0^{\infty} \sigma^2_{2l}(v)\, dv,$$

so that

$$(1/\varepsilon)^{q} a_{2l_1} \ldots a_{2l_p} \sum_{j_1,\ldots,j_p=1,m} d_{j_1} \ldots d_{j_p}$$

$$\times \prod_{k=1}^{q} \Big[\int_{t_{j_{i(2k-1)}-1}}^{t_{j_{i(2k-1)}}} \int_{t_{j_{i(2k)}-1}}^{t_{j_{i(2k)}}} [E(\dot{Y}^{\varepsilon}_u \dot{Y}^{\varepsilon}_v)]^{e(k)}\, deu\, dv \Big]$$

$$\to \prod_{k=1}^{q} a^2_{e(k)}\, 2^{q} \Big(\sum_{j=1}^{m} d_j^2 (t_j - t_{j-1}) \Big)^{q} \int_0^{\infty} \sigma^2_{e(k)}(v)\, dv.$$

We show in Lemma 11 that the contribution of the irregular diagrams tends to zero. Using now the same argument of Breuer and Major [8] (p. 434) we infer, as $\varepsilon \to 0$, that

$$E\Big(\sum_{j=1}^{m} d_j\big(g_M^{\varepsilon}(t_j)-g_M^{\varepsilon}(t_{j-1})\big)\Big)^p$$
$$\to (2q)!!\Big(\sum_{j=1}^{m} d_j^2(t_j-t_{j-1})\Big)^q\Big(2\sum_{l=1}^{M}(2l)!\,a_{2l}^2\int_0^{\infty}\sigma_{2l}^2(v)\,dv\Big)^q.$$

Remarks. (i) The random variables $\big(g_M^{\varepsilon}(t_j)-g_M^{\varepsilon}(t_{j-1})\big)$ and $\big(g_M^{\varepsilon}(t_{j'})-g_M^{\varepsilon}(t_{j'-1})\big)$ are asymptotically independent for $[t_j, t_{j-1}]\cap[t_{j'}, t_{j'-1}] = \emptyset$.

(ii) In a similar way we can show the asymptotic independence of

$$\frac{1}{\sqrt{\varepsilon}}\int_0^t H_{2k}(\dot{Y}_s^{\varepsilon})\,ds \quad \text{and} \quad \frac{1}{\sqrt{\varepsilon}}\int_0^t H_{2l}(\dot{Y}_s^{\varepsilon})\,ds \quad \text{for } k \neq l.$$

We study now the contribution of the irregular graphs. As in Breuer and Major [8] we can suppose that $l_1 \leqslant l_2 \leqslant \ldots \leqslant l_p$.

LEMMA 11. *For fixed* $j_1, j_2, \ldots, j_p$ *the following expression goes to* 0 *as* $\varepsilon \to 0$:

$$(1/\varepsilon)^{p/2}\prod_{k=1}^{p}\int_{t_{j_k-1}}^{t_{j_k}}\prod_{w\in G(V)}|\ddot{\varrho}_{\varepsilon}(s_{d_1(w)}-s_{d_2(w)}|\,ds_k. \tag{32}$$

Proof. Let $K_G(i)$ be the number of edges such that $d_1(w) = i$; then the expression (32) is less than or equal to

$$(1/\varepsilon)^{p/2}\prod_{i=1}^{p}\int_{t_{j_i-1}}^{t_{j_i}}\frac{1}{K_G(i)}\sum_{w\in G(V)}\sum_{d_1(w)=i}|\ddot{\varrho}_{\varepsilon}(s_i-s_{d_2(w)})|^{K_G(i)}\,ds_i.$$

Using the same techniques as in the proposition in [8], p. 435, we see that (32) is less than or equal to

$$\prod_{i=1}^{p}\left[\frac{1}{\sqrt{\varepsilon}}\sup_{v\in[0,1]}\int_{t_{j_i-1}}^{t_{j_i}}|\ddot{\varrho}_{\varepsilon}(s_i-v)|^{K_G(i)}\,ds_i\right] \leqslant \frac{2^p}{\varepsilon^{p/2}}\prod_{i=1}^{p}\int_0^1|\ddot{\varrho}_{\varepsilon}(s)|^{K_G(i)}\,ds.$$

Define $g(i) = K_G(i)/2l_i$. We split the domain of integration into $[0, \eta]$ and $[\eta, 1]$. We have three cases to consider.

(i) $K_G(i) \geqslant 2$.

The first part is $O(\varepsilon)$ and the second is $O(\varepsilon^{(2-2\alpha)K_G(i)})$. Since $g(i) \leqslant 1$; $2(2-2\alpha) > 1$ and $K_G(i) \geqslant 2$, we have

$$\int_0^1|\ddot{\varrho}_{\varepsilon}(s)|^{K_G(i)}\,ds \leqslant \mathrm{Const}\,\varepsilon = \mathrm{Const}\,\varepsilon^{1-g(i)}\,\varepsilon^{g(i)} \leqslant \mathrm{Const}\,\varepsilon^{g(i)}.$$

(ii) $K_G(i) = 0$.
Now we have

$$\int_0^1 |\ddot{\varrho}_\varepsilon(s)|^{K_G(i)} ds = 1 = \varepsilon^0 = \varepsilon^{g(i)}.$$

(iii) $0 < g(i) \leqslant 1/2$.
For the second integral, in the same way we obtain $O(\varepsilon^{(2-2\alpha)})$. For the first one, making $s = \varepsilon w$, we have

$$\varepsilon \int_0^{\eta/\varepsilon} |\ddot{\varrho}_\varepsilon(\varepsilon w)|^{K_G(i)} dw = \varepsilon \int_0^{M} |\ddot{\varrho}_\varepsilon(\varepsilon w)|^{K_G(i)} dw + \varepsilon \int_M^{\eta/\varepsilon} |\ddot{\varrho}_\varepsilon(\varepsilon w)|^{K_G(i)} dw.$$

The first term on the right-hand side is $O(\varepsilon)$, and for the second, using Hölder's inequality, we have

$$\int_M^{\eta/\varepsilon} |\ddot{\varrho}_\varepsilon(\varepsilon w)|^{K_G(i)} dw \leqslant \Big[\int_M^{\eta/\varepsilon} |\ddot{\varrho}_\varepsilon(\varepsilon w)|^{2l_i} dw\Big]^{g(i)} \Big[\int_M^{\eta/\varepsilon} 1 dw\Big]^{(1-g(i))}.$$

Since the first integral remains bounded when $\varepsilon \to 0$, because $2l_i \geqslant 2$, we can verify that

$$\int_M^{\eta/\varepsilon} |\ddot{\varrho}_\varepsilon(\varepsilon w)|^{K_G(i)} dw \leqslant \text{Const}\, \zeta^{g(i)} (1/\varepsilon)^{(1-g(i))}$$

with ζ small enough (the bound is obtained by making $M \to \infty$). Adding up we obtain

$$\int_0^1 |\ddot{\varrho}_\varepsilon(s)|^{K_G(i)} ds \leqslant \text{Const}\, [\varepsilon^{(2-2\alpha)} + \varepsilon + \zeta^{g(i)} \varepsilon^{g(i)}] \qquad \text{for } \varepsilon \leqslant \varepsilon(\zeta).$$

Since $g(i) \leqslant 1/2$ and $2-2\alpha > 1/2$, we have in fact

$$\int_0^1 |\ddot{\varrho}_\varepsilon(s)|^{K_G(i)} ds \leqslant \text{Const}\, \zeta^{g(i)} \varepsilon^{g(i)}.$$

Thus the expression (32) is less than or equal to

$$\text{Const}\, \varepsilon^{\Sigma_p}, \qquad \text{where } \Sigma_p = \sum_{i=1}^{p} g(i) - p/2.$$

Breuer and Major [8], p. 436, showed that either $\sum_{i=1}^{p} g(i) > p/2$ (and in this case (32) tends to 0 with ε) or there exists $1 \leqslant i_0 \leqslant p$ such that $0 < K_G(i_0) < 2l_{i_0}$, i.e. $0 < g(i_0) < 1$. Then the expression (32) is less than or equal to

$$\text{Const}\, \varepsilon^{\Sigma_p} [\zeta^{g(i_0)} + \varepsilon^{1-g(i_0)}],$$

and since $\Sigma_p = \sum_{i=1}^{p} g(i) - p/2 \geqslant 0$, the result follows. ∎

(b) *Asymptotic independence of $X(t)$ and $W(t)$.*
We want to prove that

$$A_M^\varepsilon(t) = \big(Y_{t_0}^\varepsilon = Y_0^\varepsilon, \ldots, Y_{t_m}^\varepsilon, g_M^\varepsilon(t_1), \ldots, g_M^\varepsilon(t_m) - g_M^\varepsilon(t_{m-1})\big)$$

converges weakly as $\varepsilon \to 0$ to

$$A_M(t) = \big(X_{t_0} = X_0, \ldots, X_{t_m}, \sigma'_M W(t_1), \ldots, \sigma'_M\big(W(t_m) - W(t_{m-1})\big)\big),$$

where $\sigma_M'^2 = 2\sum_{l=1}^M a_{2l}^2 (2l)! \int_0^\infty \sigma_{2l}^2(v)\,dv$; furthermore $(X_{t_1}, \ldots, X_{t_m})$ and $(W_{t_1}, \ldots, W_{t_m})$ are independent Gaussian vectors. We shall follow closely the arguments in Ho and Sun [12] with necessary modifications due to the fact that we are considering a non-ergodic situation.

Let $b_1, \ldots, b_m, d_1, \ldots, d_m$ be real constants; we are interested in the limit distribution of

$$\sum_{j=0}^m b_j Y_{t_j}^\varepsilon + \sum_{j=1}^m d_j [g_M^\varepsilon(t_j) - g_M^\varepsilon(t_{j-1})].$$

To simplify the notation we shall write

$$Z_\varepsilon(t) = \sum_{j=0}^m b_j Y_{t_j}^\varepsilon \quad \text{and} \quad W_\varepsilon(t) = \sum_{j=1}^m d_j [g_M^\varepsilon(t_j) - g_M^\varepsilon(t_{j-1})].$$

Then Z_ε is centred and Gaussian and

$$a_\varepsilon^2(t) \equiv \operatorname{Var}\big(Z_\varepsilon(t)\big) = \sum_{i,j=0,m} b_i b_j \varrho_\varepsilon(t_i - t_j).$$

The correlation between $Z_\varepsilon(t)$ and $\dot{Y}_s^\varepsilon$ is denoted by $\dot{v}_\varepsilon(s-t)$ and

$$\dot{v}_\varepsilon(s-t) = \sum_{j=0}^m b_j \dot{\varrho}_\varepsilon(s-t_j)/a_\varepsilon(t).$$

We normalize $Z_\varepsilon(t)$ defining $Z'_\varepsilon(t) = Z_\varepsilon(t)/a_\varepsilon(t)$. We have already studied the limit of

$$\begin{aligned} D_\varepsilon &= E\big(Z_\varepsilon^k(t)\, W_\varepsilon^r(t)\big) \\ &= [a_\varepsilon(t)]^k \sum_{l_1,\ldots,l_r=1,M} a_{2l_1} a_{2l_2} \ldots a_{2l_r} \sum_{j_1,\ldots,j_r=0,m} d_{j_1} d_{j_2} \ldots d_{j_r} \frac{1}{\varepsilon^{r/2}} \int_{t_{j_1-1}}^{t_{j_1}} \int_{t_{j_2-1}}^{t_{j_2}} \ldots \\ &\quad \ldots \int_{t_{j_r-1}}^{t_{j_r}} E\big[H_1^k(Z'_\varepsilon(t)) H_{2l_1}(\dot{Y}_{s_1}^\varepsilon) H_{2l_2}(\dot{Y}_{s_2}^\varepsilon) \ldots H_{2l_r}(\dot{Y}_{s_r}^\varepsilon)\big]\, ds_1\, ds_2 \ldots ds_r. \end{aligned}$$

We use again the diagram formula. In this case we have

$$E\big[H_1^k(Z'_\varepsilon(t)) H_{2l_1}(\dot{Y}_{s_1}^\varepsilon) H_{2l_2}(\dot{Y}_{s_2}^\varepsilon) \ldots H_{2l_r}(\dot{Y}_{s_r}^\varepsilon)\big] = \sum_{G\in\Gamma} \prod_{w\in G} \prod_{d_1(w)<d_2(w)} \hat{\varrho}\big(d_1(w) - d_2(w)\big).$$

The set $\Gamma = \Gamma(1, 1, \ldots, 1, 2l_1, \ldots, 2l_p)$ contains the diagrams such that the first k levels correspond to the Y_s variables. $\hat{\varrho}$ is defined as:

(i) $-\ddot{\varrho}_\varepsilon(s_i - s_j)$ if i and j are in the second level group;

(ii) $-\dot{v}_\varepsilon(t - s_j)$ if the edge w joins the first level with the second;

(iii) 1 in any other case.

We say that an *edge belongs to the first group* if it links two among the first k levels, and *to the second group* otherwise.

We shall classify the diagrams in $\Gamma(1, 1, \ldots, 1, 2l_1, 2l_2, \ldots, 2l_r)$ as in [12], p. 1166, denoting by R the set of the regular graphs, and by R^c the rest. We start with considering R.

In a regular graph, the levels are paired in such a way that it is not possible for a level in the first group to link with one of the second, yielding a factorization into two graphs, both regular. Since $k + 2l_1 + 2l_2 + \ldots + 2l_r = 2q$, k and r are both even. By part (a) of this proof the contribution of such graphs tends to

$$\Big(\sum_{i,j=0,m} b_i b_j r(t_i - t_j)\Big)^{k/2} k!!\, r!! \Big(\sum_{j=0}^{m} d_j^2 (t_j - t_{j-1})\Big)^{r/2} \Big(2 \sum_{l=1}^{M} (2l)!\, a_{2l}^2 \int_0^\infty \sigma_{2l}^2(v)\, dv\Big)^{r/2}.$$

Using the notation of [12], p. 1167, and denoting by D_ε/R^c the contribution of the irregular graphs in D_ε we obtain

$$D_\varepsilon/R^c = \sum_{G \in R^c} A_1^\varepsilon \times A_2^\varepsilon \times A_3^\varepsilon \times \varepsilon^{-r/2}.$$

Any diagram $G \in R^c$ can be partitioned into three disjoint subdiagrams: $V_{G,1}$, $V_{G,2}$ and $V_{G,3}$, which are defined as follows. $V_{G,1}$ is the maximal subdiagram of G which is regular within itself and all its edges satisfy

$$1 \leqslant d_1(w) < d_2(w) \leqslant k \quad \text{or} \quad k+1 \leqslant d_1(w) < d_2(w) \leqslant k + r.$$

Define

$$V_{G,1}^*(1) = \{j \in V_{G,1}^* \colon 1 \leqslant j \leqslant k\},$$
$$V_{G,1}^*(2) = \{j \in V_{G,1}^* \colon k+1 \leqslant j \leqslant k + r\},$$

where $V_{G,1}^*$ are the levels of $V_{G,1}$.

A_i^ε is the factor of the product corresponding to the edges of $V_{G,i}$, $i = 1, 2, 3$. The normalization for A_1^ε is therefore $\varepsilon^{-|V_{G,1}^*(2)|/2}$ and, as shown in part (a), $\varepsilon^{-|V_{G,1}^*(2)|/2} A_1^\varepsilon$ tends to

$$\Big(\sum_{i,j=0,m} b_i b_j r(t_i - t_j)\Big)^{|V_{G,1}^*(1)|/2} \frac{|V_{G,1}^*|}{2}\, q!! \Big(\sum_{j=0}^{m} d_j^2 (t_j - t_{j-1})\Big)^q$$
$$\times \Big(2 \sum_{l=1}^{M} a_{2l}^2 (2l)! \int_0^\infty \sigma_{2l}^2(v)\, dv\Big)^q$$

as $\varepsilon \to 0$, where $q = |V_{G,1}^*(2)|/2$. The limit is then $O(1)$.

Consider now A_2^ε and define $V_{G,2}$ to be the maximal subdiagram of $G-V_{G,1}$, whose edges satisfy $k+1 \leqslant d_1(w) < d_2(w) \leqslant l+r$. The normalization for A_2^ε is $\varepsilon^{-|V^*_{G,2}(2)|/2}$, where $V^*_{G,2}(2)$ are the levels of $V_{G,2}$. A graph in $V_{G,2}$ is necessarily irregular, if not it would have been taken into account in A_1^ε. As in part (a), $\varepsilon^{-|V^*_{G,2}(2)|/2} A_2^\varepsilon$ tends to zero as $\varepsilon \to 0$.

For A_3^ε define

$$V_{G,3} = G-(V_{G,1} \cup V_{G,2}),$$

$$V^*_{G,3}(1) = \{j \in V^*_{G,3}:\ 1 \leqslant j \leqslant k\}, \qquad V^*_{G,3}(2) = \{j \in V^*_{G,3}:\ k+1 \leqslant j \leqslant k+r\},$$

where $V^*_{G,3}$ are the levels of $V_{G,3}$. We need a uniform bound in t for $|\dot{\varrho}_\varepsilon(t)|$:

$$\dot{\varrho}_\varepsilon(t) = \frac{1}{\dot{\sigma}_\varepsilon}\Big[\int_{-\infty}^{\infty} e^{i\lambda t} |\hat{\varphi}(\varepsilon\lambda)|^2\, i\lambda\, h(\lambda)\, d\lambda\Big],$$

where h is the spectral density of X,

$$\dot{\sigma}_\varepsilon^2 = \varepsilon^{2\alpha-2} \int_{-\infty}^{\infty} \dot{\psi}(u)\, |u|^{2\alpha} L(\varepsilon u)\, du \sim \varepsilon^{2\alpha-2} \chi^2,$$

$$\dot{\varrho}_\varepsilon(t) \sim \frac{2}{\chi} \varepsilon^{1-\alpha} \int_0^{\infty} \lambda \sin(\lambda t)\, |\hat{\varphi}(\varepsilon\lambda)|^2\, h(\lambda)\, d\lambda.$$

If $\alpha \geqslant 1/2$, $\int_0^\infty \lambda h(\lambda)\, d\lambda < \infty$ because $\limsup_{\lambda\to\infty} \lambda^{2\alpha+1} h(\lambda) = \mathrm{Const}$ by (H1), and then

$$|\dot{\varrho}_\varepsilon(t)| \leqslant \mathrm{Const}\, \varepsilon^{1-\alpha}.$$

If $\alpha < 1/2$, splitting the domain of integration into $[0, N]$ and $[N, +\infty]$ we get

$$|\dot{\varrho}_\varepsilon(t)| \leqslant \mathrm{Const}\,[\varepsilon^{1-\alpha} + \varepsilon^\alpha] \leqslant \mathrm{Const}\, \varepsilon^\alpha.$$

Adding up, we obtain

$$|\dot{\varrho}_\varepsilon(t)| \leqslant \mathrm{Const}\, \varepsilon^{\tilde{\beta}}, \qquad \text{where } \tilde{\beta} = \inf\{\alpha, (1-\alpha)\}. \tag{33}$$

We assume now that $l_1, l_2, \ldots, l_p$ are fixed by the graph. Let $L = |V^*_{G,3}(2)|$; then

$$\varepsilon^{-|V^*_{G,3}(2)|/2} A_3^\varepsilon$$
$$= \varepsilon^{-|V^*_{G,3}(2)|/2} \sum_{j_{\xi(1)},\ldots,j_{\xi(L)}=0,m} \prod_{i=1}^{L} c_{j_{\xi(i)}} \int_{t_{j_{\xi(i)}}-1}^{t_{j_{\xi(i)}}} \prod_{e\in E(V_{G,3})} \prod_{d_1(e)\in V^*_{G,3}(1)} \dot{v}_\varepsilon(t-s_{d_2(e)})$$
$$\times \prod_{w\in E(V_{G,3})} \prod_{d_1(w)\in V^*_{G,3}(2)} [-\ddot{\varrho}_\varepsilon(s_{d_1(w)} - s_{d_2(w)})]\, ds_{\xi(i)},$$

where $E(V_{G,3})$ are the edges of $V_{G,3}$. Using (33) we get

$$\varepsilon^{-|V^*_{G,3}(2)|/2}|A_3^\varepsilon| \leqslant \mathrm{Const}\,\varepsilon^{\beta|V^*_{G,3}(1)|}\varepsilon^{-|V^*_{G,3}(2)|/2}\sum_{j_{\xi(1)},\ldots,j_{\xi(L)}=0,m}\prod_{i=1}^{L}|c_{j_{\xi(i)}}|$$

$$\times\int_{t_{j_{\xi(i)}-1}}^{t_{j_{\xi(i)}}}\prod_{w\in E(V_{G,3})}\prod_{d_1(w)\in V^*_{G,3}(2)}|-\ddot{\varrho}_\varepsilon(s_{d_1(w)}-s_{d_2(w)})|\,ds_{\xi(i)};$$

$V^*_{G,3}$ can be decomposed into three parts:

$$A_G = \{i\in V^*_{G,3}(2)\colon\ g(i)=0\},\qquad B_G=\{i\in V^*_{G,3}(2)\colon\ g(i)=1\}$$

and

$$C_G = \{i\in V^*_{G,3}(2)\colon\ g(i)>1\},$$

where $g(i)$ is the number of edges in the i-th level not connected by edges to any of the first levels. As in [12], p. 1169, we can rearrange the levels in $V^*_{G,3}(2)$ in such a way that the levels of B_G are preceded by the levels of A_G and followed by the levels of C_G. Within C_G, the levels are also rearranged so that those with smaller $g(i)$ come first. Assume first $V^*_{G,3}(2)=C_G$; then

$$(34)\quad \prod_{i=1}^{L}\int_{t_{j_{\xi(i)}-1}}^{t_{j_{\xi(i)}}}\prod_{w\in E(V_{G,3})}\prod_{d_1(w)\in V^*_{G,3}(2)}|\ddot{\varrho}_\varepsilon(s_{d_1(w)}-s_{d_2(w)})|\,ds_{\xi(i)}$$

$$=\prod_{i=1}^{L}\int_{t_{j_{\xi(i)}-1}}^{t_{j_{\xi(i)}}}\prod_{w\in E(V_{G,3})}\prod_{d_1(w)=\xi(i)}|\ddot{\varrho}_\varepsilon(s_{\xi(i)}-s_{d_2(w)})|\,ds_{\xi(i)}.$$

As in part (a), Lemma 11, the expression (34) is less than or equal to $\mathrm{Const}\,\varepsilon^{\Sigma_L}$, where $\Sigma_L=\sum_{i=1}^{L}k(i)/g(i)$ and $k(i)$ replaces $K_G(i)$; then

$$\varepsilon^{-|V^*_{G,3}(2)|/2}|A_3^\varepsilon|\leqslant\mathrm{Const}\,\varepsilon^{\beta|V^*_{G,3}(1)|}\varepsilon^{\Sigma_{C_G}}\varepsilon^{-|C_G|/2}$$

and $\Sigma_{C_G}=\sum_{i\in C_G}k(i)/g(i)\geqslant\frac{1}{2}|C_G|$. Hence

$$\varepsilon^{-|V^*_{G,3}(2)|/2}|A_3^\varepsilon|\leqslant\mathrm{Const}\,\varepsilon^{\beta|V^*_{G,3}(1)|}.$$

If $V_{G,3}\neq\emptyset$, then $|V^*_{G,3}(1)|\neq 0$ and $\varepsilon^{-|V^*_{G,3}(2)|/2}|A_3^\varepsilon|$ tends to zero as $\varepsilon\to 0$. Furthermore, if $V_{G,3}=\emptyset$, then $V_{G,2}\neq\emptyset$ (otherwise it would have been taken into account before) and

$$\varepsilon^{-|V^*_{G,2}(2)|/2}A_2^\varepsilon\to 0\qquad\text{as }\varepsilon\to 0.$$

We suppose now that $V^*_{G,3}(2)\neq C_G$, i.e. either $A_G\neq\emptyset$ or $B_G\neq\emptyset$. We have

$$|V^*_{G,3}(2)|=|A_G|+|B_G|+|C_G|.$$

If $i\in A_G$, its contribution to A_3^ε is bounded by $\mathrm{Const}\,\varepsilon^{2\beta l_i}$, and in total we shall have $\mathrm{Const}\,\varepsilon^{2\beta\Sigma_{A_G}}$, where $\Sigma_{A_G}=\sum_{i\in A_G}l_i$. On the other hand, if $i\in B_G$, we have $(2l_i-1)$ edges coming from levels in the first group, and as $g(i)=1$, there are two possibilities: either $k(i) = 1$ or $k(i) = 0$. In the second case we shall have terms of the form: $\ddot{\varrho}_\varepsilon(d_1(w)-s_i)$, which are bounded by 1. The other edges are connected to levels in the first group and their contribution is $\varepsilon^{\tilde{\beta}(2l_i-1)}$. For the first case we have

$$\int_0^1 |\ddot{\varrho}_\varepsilon(s_i - s_{d_2(w)})|\, ds_i \leqslant \sup_{v\in[0,1]} \int_0^1 |\ddot{\varrho}_\varepsilon(s_i - v)|\, ds_i$$

$$\leqslant \mathrm{Const}\big([\varepsilon \int_0^\infty \sigma_2^2(w)\, dw]^{1/2} + O(\varepsilon^{2-2\alpha})\big) = O(\zeta\sqrt{\varepsilon})$$

because $\alpha < 3/4$ and ζ is small enough. The contribution from B_G is bounded by

$$\mathrm{Const}\,\varepsilon^{\tilde{\beta}\Sigma_{B_G}}(\zeta\sqrt{\varepsilon})^{|B_G\cap\{i:k(i)=1\}|}, \quad \text{where } \Sigma_{B_G} = \sum_{i\in B_G}(2l_i-1).$$

We can prove as before that the contribution from C_G is bounded by

$$\mathrm{Const}\,\varepsilon^{\Sigma}\varepsilon^{\Sigma_{C_G}}, \quad \text{where } \Sigma = \sum_{i\in C_G}\tilde{\beta}(2l_i - g(i)),\ \Sigma_{C_G} = \sum_{i\in C_G} k(i)/g(i).$$

Hence

$$\varepsilon^{-|V^*_{G,3}|/2} A_3^\varepsilon = O\big(\varepsilon^{-(|A_G|+|B_G|+|C_G|)/2}\,\varepsilon^{2\tilde{\beta}\Sigma_{A_G}}\,\varepsilon^{\tilde{\beta}\Sigma_{B_G}}(\sqrt{\varepsilon}\,\zeta)^{|B_G\cap\{i:k(i)=1\}|}\,\varepsilon^{\Sigma}\,\varepsilon^{\Sigma_{C_G}}\big),$$

where $\Sigma_{A_G} = \sum_{i\in A_G} l_i$, $\Sigma_{B_G} = \sum_{i\in B_G}(2l_i-1)$, $\Sigma = \Sigma_{i\in C_G}\tilde{\beta}(2l_i - g(i))$, $\Sigma_{C_G} = \sum_{i\in C_G} k(i)/g(i)$.

We have the following bounds:

(35) $$2\tilde{\beta}\sum_{i\in A_G} l_i - |A_G|/2 \geqslant 2\tilde{\beta}|A_G| - |A_G|/2 \geqslant 2|A_G|[\tilde{\beta} - 1/4] \geqslant 0 \ (\text{since } l_i \geqslant 1),$$

$$\sum_{i\in C_G}\tilde{\beta}(2l_i - g(i)) \geqslant 0, \quad \sum_{i\in C_G} k(i)/g(i) - \tfrac{1}{2}|C_G| \geqslant 0 \ (\text{cf. [8]}),$$

(36) $$\tilde{\beta}\sum_{i\in B_G}(2l_i-1) + \tfrac{1}{2}[|B_G\cap\{i:\ k(i)=1\}| - |B_G|/2]$$

$$\geqslant \tilde{\beta}|B_G| + \tfrac{1}{2}[|B_G\cap\{i:\ k(i)=1\}| - |B_G|/2].$$

Define

$$X = |B_G\cap\{i:\ k(i)=0\}| \quad \text{and} \quad Y = |B_G\cap\{i:\ k(i)=1\}|;$$

for $i\in X$ there exists $j\in Y$ such that $d_1(w) = j$, $d_2(w) = i$ (by the ordering of the levels in $V^*_{G,3}(2)$). It follows that $|Y| \geqslant |X|$, and then

$$\tilde{\beta}|B_G| + \tfrac{1}{2}|B_G\cap\{i:\ k(i)=1\}| - |B_G|/2 \geqslant 2(\tilde{\beta} - \tfrac{1}{4})|X| \geqslant 0.$$

Thus, if $\tilde{\beta} > 1/4$ and $|A_G| \neq 0$, the relation (35) implies the result. If $\tilde{\beta} > 1/4$ and $|A_G| = 0$, necessarily $|B_G| \neq 0$, which implies that $|X| \neq 0$ or $|Y| \neq 0$. If $|X| \neq 0$, the inequality (36) gives the result; otherwise $|X| = 0$ and $|Y| \neq 0$ the right-hand side of (36) is equal to $\tilde{\beta}|Y|$, and since $\tilde{\beta} \neq 0$, the result follows.

As in the unidimensional case (31), we can prove that the weak convergence of the vector $(Y^\varepsilon_{t_0}, \ldots, Y^\varepsilon_{t_m}, S^\varepsilon_{t_1}, \ldots, S^\varepsilon_{t_m} - S^\varepsilon_{t_{m-1}})$ is implied by that of $(Y^\varepsilon_{t_0}, \ldots, Y^\varepsilon_{t_m}, g^\varepsilon_M(t_1), \ldots, g^\varepsilon_M(t_m) - g^\varepsilon_M(t_{m-1}))$.

(c) *Convergence of* $T_1 = \varepsilon^{-1/2} \int_0^1 f(Y^\varepsilon_s) g(\dot{Y}^\varepsilon_s)\, ds$.

We consider a discrete version of T_1, defining

$$Z^n_\varepsilon(f) = \varepsilon^{-1/2} \sum_{i=1}^{n} f(Y^\varepsilon_{(i-1)/n}) \int_{(i-1)/n}^{i/n} g(\dot{Y}^\varepsilon_s)\, ds.$$

Let d be a metric for the weak convergence and define

$$Z^n(f) = \sigma \sum_{i=1}^{n} f(X_{(i-1)/n}) \left[W_{i/n} - W_{(i-1)/n}\right].$$

We know from the previous section that $Z^n_\varepsilon(f) \to Z^n(f)$, weakly as $\varepsilon \to 0$. On the other hand,

$$\|Z^n(f) - Z^{n+p}(f)\|_2 \to 0 \quad \text{as } n \to \infty$$

for every $p > 0$. Indeed, a straightforward calculation shows that

$$E\big(Z^n(f) - Z^{n+p}(f)\big)^2 = A_1 + A_2 + A_3,$$

where

$$A_1 = \sigma^2 \sum_{i=1}^{n+p} \sum_{j=1}^{n+p} E\left[f(X_{(i-1)/(n+p)}) f(X_{(j-1)/(n+p)})\right]$$

$$\times E\left[(W_{i/(n+p)} - W_{(i-1)/(n+p)})(W_{j/(n+p)} - W_{(j-1)/(n+p)})\right] = g(n+p) \to \sigma^2 \|f\|_2^2,$$

$$A_2 = g(n) \to \sigma^2 \|f\|_2^2,$$

$$A_3 = -2\sigma^2 \sum_{i=1}^{n+p} \sum_{j=1}^{n} E\left[f(X_{(i-1)/(n+p)}) f(X_{j/n})\right]$$

$$\times E\left[(W_{i/(n+p)} - W_{(i-1)/(n+p)})(W_{j/n} - W_{(j-1)/n})\right] \to -2\sigma^2 \|f\|_2^2$$

as $n \to \infty$. This implies that there exists an r.v. $Y \in L^2(\Omega)$ such that $Z^n(f) \to Y$ in $L^2(\Omega)$ as $n \to \infty$; furthermore, we can characterize this variable using the asymptotic independence between X and W and Jensen's inequality for conditional expectations

$$\mathscr{L}(Y/X_s, 0 \leqslant s \leqslant 1) = N\Big(0; \sigma^2 \int_0^1 f^2(X_s)\, ds\Big).$$

To show the convergence of T_1 we have to prove that $d(T_1, Y) \to 0$ as $\varepsilon \to 0$, and for this, using the triangle inequality, it is enough to prove

$$\lim_{n \to \infty} \lim_{\varepsilon \to 0} \|T_1 - Z^n_\varepsilon(f)\|_2 = 0. \tag{37}$$

We have, by (5), as $\varepsilon \to 0$:

$$E(T_1 - Z_\varepsilon^n(f))^2 = E(T_1)^2 + E(Z_\varepsilon^n(f))^2 - 2E(T_1 Z_\varepsilon^n(f)),$$
$$E(T_1)^2 \to \sigma^2 \|f\|_2^2.$$

LEMMA 12. *We have*

$$E(Z_\varepsilon^n(f))^2 \to \sigma^2 \|f\|_2^2 \quad \textit{as } \varepsilon \to 0.$$

Proof. It follows that

$$E(Z_\varepsilon^n(f))^2 = \sum_{i,j=1,n} E[f(Y_{(i-1)/n}^\varepsilon) f(Y_{(j-1)/n}^\varepsilon)(S_{i/n}^\varepsilon - S_{(i-1)/n}^\varepsilon)(S_{j/n}^\varepsilon - S_{(j-1)/n}^\varepsilon)].$$

If $i = j$, we obtain

$$E[f^2(Y_{(i-1)/n}^\varepsilon)(S_{i/n}^\varepsilon - S_{(i-1)/n}^\varepsilon)^2] \to n^{-1} \|f\|_2^2 \sigma^2 \quad \text{as } \varepsilon \to 0$$

because

$$(Y_{(i-1)/n}^\varepsilon, S_{i/n}^\varepsilon - S_{(i-1)/n}^\varepsilon) \to (X_{(i-1)/n}, \sigma(W_{i/n} - W_{(i-1)/n}))$$

weakly as $\varepsilon \to 0$. By Hölder's inequality,

$$E[|f^{2p}(Y_{(i-1)/n}^\varepsilon)(S_{i/n}^\varepsilon - S_{(i-1)/n}^\varepsilon)^{2p}|] \leqslant \|f^2\|_2^{2/p}(E|S_{i/n}^\varepsilon - S_{(i-1)/n}^\varepsilon|^{2pq})^{1/q}$$

with $p = \sqrt{2}$, $q = \sqrt{2}/(\sqrt{2}-1)$, and this is uniformly bounded by the results in part (a), for ε small enough. If $i \neq j$, a similar argument shows that this term tends to zero when $\varepsilon \to 0$. ∎

To complete the proof of (37), it is necessary to show that

(38) $$\lim_{n\to\infty} \lim_{\varepsilon\to 0} E(T_1 Z_\varepsilon^n(f)) = \sigma^2 \|f\|_2^2.$$

Define

$$f_m(x) = \sum_{l=1}^{m} c_l H_l(x), \qquad Z_m^{M,\varepsilon}(f) = \varepsilon^{-1/2} \int_0^1 f_m(Y_s^\varepsilon)\, g_M(\dot{Y}_s^\varepsilon)\, ds,$$

$$Z_{n,m}^{M,\varepsilon}(f) = \varepsilon^{-1/2} \sum_{i=1}^{m} f_m(Y_{(i-1)/n}^\varepsilon) \int_{(i-1)/n}^{i/n} g_M(\dot{Y}_s^\varepsilon)\, ds.$$

We have

(39) $$E|Z_\varepsilon^n(f) T_1 - Z_{n,m}^{M,\varepsilon}(f) Z_m^{M,\varepsilon}(f)|$$
$$\leqslant \big(E(Z_\varepsilon^n(f) - Z_{n,m}^{M,\varepsilon}(f))^2 E(T_1^2)\big)^{1/2} + \big(E(T_1 - Z_m^{M,\varepsilon}(f))^2 E(Z_{n,m}^{M,\varepsilon}(f))^2\big)^{1/2}.$$

Let us show that the right-hand side of (39) goes to 0 as $\varepsilon \to 0$, $n \to \infty$, $(M, m) \to \infty$ in this order. As we will show in Lemma 14 that

(40) $$\lim_{n\to\infty} \lim_{\varepsilon\to 0} E(Z_{n,m}^{M,\varepsilon}(f) Z_m^{M,\varepsilon}(f)) = \|f_m\|_2^2 \sigma_M'^2,$$

where $\sigma_M'^2 = \sum_{k=1}^{M} 2a_{2k}^2 (2k)! \int_0^\infty \sigma_{2k}^2(w)\, dw$, this will imply (38).

LEMMA 13. *The right-hand side of* (39) *goes to* 0 *as* $\varepsilon \to 0$, $n \to \infty$, $(M, m) \to \infty$ *in this order.*

Proof. To show that the right-hand side of (39) goes to zero it is enough to see that the both terms on the right-hand side go to zero. We know that as $\varepsilon \to 0$

$$E(T_1)^2 \to \sigma^2 \|f\|_2^2, \quad E\big(Z_{n,m}^{M,\varepsilon}(f)\big)^2 \to \|f_m\|_2^2 \sigma_M'^2,$$
$$E\big(Z_\varepsilon^n(f)\big)^2 \to \sigma^2 \|f\|_2^2, \quad E\big(Z_m^{M,\varepsilon}(f)\big)^2 \to \|f_m\|_2^2 \sigma_M'^2,$$
$$E\big(Z_m^\varepsilon(f) Z_{n,m}^{M,\varepsilon}(f)\big) \to E[f(X_0) f_m(X_0)] \sigma_M'^2.$$

This last result comes from the asymptotic independence between

$$\frac{1}{\sqrt{\varepsilon}} \int_0^t H_{2l}(\dot{Y}_s^\varepsilon) ds \quad \text{and} \quad \frac{1}{\sqrt{\varepsilon}} \int_0^t H_{2k}(\dot{Y}_s^\varepsilon) ds$$

(see remarks in section (a)), by using the technique employed for proving the convergence of $E\big(Z_\varepsilon^n(f)\big)^2$. To finish the proof that the right-hand side of (39) goes to zero it is enough to show that

$$E\big(T_1 Z_m^{M,\varepsilon}(f)\big) \to \|f_m\|_2^2 \sigma_M'^2 \quad \text{as } \varepsilon \to 0,$$

but

$$E\big(T_1 Z_m^{M,\varepsilon}(f)\big) = \frac{2}{\varepsilon} \int_0^1 (1-v) E\big(f(Y_0^\varepsilon) g(\dot{Y}_0^\varepsilon) f_m(Y_v^\varepsilon) g_M(\dot{Y}_v^\varepsilon)\big) dv.$$

By the Schwarz inequality,

$$\big|E\big(f(Y_0^\varepsilon) g(\dot{Y}_0^\varepsilon) f_m(Y_v^\varepsilon) g_M(\dot{Y}_v^\varepsilon)\big) - E\big(f_{m_1}(Y_0^\varepsilon) g(\dot{Y}_0^\varepsilon) f_m(Y_v^\varepsilon) g_M(\dot{Y}_v^\varepsilon)\big)\big| \leqslant b \|f - f_{m_1}\|_2,$$

where $b = \|f_m\|_4 \|g\|_8^2 < +\infty$. Furthermore,

$$(41) \quad E\big(f_{m_1}(Y_0^\varepsilon) g(\dot{Y}_0^\varepsilon) f_m(Y_v^\varepsilon) g_M(\dot{Y}_v^\varepsilon)\big) = \sum_{k_1=1}^{+\infty} a_{2k_1} E\big(f_{m_1}(Y_0^\varepsilon) H_{2k_1}(\dot{Y}_0^\varepsilon) f_m(Y_v^\varepsilon) g_M(\dot{Y}_v^\varepsilon)\big).$$

The last identity comes from the fact that

$$\sum_{k_1=1}^{+\infty} |a_{2k_1}| E[|f_{m_1}(Y_0^\varepsilon) H_{2k_1}(\dot{Y}_0^\varepsilon) f_m(Y_v^\varepsilon) g_M(\dot{Y}_v^\varepsilon)|]$$
$$\leqslant \sum_{k_1=1}^{+\infty} |a_{2k_1}| \sqrt{(2k_1)!} \|f_{m_1}\|_4 \|f_m\|_8 \|g_M\|_8 < +\infty$$

and the identity (41) is equal to

$$\sum_{k_1=1}^{+\infty} \sum_{l_1=0}^{m_1} \sum_{l_2=0}^{m} \sum_{k_2=1}^{M} c_{l_1} c_{l_2} a_{2k_1} a_{2k_2} E\big(H_{l_1}(Y_0^\varepsilon) H_{2k_1}(\dot{Y}_0^\varepsilon) H_{l_2}(Y_v^\varepsilon) H_{2k_2}(\dot{Y}_v^\varepsilon)\big).$$

The orthogonality properties imply that $2k_1 = l_2 + 2k_2 - l_1$. Hence $2k_1 \leqslant 2M+m$ and (41) does not depend on m_1, and so

$$E\big(T_1 Z_m^{M,\varepsilon}(f)\big) = \frac{2}{\varepsilon}\int_0^1 (1-v) \sum_{r=2}^{2M+m} \sum_{l_1=0}^{2M+m-2} \sum_{l_2=0, l_1+l_2 \text{ even}}^{m} c_{l_1} c_{l_2} a_{r-l_1} a_{r-l_2} \times E\{H_{l_1}(Y_0^\varepsilon) H_{r-l_1}(\dot{Y}_0^\varepsilon) H_{l_2}(Y_v^\varepsilon) H_{r-l_2}(\dot{Y}_v^\varepsilon)\}.$$

By an application of the diagram formula, we get

$$\lim_{\varepsilon \to 0} E\big(T_1 Z_m^{M,\varepsilon}(f)\big) = \|f_m\|_2^2 \sigma_M'^2. \ \blacksquare$$

LEMMA 14. *The equality* (40) *holds true.*

Proof. We have

$$E\big(Z_{n,m}^{M,\varepsilon}(f) Z_m^{M,\varepsilon}(f)\big) = \varepsilon^{-1} \sum_{i,j=1,n} \int_{(i-1)/n}^{i/n} \int_{(j-1)/n}^{j/n} E\{f_m(Y_{(i-1)/n}^\varepsilon) g_M(\dot{Y}_s^\varepsilon) f_m(Y_{s'}^\varepsilon) g_M(\dot{Y}_{s'}^\varepsilon)\} \, ds \, ds'.$$

We consider separately the terms corresponding to $i = j$ and $i \neq j$. For the diagonal terms we develop f_m and g_M in Hermite's basis. We have, by stationarity and a change of variable in the integral above, that each term of the multiple sum is equal to

$$\varepsilon^{-1} \int_0^{1/n} \int_0^{1/n} E\{H_{l_1}(Y_0^\varepsilon) H_{2k_1}(\dot{Y}_u^\varepsilon) H_{l_2}(Y_v^\varepsilon) H_{2k_2}(\dot{Y}_v^\varepsilon)\} \, du \, dv.$$

Using the diagram formula again we have to look at terms of the form

$$\varepsilon^{-1} \int_0^{1/n} \int_0^{1/n} \varrho_\varepsilon^{d_1}(v) \dot{\varrho}_\varepsilon^{d_2}(u) \dot{\varrho}_\varepsilon^{d_3}(u-v) \dot{\varrho}_\varepsilon^{d_4}(v) \big(-\ddot{\varrho}_\varepsilon(u-v)\big)^{d_5} \, du \, dv. \tag{42}$$

Four different cases must be considered:

1. $d_5 \geqslant 2$ and $d_2 + d_3 + d_4 \neq 0$. Now $|\dot{\varrho}_\varepsilon(\cdot)|$ is bounded by $\varepsilon^{\tilde{\beta}}$, and it follows that (42) is less than or equal to

$$\text{Const}\, n^{-1} \varepsilon^{\tilde{\beta}(d_2+d_3+d_4)} \int_0^\infty |\ddot{\varrho}_\varepsilon(\varepsilon x)|^{d_5} \, dx \to 0 \quad \text{as } \varepsilon \to 0.$$

2. $d_5 \geqslant 2$ and $d_2 + d_3 + d_4 = 0$. In this case $l_1 = l_2 = d_1 = l$ and $2k_1 = 2k_2 = d_5 = 2k$. It is easy to show that (42) goes to $2\big[\int_0^{1/n} r^l(v)\, dv\big]\big[\int_0^\infty \sigma_{2k}^2(v)\, dv\big]$ as $\varepsilon \to 0$.

3. $d_5 = 0$. As in the case 1, the term (42) is less than or equal to $\text{Const}\, \varepsilon^{\tilde{\beta}(d_2+d_3+d_4)-1} \to 0$ because $\tilde{\beta} > 1/4$.

4. $d_5 = 1$. By the same argument we obtain the bound

$$n^{-1} \varepsilon^{\tilde{\beta}(2k_1+2k_2-1)-1} \int_0^{1/n} |\ddot{\varrho}_\varepsilon(z)| \, dz,$$

and since

$$\int_0^{1/n\varepsilon} |\ddot{\varrho}_\varepsilon(\varepsilon z)|\, dz \simeq (1/n\varepsilon)^{2\alpha-1},$$

the term (42) is less than or equal to

$$\mathrm{Const}(1/n)^{2\alpha}\varepsilon^{\beta(2k_1+2k_2-1)-(2\alpha-1)} \to 0$$

since $\alpha < 3/4$. Summing up we obtain

$$\varepsilon^{-1}\sum_{i=1}^{n}\int_{(i-1)/n}^{i/n}\int_{(i-1)/n}^{i/n} E\{f_m(Y^\varepsilon_{(i-1)/n})g_M(\dot{Y}^\varepsilon_s)f_m(Y^\varepsilon_{s'})g_M(\dot{Y}^\varepsilon_{s'})\}\, ds\, ds'$$

$$\to \sigma_M'^2\Big(\sum_{l=0}^{m} c_l^2\, l!\, n\int_0^{1/n} r^l(v)\, dv\Big) \quad \text{as } \varepsilon \to 0,$$

and this goes to $\|f_m\|_2^2\, \sigma_M'^2$ as $n \to \infty$.

For the non-diagonal terms the same cases have to be examined and the only difference is in the treatment of $\ddot{\varrho}_\varepsilon$, considering if it is near the diagonal or not. ▪

Remark. It is interesting to note that if we define

$$Z_2^\varepsilon(f) \equiv \varepsilon^{-1/2}\int_0^1 f(Y_s^\varepsilon)H_2(\dot{Y}_s^\varepsilon)\, ds$$

(recall that $H_2(x) = x^2 - 1$), then

$$\varepsilon^{-1/2}\Big[\int_0^1 f(Y_s^\varepsilon)(\dot{Y}_s^\varepsilon)^2\, ds - \int_0^1 f(X_s)\, ds\Big] \equiv Z_2^\varepsilon(f) + \hat{T}_2.$$

We know that $E(\hat{T}_2^2) = o(1)$, so that if $1/4 < \alpha$, then $\hat{T}_2$ converges in probability to zero as $\varepsilon \to 0$. Furthermore, it can be shown by using an easier argument that we have weak convergence for $Z_2^\varepsilon(f)$ to a random variable $Y \in L^2$ such that

$$\mathscr{L}(Y/X_s,\, 0 \leqslant s \leqslant 1) = N\Big(0, \Big(\int_0^1 f^2(X_s)\, ds\Big)\sigma_2^2\Big).$$

Given the σ-algebra generated by $\{X_s, 0 \leqslant s \leqslant 1\}$, this limit is in fact a stochastic integral with respect to the Brownian motion which is the limit of $S_t'^\varepsilon = \varepsilon^{-1/2}\int_0^t H_2(\dot{Y}_s^\varepsilon)\, ds$. This last result comes from the tightness of the sequence S', which can be proved as follows. Using a result in Billingsley [7] it is enough to show that

$$E|S_t'^\varepsilon|^4 \leqslant \mathrm{Const}\, |t|^{1+\gamma} \quad \text{with } \gamma > 0$$

but

$$E|S_t'^\varepsilon|^4 = \varepsilon^{-2}\int_0^t\int_0^t\int_0^t\int_0^t E\{H_2(\dot{Y}^\varepsilon_{s_1})H_2(\dot{Y}^\varepsilon_{s_2})H_2(\dot{Y}^\varepsilon_{s_3})H_2(\dot{Y}^\varepsilon_{s_4})\}\, ds_1\, ds_2\, ds_3\, ds_4.$$

Because of the symmetry in the variables s_i, we have only to consider two types of graphs. First, there are those which have levels 1 and 2 connected and levels 3 and 4 also connected. The graphs of second type have levels 1, 2, 3, 4, 1 connected in this order. For the first case we have

$$\frac{1}{\varepsilon^2}\int_0^t\int_0^t\int_0^t\int_0^t [\ddot{\varrho}_\varepsilon(s_1-s_2)]^2 [\ddot{\varrho}_\varepsilon(s_3-s_4)]^2\, ds_1\, ds_2\, ds_3\, ds_4$$

$$= \frac{1}{\varepsilon^2}\Big[\int_0^t\int_0^t [\ddot{\varrho}_\varepsilon(u-v)]^2\, du\, dv\Big]^2 = \frac{4}{\varepsilon^2}\Big[\int_0^t (t-v)[\ddot{\varrho}_\varepsilon(v)]^2\, dv\Big]^2 \leqslant \frac{4t^2}{\varepsilon^2}\Big[\int_0^t [\ddot{\varrho}_\varepsilon(v)]^2\, dv\Big]^2$$

$$\leqslant 4t^2\Big[\int_0^\infty [\ddot{\varrho}_\varepsilon(\varepsilon w)]^2\, dw\Big]^2 \leqslant \mathrm{Const}\, t^2.$$

For the second case we consider

$$\varepsilon^{-2}\int_0^t\int_0^t\int_0^t\int_0^t \big(\ddot{\varrho}_\varepsilon(s_1-s_2)\big)\big(\ddot{\varrho}_\varepsilon(s_2-s_3)\big)\big(\ddot{\varrho}_\varepsilon(s_3-s_4)\big)\big(\ddot{\varrho}_\varepsilon(s_4-s_1)\big)\, ds_1\, ds_2\, ds_3\, ds_4.$$

By the Schwarz inequality, and in the same way as before,

$$\int_0^t |\ddot{\varrho}_\varepsilon(s_2-s_3)|\, |\ddot{\varrho}_\varepsilon(s_3-s_4)|\, ds_3 \leqslant \mathrm{Const}\,\varepsilon\Big[\int_0^\infty \big(\ddot{\varrho}_\varepsilon(\varepsilon w)\big)^2\, dw\Big],$$

which implies that the first integral is bounded by

$$\frac{\mathrm{Const}}{\varepsilon}\int_0^t\int_0^t\int_0^t |\ddot{\varrho}_\varepsilon(s_1-s_2)|\, |\ddot{\varrho}_\varepsilon(s_4-s_1)|\, ds_1\, ds_2\, ds_4.$$

Using again the same argument we obtain the bound

$$\mathrm{Const}\int_0^t\int_0^t ds_2\, ds_4 \leqslant \mathrm{Const}\, t^2. \ \blacksquare$$

Case 3. $3/4 < \alpha < 1$.
Now we have

$$T_1 = \frac{1}{\varepsilon^{2(1-\alpha)}}\int_0^1 f(Y_s^\varepsilon)\, g(\dot{Y}_s^\varepsilon)\, ds.$$

Define

$$W_\varepsilon(f) = \frac{1}{\varepsilon^{2(1-\alpha)}}\, a_2 \int_0^1 f(Y_s^\varepsilon)\, H_2(\dot{Y}_s^\varepsilon)\, ds,$$

where a_2 is the second Hermite coefficient of g. We shall show that

$$E\big(W_\varepsilon(f)-T_1\big)^2 \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus, to study the convergence of T_1 it is enough to consider that of $W_\varepsilon(f)$. First, it is easy to see that

$$\lim_{\varepsilon\to 0} E(T_1^2) = \lim_{\varepsilon\to 0} E(W_\varepsilon^2(f)).$$

The limit on the left-hand side was calculated in the proof of Theorem 1 and for the right-hand side the only difference is that g is replaced by H_2. They are equal to (4). To study the convergence of the covariance term a similar argument can be used: write

$$E(W_\varepsilon(f)\,T_1) = \frac{1}{\varepsilon^{4(1-\alpha)}} 2a_2 \int_0^1 (1-u)\left[E\left(f(Y_0^\varepsilon)\,f(Y_u^\varepsilon)\,g(\dot{Y}_0^\varepsilon)\,H_2(\dot{Y}_u^\varepsilon)\right)\right] du.$$

As in the proof of Theorem 1 we split the domain of integration into $[0, M\varepsilon]$, $[M\varepsilon, \eta]$ and $[\eta, 1]$. It is easy to show that the contribution from the first integral is $o(\varepsilon^{4\alpha-3})$; for the second, using an argument similar to that of Lemma 7 we infer that it is $O(\eta^{4\alpha-3})$, and for the third, using the Dominated Convergence Theorem and making $\varepsilon, \eta \to 0$ we see that this goes to (4), so that

$$\lim_{\varepsilon\to 0} E(W_\varepsilon(f) - T_1)^2 = 0.$$

To simplify the notation we shall study the asymptotic behaviour of $W_\varepsilon'(f) = W_\varepsilon(f)/a_2$. We have the following expression for the regularized process and its derivatives where $dZ_X(\lambda)$ is the spectral random measure corresponding to X_t and $dZ_\varepsilon(\lambda) = \hat{\varphi}(\varepsilon\lambda)\,dZ_X(\lambda)$,

$$Y_t^\varepsilon = \frac{1}{\sigma_\varepsilon}\int_{-\infty}^{+\infty} e^{it\lambda}\,\hat{\varphi}(\varepsilon\lambda)\,dZ_X(\lambda) = \frac{1}{\sigma_\varepsilon}\int_{-\infty}^{+\infty} e^{it\lambda}\,dZ_\varepsilon(\lambda),$$

$$\dot{Y}_t^\varepsilon = \frac{1}{\dot{\sigma}_\varepsilon}\int_{-\infty}^{+\infty} e^{it\lambda}\,i\lambda\hat{\varphi}(\varepsilon\lambda)\,dZ_X(\lambda) = \frac{1}{\dot{\sigma}_\varepsilon}\int_{-\infty}^{+\infty} e^{it\lambda}\,i\lambda\,dZ_\varepsilon(\lambda).$$

We can write $W_\varepsilon'(f)$ as

$$W_\varepsilon'(f) = \frac{1}{\varepsilon^{2(1-\alpha)}}\sum_{k=2}^{\infty} c_{k-2}\int_0^1 H_{k-2}\left(\frac{1}{\sigma_\varepsilon}\int_{-\infty}^{+\infty} e^{is\lambda}\,dZ_\varepsilon(\lambda)\right) H_2\left(\frac{1}{\dot{\sigma}_\varepsilon}\int_{-\infty}^{+\infty} e^{is\lambda}\,i\lambda Z_\varepsilon(\lambda)\right) ds.$$

To prove this identity we have to justify the interchange between the sum and the integral, but this is a consequence of the following facts. On the one hand,

$$D_M = \sum_{k=2}^{M} c_{k-2}\int_0^1 H_{k-2}(Y_s^\varepsilon)\,H_2(\dot{Y}_s^\varepsilon)\,ds$$

is a Cauchy sequence in $L^2(\Omega)$, and on the other, defining in L^2

$$f_M = f - \sum_{k=2}^{M} c_{k-2}\,H_{k-2},$$

it is easy to see that

$$E\Big[\int_0^1 f_M(Y_s^\varepsilon) H_2(\dot{Y}_s^\varepsilon)\,ds\Big]^2 \leqslant E\big(f_M^2(Y_0^\varepsilon)\big) E\big(H_2^2(\dot{Y}_s^\varepsilon)\big) = 2!\,\|f_M\|_2^2$$

and this goes to zero as $M \to \infty$. Going back to the asymptotic study of W_ε' we consider now one of the terms in its expansion, which will be denoted by $T_k(\varepsilon)$. Using Itô's formula for the Wiener–Itô integral [10], we get

$$T_k(\varepsilon) = \frac{1}{\varepsilon^{2(1-\alpha)}} \int_0^1 H_{k-2}\left(\frac{1}{\sigma_\varepsilon}\int_{-\infty}^{+\infty} e^{is\lambda}\,dZ_\varepsilon(\lambda)\right) H_2\left(\frac{1}{\dot{\sigma}_\varepsilon}\int_{-\infty}^{+\infty} e^{is\lambda}\, i\lambda\, dZ_\varepsilon(\lambda)\right) ds$$

$$= \frac{1}{\varepsilon^{2(1-\alpha)}} \int_0^1 \frac{1}{k!} \int_{R^k} \sum_{\pi\in\Pi_k} \omega_{\xi(\pi(1))}(\lambda_1, s) \ldots \omega_{\xi(\pi(k))}(\lambda_k, s)\, dZ_\varepsilon(\lambda_1) \ldots dZ_\varepsilon(\lambda_k)\, ds,$$

where

$$\omega_1(\lambda, s) = \frac{1}{\sigma_\varepsilon}\exp(is\lambda), \qquad \omega_2(\lambda, s) = \frac{i\lambda}{\dot{\sigma}_\varepsilon}\exp(is\lambda),$$

$\xi(j)$ is 1 if $j \leqslant k-2$ and 2 otherwise, and Π_k is the set of permutations of $\{1, 2, \ldots, k\}$. We can use Itô's formula since the functions $\omega_i(\lambda, s)$ are orthogonal with respect to the measure $|\hat{\varphi}(\varepsilon\lambda)|^2 h(\lambda)\,d\lambda$, where h is the spectral density of the process. As in [9], p. 330, integrating the expression above with respect to s and defining $K(\lambda) = (i\lambda)^{-1}\big(\exp(i\lambda) - 1\big)$ we obtain

$$T_k(\varepsilon) = \frac{-\varepsilon^{2(\alpha-1)}}{\dot{\sigma}_\varepsilon^2\,\sigma_\varepsilon^{k-2}} \frac{1}{k!} \int_{R^k} K(\lambda_1 + \ldots + \lambda_k)$$

$$\times \sum_{\pi\in\Pi_k} \lambda_{\pi^{-1}(k-1)} \lambda_{\pi^{-1}(k)}\, dZ_\varepsilon(\lambda_1) \ldots dZ_\varepsilon(\lambda_k) \text{ a.s.}$$

But since $\varepsilon^{2(1-\alpha)}\dot{\sigma}_\varepsilon^2 \to C\chi^2$ and $\sigma_\varepsilon^{k-2} \to 1$ as $\varepsilon \to 0$, to obtain the asymptotic behaviour of $T_k(\varepsilon)$ it is enough to consider the rest of the expression above. Define

$$M_k(\varepsilon) = \frac{-1}{k!} \int_{R^k} K(\lambda_1 + \ldots + \lambda_k) \sum_{\pi\in\Pi_k} \lambda_{\pi^{-1}(k-1)} \lambda_{\pi^{-1}(k)}\, dZ_\varepsilon(\lambda_1) \ldots dZ_\varepsilon(\lambda_k).$$

Using Lemma 15 below we can prove that

$$E\big(M_k^2(\varepsilon)\big) = \frac{1}{k!} \int_{R^k} |K(\lambda_1 + \ldots + \lambda_k)|^2 \sum_{\pi\in\Pi_k} \sum_{\nu\in\Pi_k} \lambda_{\pi^{-1}(k-1)} \lambda_{\pi^{-1}(k)} \lambda_{\nu^{-1}(k-1)} \lambda_{\nu^{-1}(k)}$$

$$\times |\hat{\varphi}(\varepsilon\lambda_1)|^2 \ldots |\hat{\varphi}(\varepsilon\lambda_k)|^2 h(\lambda_1) \ldots h(\lambda_k)\, d\lambda_1 \ldots d\lambda_k$$

converges, as $\varepsilon \to 0$, to

$$E\big(M_k^2(0)\big) = \frac{1}{k!} \int_{R^k} |K(\lambda_1 + \ldots + \lambda_k)|^2 \sum_{\pi\in\Pi_k} \sum_{\nu\in\Pi_k} \lambda_{\pi^{-1}(k-1)} \lambda_{\pi^{-1}(k)} \lambda_{\nu^{-1}(k-1)} \lambda_{\nu^{-1}(k)}$$

$$\times h(\lambda_1) \ldots h(\lambda_k)\, d\lambda_1 \ldots d\lambda_k < \infty.$$

Since the last integral above is finite, we can define the following Itô–Wiener integral:

$$M_k(0) = \frac{1}{k!}\int_{R^k} K(\lambda_1 + \dots + \lambda_k) \sum_{\pi\in\Pi_k} \lambda_{\pi^{-1}(k-1)}\lambda_{\pi^{-1}(k)}\, dZ_X(\lambda_1)\dots dZ_X(\lambda_k).$$

Consider now

$$D(k,\varepsilon) = \frac{1}{k!} K(\lambda_1 + \dots + \lambda_k) \times \sum_{\pi\in\Pi_k} \lambda_{\pi^{-1}(k-1)}\lambda_{\pi^{-1}(k)}\,\hat{\varphi}(\varepsilon\lambda_1)\dots\hat{\varphi}(\varepsilon\lambda_k)\sqrt{h(\lambda_1)}\dots\sqrt{h(\lambda_k)}.$$

$D(k,\varepsilon)$ converges pointwise to $D(k,0)$ as $\varepsilon\to 0$ and the calculations above imply that $\|D(k,\varepsilon)\|^2 \to \|D(k,0)\|^2$ as $\varepsilon\to 0$ in L^2 with respect to Lebesgue measure in R^k, and the Lebesgue theorem implies that $\|D(k,\varepsilon)-D(k,0)\|\to 0$ as $\varepsilon\to 0$ in the L^2-norm with respect to Lebesgue measure. We have in fact $M_k(\varepsilon)\to M_k(0)$ in $L^2(\Omega)$, i.e. $T_k(\varepsilon)\to T_k(0)$ in $L^2(\Omega)$, where $T_k(0)=\chi^{-2}M_k(0)$. We will now prove that, as $\varepsilon\to 0$,

$$W_\varepsilon'(f) = \sum_{k=2}^{\infty} T_k(\varepsilon) \simeq \frac{1}{C\chi^2}\sum_{k=2}^{\infty}\sigma_\varepsilon^{2-k} M_k(\varepsilon) \to \sum_{k=2}^{\infty} T_k(0).$$

First, note that

$$\operatorname{Var}(W_\varepsilon') = 4\varepsilon^{-4(1-\alpha)}\frac{2}{\dot{\sigma}_\varepsilon^2}\sum_{k=2}^{\infty} c_{k-2}^2 (k-2)! \sum_{j=(k-4)\vee 0}^{k-2} \frac{(k-2)!}{j!\,[(k-2-j)!]^2 (j-k+4)!} \times \frac{1}{\sigma_\varepsilon^{k-2}}\int_0^1 (1-u)[\varrho_\varepsilon(u)]^j[\dot{\varrho}_\varepsilon(u)]^{2(k-2-j)}[-\ddot{\varrho}_\varepsilon(u)]^{j-k+4}\,du$$

tends, as $\varepsilon\to 0$, to

$$4\frac{2}{\chi^2}\sum_{k=2}^{\infty} c_{k-2}^2 (k-2)! \sum_{j=(k-4)\vee 0}^{k-2} \frac{(k-2)!}{j!\,[(k-2-j)!]^2 (j-k+4)!} \times \int_0^1 (1-u)[r(u)]^j[\dot{r}(u)]^{2(k-2-j)}[-\ddot{r}(u)]^{j-k+4}\,du.$$

A similar argument gives the result when the sum k begins with $M+1$. This implies that

$$\lim_{M\to\infty}\limsup_{\varepsilon\to 0}\sum_{k=M+1}^{\infty} E(T_k^\varepsilon)^2 = 0,$$

since

$$\limsup_{\varepsilon\to 0}\Big|\sum_{k=M+1}^{\infty}\big(E(T_k^\varepsilon)^2 - E(T_k(0))^2\big)\Big| = 0 \quad\text{and}\quad \lim_{M\to\infty}\sum_{k=M+1}^{\infty} E\big(T_k(0)\big)^2 = 0.$$

On the other hand,

$$E\Big(\sum_{k=2}^{\infty}(T_k^{\varepsilon}-T_k(0))\Big)^2$$
$$\leqslant \mathrm{Const}\Big[E\Big(\sum_{k>M}^{\infty}T_k^{\varepsilon}\Big)^2+E\Big(\sum_{k=2}^{M}(T_k^{\varepsilon}-T_k(0))\Big)^2+E\Big(\sum_{k>M}^{\infty}T_k(0)\Big)^2\Big].$$

Consequently, using the orthogonality relations and previous calculations we obtain

$$\lim_{\varepsilon\to 0}E\Big(\sum_{k=2}^{\infty}(T_k(\varepsilon)-T_k(0))\Big)^2=0,$$

and the result follows. ∎

LEMMA 15. *Let* (μ_ε) *be a sequence of finite measures in* $\mathbf{R}^k$ *such that their density with respect to Lebesgue measure can be written as*

$$|K(\lambda_1+\ldots+\lambda_k)|^2\sum_{\pi\in\Pi_k}\sum_{\nu\in\Pi_k}\lambda_{\pi^{-1}(k-1)}\lambda_{\pi^{-1}(k)}\lambda_{\nu^{-1}(k-1)}\lambda_{\nu^{-1}(k)}$$
$$\times|\hat{\varphi}(\varepsilon\lambda_1)|^2\ldots|\hat{\varphi}(\varepsilon\lambda_k)|^2h(\lambda_1)\ldots h(\lambda_k).$$

Then $\mu_\varepsilon\to\mu$ *weakly as* $\varepsilon\to 0$, *where* μ *is a finite measure on* $\mathbf{R}^k$. *This implies that*

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^k}d\mu_\varepsilon(x)=\int_{\mathbf{R}^k}d\mu(x).$$

Proof. Define $\Psi_\varepsilon(\gamma_1,\ldots,\gamma_k)$ to be the Fourier transform of μ_ε. Since $K(\lambda)$ is the Fourier transform of Pólya's function, i.e. $|K(\lambda)|^2=\hat{q}(\lambda)$, where $q(u)=\max(1-|u|,0)$, we get

$$\Psi_\varepsilon(\gamma_1,\ldots,\gamma_k)$$
$$=\sum_{\pi\in\Pi_k}\sum_{\nu\in\Pi_k}\int_{\mathbf{R}^k}\int_{-1}^{1}q(u)\exp\big(i[\lambda_1(u+\gamma_1)+\ldots+\lambda_k(u+\gamma_k)]\big)\lambda_{\pi^{-1}(k-1)}\lambda_{\pi^{-1}(k)}$$
$$\times\lambda_{\nu^{-1}(k-1)}\lambda_{\nu^{-1}(k)}|\hat{\varphi}(\varepsilon\lambda_1)|^2\ldots|\hat{\varphi}(\varepsilon\lambda_k)|^2h(\lambda_1)\ldots h(\lambda_k)\,du\,d\lambda_1\ldots d\lambda_k.$$

We have to consider three cases depending on the permutations that appear in this expression.

Case 1. For two different integers the permutations are the same. We may assume, without loss of generality, that $\pi^{-1}(k)=\nu^{-1}(k)=1$ and $\pi^{-1}(k-1)=\nu^{-1}(k-1)=2$. The integral is then

$$\int_{\mathbf{R}^k}\int_{-1}^{1}q(u)\exp\big(i[\lambda_1(u+\gamma_1)+\ldots+\lambda_k(u+\gamma_k)]\big)\,du$$
$$\times\lambda_1^2\lambda_2^2|\hat{\varphi}(\varepsilon\lambda_1)|^2\ldots|\hat{\varphi}(\varepsilon\lambda_k)|^2h(\lambda_1)\ldots h(\lambda_k)\,d\lambda_1\ldots d\lambda_k$$
$$=\int_{-1}^{1}q(u)\ddot{\varrho}_\varepsilon(\gamma_1+u)\ddot{\varrho}_\varepsilon(\gamma_2+u)\varrho_\varepsilon(\gamma_3+u)\ldots\varrho_\varepsilon(\gamma_k+u)\,du.$$

There are $2k(k-1)[(k-2)!]^2$ such cases.

Case 2. The permutations coincide for one index and differ for the other two indices. We may assume without loss of generality that $\pi^{-1}(k)=\nu^{-1}(k)=1$ and $\pi^{-1}(k-1)=2$, $\nu^{-1}(k-1)=3$. As before we get

$$\int_{-1}^{1} q(u)\,[\ddot{\varrho}_\varepsilon(\gamma_1+u)]\,\dot{\varrho}_\varepsilon(\gamma_2+u)\,\dot{\varrho}_\varepsilon(\gamma_3+u)\,\varrho_\varepsilon(\gamma_4+u)\ldots\varrho_\varepsilon(\gamma_k+u)\,du$$

and there are $4k!(k-2)!(k-2)$ possible cases.

Case 3. All indices are different, the integral is

$$\int_{-1}^{1} q(u)\,\dot{\varrho}_\varepsilon(\gamma_1+u)\,\dot{\varrho}_\varepsilon(\gamma_2+u)\,\dot{\varrho}_\varepsilon(\gamma_3+u)\,\dot{\varrho}_\varepsilon(\gamma_4+u)\,\varrho_\varepsilon(\gamma_5+u)\ldots\varrho_\varepsilon(\gamma_k+u)\,du,$$

and there are $k!(k-2)!(k-2)(k-3)$ possible cases. We will only consider in detail one integral belonging to Case 1:

$$L_\varepsilon=\int_{-1}^{1} q(u)\,\ddot{\varrho}_\varepsilon(\gamma_1+u)\,\ddot{\varrho}_\varepsilon(\gamma_2+u)\,\varrho_\varepsilon(\gamma_3+u)\ldots\varrho_\varepsilon(\gamma_k+u)\,du.$$

We will prove that L_ε converges uniformly for $(\gamma_1,\ldots,\gamma_k)\in\kappa$, where κ is compact in $\boldsymbol{R}^k$, as $\varepsilon\to 0$, to

$$L_0=\int_{-1}^{1} q(u)\,\ddot{r}(\gamma_1+u)\,\ddot{r}(\gamma_2+u)\,r(\gamma_3+u)\ldots r(\gamma_k+u)\,du.$$

We have

$$\begin{aligned}|L_\varepsilon(\gamma_1,\ldots,\gamma_k)-L_0(\gamma_1,\ldots,\gamma_k)|&\\ \leqslant \|\varrho_\varepsilon\|_\infty^{k-2}\int_{-1}^{1} q(u)\,|\ddot{\varrho}_\varepsilon(\gamma_1+u)\,\ddot{\varrho}_\varepsilon(\gamma_2+u)-\ddot{r}(\gamma_1+u)\,\ddot{r}(\gamma_2+u)|\,du&\\ +\sup_{u\in[-1,1]}|\varrho_\varepsilon(\gamma_3+u)\ldots\varrho_\varepsilon(\gamma_k+u)-r(\gamma_3+u)\ldots r(\gamma_k+u)|\,\|\ddot{r}\|_2^2.&\end{aligned}$$

The integral above is bounded by

$$\|\ddot{\varrho}_\varepsilon\|_2\Big[\Big[\int_{-1}^{1}|\ddot{\varrho}_\varepsilon(\gamma_1+u)-\ddot{r}(\gamma_1+u)|^2\,du\Big]^{1/2}+\Big[\int_{-1}^{1}|\ddot{\varrho}_\varepsilon(\gamma_2+u)-\ddot{r}(\gamma_2+u)|^2\,du\Big]^{1/2}\Big]$$

and since $\ddot{r}\in L^2$, the two terms tend to zero as $\varepsilon\to 0$. Hence $\lim_{\varepsilon\to 0}\Psi_\varepsilon(\gamma_1,\ldots,\gamma_k)$ is continuous and, in particular, at $(\gamma_1,\ldots,\gamma_k)=(0,\ldots,0)$. Therefore there exists a finite measure μ such that $\hat{\mu}_\varepsilon\to\hat{\mu}$ and, by Levy's theorem, $\mu_\varepsilon\to\mu$ weakly. ∎

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Corinne Berzin, Université Paris XI, Laboratoire de Statistique Appliquée
Centre d'Orsay, Bat. 425. 91405 Orsay Cédex
e-mail: Corinne.Berzin@math.u-psud.fr

José R. León, Depto. de Matemáticas, Facultad de Ciencias
U.C.V., Apartado 47197, Caracas 1047, Venezuela
e-mail: jleon@euler.ciens.ucv.ve

Joaquín Ortega, IVIC-Matemáticas, Apartado 21.827,
Caracas 1020-A, Venezuela
e-mail: jortega@cauchy.iric.ve

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