PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 18, Fasc. 1 (1998), pp. 173–183

INDEPENDENT MARGINALS OF OPERATOR-SEMISTABLE AND OPERATOR-STABLE PROBABILITY MEASURES*

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Abstract. We investigate independent marginals of full operator-semistable and operator-stable probability measures on finite-dimensional vector spaces. In particular, it is shown that for purely Poissonian operator-semistable and operator-stable distributions their independent marginals have decomposability properties of the same kind. Operator-semistability and operator-stability of independent marginals of Gaussian measures are studied in detail, and a description of independent marginals of an arbitrary operator-semistable or operator-stable distribution is obtained.

Introduction. Let μ be a probability measure on a finite-dimensional real vector space V with σ-algebra $\mathscr{B}(V)$ of its Borel subsets. A projection T on V will be called an *independent marginal* of μ if

$$\mu = T\mu * (I-T)\mu \quad (I \text{ — the identity operator}),$$

i.e. if T and $I-T$ are independent random variables from probability space $(V, \mathscr{B}(V), \mu)$ into V (the same name will be sometimes applied also to the measure $T\mu$). The aim of the paper is to investigate properties of measure $T\mu$ for T being an independent marginal of μ, and μ being a full operator-semistable or operator-stable probability distribution on V. Problems of this type have been considered in [2], [6], and [9], and in this work we generalize and complete some of the earlier results. In particular, we show that for purely Poissonian operator-semistable and operator-stable distributions their independent marginals follow, in principle, the same pattern of decomposability. Operator-semistability and operator-stability of independent marginals of Gaussian measures are studied in detail, and, finally, a description of independent marginals of an arbitrary operator-semistable or operator-stable distribution is obtained.

* Work supported by KBN grant 2 1020 91 01.

1. Preliminaries and notation. Throughout the paper, V will stand for an r-dimensional real vector space with an inner product $(\cdot, \cdot)$ yielding a norm $\|\cdot\|$, and the algebra $\mathscr{B}(V)$ of its Borel subsets.

An infinitely divisible measure μ on V has the unique representation $[m, D, M]$, where $m \in V$, D is a non-negative linear operator on V, and M is the Lévy spectral measure of μ, i.e. a Borel measure defined on $V_0 = V - \{0\}$ such that

$$\int_{V_0} \|u\|^2/(1+\|u\|^2)\, M(du) < \infty.$$

The characteristic function $\hat{\mu}$ of μ takes then the form

$$\hat{\mu}(v) = \exp\left\{i(m, v) - \frac{1}{2}(Dv, v) + \int_{V_0} \left[e^{i(v,u)} - 1 - \frac{i(v, u)}{1+\|u\|^2}\right] M(du)\right\}$$

(see e.g. [7]). The measure $[m, D, 0]$ is called the *Gaussian part* of μ, the measure $[0, 0, M]$ is called its *Poissonian part*; μ is called *purely Gaussian* if $M = 0$, and *purely Poissonian* if $D = 0$.

A probability measure on V is called *full* if it is not concentrated on any proper hyperplane of V.

The main objects of our considerations will be full operator-semistable and operator-stable probability measures on V and their independent marginals as defined in the Introduction. For a more detailed description of these measures, the reader is referred to [3] and [5] (operator-semistable) and [1], [4] and [8] (operator-stable). Here we only recall that if μ is a full operator-semistable measure, then it is infinitely divisible and

$$\mu^a = A\mu * \delta(h) \tag{1}$$

for some $0 < a < 1$, $h \in V$, and a non-singular linear operator A in V. Measures satisfying (1) will be called *(a, A)-quasi-decomposable*, and for full measures quasi-decomposability is equivalent to operator-semistability. Furthermore, there are decompositions

$$\mu = \mu_1 * \mu_2, \qquad V = V_1 \oplus V_2 \tag{2}$$

such that V_1 and V_2 are A-invariant subspaces of V, μ_1 is a purely Poissonian (a, A)-quasi-decomposable measure concentrated (and full) on V_1, and μ_2 is a Gaussian (a, A)-quasi-decomposable measure concentrated (and full) on V_2.

We let $G_a(\mu)$ denote the set of the operators A's which can occur in equation (1).

Full operator-stable measures are characterized by the following condition:

There exists a non-singular operator B in V, called an *exponent* of μ, such that for each $t > 0$

$$t^B \in G_t(\mu), \qquad \text{where } t^B = e^{(\log t)B}.$$

Moreover, decompositions (2) also hold with V_1 and V_2 being B-invariant, μ_1 — purely Poissonian concentrated on V_1, μ_2 — Gaussian concentrated (and full) on V_2, and for $i = 1, 2$

$$\mu_i^t = t^B \mu_i * \delta(h_t^{(i)}), \qquad t > 0,$$

with some $h_t^{(i)} \in V_i$.

2. Marginals of operator-semistable measures. We begin with the following generalization of Theorem 6 of [6].

THEOREM 1. *Let $\mu = [m, 0, M]$ be a full (a, A)-quasi-decomposable probability measure on V, and let T be an independent marginal of μ. Then there exists a positive integer n such that $TA^n = A^n T$, and, consequently, $T\mu$ is (a^n, A^n)-quasi--decomposable.*

Proof. Put

$$U = T(V), \qquad W = (I-T)(V),$$

and let S_M be the support of the Lévy measure M. By virtue of [6] and [9] we have

$$S_M \subset U \cup W. \tag{3}$$

From the fullness of μ, and thus M, it follows that $\operatorname{Lin} S_M = V$ and, consequently,

$$\operatorname{Lin}(S_M \cap U) = U, \qquad \operatorname{Lin}(S_M \cap W) = W.$$

Equality (1) implies that $aM = AM$, which in turn yields the A-invariance of S_M.

Let $\{v_1, \ldots, v_k\} \subset S_M \cap U$ be a basis in U, and let $\{v_{k+1}, \ldots, v_r\} \subset S_M \cap W$ be a basis in W (we have assumed that $\dim U = k$ and $\dim W = r-k$). According to (3) and the A-invariance of S_M, for each $m = 0, 1, \ldots$ and each $i = 1, \ldots, r$, $A^m v_i$ is either in $S_M \cap U$ or in $S_M \cap W$. Let us represent the sequence $\{A^m v_1, \ldots, A^m v_k, A^m v_{k+1}, \ldots, A^m v_r\}$ as a sequence of 0's and 1's, where 0 at the i-th place means that $A^m v_i \in S_M \cap U$ and 1 at the i-th place means that $A^m v_i \in S_M \cap W$ (for instance, if $m = 0$, we have the sequence $\{0, \ldots, 0, 1, \ldots, 1\}$). Condition (3) together with the fullness of M implies that exactly k elements of $\{A^m v_1, \ldots, A^m v_r\}$ are in $S_M \cap U$, and $r-k$ elements are in $S_M \cap W$; in other words, in our representing sequences there will be exactly k zeros and $r-k$ ones. Since there are only $\binom{r}{k}$ such different sequences, we can find elements $v_{i_1}, \ldots, v_{i_k}$ and two positive integers m_1, m_2,

$$m_1 < m_2, \qquad m_2 - m_1 \leqslant \binom{r}{k},$$

such that

$$A^{m_1} v_{i_1}, \ldots, A^{m_1} v_{i_k} \in U \qquad \text{(the zeros)},$$

$$A^{m_1} v_j \in W \text{ for } j \notin \{i_1, \ldots, i_k\} \qquad \text{(the ones)}$$

and

$$A^{m_2} v_{i_1}, \ldots, A^{m_2} v_{i_k} \in U, \qquad A^{m_2} v_j \in W \text{ for } j \notin \{i_1, \ldots, i_k\}.$$

Putting

$$u_1 = A^{m_1} v_{i_1}, \ldots, u_k = A^{m_1} v_{i_k}, \quad w_j = A^{m_1} v_j \text{ for } j \notin \{i_1, \ldots, i_k\}$$

and $n = m_2 - m_1$, we get

$$u_1, \ldots, u_k \in U, \quad A^n u_1, \ldots, A^n u_k \in U$$

and

$$w_j \in W, \quad A^n w_j \in W \text{ for } j \notin \{i_1, \ldots, i_k\}.$$

Since $\{u_1, \ldots, u_k\}$ form a basis in U and $\{w_j\}$ form a basis in W, we obtain

$$A^n(U) = U, \quad A^n(W) = W,$$

showing that $TA^n = A^n T$.

Iterating equality (1) gives the formula

$$\mu^{a^n} = A^n \mu * \delta(h_n),$$

and, consequently,

$$(T\mu)^{a^n} = T\mu^{a^n} = TA^n \mu * \delta(Th_n) = A^n T\mu = A^n T\mu * \delta(Th_n),$$

so $T\mu$ is (a^n, A^n)-quasi-decomposable. ■

Our next aim is to investigate (a, A)-quasi-decomposable Gaussian measures. We begin with a simple characterization of operators A's for which a full Gaussian distribution can be (a, A)-quasi-decomposable.

PROPOSITION 2. *Let $\mu = [m, D, 0]$ be a full Gaussian measure on V, and let $a > 0$. Then*

$$G_a(\mu) = \sqrt{a} D^{1/2} \boldsymbol{O} D^{-1/2},$$

where $\boldsymbol{O}$ is the orthogonal group on V.

Proof. It is easy to verify that a Gaussian measure $\mu = [m, D, 0]$ satisfies equation (1) if and only if

$$aD = ADA^*. \tag{4}$$

It is immediately seen that for any orthogonal H and the operator A defined as

$$A = \sqrt{a} D^{1/2} H D^{-1/2}$$

equality (4) holds, which proves the inclusion

$$\sqrt{a} D^{1/2} \boldsymbol{O} D^{-1/2} \subset G_a(\mu).$$

Assume now that (4) holds. The fullness of μ implies the invertibility of D, and we have

$$aI = D^{-1/2} ADA^* D^{-1/2} = (D^{1/2} A^* D^{-1/2})^* D^{1/2} A^* D^{-1/2},$$

which means that the absolute value of the operator $D^{1/2} A^* D^{-1/2}$ is $\sqrt{a} I$. The polar decomposition formula gives the equality

$$D^{1/2} A^* D^{-1/2} = H |D^{1/2} A^* D^{-1/2}| = \sqrt{a} H$$

for some orthogonal H, so

$$A = (\sqrt{a}D^{-1/2} HD^{-1/2})^* = \sqrt{a}D^{1/2} H^* D^{-1/2},$$

showing that $A \in \sqrt{a}D^{1/2} \mathcal{O} D^{-1/2}$. ∎

Remark. The above proposition can be thought of as an "operator-semi-stable" counterpart of Theorem 4.6.10 from [4], which gives a characterization of the set of exponents of Gaussian measures.

Now we shall analyse conditions of quasi-decomposability of independent marginals of full Gaussian measures.

PROPOSITION 3. *Let $\mu = [m, D, 0]$ be a full (a, A)-quasi-decomposable Gaussian measure on V, and let T be an independent marginal of μ. Then $T\mu$ is (a, A)-quasi-decomposable if and only if A and T commute.*

Proof. Put $P = I - T$. Then

$$\mu^a = (T\mu * P\mu)^a = (T\mu)^a * (P\mu)^a = T\mu^a * P\mu^a$$

and

$$A\mu = AT\mu * AP\mu.$$

From equality (1) we get

$$T\mu^a * P\mu^a = AT\mu * AP\mu * \delta(h);$$

thus

$$T\mu^a = TAT\mu * TAP\mu * \delta(Th). \tag{5}$$

If A and T commute, we have $TAP = 0$, so (5) becomes

$$T\mu^a = AT\mu * \delta(Th),$$

which means that $T\mu$ is (a, A)-quasi-decomposable.

Now, assume that $T\mu$ is (a, A)-quasi-decomposable. Then

$$T\mu^a = AT\mu * \delta(h'),$$

so

$$T\mu^a = TAT\mu * \delta(Th'),$$

which together with (5) leads to the equality

$$TAT\mu * \delta(Th') = TAT\mu * TAP\mu * \delta(Th).$$

Since all the measures involved are Gaussian, the above equality shows that $TAP\mu$ is a degenerate measure and, consequently,

$$(TAP)D(TAP)^* = 0. \tag{6}$$

By Proposition 2, A takes the form $A = \sqrt{a}D^{1/2} HD^{-1/2}$ for some orthogonal H, so (6) leads to the equality

$$aTD^{1/2} HD^{-1/2} PDP^* D^{-1/2} H^* D^{1/2} T^* = 0,$$

and multiplying on the left by $D^{-1/2}$ and on the right by $D^{1/2}$, we get

(7) $$D^{-1/2} TD^{1/2} HD^{-1/2} PD^{1/2} D^{1/2} P^* D^{-1/2} H^* D^{1/2} T^* D^{-1/2} = 0.$$

Put

$$D^{-1/2} TD^{1/2} = R.$$

Then $R = R^2$; moreover,

(8) $$R^* = D^{1/2} T^* D^{-1/2} = D^{-1/2} DT^* D^{-1/2}.$$

Since T is an independent marginal, we have, according to [6] and [9],

(9) $$D = TDT^* + PDP^*,$$

so

$$TD = TDT^* = DT^*.$$

Thus (8) leads to the equality

$$R^* = D^{-1/2} TDD^{-1/2} = D^{-1/2} TD^{1/2} = R,$$

showing that R is an orthogonal projection. Furthermore,

$$R^{\perp} = I - R = D^{-1/2}(I-T) D^{1/2} = D^{-1/2} PD^{1/2}.$$

Consequently, equality (7) takes the form $RHR^{\perp} H^* R = 0$, so

$$RHR^{\perp}(RHR^{\perp})^* = 0,$$

which means that

$$RHR^{\perp} = 0, \quad \text{i.e.} \quad RH = RHR.$$

Since H is orthogonal and R is an orthogonal projection, the last equality means that H and R commute. Thus we have

$$D^{-1/2} TD^{1/2} H = HD^{-1/2} TD^{1/2},$$

which, in turn, gives

$$TD^{1/2} HD^{-1/2} = D^{1/2} HD^{-1/2} T.$$

Multiplying both sides by $\sqrt{a}$, we finally obtain $TA = AT$, which completes the proof. ∎

The last two results lead us to an example of a full (a, A)-quasi-decomposable Gaussian measure having r independent one-dimensional marginals which are not (a^n, A^n)-quasi-decomposable for any n.

EXAMPLE. Let $T_1, \ldots, T_r$ be one-dimensional orthogonal projections, and let $0 < \lambda_1 < \ldots < \lambda_r$. Put

$$D = \sum_{i=1}^{r} \lambda_i T_i,$$

and let $\mu = [0, D, 0]$. We have

$$D = \sum_{i=1}^{r} T_i D T_i = \sum_{i=1}^{r} T_i D T_i^*;$$

thus $T_1, \ldots, T_r$ are independent marginals of μ. Let H be an orthogonal operator, and put

$$A = \sqrt{a} D^{1/2} H D^{-1/2} \quad \text{for some } a > 0.$$

By Proposition 2, μ is (a, A)-quasi-decomposable. Now, for any integer n,

$$A^n = a^{n/2} D^{1/2} H^n D^{-1/2},$$

so A^n commutes with T_i if and only if H^n does. Hence, if we have chosen H in such a way that

$$H^n T_i \neq T_i H^n, \quad i = 1, \ldots, r, \text{ all } n,$$

then by Proposition 3 none of the marginals T_i's will be (a^n, A^n)-quasi-decomposable for any n. ∎

Our final goal in this chapter is to give a description of independent marginals of an arbitrary full (a^n, A^n)-quasi-decomposable measure. We have

THEOREM 4. *Let $\mu = [m, D, M]$ be a full (a, A)-quasi-decomposable measure on V, and let T be an independent marginal of μ with $T(V) = U$. Then there are decompositions*

$$U = U_1 \oplus U_2, \quad T\mu = \nu_1 * \nu_2$$

such that ν_1 is a purely Poissonian (a^n, A^n)-quasi-decomposable (for some n) measure concentrated on U_1, and ν_2 is a Gaussian measure concentrated on U_2.

Proof. Put $P = I - T$, $W = P(V)$, and let again S_M stand for the support of M. For S_M relation (3) holds; thus putting

$$U_1 = \mathrm{Lin}(S_M \cap U), \quad W_1 = \mathrm{Lin}(S_M \cap W),$$

we get

$$\mathrm{Lin}\, S_M = U_1 \oplus W_1.$$

Now, let us take into account decompositions (2). The Poissonian part μ_1 lives on V_1, so we have $V_1 = U_1 \oplus W_1$. Restrict for the moment our attention to the subspace V_1 and the measure μ_1. We have $S_M \subset U_1 \cup W_1$. Thus denoting by T_1 the projection onto U_1 with kernel W_1, and by P_1 the projection onto W_1 with kernel U_1, we infer from [6] and [9] that T_1 and P_1 are independent marginals of μ_1, so by Theorem 1 we have

$$T_1 A^n = A^n T_1, \quad P_1 A^n = A^n P_1 \quad \text{for some } n,$$

and $T_1 \mu_1$, $P_1 \mu_1$ are (a^n, A^n)-quasi-decomposable.

Now we shall analyse the Gaussian part. It is concentrated on V_2, so we have

$$D(V) = D(V_2) = V_2.$$

Since T and P are independent marginals of μ, relation (9) holds. Thus

$$T(V_2) = TD(V_2) = DT^*(V_2) \subset V_2$$

and, similarly,

$$P(V_2) \subset V_2.$$

Putting $T(V_2) = U_2$ and $P(V_2) = W_2$, we obtain the decomposition $V_2 = U_2 \oplus W_2$. Let R be the orthogonal projection onto V_2. We have $D = RD$, so R and D commute. Furthermore,

$$(T \mid V_2)^* = RT^* \mid V_2, \quad (P \mid V_2)^* = RP^* \mid V_2,$$

which together with the equality

$$D = TDRT^* + PDRP^*$$

gives

$$\begin{aligned} D \mid V_2 &= TDRT^* \mid V_2 + PDRP^* \mid V_2 \\ &= (T \mid V_2)(D \mid V_2)(T \mid V_2)^* + (P \mid V_2)(D \mid V_2)(P \mid V_2)^*. \end{aligned}$$

Now restricting our attention to the subspace V_2 and the measure μ_2, and denoting by T_2 the projection onto U_2 with kernel W_2, and by P_2 the projection onto W_2 with kernel U_2, we get

$$D = T_2 D T_2^* + P_2 D P_2^*,$$

which means that T_2 and P_2 are independent marginals of μ_2. Finally, we have

$$V = V_1 \oplus V_2 = (U_1 \oplus W_1) \oplus (U_2 \oplus W_2) = (U_1 \oplus U_2) \oplus (W_1 \oplus W_2) = U \oplus W,$$

and since

$$U_1 \oplus U_2 \subset U, \quad W_1 \oplus W_2 \subset W,$$

we obtain

$$U = U_1 \oplus U_2, \quad W = W_1 \oplus W_2.$$

Extending the projections T_1, T_2, P_1, P_2 in the natural way to the whole V (i.e. for instance T_1 will be the projection onto U_1 with kernel $U_2 \oplus W_1 \oplus W_2$) we shall get

$$T = T_1 + T_2, \quad P = P_1 + P_2$$

and

$$\mu = \mu_1 * \mu_2 = T_1 \mu_1 * P_1 \mu_1 * T_2 \mu_2 * P_2 \mu_2,$$

which gives

$$T_i \mu = T_i \mu_i, \quad P_i \mu = P_i \mu_i, \quad i = 1, 2.$$

Thus we have

$$\mu = T\mu * P\mu = T_1 \mu * P_1 \mu * T_2 \mu * P_2 \mu,$$

and applying T to both sides of the above equality we obtain $T\mu = T_1\mu * T_2\mu$. Putting $\nu_1 = T_1\mu$ and $\nu_2 = T_2\mu$, we obtain the desired decomposition. ∎

Remark. Neither the measure ν_2 nor the measure $P_2\mu$ need not be (a^m, A^m)-quasi-decomposable for any m (however, their convolution being the Gaussian part μ_2 of μ is (a, A)-quasi-decomposable). Nevertheless, this fact does not affect operator-semistability of the marginal $T\mu$ as is seen in the following corollary.

COROLLARY. *Let T be an independent marginal of a full (a, A)-quasi--decomposable measure μ on V. Then $T\mu$ is operator-semistable.*

Proof. In the course of the proof of Theorem 4 it was shown that $T\mu = T_1\mu * T_2\mu$ with $T_1 A^n = A^n T_1$ for some n, which means that $A^n(U_1) = U_1$. Define an operator A_n by

$$A_n = \begin{cases} A^n & \text{on } U_1, \\ \sqrt{a^n} I & \text{on } U_2, \\ \text{arbitrary} & \text{on } W. \end{cases}$$

Since $T_2\mu$ is Gaussian, it is $(a^n, \sqrt{a^n} I)$-quasi-decomposable, and we have

$$\begin{aligned}(T\mu)^{a^n} &= (T_1\mu)^{a^n} * (T_2\mu)^{a^n} = A^n T_1\mu * \delta(h_1) * \sqrt{a^n}\, T_2\mu * \delta(h_2) \\ &= A_n T_1\mu * A_n T_2\mu * \delta(h_1 + h_2) = A_n(T_1\mu * T_2\mu) * \delta(h_1 + h_2) \\ &= A_n T\mu * \delta(h_1 + h_2),\end{aligned}$$

showing that $T\mu$ is (a^n, A_n)-quasi-decomposable, hence operator-semistable. ∎

3. Marginals of operator-stable measures. In general, operator-stability exhibits much more regular behaviour as will be seen in the following counterparts of results about operator-semistability. In particular, we have

THEOREM 5. *Let $\mu = [m, 0, M]$ be a full operator-stable probability measure on V with exponent B, and let T be an independent marginal of μ. Then T and B commute, and $T\mu$ is operator-stable with exponent TB.*

Proof. Putting $U = T(V)$ and $W = (I - T)(V)$, we have again relation (3), and the equality $\mu^t = t^B\mu * \delta(h_t)$ yields the inclusion $t^B(S_M) \subset S_M$. Thus, for an arbitrary $u \in S_M \cap U$, $t^B u \in S_M \cap U$, and the same is true for $w \in W$. From the fullness of M we infer that $t^B(U) \subset U$ and $t^B(W) \subset W$, and differentiation at 1 gives $B(U) \subset U$ and $B(W) \subset W$. Since B is invertible, we get $B(U) = U$ and $B(W) = W$, which means that T and B commute. Accordingly,

$$(T\mu)^t = Tt^B\mu * \delta(Th_t) = t^{TB}\, T\mu * \delta(Th_t),$$

showing that TB is an exponent of $T\mu$. ∎

PROPOSITION 6. *Let $\mu = [m, D, 0]$ be a full operator-stable Gaussian measure on V with exponent B, and let T be an independent marginal of μ. Then $T\mu$ is operator-stable with exponent TBT.*

Proof. According to Propositions 4.3.2 and 4.3.3 of [4], B is an exponent of μ if and only if

$$D = BD + DB^*.$$

Multiplying the above equality by T on the left and by T^* on the right and taking into account the relations $TD = DT^* = TDT^*$ which follow from (9), we obtain

$$TDT^* = TBDT^* + TDB^*T^* = (TBT)(TDT^*) + (TDT^*)(TBT)^*.$$

Since TDT^* is the covariance operator of the measure $T\mu$, applying again the above-mentioned propositions from [4], we see that TBT is an exponent of $T\mu$. ∎

By reasoning in a similar fashion to that in the proof of Theorem 4, we obtain the following result:

THEOREM 7. *Let μ be a full operator-stable measure on V with exponent B, and let T be an independent marginal of μ with $T(V) = U$. Then there are decompositions*

$$U = U_1 \oplus U_2, \qquad T\mu = \nu_1 * \nu_2$$

such that ν_1 is a purely Poissonian operator-stable measure concentrated on U_1 with exponent $T_1 B = BT_1$, and ν_2 is an operator-stable Gaussian measure concentrated on U_2 with exponent $T_2 BT_2$, where T_1 and T_2 are projections onto U_1 and U_2, respectively, with kernels $\ker T_1 = U_2 \oplus W$, $\ker T_2 = U_1 \oplus W$, $W = (I-T)(V)$. ∎

REFERENCES

[1] W. N. Hudson and J. D. Mason, *Operator-stable laws*, J. Multivariate Anal. 11 (1981), pp. 434–447.

[2] — and H. G. Tucker, *Operator-stable distributions with independent marginals*, Z. Wahrsch. verw. Gebiete 58 (1981), pp. 285–297.

[3] R. Jajte, *Semi-stable probability measures on R^N*, Studia Math. 61 (1977), pp. 29–39.

[4] Z. J. Jurek and J. D. Mason, *Operator-Limit Distributions in Probability Theory*, Wiley, New York 1993.

[5] A. Łuczak, *Operator semi-stable probability measures on R^N*, Colloq. Math. 45 (1981), pp. 287–300; *Corrigenda to "Operator semi-stable probability measures on R^N"*, ibidem 52 (1987), pp. 167–169.

[6] — *Independent marginals of infinitely divisible and operator semi-stable measures*, J. Multivariate Anal. 28 (1989), pp. 9–19.

[7] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York 1967.
[8] M. Sharpe, *Operator stable probability distributions on vector groups*, Trans. Amer. Math. Soc. 136 (1969), pp. 51–65.
[9] J. A. Veeh, *Infinitely divisible measures with independent marginals*, Z. Wahrsch. verw. Gebiete 61 (1982), pp. 303–308.

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Received on 9.5.1997