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SLOW CONVERGENCE TO NORMALITY: AN EDGEWORTH EXPANSION WITHOUT THIRD MOMENT

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Abstract. Let F be a non-lattice distribution function which lies in the domain of attraction of a normal distribution. Exact uniform convergence rates are obtained for the convergence of the normalized partial sums of i.i.d. random variables with distribution F. The assumptions are

$$1-F(x)+F(-x)\in RV_{\varrho-2} \quad (-1\leqslant \varrho\leqslant 0)$$

and

$$(1-F(x))/(1-F(x)+F(-x))\to p\in[0, 1] \quad (\text{as } x\to\infty).$$

For $\varrho=-1$ somewhat weaker conditions are sufficient.

1. Introduction. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution function F which lies in the domain of attraction of a normal law, i.e., the function $\int_{-x}^{x} y^2\,dF(y)$ $(x>0)$ is slowly varying at infinity. An equivalent condition is: the function

$$H(x):=\int_0^x \big(1-F(u)+F(-u)\big)u\,du$$

is slowly varying at infinity, that is,

$$\lim_{t\to\infty}\frac{H(tx)}{H(t)}=1 \quad \text{for all } x>0. \tag{1.1}$$

Then there exist constants $a_n>0$ and $b_n\in R$ such that

$$P\Big(\big(\sum_{i=1}^{n} X_i-b_n\big)/a_n\leqslant x\Big)\to \Phi(x):=(2\pi)^{-1/2}\int_{-\infty}^{x}\exp\{-u^2/2\}\,du \quad \text{for all } x\in R. \tag{1.2}$$

If the third moment is finite and F is non-lattice, the difference of the two terms in (1.2), multiplied by $\sqrt{n}$, converges $(n\to\infty)$ uniformly in x (see Petrov [9]).

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Assume now that the third moment does not exist. We are going to relate the uniform rate of convergence in (1.2) to the (pointwise) rate of convergence in (1.1).

A natural rate of convergence condition for (1.1) is the following. Suppose that there is a positive function $A^*(t)$ ($A^*(t) \to 0$ as $t \to \infty$) such that

$$\lim_{t\to\infty} \frac{H(tx)/H(t)-1}{A^*(t)}$$

exists for every $x > 0$. Then the limit function must be of the form

$$c' \frac{x^\varrho - 1}{\varrho}$$

for constants $\varrho \leq 0$ and $c' \in R$ (see Theorem 1.9 of Geluk and de Haan [3] or Lemma 3.2.1 of Bingham et al. [1]; $(x^0-1)/0$ is defined as $\log x$). Without loss of generality we can assume $c' = -1, 0$, or 1. The case $c' = 0$ is somewhat less informative, so we shall henceforth assume $c' = \pm 1$. So suppose there is a function A with $\lim_{t\to\infty} A(t) = 0$ and not changing sign near infinity, such that

$$\lim_{t\to\infty} \frac{H(tx)H(t)-1}{A(t)} = \frac{x^\varrho - 1}{\varrho} \quad \text{for all } x > 0. \tag{1.3}$$

The function $|A|$ is then regularly varying with index ϱ ($|A| \in RV_\varrho$). It can be proved (see the Appendix) that (1.3) is equivalent to the regular variation of $1-F(x)+F(-x)$ at infinity with index $\varrho - 2$. We shall prove that if this is the case and if the balance condition

$$\lim_{x\to\infty} \frac{1-F(x)}{1-F(x)+F(-x)} = p \in [0, 1] \tag{1.4}$$

is satisfied, then for a suitable choice of the sequences a_n and b_n the limit

$$\lim_{n\to\infty} \Delta_n / \{n(1-F(a_n)+F(-a_n))\}$$

exists, where

$$\Delta_n := \sup_{x\in R} \left|P\left(\left(\sum_{i=1}^n X_i - b_n\right)/a_n \leq x\right) - \Phi(x)\right|.$$

This will follow from the uniform convergence of

$$\frac{P\left(\left(\sum_{i=1}^n X_i - b_n\right)/a_n \leq x\right) - \Phi(x)}{n\left(1-F(a_n)+F(-a_n)\right)} \quad \text{as } n \to \infty,$$

a first order expansion of Edgeworth type.

In fact, in the case $\varrho = -1$, somewhat weaker conditions are sufficient (see Theorem 2). These conditions are implied by the condition $E|X|^3 < \infty$, so that the classical result is a special case of ours.

Our results are closest in spirit to the results of Hall [5] (cf. also [6] and [7]). We require three conditions: non-lattice distribution, regular variation of the combined tails, and the balance condition. Hall [5] only requires regular variation of both tails. The conclusions of Hall [5] are somewhat weaker: he proves upper and lower bounds whereas we have an actual limit. The balance condition and the non-lattice condition are necessary for our results. For further references see Hall [6].

It may be emphasized that the case $\varrho = 0$ allows for extremely low convergence rates (whether or not the variance exists, is immaterial).

The well-known inequalities of Berry and Esséen (cf. Feller [2]) are of a different type: they hold for any x but also for any n.

2. Results and proofs. Throughout most of this paper we assume

$$1-F(x)+F(-x)\in RV_{\varrho-2} \qquad (-1 \leqslant \varrho \leqslant 0).$$

Note that $1-F(x)+F(-x)\in RV_{\varrho-2}$ $(-1 \leqslant \varrho \leqslant 0)$ implies that $\int_{-x}^{x} y^2\,dF(y)$ $(x > 0)$ is a slowly varying function at infinity, so that F is in the domain of attraction of the normal distribution (see p. 83 of Ibragimov and Linnik [8]). An equivalent condition is: the function $H(x)$ is slowly varying at infinity. This implies $E|X| < \infty$, and so there is no loss of generality in supposing that $EX = 0$. We make this assumption throughout.

Since $x^{-2}H(x)\to 0$ as $x\to\infty$, the function

$$a(x) := \sup\{a\colon\ 2a^{-2}H(a) \geqslant x^{-1}\}$$

is well defined for all large x. For such values of x we have

$$2x\big(a(x)\big)^{-2}H\big(a(x)\big) = 1. \tag{2.1}$$

For large n define

$$a_n := a(n) \tag{2.2}$$

and $b_n = 0$. Relation (1.2) holds for such choices of a_n and b_n.

THEOREM 1. *Assume*

$$1-F(x)+F(-x)\in RV_{\varrho-2} \qquad (-1 < \varrho \leqslant 0),$$

$$\big(1-F(x)\big)/\big(1-F(x)+F(-x)\big)\to p\in[0,\,1] \qquad (as\ x\to\infty),$$

$EX = 0$ *and* F *is a non-lattice distribution function. Let* a_n *be defined by* (2.2). *Then*

$$\lim_{n\to\infty}\frac{P\big(\sum_{i=1}^{n}X_i/a_n \leqslant x\big)-\Phi(x)}{n\big(1-F(a_n)+F(-a_n)\big)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx}}{-it}\,C_\varrho(t)\exp\{-t^2/2\}\,dt \tag{2.3}$$

uniformly for all $x\in R$, *where*

$$C_\varrho(t) = \operatorname{sgn}(t)\,|t|^{2-\varrho}A_\varrho i+|t|^{2-\varrho}B_\varrho+t^2\,\frac{1-|t|^{-\varrho}}{\varrho},$$

$$A_\varrho = \frac{2p-1}{\varrho(\varrho-1)}\,\Gamma(1+\varrho)\sin\frac{\varrho\pi}{2}, \qquad B_\varrho = \frac{1}{\varrho(\varrho-1)}\,\Gamma(1+\varrho)\cos\frac{\varrho\pi}{2}+\frac{1}{\varrho}$$

(define $A_0 := \lim_{\varrho\to 0} A_\varrho = (1-2p)\pi/2$ *and* $B_0 := \lim_{\varrho\to 0} B_\varrho = \gamma-1$, γ *is the Euler constant), and*

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{for } t \geqslant 0, \\ -1 & \text{for } t < 0. \end{cases}$$

THEOREM 2. *Let*

$$K(y) := \int_0^y (1-F(x)+F(-x))\,x^2\,dx,$$

$$K_1(y) := \int_0^y (1-F(x))\,x^2\,dx \quad \text{and} \quad K_2(y) := \int_0^y F(-x)\,x^2\,dx.$$

Assume $\int_{-y}^{y} |x|^3\,dF(x) \in RV_0$, $K_1(x)/K(x) \to q \in [0, 1]$ *(as* $x\to\infty$*),* $EX = 0$, $EX^2 = 1$, *and* F *is a non-lattice distribution function. Then*

$$\lim_{n\to\infty} \frac{P\left(\sum_{i=1}^n X_i/\sqrt{n} \leqslant x\right) - \Phi(x)}{n^{-1/2} K(\sqrt{n})} = \frac{2q-1}{2\sqrt{2\pi}}(1-x^2)\exp\{-x^2/2\} \tag{2.4}$$

uniformly for all $x \in R$.

COROLLARY 1. *It follows that the supremum of the norm of the left-hand sides of* (2.3) *and* (2.4) *for* $x \in R$ *converges to the corresponding supremum of the norm of the right-hand sides, hence the uniform convergence rate.*

Remark 1. Note that $B_\varrho < 0$ for $-1 < \varrho \leqslant 0$ in Theorem 1 so that the limit in (2.3) is not identically zero.

Remark 2. $1-F(x)+F(-x) \in RV_{-3}$ implies $\int_{-y}^{y} |x|^3\,dF(x) \in RV_0$.

Remark 3. Theorem 1 implies that the sequence Δ_n is regularly varying with index $\varrho/2$. Theorem 2 implies that the sequence Δ_n is regularly varying with index $-1/2$. So the range of the index is $[-1/2, 0]$.

Remark 4. If F is non-lattice and $E|X|^3 < \infty$, the conditions of Theorem 2 are fulfilled and the classical result ensues.

Remark 5. The conditions of Theorem 1 for $\varrho = -1$ imply those of Theorem 2. But (2.3) does not hold for $\varrho = -1$.

THEOREM 3. *Assume*

$$1-F(x)+F(-x) \in RV_{\varrho-2} \quad (-1 < \varrho \leqslant 0),$$

$$(1-F(x))/(1-F(x)+F(-x)) \to p \in [0, 1] \quad (\text{as } x \to \infty),$$

$EX = 0$, *and* $|\mu|^k$ *is integrable for some* $k \geqslant 1$, *where* μ *denotes the characteristic function of* F. *Then*

$$\frac{\partial}{\partial x} P\left(\sum_{i=1}^n X_i/a_n \leqslant x\right) \text{ exists for } n \geqslant k$$

and

$$\lim_{n\to\infty} \frac{(\partial/\partial x) P\left(\sum_{i=1}^{n} X_i/a_n \leqslant x\right)-(\partial/\partial x)\Phi(x)}{n\left(1-F(a_n)+F(-a_n)\right)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} C_\varrho(t)\exp\{-t^2/2\}\,dt$$

uniformly for all $x\in R$, *where* $C_\varrho(t)$ *is defined in Theorem* 1.

THEOREM 4. *Assume that* $\int_{-y}^{y}|x|^3\,dF(x)\in RV_0$, $K_1(x)/K(x)\to q\in[0, 1]$ (*as* $x\to\infty$), $EX=0$, $EX^2=1$, *and* $|\mu|^k$ *is integrable for some* $k\geqslant 1$, *where* μ *denotes the characteristic function of* F. *Then*

$$\frac{\partial}{\partial x} P\left(\sum_{i=1}^{n} X_i/\sqrt{n}\leqslant x\right) \text{ exists for } n\geqslant k$$

and

$$\lim_{n\to\infty} \frac{(\partial/\partial x) P\left(\sum_{i=1}^{n} X_i/\sqrt{n} \leqslant x\right)-(\partial/\partial x)\Phi(x)}{n^{-1/2}K(\sqrt{n})} = -\frac{2q-1}{2\sqrt{2\pi}}(3x-x^3)\exp\{-x^2/2\}$$

uniformly for all $x\in R$.

For the proofs we need four lemmas.

LEMMA 1. *Assume*

$$1-F(x)+F(-x)\in RV_{\varrho-2}\qquad(-1<\varrho\leqslant 0),$$
$$\left(1-F(x)\right)/\left(1-F(x)+F(-x)\right)\to p\in[0, 1]\qquad(\text{as } x\to\infty),$$

and $EX=0$. *Let* μ *denote the characteristic function of* F. *Then*

$$\text{(2.5)}\quad \lim_{y\to\infty}\frac{\log\mu(t/y)+(t/y)^2H(y)}{1-F(y)+F(-y)} = |t|^{2-\varrho}C_\varrho(\operatorname{sgn}(t))+t^2\,\frac{1-|t|^{-\varrho}}{\varrho} = C_\varrho(t),$$

where $C_\varrho(t)$ *is defined in Theorem* 1.

Proof. Note

$$\mu(1/y)-1+y^{-2}H(y)$$
$$= \int_{-\infty}^{\infty}(e^{ix/y}-1-ix/y)\,dF(x)+y^{-2}\int_0^y\left(1-F(x)+F(-x)\right)x\,dx$$
$$= -\int_0^{\infty}(e^{ix}-1-ix)\,d\left(1-F(xy)\right)-\int_0^{\infty}(e^{-ix}-1+ix)\,dF(-xy)$$
$$+\int_0^1\left(1-F(xy)+F(-xy)\right)x\,dx$$
$$= i\int_0^{\infty}\left(1-F(xy)\right)(e^{ix}-1)\,dx-i\int_0^{\infty}F(-xy)(e^{-ix}-1)\,dx$$
$$+\int_0^1\left(1-F(xy)+F(-xy)\right)x\,dx$$

$$= i\int_1^\infty (1-F(xy))(e^{ix}-1)\,dx + i\int_0^1 (1-F(xy))(e^{ix}-1-ix)\,dx$$

$$-i\int_1^\infty F(-xy)(e^{-ix}-1)\,dx - i\int_0^1 F(-xy)(e^{-ix}-1+ix)\,dx$$

and

$$|e^{itx}-1| \leqslant 2, \qquad |e^{itx}-1-itx| \leqslant |tx|^2.$$

Combining

$$1-F(x)+F(-x) \in RV_{\varrho-2}, \qquad (1-F(x))/(1-F(x)+F(-x)) \to p$$

and Theorem 1.8 of Geluk and de Haan [3] (with $k(t) = (e^{\pm it}-1\mp it)/t^2$ for part (i) and $k(t) = (e^{\pm it}-1)/t$ for part (ij)), we get

$$\text{(2.6)} \quad \lim_{y\to\infty} \frac{\mu(1/y)-1+y^{-2}H(y)}{1-F(y)+F(-y)}$$

$$= i\int_1^\infty (e^{ix}-1)\,px^{\varrho-2}\,dx - i\int_1^\infty (e^{-ix}-1)(1-p)\,x^{\varrho-2}\,dx$$

$$+ i\int_0^1 (e^{ix}-1-ix)\,px^{\varrho-2}\,dx - i\int_0^1 (e^{-ix}-1+ix)(1-p)\,x^{\varrho-2}\,dx$$

$$= i(2p-1)\int_0^\infty (\cos x-1)\,x^{\varrho-2}\,dx - \int_1^\infty x^{\varrho-2}\sin x\,dx - \int_0^1 x^{\varrho-2}(\sin x - x)\,dx$$

$$:= C_\varrho(1).$$

We now work out the value of $C_\varrho(1)$ for $-1 < \varrho \leqslant 0$. If $\varrho = 0$, then

$$C_\varrho(1) = i(1-2p)\int_0^\infty \frac{1-\cos x}{x^2}\,dx - \sin 1 - \int_1^\infty \frac{\cos x}{x}\,dx + \sin 1 - 1 + \int_0^1 \frac{1-\cos x}{x}\,dx$$

$$= i(1-2p)\int_0^\infty \frac{1-\cos x}{x^2}\,dx + \int_0^1 \frac{1-\cos x}{x}\,dx - \int_1^\infty \frac{\cos x}{x}\,dx - 1$$

$$= i(1-2p)\,\pi/2 + \gamma - 1$$

(the last equation comes from 3.782 of Gradshtein and Ryzhik [4]), where γ is Euler constant.

If $\varrho < 0$, then

$$C_\varrho(1) = i\,\frac{2p-1}{\varrho-1}\int_0^\infty x^{\varrho-1}\sin x\,dx + \frac{1}{\varrho-1}\sin 1 + \frac{1}{\varrho-1}\int_1^\infty x^{\varrho-1}\cos x\,dx$$

$$-\frac{1}{\varrho-1}(\sin 1 - 1) + \frac{1}{\varrho-1}\int_0^1 x^{\varrho-1}(\cos x - 1)\,dx$$

$$= -i\frac{2p-1}{(\varrho-1)\varrho}\int_0^\infty x^\varrho \cos x\,dx - \frac{\cos 1}{\varrho(\varrho-1)} + \frac{1}{\varrho(\varrho-1)}\int_1^\infty x^\varrho \sin x\,dx$$

$$+\frac{\cos 1 - 1}{\varrho(\varrho-1)} + \frac{1}{\varrho(\varrho-1)}\int_0^1 x^\varrho \sin x\,dx + \frac{1}{\varrho-1}$$

$$= i\frac{1-2p}{\varrho(\varrho-1)}\Gamma(1+\varrho)\cos\frac{(1+\varrho)\pi}{2} + \frac{1}{\varrho(1-\varrho)}\Gamma(1+\varrho)\sin\frac{(1+\varrho)\pi}{2} + \frac{1}{\varrho}$$

(the last equation comes from 3.761 of Gradshtein and Ryzhik [4]).

Note that (2.6) implies

$$\lim_{y\to\infty}\frac{\mu(1/y)-1}{y^{-2}H(y)} = 1;$$

hence

$$\lim_{y\to\infty}\frac{|\mu(1/y)-1|^2}{1-F(y)+F(-y)} = \lim_{y\to\infty}\frac{|\mu(1/y)-1|^2}{(y^{-2}H(y))^2}\,\frac{(y^{-2}H(y))^2}{1-F(y)+F(-y)} = 0.$$

Since $\log(1+x) = x + O(x^2)$ as $x\to 0$, we thus find

$$\text{(2.7)}\qquad \lim_{y\to\infty}\frac{\log\mu(1/y)+y^{-2}H(y)}{1-F(y)+F(-y)} = C_\varrho(1) = A_\varrho i + B_\varrho,$$

where A_ϱ and B_ϱ are defined in Theorem 1. Similarly we can prove

$$\text{(2.8)}\qquad \lim_{y\to\infty}\frac{\log\mu(-1/y)+y^{-2}H(y)}{1-F(y)+F(-y)} = C_\varrho(-1) = -A_\varrho i + B_\varrho.$$

More generally, for $t\neq 0$ we have

$$\text{(2.9)}\qquad \frac{\log\mu(t/y)+(t/y)^2H(y)}{1-F(y)+F(-y)}$$

$$= \frac{\log\mu(t/y)+(t/y)^2H(y/|t|)}{1-F(y/|t|)+F(-y/|t|)}\,\frac{1-F(y/|t|)+F(-y/|t|)}{1-F(y)+F(-y)}$$

$$+t^2\int_{1/|t|}^{1}\frac{1-F(yx)+F(-yx)}{1-F(y)+F(-y)}\,x\,dx;$$

hence, by (2.7), (2.8) and $1-F(x)+F(-x)\in RV_{\varrho-2}$, (2.5) is proved. ∎

LEMMA 2. *Assume that* $\int_{-y}^{y}|x|^3\,dF(x)\in RV_0$, $K_1(x)/K(x)\to q\in[0, 1]$ *(as* $x\to\infty$*),* $EX=0$, *and* $EX^2=1$. *Let* μ *denote the characteristic function of* F. *Then*

$$\text{(2.10)}\qquad \lim_{y\to\infty}\frac{\log\mu(t/y)+t^2/(2y^2)}{y^{-3}K(y)} = -|t|^3\,\mathrm{sgn}(t)(q-1/2)\,i.$$

Proof. From the proof of Theorem 2 of Feller VIII.9 (cf. [2]) and $\int_{-y}^{y}|x|^3\,dF(x)\in RV_0$, we have

(2.11) $$\lim_{y\to\infty}\frac{y^3(1-F(y)+F(-y))}{\int_{-y}^{y}|x|^3\,dF(x)}=0.$$

Since

$$\int_{-y}^{y}|x|^3\,dF(x)=3\int_0^y(1-F(x)+F(-x))\,x^2\,dx-y^3(1-F(y)+F(-y)),$$

we have

$$\lim_{y\to\infty}\frac{K(y)}{\int_{-y}^{y}|x|^3\,dF(x)}=\frac{1}{3}.$$

Therefore

(2.12) $$K(y)\in RV_0 \quad\text{and}\quad \lim_{y\to\infty}\frac{y^3(1-F(y)+F(-y))}{K(y)}=0,$$

which implies

(2.13) $$\frac{y^3(1-F(y))}{K(y)}\to 0 \quad\text{and}\quad \frac{y^3F(-y)}{K(y)}\to 0.$$

Note that

$$\mu(1/y)-1+\frac{1}{2y^2}+\frac{i}{2y^3}(K_1(y)-K_2(y))$$

$$=-\int_0^\infty(e^{ix}-1-ix+x^2/2)\,d(1-F(xy))-\int_0^\infty(e^{-ix}-1+ix+x^2/2)\,dF(-xy)$$

$$+\frac{i}{2}\int_0^1(1-F(xy))\,x^2\,dx-\frac{i}{2}\int_0^1F(-xy)\,x^2\,dx$$

$$=i\int_1^\infty(e^{ix}-1-ix)(1-F(xy))\,dx+i\int_0^1(e^{ix}-1-ix+x^2/2)(1-F(xy))\,dx$$

$$-i\int_1^\infty(e^{-ix}-1+ix)\,F(-xy)\,dx-i\int_0^1(e^{-ix}-1+ix+x^2/2)\,F(-xy)\,dx$$

and

$$y^3(1-F(y)+F(-y))\leqslant 3K(y)\quad\text{for all } y>0.$$

Using (2.12) and (2.13), similar to the proof of Lemma 1, we can prove Lemma 2. ∎

LEMMA 3. *Assume*

$$1-F(x)+F(-x)\in RV_{\varrho-2}\quad(-1<\varrho\leqslant 0),$$

$$(1-F(x))/(1-F(x)+F(-x))\to p\in[0,1]\quad(\text{as } x\to\infty),$$

and $EX=0$. *Let* $\mathrm{Re}(z)$ *and* $\mathrm{Im}(z)$ *denote the real part and the imaginary part of a complex variable* z, *respectively. Then*

(i) *for any* $0<\varepsilon<\varepsilon_0:=(-B_\varrho)\wedge 1$, *there exists* $y_0>0$ *such that for* $y\geqslant y_0$, $y/|t|\geqslant y_0$

$$(B_\varrho-\varepsilon)(1+\varepsilon)|t|^{2-\varrho}\exp\{\varepsilon|\log|t||\}+t^2(1-\varepsilon)\frac{1-|t|^{-\varrho-\varepsilon}}{\varrho+\varepsilon}$$

$$\leqslant \operatorname{Re}\left(\frac{\log\mu(t/y)+(t/y)^2H(y)}{1-F(y)+F(-y)}\right)$$

$$\leqslant (B_\varrho+\varepsilon)(1-\varepsilon)|t|^{2-\varrho}\exp\{-\varepsilon|\log|t||\}+t^2(1+\varepsilon)\frac{1-|t|^{-\varrho+\varepsilon}}{\varrho-\varepsilon};$$

(ii) *for any*

$$0<\varepsilon<\varepsilon_1:=\begin{cases}1 & \textit{for } p=1/2,\\ |A_\varrho|\wedge 1 & \textit{for } p\neq 1/2,\end{cases}$$

there exists $y_0>0$ *such that for* $y\geqslant y_0$, $y/|t|\geqslant y_0$

$$\big(\operatorname{sgn}(t)A_\varrho-\varepsilon\big)\big(1-\varepsilon\operatorname{sgn}(\operatorname{sgn}(t)A_\varrho-\varepsilon)\big)|t|^{2-\varrho}\exp\{-\operatorname{sgn}(\operatorname{sgn}(t)A_\varrho-\varepsilon)\varepsilon|\log|t||\}$$

$$\leqslant \operatorname{Im}\left(\frac{\log\mu(t/y)+(t/y)^2H(y)}{1-F(y)+F(-y)}\right)$$

$$\leqslant \big(\operatorname{sgn}(t)A_\varrho+\varepsilon\big)\big(1+\varepsilon\operatorname{sgn}(t)\operatorname{sgn}(A_\varrho)\big)|t|^{2-\varrho}\exp\{\operatorname{sgn}(t)\operatorname{sgn}(A_\varrho)\varepsilon|\log|t||\}.$$

Proof. Using (2.7)–(2.9), $1-F(x)+F(-x)\in RV_{\varrho-2}$ and Potter bounds (see Bingham et al. [1]), we easily obtain the lemma. ∎

LEMMA 4. *Assume that* $\int_{-y}^{y}|x|^3\,dF(x)\in RV_0$, $K_1(x)/K(x)\to q\in[0,1]$ (*as* $x\to\infty$), $EX=0$, *and* $EX^2=1$. *Let* Re(z) *and* Im(z) *denote the real part and the imaginary part of a complex variable z, respectively. Then*

(i) *for any* $0<\varepsilon<1$, *there exists* $y_0>0$ *such that for* $y\geqslant y_0$, $y/|t|\geqslant y_0$

$$-\varepsilon(1+\varepsilon)|t|^3\exp\{\varepsilon|\log|t||\}$$

$$\leqslant \operatorname{Re}\left(\frac{\log\mu(t/y)+t^2/(2y^2)}{y^{-3}K(y)}\right)\leqslant \varepsilon(1+\varepsilon)|t|^3\exp\{\varepsilon|\log|t||\};$$

(ii) *for any*

$$0<\varepsilon<\varepsilon_1:=\begin{cases}|q-1/2| & \textit{for } q\neq 1/2,\\ 1 & \textit{for } q=1/2,\end{cases}$$

there exists $y_0>0$ *such that for* $y\geqslant y_0$, $y/|t|\geqslant y_0$

$$\big(\operatorname{sgn}(t)(1/2-q)-\varepsilon\big)\big(1-\varepsilon\operatorname{sgn}((1/2-q)\operatorname{sgn}(t)-\varepsilon)\big)|t|^3$$
$$\times\exp\{-\operatorname{sgn}((1/2-q)\operatorname{sgn}(t)-\varepsilon)\varepsilon|\log|t||\}$$

$$\leqslant \operatorname{Im}\left(\frac{\log\mu(t/y)+t^2/(2y^2)}{y^{-3}K(y)}\right)\leqslant \big(\operatorname{sgn}(t)(1/2-q)+\varepsilon\big)\big(1+\varepsilon\operatorname{sgn}(t)\operatorname{sgn}(1/2-q)\big)|t|^3$$
$$\times\exp\{\operatorname{sgn}(t)\operatorname{sgn}(1/2-q)\varepsilon|\log|t||\}.$$

The proof is similar to the proof of Lemma 3 by using Lemma 2.

Proof of Theorem 1. Define

$$A_n := n\big(1-F(a_n)+F(-a_n)\big), \qquad m_n := (A_n)^{(-1+\varepsilon)/(2-\varrho+\varepsilon)},$$

and

$$R(x) := \frac{A_n}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx}}{-it}\,C_\varrho(t)\exp\{-t^2/2\}\,dt.$$

Note that $m_n \to \infty$ and $n^{(1+\varepsilon)/2}m_n/a_n \to \infty$ since both sequences are regularly varying with positive indices. By Lemma 3,

$$n\log\mu(t/a_n)+t^2/2 \to 0 \tag{2.14}$$

uniformly for $|t| \leqslant m_n$ as $n \to \infty$.

Now

$$\begin{aligned}(2.15)\quad &\mu^n(t/a_n)-\exp\{-t^2/2\}\\ &= \big(n\log\mu(t/a_n)+t^2/2\big)\exp\{-t^2/2\}\exp\{\theta_n\big(n\log\mu(t/a_n)+t^2/2\big)\}\end{aligned}$$

for some θ_n with $|\theta_n| \in [0, 1]$, depending on t.

Since F is a non-lattice distribution, for any $\delta > 0$ there exists a sequence $\lambda(n)$ with $\lambda(n) \to \infty$ as $n \to \infty$ such that

$$\int_{\delta}^{\lambda(n)} |\mu^n(t)|t^{-1}\,dt = o\big(\exp\{-\sqrt{n}/2\}\big) \tag{2.16}$$

(see Lemma 3.3.1 of Ibragimov and Linnik [8]). It is easy to know that $A = \sup|\Phi'(x)+R'(x)| < \infty$ and $\exp\{-t^2/2\}+A_nC_\varrho(t)\exp\{-t^2/2\}$ is the Fourier Stieltjes transform of $\Phi - R$. Using the smoothing lemma with $T = \lambda(n)a_n$ (see Feller [2], XVI.3, Lemma 2), we get

$$\begin{aligned}&\sup_x\Big|P\big(\textstyle\sum_{i=1}^n X_i/a_n \leqslant x\big)-\Phi(x)-R(x)\Big|\\
&\leqslant \frac{1}{\pi}\int_{|t|\leqslant T}\left|\frac{\mu^n(t/a_n)-\exp\{-t^2/2\}-A_nC_\varrho(t)\exp\{-t^2/2\}}{t}\right|dt+\frac{24A}{\pi T}\\
&\leqslant \frac{1}{\pi}\int_{|t|\leqslant m_n}\left|\frac{\mu^n(t/a_n)-\exp\{-t^2/2\}-A_nC_\varrho(t)\exp\{-t^2/2\}}{t}\right|dt\\
&\quad+\frac{1}{\pi}\int_{m_n\leqslant|t|\leqslant T}\left|\frac{\mu^n(t/a_n)}{t}\right|dt\\
&\quad+\frac{1}{\pi}\int_{m_n\leqslant|t|\leqslant T}\left|\frac{\exp\{-t^2/2\}+A_nC_\varrho(t)\exp\{-t^2/2\}}{t}\right|dt+\frac{24A}{\pi T}.\end{aligned}$$

It is obvious that

$$\frac{1}{A_n}\int_{m_n\leqslant|t|\leqslant T}\left|\frac{\exp\{-t^2/2\}+A_nC_\varrho(t)\exp\{-t^2/2\}}{t}\right|dt \to 0$$

and

$$\frac{1}{A_n T} = \frac{1}{\lambda(n) a_n A_n} \to 0.$$

By (2.14), (2.15) and Lemma 3, we have

$$\frac{1}{A_n} \int_{|t| \leqslant m_n} \left| \frac{\mu^n(t/a_n) - \exp\{-t^2/2\} - A_n C_\varrho(t) \exp\{-t^2/2\}}{t} \right| dt \to 0.$$

In order to complete the proof of Theorem 1, we only need to prove

$$\text{(2.17)} \qquad \frac{1}{A_n} \int_{m_n \leqslant |t| \leqslant T} \left| \frac{\mu^n(t/a_n)}{t} \right| dt \to 0.$$

Since there exists $\delta > 0$ such that for $|t| \leqslant \delta a_n$

$$|\mu(t)| \leqslant \exp\{-\delta |t|^{d_0}\}, \quad d_0 = 2/(1+\varepsilon)$$

(see relation (4.2.7) of Ibragimov and Linnik [8]), note that $m_n/a_n \to 0$ $(n \to \infty)$. As n is large enough, we have

$$\frac{1}{A_n} \int_{m_n \leqslant |t| \leqslant \delta a_n} |\mu(t/a_n)|^n |t|^{-1} dt \leqslant \frac{1}{A_n} \int_{|t| \geqslant m_n} \exp\{-\delta n (a_n)^{-d_0} |t|^{d_0}\} |t|^{-1} dt$$

$$= \frac{1}{A_n} \int_{|t| \geqslant n^{1/d_0} m_n (a_n)^{-1}} \exp\{-\delta |t|^{d_0}\} |t|^{-1} dt \to 0 \quad (n \to \infty)$$

since $n^{1/d_0} m_n (a_n)^{-1} \to \infty$ $(n \to \infty)$. By (2.16) we have

$$\frac{1}{A_n} \int_{\delta a_n \leqslant |t| \leqslant T} \left| \frac{\mu^n(t/a_n)}{t} \right| dt = \frac{2}{A_n} \int_{\delta}^{\lambda(n)} \left| \frac{\mu^n(t)}{t} \right| dt \to 0.$$

Thus (2.17) holds. ∎

Proof of Theorem 2. Note that (see 3.952.4 of Gradshtein and Ryzhik [4])

$$\int_0^\infty t^2 \exp\{-t^2/2\} \cos(tx)\, dt = \sqrt{\pi/2}\,(1-x^2) \exp\{-x^2/2\}.$$

The proof is similar to the proof of Theorem 1 by using Lemma 4 instead of Lemma 3. ∎

Proof of Theorem 3. By the Fourier inversion theorem of Feller [2], XV.3,

$$\frac{\partial}{\partial x} P\Big(\sum_{i=1}^n X_i/a_n \leqslant x\Big) \text{ exists for all } n \geqslant k$$

and

$$\text{(2.18)} \quad \frac{\partial}{\partial x} P\Big(\sum_{i=1}^n X_i/a_n \leqslant x\Big) - \frac{\partial}{\partial x} \Phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \big(\mu^n(t/a_n) - \exp\{-t^2/2\}\big) dt.$$

The proof is similar to the proof of Theorem 1. ∎

Proof of Theorem 4. Note that (see 3.942.5 of Gradshtein and Ryzhik [4])

$$\int_0^\infty t^3 \exp\{-t^2/2\} \sin(tx)\,dt = \sqrt{\pi/2}(3x-x^3)\exp\{-x^2/2\}.$$

The proof is similar to the proof of Theorem 3. ■

Appendix. We now prove that (1.3) is equivalent to the regular variation of $1-F(x)+F(-x)$ at infinity with index $\varrho-2$. By Theorems 1.9 and 1.10 of Geluk and de Haan [3] or Lemma 3.2.1 of Bingham et al. [1], relation (1.3) is equivalent to the following:

For $\varrho<0$, $\lim_{x\to\infty} H(x)$ exists and

$$H(\infty)-H(x) = \int_x^\infty \big(1-F(u)+F(-u)\big)u\,du \in RV_\varrho.$$

For $\varrho = 0$ the function H is in the class Π. Equivalently,

$$\int_x^\infty \big(1-F(\sqrt{u})+F(-\sqrt{u})\big)\,du \in RV_{\varrho/2} \quad \text{or} \quad \in \Pi,$$

respectively. An application of the monotone density theorem (Propositions 1.7.11 and 1.19.5 of Geluk and de Haan [3]) completes the proof.

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