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REMARKS ON THE POSITIVITY OF DENSITIES OF STABLE LAWS*

BY

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Abstract. Let $0 < \alpha < \infty$, $\alpha \neq 1$, and $\mathcal{S}$ be a non-empty subset of $\boldsymbol{R}^d$, the d-dimensional Euclidean space. It is shown that if $\mathcal{S}$ satisfies $a\mathcal{S}+b\mathcal{S} = \mathcal{S}$ whenever $a, b \geqslant 0$ with $a^\alpha+b^\alpha = 1$, then $\mathcal{S}$ is a convex cone with vertex at 0. This, in particular, confirms a conjecture of Port and Vitale [4]. Using this result, an elementary, completely geometric and unified proof is provided for the following known result concerning the positivity properties of densities of α-stable laws on $\boldsymbol{R}^d$, $0 < \alpha < 2$, $\alpha \neq 1$: Let X be a strictly α-stable random vector in $\boldsymbol{R}^d$ with truly d-dimensional law μ, and let $p(t, \cdot)$ and σ be the density of $t^{1/\alpha}\cdot\mu$, the law of $t^{1/\alpha}X$, and the spectral measure of μ, respectively. If $0 < \alpha < 1$ and the support of σ is contained in a half-space, then, for any $t > 0$, $p(t, x) > 0$ if and only if x belongs to the interior of the convex cone generated by support of σ; and, in all other cases, $p(t, x) > 0$ for all $t > 0$ and $x \in \boldsymbol{R}^d$.

Let X be a strictly α-stable random vector in $\boldsymbol{R}^d$ with a truly d-dimensional law μ, where $0 < \alpha < 2$ and $\alpha \neq 1$; and let $p(t, \cdot)$ be the bounded continuous density of $t^{1/\alpha}\cdot\mu$, the law of $t^{1/\alpha}X$, $t > 0$. Let σ and $\mathscr{C}(\sigma)$ denote, respectively, the spectral measure of μ and the interior of the convex cone (equivalently, the closed convex cone) generated by $\operatorname{supp}(\sigma)$, the support of σ. Finally, let $S(t) = \{x\colon p(t, x) > 0\}$ for any $t > 0$.

It follows from the definition of $p(\cdot, \cdot)$ that

(1) $$p(t, x) = p(1, t^{-1/\alpha}x)t^{-d/\alpha}$$

for all $t > 0$ and $x \in \boldsymbol{R}^d$; and using the characteristic property

$$(t+s)^{1/\alpha}\cdot\mu = t^{1/\alpha}\cdot\mu * s^{1/\alpha}\cdot\mu \quad \text{for } t > 0,\ s > 0$$

of the law μ, we obtain

(2) $$p(t+s, \cdot) = p(t, \cdot) * p(s, \cdot)$$

for all $t, s > 0$. As noted in [4, p. 1019], it follows from (1), (2) and the continuity of $p(t, \cdot)$ that

$$(s+t)^{1/\alpha}S = s^{1/\alpha}S+t^{1/\alpha}S \quad \text{for all } s, t > 0;$$

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equivalently,

$$aS + bS = S \tag{3}$$

for all $a, b \geqslant 0$ with $a^\alpha + b^\alpha = 1$, where $S = S(1)$.

In [7], Taylor proved that if $0 < \alpha < 1$ and $\mathrm{supp}(\sigma)$ is contained in a half-space (throughout, by a *half-space* we mean a set of the type $\{y \geqslant 0\}$ for some $y \neq 0$), then $p(t, x) = 0$ for all $t > 0$ and $x \notin \mathscr{C}(\sigma)$; and he conjectured that $p(t, x) > 0$ for all $t > 0$ and $x \in \mathscr{C}(\sigma)$. Further, he formulated a theorem stating that in all other cases (e.g., $1 < \alpha < 2$ or $0 < \alpha < 1$ and $\mathrm{supp}(\sigma)$ is not contained in a half-space) $p(t, x) > 0$ for all $t > 0$ and $x \in \mathbf{R}^d$.

Port [3] pointed out that Taylor's proof of the above-stated theorem is incomplete and he provided a complete proof. Later, Kesten [2], using the Lévy–Itô representation of d-dimensional Lévy processes in an essential way, supplied a proof of Taylor's conjecture and gave also a different proof of the theorem for the case $0 < \alpha < 1$; but, for the case $1 < \alpha < 2$, Kesten could provide a proof of the theorem only under the additional *restrictive* condition that $\mathrm{supp}(\sigma)$ is not contained in a half-space. Port's proof of the theorem seems simpler and, unlike Kesten's proof, does not depend on properties of stochastic processes. Recently, Port and Vitale [4] provided an alternate proof of Taylor's conjecture; in addition, they showed that the analog of the theorem also holds for $\alpha = 1$. Port's proof of the theorem in [3] uses only (3) and some geometric arguments; and, as noted above, dos not make use of any ideas from the theory of stochastic processes. Similar remarks apply for the proof of the analog of the theorem for $\alpha = 1$ given in [4], except that, in this case, Port and Vitale use a modified version of (3). On the other hand, both the proofs of the conjecture given in [2] and [4] use, in an essential way, certain properties of stochastic processes. Port and Vitale [4] asked if a proof of the conjecture can be provided which (like the proofs of the theorem and its analog for $\alpha = 1$ given in [3] and [4], respectively) is based entirely on purely geometric methods, and which does not rely on any properties of processes.

To state the above question more precisely, we recall the definition of the open set S ($\equiv S(1)$). The crucial step in the Port–Vitale proof of Taylor's conjecture is to prove the following result: If $0 < \alpha < 1$, then (3) implies that S is a convex cone with vertex at 0. In one stage of the proof of this result they use properties of processes together with certain geometric arguments (see [4, p. 1020] for more on this point). In order to make their proof of Taylor's conjecture free of any properties of processes, they asked for a proof which is based on purely geometric arguments. In fact, they conjectured that the conclusion of this result is valid for *any* open subset $\mathscr{S}$ (i.e., one which is not necessarily of the type $S(1)$) satisfying (3) for any $\alpha \in (0, 1)$, and they asked for a geometric proof of this conjecture. (It must be pointed out that, except in the special case where $\mathscr{S} = S(1)$, no proof, geometric or otherwise, of this conjecture appears to be available in the literature. There seems to be some ambiguity about this point in [4].)

The *main* purpose of this paper is to provide an elementary and simple geometric proof of the Port–Vitale conjecture; in fact, a slightly more general result is proved in which we do not require that $\mathscr{S}$ be open, and, moreover, α can be any positive real number not equal to 1. Specifically, we prove the following result:

THEOREM 1. *Let $\mathscr{S}$ be a non-empty subset of $\boldsymbol{R}^d$ and $\alpha > 0$, $\alpha \neq 1$. If $\mathscr{S}$ satisfies*

$$(*) \qquad a\mathscr{S} + b\mathscr{S} = \mathscr{S}$$

for any $a, b \geqslant 0$ with $a^\alpha + b^\alpha = 1$, then $\mathscr{S}$ is a convex cone with vertex at 0.

Using Theorem 1, some separation arguments and known properties of α-stable densities on $\boldsymbol{R}$, we obtain easily the positivity properties of densities of α-stable laws on $\boldsymbol{R}^d$, $0 < \alpha < 2$, $\alpha \neq 1$ (Theorem 2). As noted above, the results included in Theorem 2 are known; however, since our methods provide a unified proof for both $0 < \alpha < 1$ and $1 < \alpha < 2$ cases, as opposed to the earlier known proofs, and since our proof is more succinct, at least for the case $\alpha > 1$, we have included this proof here. In the statement and proof of Theorem 2, we have used the notation introduced above.

THEOREM 2. *Let μ be a truly d-dimensional strictly α-stable law on $\boldsymbol{R}^d$ and let σ be the spectral measure of μ. If $1 < \alpha < 2$ or if $0 < \alpha < 1$ and* $\operatorname{supp}(\sigma)$ *is not contained in a half-space, then $p(t, x) > 0$ for all $t > 0$ and $x \in \boldsymbol{R}^d$. If $0 < \alpha < 1$ and* $\operatorname{supp}(\sigma)$ *is contained in a half-space, then, for any $t > 0$, $p(t, x) > 0$ if and only if x belongs to $\mathscr{C}(\sigma)$.*

We now proceed to prove Theorem 1. For the proof we need the following lemma:

LEMMA. *Let n be a positive integer and let $0 < \alpha < \infty$, $\alpha \neq 1$, and $\beta = \alpha(\alpha - 1)^{-1}$. Let a_k, b_k $(k = 1, \ldots, n)$ be real numbers satisfying $a_k > 0$, $b_k \geqslant 0$ and $\sum_{k=1}^n b_k^\alpha = 1$. Then we have the following*:

(i) *If $0 < \alpha < 1$, then $\sum_{k=1}^n a_k b_k \geqslant (\sum_{k=1}^n a_k^\beta)^{1/\beta}$.*

(ii) *If $\alpha > 1$, then $\sum_{k=1}^n a_k b_k \leqslant (\sum_{k=1}^n a_k^\beta)^{1/\beta}$.*

Proof of the Lemma. (i) Let $\alpha_1 = 1/\alpha$ and $\beta_1 = 1/(1-\alpha)$; then $1 < \alpha_1 < \infty$ and $1/\alpha_1 + 1/\beta_1 = 1$. Hence, we have

$$1 = \sum_{k=1}^n b_k^\alpha = \sum_{k=1}^n (a_k b_k)^\alpha a_k^{-\alpha}$$

$$\leqslant \Big(\sum_{k=1}^n (a_k b_k)^{\alpha\alpha_1}\Big)^{1/\alpha_1} \Big(\sum_{k=1}^n a_k^{-\alpha\beta_1}\Big)^{1/\beta_1} = \Big(\sum_{k=1}^n a_k b_k\Big)^\alpha \Big(\sum_{k=1}^n a_k^\beta\Big)^{1-\alpha},$$

which yields (i). The proof of (ii) follows by Hölder's inequality.

Proof of Theorem 1. We first show that if $0 < \alpha < 1$ (resp. $\alpha > 1$), then $s\mathscr{S} \subseteq \mathscr{S}$ whenever $0 < s \leqslant 1$ (resp. $s \geqslant 1$). Both of these facts are known and are proved, for instance, in [3, p. 368]. The proofs of these facts are easy and are included here for completeness. First note that

$$\{a+(1-a^\alpha)^{1/\alpha}\colon\ 0 \leqslant a \leqslant 1\} = [2^{1-1/\alpha}, 1] \text{ (resp. } [1, 2^{1-1/\alpha}])$$

whenever $0 < \alpha < 1$ (resp. $\alpha > 1$). Hence, from (*) it follows that if $0 < \alpha < 1$ (resp. $\alpha > 1$), then $s\mathcal{S} \subseteq \mathcal{S}$ whenever $2^{1-1/\alpha} \leqslant s \leqslant 1$ (resp. $1 \leqslant s \leqslant 2^{1-1/\alpha}$). The proof of these two facts now follows by iteration.

Next we shall prove that if $0 < \alpha < 1$, then

$$(4)\qquad s\mathcal{S} \subseteq \mathcal{S} \quad \text{for some } s \geqslant [2^d/(2^d-1)]^{1/\alpha-1} > 1,$$

and if $\alpha > 1$, then

$$(5)\qquad s\mathcal{S} \subseteq \mathcal{S} \quad \text{for some } s \leqslant [2^d/(2^d-1)]^{1/\alpha-1} < 1.$$

Once this is done, the proof of the theorem follows easily. Indeed, (4) and (5) and what we have proved above show that $\mathcal{S}$ is a cone; and this fact along with (*) implies that $\mathcal{S} = 2^{-1/\alpha}(\mathcal{S}+\mathcal{S}) = \mathcal{S}+\mathcal{S}$, and hence $\mathcal{S}$ is a convex cone with vertex at 0.

Now we proceed to prove (4) and (5). Let x be an arbitrary non-zero element of $\mathcal{S}$. Then, using (*) with $a = b = 2^{-1/\alpha}$, and a recursive argument, we can choose a positive integer $n \leqslant d$ and vectors u_j, v_j $(j = 1, 2, \ldots, n)$ from $\mathcal{S}$ such that $u_{k-1} = 2^{-1/\alpha}(u_k+v_k)$, $1 \leqslant k \leqslant n$, $u_k \notin \mathrm{sp}\{u_0, u_1, \ldots, u_{k-1}\}$, $1 \leqslant k \leqslant n-1$, and u_n (hence v_n) $\in \mathrm{sp}\{u_0, u_1, \ldots, u_{n-1}\}$, where $u_0 = x$ and $\mathrm{sp}(A)$ denotes the linear span of the set A. Set λ $(\equiv \lambda(\alpha)) = 2^{1-1/\alpha}$, $w_1 = u_0$, and $w_k = \lambda^{k-1}u_{k-1} - \lambda^{k-2}u_{k-2}$, $2 \leqslant k \leqslant n$. Then, clearly, $w_1, w_2, \ldots, w_n$ are linearly independent,

$$(6)\qquad u_0, u_k, v_k \in \mathrm{sp}\{w_1, w_2, \ldots, w_n\}, \quad 1 \leqslant k \leqslant n,$$

$$(7)\qquad u_k = \lambda^{-k}(w_1+w_2+\ldots+w_k+w_{k+1}),$$

and

$$(8)\qquad v_k = \lambda^{-k}(w_1+w_2+\ldots+w_k-w_{k+1})$$

for $1 \leqslant k \leqslant n-1$.

From (6) we have

$$(9)\qquad u_n = \lambda^{-n}\sum_{k=1}^{n} b_k w_k \quad \text{and} \quad v_n = \lambda^{-n}\sum_{k=1}^{n} c_k w_k$$

for some (unique) b_k's and c_k's. We shall show that we can choose $t_k, t'_k \geqslant 0$, $\varepsilon_k, \varepsilon'_k \in \{-1, 1\}$, $k = 1, 2, \ldots, n-1$, such that the numbers

$$b \equiv \lambda^{-n}b_1 + \sum_{k=1}^{n-1} t_k \quad \text{and} \quad c \equiv \lambda^{-n}c_1 + \sum_{k=1}^{n-1} t'_k$$

satisfy the equations

$$(10)\qquad u_n + \sum_{k=1}^{n-1} t_k(w_1+w_2+\ldots+w_k+\varepsilon_k w_{k+1}) = bw_1$$

and

$$v_n + \sum_{k=1}^{n-1} t'_k(w_1 + w_2 + \ldots + w_k + \varepsilon'_k w_{k+1}) = cw_1, \tag{11}$$

respectively. Recalling (9) and comparing the coefficients of w_j's in (10), we see that (10) is equivalent to the following set of equations:

$$\lambda^{-n} b_1 + \sum_{j=1}^{n-1} t_j = b, \quad \lambda^{-n} b_n + t_{n-1}\varepsilon_{n-1} = 0,$$
$$\lambda^{-n} b_k + \sum_{j=k}^{n-1} t_j + t_{k-1}\varepsilon_{k-1} = 0, \quad 2 \leqslant k \leqslant n-1. \tag{12}$$

The choice of t_k's and ε_k's is now obvious. In fact, take $t_{n-1} = \lambda^{-n}|b_n|$ and $\varepsilon_{n-1} = -\operatorname{sgn}(\lambda^{-n} b_n/t_{n-1})$, with the convention that $\operatorname{sgn}(0/0) = 1$; having specified $t_{n-1}, t_{n-2}, \ldots, t_k$ and $\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_k$, we take

$$t_{k-1} = \Big|\lambda^{-n} b_k + \sum_{j=k}^{n-1} t_j\Big| \quad \text{and} \quad \varepsilon_{k-1} = \Big\{\Big[\lambda^{-n} b_k + \sum_{j=k}^{n-1} t_j\Big]/t_{k-1}\Big\}, \quad 2 \leqslant k \leqslant n.$$

Then, clearly, b satisfies (12) and hence also (10); a similar choice of t'_k and ε'_k is used to show that c satisfies (11).

From (*) we clearly have $a_1\mathscr{S} + a_2\mathscr{S} + \ldots + a_n\mathscr{S} = \mathscr{S}$ whenever $a_j \geqslant 0$ with $\sum_{j=1}^{n} a_j^{\alpha} = 1$. Using this fact, (10), (11) and the fact that u_j and v_j belong to $\mathscr{S}$, we infer from (7) and (8) that $(1 + \sum_{k=1}^{n-1} \lambda^{k\alpha} t_k^{\alpha})^{-1/\alpha} b w_1$ belongs to $\mathscr{S}$. Thus, recalling that

$$b = \lambda^{-n} b_1 + \sum_{k=1}^{n-1} t_k$$

and writing

$$d_k = t_k \lambda^k \Big[1 + \sum_{j=1}^{n-1} t_j^{\alpha} \lambda^{j\alpha}\Big]^{-1/\alpha} \quad \text{if } 1 \leqslant k \leqslant n-1$$

and

$$d_n = \Big[1 + \sum_{k=1}^{n-1} t_k^{\alpha} \lambda^{k\alpha}\Big]^{-1/\alpha},$$

we obtain

$$s_0 \equiv \Big(1 + \sum_{k=1}^{n-1} \lambda^{k\alpha} t_k^{\alpha}\Big)^{-1/\alpha} b = \sum_{k=1}^{n-1} d_k \lambda^{-k} + b_1 d_n \lambda^{-n} \tag{13}$$

and so $s_0 x$ belongs to $\mathscr{S}$ (recall that $w_1 = u_0 = x$). Similarly, $s'_0 x$ belongs to $\mathscr{S}$, where

$$s'_0 \equiv \Big(1 + \sum_{k=1}^{n-1} \lambda^{k\alpha} t_k'^{\alpha}\Big)^{-1/\alpha} c = \sum_{k=1}^{n-1} d'_k \lambda^{-k} + c_1 d'_n \lambda^{-n}, \tag{14}$$

and d'_k's are defined as d_k's by replacing t_j by t'_j.

Next, recalling that $u_n + v_n = 2^{1/\alpha} u_{n-1}$, using (7) and (9), and comparing the coefficient of w_1, we have $b_1 + c_1 = 2^{1/\alpha} \cdot 2^{1-1/\alpha} = 2$. Therefore, either $b_1 \geqslant 1$ and $c_1 \leqslant 1$ or $b_1 \leqslant 1$ and $c_1 \geqslant 1$. In the first case, writing $\beta = \alpha(\alpha-1)^{-1}$ and noting that $\sum_{k=1}^{n} d_k^{\alpha} = 1$ and that $n \leqslant d$, we infer from the lemma and (12) and (13) that

$$s_0 \geqslant \sum_{k=1}^{n} d_k \lambda^{-k} \geqslant \Big(\sum_{k=1}^{n} \lambda^{-k\beta}\Big)^{1/\beta} = (1-1/2^n)^{1-1/\alpha} \geqslant (1-1/2^d)^{1-1/\alpha}$$

$$= (2^d/(2^d-1))^{(1/\alpha)-1} \quad \text{if } 0 < \alpha < 1;$$

similarly, using (14) rather than (13), we see that

$$s_0' \leqslant (2^d/(2^d-1))^{(1/\alpha)-1} \quad \text{if } \alpha > 1.$$

In the second case also (using (14) for the case $0 < \alpha < 1$ and (13) for the case $\alpha > 1$) one gets similar inequalities. This proves (4) and (5); and the proof is completed.

As noted above, for the proof of Theorem 2, we shall need positivity properties of α-stable densities on $\boldsymbol{R}$. These properties are known and are summarized (without proof), e.g., in [6]; these can also be deduced from results proved in [5] (note that the proofs of the results in [5] do not make use of any properties of 1-dimensional stable laws). In order to make our proof of Theorem 2 completely self-contained, we provide here elementary proofs of these positivity properties of α-stable densities on $\boldsymbol{R}$, $0 < \alpha < 2$, $\alpha \neq 1$. Our proof of these properties depends only on Theorem 1 and some standard facts about α-stable laws on $\boldsymbol{R}$. Let μ be a strictly α-stable law on $\boldsymbol{R}$, $0 < \alpha < 1$, $\alpha \neq 1$, and let

$$dF(x) = c_1 I(x > 0) x^{-(1+\alpha)} dx + c_2 I(x < 0)|x|^{-(1+\alpha)} dx$$

denote the Lévy measure of μ, where $0 \leqslant c_1,\ c_2 < \infty$, $c_1 + c_2 > 0$. We shall prove that if $0 < \alpha < 1$ and $c_2 = 0$ (resp. $c_1 = 0$), then $\boldsymbol{S} = \{p(1, x) > 0\} = \boldsymbol{R}^+ \equiv (0, \infty)$ (resp. $\boldsymbol{S} = \boldsymbol{R}^- \equiv (-\infty, 0)$); and, in all other cases, $\boldsymbol{S} = \boldsymbol{R}$.

To prove the above result, first we note that from Theorem 1 and (3) we have

$$\text{(15)} \qquad \boldsymbol{S} = \boldsymbol{R}^+ \text{ or } \boldsymbol{R}^- \text{ or } \boldsymbol{R}.$$

Let now $0 < \alpha < 1$ and $c_2 = 0$; then $\hat{\mu}$, the characteristic function of μ, is given by

$$\hat{\mu}(t) = \exp\Big[\int_{\boldsymbol{R}} (e^{itx} - 1) \frac{c_1 dx}{x^{1+\alpha}}\Big];$$

hence μ is the weak limit of the sequence $\{v_n\}$ of Poisson laws with

$$\hat{v}_n(t) = \exp\Big[\int_{n^{-1}}^{\infty} (e^{itx} - 1) \frac{c_1 dx}{x^{1+\alpha}}\Big].$$

Thus, since $\mathrm{supp}(\nu_n) \subseteq [0, \infty)$, we have $\mathrm{supp}(\mu) \subseteq [0, \infty)$. Therefore, by (15), we have $S = \boldsymbol{R}^+$. If $c_2 = 0$, a similar argument shows that $S = \boldsymbol{R}^-$; if both c_1 and c_2 are positive, then $\mu = \mu_1 * \mu_2$, where μ_1 (resp. μ_2) is the α-stable law with Lévy measure $c_1 I(x > 0)x^{-(1+\alpha)}dx$ (resp. $c_2 I(x < 0)\,|x|^{-(1+\alpha)}dx$). Hence, since (as shown above) the density of μ_1 (resp. of μ_2) is positive on $\boldsymbol{R}^+$ (resp. on $\boldsymbol{R}^-$), it follows that the density of μ is positive on $\boldsymbol{R}$. Now, let $\alpha > 1$. Then, in view of (15), in order to show that $S = \boldsymbol{R}$, it is enough to prove that $\mathrm{supp}(\mu)$ intersects both $\boldsymbol{R}^+$ and $\boldsymbol{R}^-$. This fact can be deduced from a result in [1, p. 539], which gives a criterion for the support of an i.d. probability measure to be contained in $[0, \infty)$. We shall, however, provide an alternative, more elementary proof for the above fact: If this fact were not true, then either $\mathrm{supp}(\mu) \subseteq [0, \infty)$ or $\mathrm{supp}(\mu) \subseteq (-\infty, 0]$. Suppose $\mathrm{supp}(\mu) \subseteq [0, \infty)$. Let X_j's be i.i.d. with the law of $X_1 = \mu$. Then, using the characteristic property of μ, the fact that $\alpha > 1$ and that $X_j \geqslant 0$ a.s., we have

$$X_1 \stackrel{\mathrm{d}}{=} \frac{X_1 + X_2 + \ldots + X_n}{n^{1/\alpha}} \geqslant \frac{X_1 + X_2 + \ldots + X_{[n^{1/\alpha}]}}{n^{1/\alpha}}$$

$$\geqslant \frac{X_1 \wedge N + X_2 \wedge N + \ldots + X_{[n^{1/\alpha}]} \wedge N}{n^{1/\alpha}} \quad \text{a.s.,}$$

where $X_j \wedge N$ denotes $\min(X_j, N)$. But, by Chebyshev's inequality, the right-hand side of this inequality converges in probability to $\mathrm{E}(X_1 \wedge N)$ as $n \to \infty$ for any N. Therefore, using the above inequalities, we obtain $X_1 \geqslant \mathrm{E}(X_1 \wedge N)$ a.s. for every N. This, along with the monotone convergence theorem, implies that $X_1 \geqslant \mathrm{E}(X_1)$ a.s.; which, in turn, implies that $X_1 = \mathrm{E}(X_1)$ a.s., a contradiction. Similarly, the assumption $\mathrm{supp}(\mu) \subseteq (-\infty, 0]$ leads to a contradiction.

Proof of Theorem 2. First we note that, by Theorem 1 and equation (3), $S \equiv S(1)$ is a convex cone with vertex at 0. Using this and (1), we have $S(t) = S$; thus, it is enough to prove the theorem for $p(1, \cdot)$. Now consider the first case; since S is a convex cone with vertex at 0, either $S = \boldsymbol{R}^d$ or $S \subseteq \{x\colon\ y(x) \geqslant 0\}$ for some $y \neq 0$. The second alternative is not possible; for, if it were true, then, on the one hand, the non-degenerate strictly α-stable law $\mu_y \equiv \mu \circ y^{-1}$ on $\boldsymbol{R}$ will be supported by $[0, \infty)$. On the other hand, as shown below, $\mathrm{supp}(\mu_y) = \boldsymbol{R}$. In fact, if $\alpha > 1$, then the fact that $\mathrm{supp}(\mu_y) = \boldsymbol{R}$ is proved above. If $0 < \alpha < 1$, then F_y, the Lévy measure of μ_y, is

$$c_1 I(x > 0)x^{-(1+\alpha)}dx + c_2 I(x < 0)|x|^{-(1+\alpha)}dx,$$

where $c_1 = \alpha F_y([1, \infty))$ and $c_2 = \alpha F_y((-\infty, -1])$. Using the formula

$$F(A) = \int_{\{\|u\|=1\}} \int_0^\infty I_A(su)s^{-(1+\alpha)}ds\,\sigma(du)$$

for the Lévy measure F of μ, we obtain

$$c_1 = \int_{\{y>0\}\cap\{\|u\|=1\}} y^\alpha(u)\sigma(du) \quad \text{and} \quad c_2 = \int_{\{y<0\}\cap\{\|u\|=1\}} |y(u)|^\alpha \sigma(du).$$

Since, by the hypothesis on $\operatorname{supp}(\sigma)$,

$$\sigma(\{y>0\}\cap\{\|u\|=1\})>0 \quad \text{and} \quad \sigma(\{y<0\}\cap\{\|u\|=1\})>0,$$

c_1 and c_2 are both positive. Hence, from the 1-dimensional result proved above, again $\operatorname{supp}(\mu_y) = \boldsymbol{R}$.

For the second case, one needs to prove that $\mathscr{C}(\sigma) = \boldsymbol{S}$. We first show that $\mathscr{C}(\sigma) \subseteq \boldsymbol{S}$ or, equivalently, $\mathscr{C}(\sigma) \subseteq \bar{\boldsymbol{S}}$, the closure of $\boldsymbol{S}$, since $\mathscr{C}(\sigma)$ is open and $\operatorname{Int}(\bar{\boldsymbol{S}}) = \operatorname{Int}(\boldsymbol{S}) = \boldsymbol{S}$. If the inclusion $\mathscr{C}(\sigma) \subseteq \bar{\boldsymbol{S}}$ were not true, then there would be a non-zero z in $\mathscr{C}(\sigma)$ such that $z \notin \bar{\boldsymbol{S}}$. Then, using a standard separation result, we can find a $y \in \boldsymbol{R}^d$ such that $y(z) < 0$ and $y(x) \geqslant 0$ for all $x \in \bar{\boldsymbol{S}}$. Thus, on the one hand, $\mu_y([0, \infty)) = 1$; therefore, using the 1-dimensional result proved above, we infer that the constant c_2 in the Lévy measure $c_1 I(x>0)x^{-(1+\alpha)}dx + c_2 I(x<0)|x|^{-(1+\alpha)}dx$ of μ_y must be 0. On the other hand, since $z = \sum t_j z_j$, $t_j > 0$ and $z_j \in \operatorname{supp}(\sigma)$, there exists a $z_0 \in \operatorname{supp}(\sigma)$ such that $y(z_0) < 0$. Therefore, as in the previous paragraph,

$$c_2 = \int_{\{y<0\}\cap\{\|u\|=1\}} |y(u)|^\alpha \sigma(du) > 0.$$

Hence we must have $\mathscr{C}(\sigma) \subseteq \bar{\boldsymbol{S}}$, and so $\mathscr{C}(\sigma) \subseteq \boldsymbol{S}$. A similar and, in fact, simpler argument shows that $\boldsymbol{S} \subseteq \mathscr{C}(\sigma)$, thus completing the proof.

Remarks. (i) If $\{X(t)\colon t \geqslant 0\}$ is a strictly α-stable process, $0 < \alpha < 2$, $\alpha \neq 1$, with stationary and independent increments such that $X(0) = 0$ and such that the law of $X(t)$ is equal to $t^{1/\alpha} \cdot \mu$, where μ is a truly d-dimensional strictly α-stable law on $\boldsymbol{R}^d$, then, clearly, the density of $X(t)$ is $p(t, \cdot)$ as defined in this note. The authors of [3], [4] and [7] defined the densities $p(t, \cdot)$, $t > 0$, by introducing the process $\{X(t)\colon t \geqslant 0\}$ as above. However, since we are not concerned with any aspects of stable processes in this note, we have introduced these densities in an alternate and more direct manner.

(ii) As noted earlier, Port and Vitale [4] proved that if $\{X(t)\colon t \geqslant 0\}$ is a Cauchy (i.e., α-stable, $\alpha = 1$) process in $\boldsymbol{R}^d$ and $p(t, \cdot)$ is the density of $X(t)$, then $p(t, x) > 0$ for all $t > 0$ and $x \in \boldsymbol{R}^d$. This, along with Theorem 2, provides purely geometric proofs for the positivity properties of densities of α-stable laws on $\boldsymbol{R}^d$ for all $\alpha \in (0, 2)$. In this regard, we would like to mention here that Rajput [5], independently of [4], proved several positivity and analyticity properties of densities of more general i.d. laws on $\boldsymbol{R}^d$, and deduced the above-noted positivity properties of stable densities as corollaries to these more general results. However, his methods of proof, unlike those used in Theorem 2 and Theorem 2 of [4], are non-geometric and use advanced results from measure theory (via known results on supports of i.d. laws) and the theory of complex variables.

(iii) We conclude this note by making two minor observations concerning the Port–Vitale proof of Theorem 2 of [4]. First, we note that, by using the elementary (and easy to prove) fact that the only dense convex subset of $\boldsymbol{R}^d$ is $\boldsymbol{R}^d$ itself, their proof can be simplified somewhat. This is pointed out next where we have used the same notation as in the proof of Theorem 2 of [4].

As noted in [4], using the property of 1-dimensional Cauchy density and a separation argument, we infer that $\overline{\mathrm{conv}(S)}$, and hence $\mathrm{conv}(S) = \boldsymbol{R}^d$. Hence, from (15) of [4] we have

$$\boldsymbol{R}^d = S - L\,(= S - [0,1]b). \tag{1'}$$

From (13) of [4] we have $S \supseteq S - [0, (2/\pi)\ln m]a$ for any m; hence

$$S \supseteq S + [0, \infty)b, \tag{2'}$$

where (recall) $b = -[(2/\pi)\ln(d+1)]a$. Now, using (1′), (2′) and an argument similar to the one used towards the end in the last but one paragraph of [4], one sees easily that $S = \boldsymbol{R}^d$.

The second observation concerning the Port–Vitale proof is that there seems to be a minor oversight in their proof: They seem to imply that $b = b(d) = -[(2/\pi)\ln(d+1)]a$, used in equation (15) of [4], and $b(2)$, used in the end of the last but one paragraph of the proof, are equal. But, as obviously it is not so, one cannot necessarily conclude, contrary to the assertion in [4], that $(1-\lambda)b(d)$ belongs to $L(2) \equiv [0, 1]b(2)$; hence a minor modification in the proof is needed: By iterating $\bar{S} + L(2) \subseteq \bar{S}$, one obtains $\bar{S} + [0, \infty)b(2)$ $(= \bar{S} + [0, \infty)b(d)) \subseteq \bar{S}$; therefore, necessarily $z + b(d) = z + \lambda b(d) + (1-\lambda)b(d) \in \bar{S}$. This completes the proof of the fact that $\bar{S} = \boldsymbol{R}^d$.

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