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CONDITIONED FUNCTIONAL CENTRAL LIMIT THEOREM FOR RANDOM PARTIAL SUMS

BY

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Abstract. Generalization of a conditioned functional central limit theorem of Szubarga and Szynal [6] is proved. It is shown that on a natural condition for random index randomly selected partial sums of independent, identically distributed random variables with zero mean and finite variance, suitably scaled, normed and conditioned to stay positive converge to the Brownian meander process.

1. Introduction. Let $\{\xi_n, n \geqslant 1\}$ be a sequence of independent, identically distributed random variables with $E\xi_1 = 0$ and $E\xi_1^2 = \sigma^2$, $0 < \sigma^2 < \infty$. Put $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$, $n \geqslant 1$. For the random walk $\{S_n, n \geqslant 0\}$ let $T = \inf\{n > 0\colon S_n \leqslant 0\}$, where the infimum of the empty set is taken to be $+\infty$. Since $P\{T > n\} > 0$ for each n, probabilities conditioned by this event are well defined. Finally, let X_n^+ be the random element of the space $D[0, 1]$ which is endowed with the Skorokhod topology, defined by

$$X_n^+(t) = (S_{[nt]}/\sigma n^{1/2} \mid T > n), \quad 0 \leqslant t \leqslant 1,$$

where $[x]$ is integer part of x.

Under the additional assumption

(1) $E|\xi_1|^3 < \infty$ and ξ_1 is nonlattice or integer valued with span 1,

Iglehart [3] (Theorem 3.4) showed the weak convergence of the sequence $\{X_n^+, n \geqslant 1\}$ to the Brownian meander process W^+ defined by

$$W^+(t) = |W(\tau + \beta t)|/\beta^{1/2}, \quad 0 \leqslant t \leqslant 1,$$

where W is the Brownian process, $\tau = \sup\{t \in [0, 1]\colon W(t) = 0\}$ and $\beta = 1 - \tau$. Bolthausen [2] (Theorem 3.2) demonstrated that the additional assumption (1) is superfluous in the above-mentioned convergence.

The main result of this paper shows that the assumption (1) may also be omitted in Theorem 4.8 of Iglehart [3] and in Theorem 3 of Szubarga and Szynal [6]. The proof presented here is inspired by the method of random change of time in Billingsley [1], Chapter 17, and it is simpler and more natural than proofs already known.

2. Main result. Let $D[0, \infty)$ be the space of real-valued right-continuous functions on $[0, \infty)$ having left limits. This space is considered with Lindval's metric defined in [4]. We define the random element Y_n of the space $D[0, \infty)$ by setting

$$Y_n(t) = S_{[nt]}/\sigma n^{1/2}, \quad 0 \leqslant t < \infty.$$

Let Y be a process defined as

$$Y(t) = \begin{cases} W^+(t), & 0 \leqslant t \leqslant 1, \\ W^+(1) + W(t-1), & t \geqslant 1, \end{cases}$$

where W^+ is a Brownian meander and W is an ordinary Brownian motion independent of W^+. By Donsker's theorem and Bolthausen result (cf. [2]) we have

$$(Y_n \mid T > n) \Rightarrow Y \quad \text{in } D[0, \infty),$$

where the symbol $\Rightarrow$ means weak convergence. Now we formulate the main result of this paper.

THEOREM 1. *Let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued random variables, defined on the same probability space as the sequence $\{\xi_n, n \geqslant 1\}$. If $\{\alpha_n, n \geqslant 1\}$ is a sequence of positive real numbers tending to infinity, such that*

$$\lim_{n \to \infty} \alpha_n^{1/2} P\{|N_n/\alpha_n - \alpha| \geqslant \varepsilon\} = 0 \quad \textit{for constant } \alpha > 0, \tag{2}$$

then

$$(Y_{N_n} \mid T > N_n) \Rightarrow Y \quad \textit{in } D[0, \infty). \ ■$$

Let r_1: $D[0, \infty) \to D[0, 1]$ be the restriction to $[0, 1]$ defined for $x \in D[0, \infty)$ as $r_1(x)(t) = x(t)$, $0 \leqslant t \leqslant 1$. This mapping is continuous for each $x \in D[0, \infty)$ for which $t = 1$ is a continuity point. Observe that $r_1(Y_n)$ and $r(Y)$ are equal in distribution to X_n and W^+, respectively. Thus from Theorem 5.1 in [1] and our Theorem 1 we have a generalization of Theorem 3 from [6] without the additional assumption (1).

COROLLARY 1. *If the assumptions of Theorem 1 hold, then*

$$(X_{N_n} \mid T > N_n) \Rightarrow W^+ \quad \textit{in } D[0, 1]. \ ■$$

3. Proof of Theorem 1. The proof of the main result requires two lemmas. The first lemma is of technical character only. The second one is a conditioned analogue of Theorem 17.1 in [1].

LEMMA 1. *For the process Y and a sequence $\{a_n; n \geqslant 1\}$ of positive real numbers tending to one, let*

$$Z_n(t) = a_n^{1/2} Y(t/a_n), \quad 0 \leqslant t < \infty.$$

Then

$$Z_n \Rightarrow Y \quad \text{in } D[0, \infty). \blacksquare$$

Proof. Denote by h_n, $n \geqslant 1$, mappings from $D[0, \infty)$ into $D[0, \infty)$ defined as

$$h_n(x)(t) = a_n^{1/2} x(t/a_n), \quad 0 \leqslant t < \infty.$$

Let E be the set of points x such that $h_n(x_n) \to x$ fails to hold for some sequence x_n approaching x. Since for every $x_n \to x$ with $x \in C[0, \infty)$ and for every $r \geqslant 0$ we have

$$\lim_{n\to\infty} \sup_{0\leqslant t\leqslant r} |h_n(x_n)(t) - x(t)| = 0,$$

by $P\{Y \in C[0, \infty)\} = 1$ we get $P\{Y \in E\} = 0$. Now using the relation $Z_n = h_n(Y)$ and Theorem 5.5 from [1] we get the assertion of the lemma. ■

Let $\{v_n, n \geqslant 1\}$ be a sequence of positive integer-valued random variables defined on the same probability space as the sequence $\{Y_n, n \geqslant 1\}$ and let $\{A_n, n \geqslant 1\}$ be a sequence of events from this common probability space, with positive probabilities.

LEMMA 2. *Let $\{\theta_n, n \geqslant 1\}$ be a sequence of positive real numbers tending to infinity such that*

$$\lim_{n\to\infty} P\{|v_n/\theta_n - \theta| \geqslant \varepsilon \mid A_{\theta_n}\} = 0 \tag{3}$$

for some constant $\theta > 0$. Furthermore, let

$$(Y_n \mid A_n) \Rightarrow K. \tag{4}$$

Then

$$(Y_{v_n} \mid A_{\theta_n}) \Rightarrow \tilde{K}, \quad \text{where } \tilde{K}(t) = \theta^{-1/2} K(\theta t). \blacksquare$$

Proof. As in the proof of Theorem 17.1 in [1] we may assume without restriction that θ_n, $n \geqslant 1$, are integer and define the random change of time

$$\Phi_n(t) = \frac{v_n}{\theta_n} t, \quad 0 \leqslant t < \infty.$$

From the equality

$$P\left\{\sup_{0\leqslant t\leqslant r} \left|\frac{v_n}{\theta_n} t - \theta t\right| \geqslant \varepsilon \mid A_{\theta_n}\right\} = P\left\{\left|\frac{v_n}{\theta_n} - \theta\right| \geqslant \varepsilon \mid A_{\theta_n}\right\}$$

and from the assumption (3) we get the convergence

$$((Y_{\theta_n}, \Phi_n) \mid A_{\theta_n}) \Rightarrow (K, \Phi),$$

where $\Phi(t) = \theta t$, $0 \leqslant t < \infty$. Therefore, from [1], Chapter 17, p. 144, we obtain

$$(Y_{\theta_n} \circ \Phi_n \mid \Lambda_{\theta_n}) \Rightarrow K \circ \Phi \quad \text{as } n \to \infty. \tag{5}$$

Now from the assumption (3) and from (5) we have

$$\lim_{n\to\infty} P\left\{\sup_{0\leqslant t\leqslant r}\left|\frac{1}{\sqrt{\theta}} Y_{\theta_n}\circ\Phi_n(t) - Y_{v_n}\right| \geqslant \varepsilon \mid \Lambda_{\theta_n}\right\}$$

$$= \lim_{n\to\infty} P\left\{\sup_{0\leqslant t\leqslant r}\left|\frac{1}{\sqrt{\theta}}\frac{1}{\sigma\sqrt{\theta_n}} S_{[v_n t]} - \frac{1}{\sigma\sqrt{v_n}} S_{[v_n t]}\right| \geqslant \varepsilon \mid \Lambda_{\theta_n}\right\}$$

$$= \lim_{n\to\infty} P\left\{\left|\frac{1}{\sqrt{\theta}} - \sqrt{\frac{\theta_n}{v_n}}\right| \sup_{0\leqslant t\leqslant r}\left|\frac{1}{\sigma\sqrt{\theta_n}} S_{[v_n t]}\right| \geqslant \varepsilon \mid \Lambda_{\theta_n}\right\} = 0,$$

which implies $(Y_{v_n} \mid \Lambda_{\theta_n}) \Rightarrow \theta^{-1/2} K \circ \Phi = \tilde{K}$. This completes the proof. ∎

Notice that for the assertion of Lemma 2 it is not necessary for $\{\xi_n, n \geqslant 1\}$ to be the sequence of independent random variables. It suffices that the assumption (4) holds for this sequence.

Using Lemma 2 with $\Lambda_{\theta_n} = P\{T > n\}$ from Bolthausen's result [2] and the asymptoticity of $P\{T > n\}$ (see (8)) we get

COROLLARY 2. *If the conditions of Theorem* 1 *with* $\alpha = 1$ *hold, then*

$$(X_{N_n} \mid T > n) \Rightarrow W^+ \quad \textit{in } D[0, 1]. \ ∎$$

Proof of Theorem 1. There is no loss of generality in assuming that α_n are integer and that $\alpha = 1$, since this can be arranged by passing to new constants α_n if necessary. Applying Lemma 2 with $\theta_n = \alpha_n$ and $\Lambda_{\theta_n} = \{T > N_n\}$, for the proof of our theorem it is enough to show that the following conditions hold:

$$\lim_{n\to\infty} P\{|N_n/\alpha_n - 1| \geqslant \varepsilon \mid T > N_n\} = 0 \tag{6}$$

and

$$(Y_{\alpha_n} \mid T > N_n) \Rightarrow Y. \tag{7}$$

From (2) and from the asymptoticity

$$\lim_{n\to\infty} n^{1/2} P\{T > n\} = c, \quad c > 0 \tag{8}$$

(see [5]) we get

$$\lim_{n\to\infty} \alpha_n^{1/2} P\{T > N_n\} = c \tag{9}$$

(see [6]). Hence the convergence (6) is a consequence of the assumption (2). By

Theorem 2.1 in [1] the convergence in (7) is equivalent to the condition that for all closed F of the space $D[0, \infty)$

$$\limsup_{n\to\infty} P\{Y_{\alpha_n}\in F \mid T > N_n\} \leqslant P\{Y\in F\}. \tag{10}$$

For fixed ε, $0<\varepsilon<1$, set $a_n = [(1-\varepsilon)\alpha_n]$, $b_n = [(1+\varepsilon)\alpha_n]$ and $A_n = \{k\colon a_n \leqslant k \leqslant b_n\}$. From (6) and the inequality

$$\begin{aligned} P\{Y_{\alpha_n}\in F,\ T>N_n\} &= P\{Y_{\alpha_n}\in F,\ T>N_n,\ N_n\in A_n\} \\ &\quad + P\{Y_{\alpha_n}\in F,\ T>N_n,\ N_n\in A_n^c\} \\ &\leqslant P\{Y_{\alpha_n}\in F,\ T>a_n\} + P\{N_n\in A_n^c,\ T>N_n\} \end{aligned}$$

we get

$$\limsup_{n\to\infty} P\{Y_{\alpha_n}\in F \mid T>N_n\} \leqslant \limsup_{n\to\infty} \frac{P\{T>a_n\}}{P\{T>N_n\}} P\{Y_{\alpha_n}\in F \mid T>a_n\}.$$

It follows from (8) and (9) that

$$\lim_{n\to\infty} \frac{P\{T>a_n\}}{P\{T>N_n\}} = \sqrt{\frac{1}{1-\varepsilon}}.$$

Applying Lemma 2 with $v_n = \alpha_n$, $\theta_n = a_n$ and $A_{\theta_n} = \{T>a_n\}$ we get

$$(Y_{\alpha_n} \mid T>a_n) \Rightarrow Y^\varepsilon,$$

where $Y^\varepsilon(t) = (1-\varepsilon)^{1/2}\, Y\big(t/(1-\varepsilon)\big)$ but from Lemma 1 we know that $Y^\varepsilon \Rightarrow Y$ as $\varepsilon\to 0$. Thus

$$\begin{aligned} \limsup_{n\to\infty} P\{Y_{\alpha_n}\in F \mid T>N_n\} &\leqslant \limsup_{\varepsilon\to 0}\limsup_{n\to\infty} \sqrt{1/(1-\varepsilon)}\, P\{Y_{\alpha_n}\in F \mid T>a_n\} \\ &\leqslant \limsup_{\varepsilon\to 0} P\{Y^\varepsilon\in F\} \leqslant P\{Y\in F\}. \end{aligned}$$

This completes the proof of the theorem. ■

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