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ON THE CENTRAL LIMIT THEOREM WITH ALMOST SURE CONVERGENCE

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Abstract. Let S_n be the partial sums of i.i.d.r.v.'s with zero means and variance one and let $a(x)$ be a real function. In this paper, sufficient conditions are given under which $a(S_n/\sqrt{n})$ converges almost surely to $\int_{-\infty}^{\infty} a(x)\,d\Phi(x)$. Two variants of convergence are considered: limitation of $a(S_n/\sqrt{n})$ by logarithmic means and limitation of $a(S_{n_k}/\sqrt{n_k})$ by arithmetic means, where $n_k = c^{k^\alpha}$, $\alpha > 0$, $c > 1$. Under the same assumptions in the same sense, $a(\max_{1 \leqslant m \leqslant n} S_m/\sqrt{n})$ converges almost surely to $2\int_0^{\infty} a(x)\,d\Phi(x)$.

1. Introduction and results. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables. Suppose that $EX_1 = 0$, $EX_1^2 = 1$, and let $S_n = X_1 + X_2 + \ldots + X_n$. Then $S_n/\sqrt{n}$ converges in distribution to the standard normal distribution $\Phi(x)$. In what follows the almost sure convergence of $S_n/\sqrt{n}$ shall be considered. If the sums S_n are reduced mod 1 to the interval $0 \leqslant S_n < 1$, then under certain assumptions the relation

$$\frac{1}{N}\sum_{n=1}^{N} a(S_n) \to \int_0^1 a(x)\,dx \qquad (N \to \infty) \tag{1}$$

holds almost surely (see [5], [3], [7] and [9]). Unfortunately, because of the normalizing factor $1/\sqrt{n}$, an analogous statement fails for the customary sums $S_n/\sqrt{n}$ (see [6], Theorem 1, and [8], Theorem 1). In order to overcome the strong dependencies among the $S_n/\sqrt{n}$, logarithmic means are applied.

THEOREM 1. *Let $a(x)$ be a real function which is a.e. continuous and for which $|a(x)| \leqslant e^{\gamma x^2}$, $\gamma < 1/4$. Then under the assumption $E|X_1|^{2+\delta} < \infty$, $\delta > 0$, we have*

$$P\left\{\lim_{N\to\infty} \frac{1}{\log N}\sum_{n=1}^{N} \frac{1}{n} a(S_n/\sqrt{n}) = \int_{-\infty}^{\infty} a(x)\,d\Phi(x)\right\} = 1. \tag{2}$$

COROLLARY 1. *Let* $\mathrm{E}|X_1|^{2+\delta} < \infty$, $\delta > 0$, *and* $\varrho > 0$. *Then*

$$P\left\{\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} n^{-1-\varrho} |S_n|^{2\varrho} = \frac{1}{\sqrt{\pi}} 2^{\varrho} \Gamma\left(\varrho + \frac{1}{2}\right)\right\} = 1, \tag{3}$$

e.g.

$$P\left\{\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \left(\frac{S_n}{n}\right)^2 = 1\right\} = 1. \tag{3a}$$

Another way to remove the strong dependencies among the $S_n/\sqrt{n}$ consists in the consideration of subsequences $n_k = c^{k^\alpha}$, $\alpha > 0$, $c > 1$.

THEOREM 2. *Under the assumptions of Theorem* 1 *we have*

$$P\left\{\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} a(S_{n_k}/\sqrt{n_k}) = \int_{-\infty}^{\infty} a(x)\, d\Phi(x)\right\} = 1. \tag{4}$$

COROLLARY 2. *Under the assumptions of Corollary* 1 *we have*

$$P\left\{\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} n_k^{-\varrho} |S_{n_k}|^{2\varrho} = \frac{1}{\sqrt{\pi}} 2^{\varrho} \Gamma\left(\varrho + \frac{1}{2}\right)\right\} = 1. \tag{5}$$

Theorems 1 and 2 can be found in [6] and Theorem 1 in [13]–[15] for bounded $a(x)$ only, of course without the corollaries. The proofs given here are straightforward and simpler than in [6]. The corollaries can be compared with Strassen's [12] result

$$P\left\{\limsup_{N\to\infty} N^{-1-\varrho} (2 \log\log N)^{-\varrho} \sum_{n=1}^{N} |S_n|^{2\varrho} = 2(2\varrho+2)^{\varrho-1} (2\varrho)^{-\varrho} \left(\int_0^1 \frac{dt}{\sqrt{1-t^{2\varrho}}}\right)^{-2\varrho}\right\} = 1, \tag{6}$$

where $\varrho \geqslant 1/2$ (cf. also [11], p. 296).

For the special case $a(x) = 1_{(-\infty,u)}(x)$, where $1_{(-\infty,u)}(\cdot)$ is the indicator function of the interval $-\infty < x < u$, the rate of convergence in (4) is estimated in [8]. Note that

$$F_n(u) = \sum_{k=1}^{K} 1_{(-\infty,u)}(S_{n_k}/\sqrt{n_k})$$

is the empirical distribution function of the weakly dependent random variables $S_{n_k}/\sqrt{n_k}$. Further assertions on $1_{(-\infty,u)}(S_n/\sqrt{n})$ can be found in [10]. In [6] an example is given that shows that Theorems 1 and 2 do not hold in general if $a(x)$ is only measurable.

Under the assumptions concerning X_i, the sequence $M_n/\sqrt{n}$,

$$M_n = \max_{1 \leqslant m \leqslant n} S_m,$$

converges in distribution to $2\Phi(x) - 1$. Also for this fact two almost sure variants can be given.

THEOREM 3. *Under the assumptions of Theorem* 1 *we have*

$$P\left\{\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}a(M_n/\sqrt{n}) = 2\int_0^\infty a(x)\,d\Phi(x)\right\} = 1 \tag{7}$$

and

$$P\left\{\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}a(M_{n_k}/\sqrt{n_k}) = 2\int_0^\infty a(x)\,d\Phi(x)\right\} = 1. \tag{8}$$

2. Proof of Theorem 1. We shall repeatedly assume that the X_n are normally distributed. In this case we denote their partial sums $X_1+\ldots+X_n$ by W_n. Then $W_n/\sqrt{n}$ is also normally distributed. The proof of Theorem 1 is performed in three steps.

2.1. Firstly, we assume that $a(x) = 1_{(-\infty,u)}(x)$ is the indicator function of the interval $-\infty < x < u$. We estimate the quantities

$$g_{jn} = \mathrm{E}\left\{\left(1_{(-\infty,u)}(W_j/\sqrt{j})-\Phi(u)\right)\left(1_{(-\infty,u)}(W_n/\sqrt{n})-\Phi(u)\right)\right\}$$

for $j < n$. Then

$$W_n/\sqrt{n} = (W_j/\sqrt{j})\sqrt{j/n}+(W_{j,n}/\sqrt{n-j})\sqrt{(n-j)/n},$$

where $W_{j,n} = X_{j+1}+\ldots+X_n$ is independent of W_j and normally distributed. Therefore

$$g_{jn} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-x^2/2}\Big\{\int_{-\infty}^{(\sqrt{n}u-\sqrt{j}x)/\sqrt{n-j}} e^{-y^2/2}dy\Big\}\,dx-\Phi^2(u)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-x^2/2}\left\{\Phi\left(\frac{\sqrt{n}u-\sqrt{j}x}{\sqrt{n-j}}\right)-\Phi(u)\right\}dx.$$

Since $\sqrt{n}/\sqrt{(n-j)}-1 \leqslant \sqrt{j}/\sqrt{n-j}$, we obtain

$$|g_{jn}| \leqslant \frac{1}{2\pi}\sqrt{\frac{j}{n-j}}\int_{-\infty}^{u}(|u|+|x|)\,e^{-x^2/2}dx \leqslant C_1\sqrt{\frac{j}{n-j}} \tag{9}$$

with a constant C_1. Moreover, we have trivially $|g_{jn}| \leqslant 1$, and it follows that

$$\mathrm{E}\left\{\sum_{n=1}^{N}\frac{1}{n}\left(1_{(-\infty,u)}(W_n/\sqrt{n})-\Phi(u)\right)\right\}^2 \leqslant 2\sum_{n=1}^{N}\sum_{j=1}^{n}\frac{|g_{jn}|}{jn}$$

$$\leqslant 2C_1\sum_{n=1}^{N}\frac{1}{n}\sum_{j=1}^{[n/2]-1}\sqrt{\frac{1}{j(n-j)}}+2\sum_{n=1}^{N}\frac{1}{n}\sum_{j=[n/2]}^{n}\frac{1}{j}.$$

If we apply the simple estimate

$$\sqrt{\frac{1}{j(n-j)}} \leqslant \int_{(j-1)/n}^{j/n}\frac{dt}{\sqrt{t(1-t)}},$$

we arrive at

$$\mathrm{E}\left\{\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(1_{(-\infty,u)}(W_n/\sqrt{n})-\Phi(u)\right)\right\}^2 \leqslant C_2/\log N.$$

Now we put $N_k = 2^{k^2}$. Then, by using standard arguments, we conclude from Chebyshev's inequality and the Borel–Cantelli lemma that

$$\frac{1}{\log N_k}\sum_{n=1}^{N_k}\frac{1}{n}\left(1_{(-\infty,u)}(W_n/\sqrt{n})-\Phi(u)\right)\to 0 \qquad (k\to\infty)$$

with probability one. On the other hand,

$$\frac{1}{\log N_k}\sum_{n=N_k+1}^{N}\ldots \leqslant C_3/k\to 0 \qquad (k\to\infty)$$

for $N_k < N < N_{k+1}$ and, consequently,

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(1_{(-\infty,u)}(W_n/\sqrt{n})-\Phi(u)\right) = 0 \text{ a.s.}$$

Thus the first step is completed if the X_n are normally distributed. If not, we approximate and obtain

$$S_n - W_n = \varepsilon_n(\omega)\, n^{1/(2+\delta)}, \tag{10}$$

where $\varepsilon_n(\omega)\to 0$ as $n\to\infty$ for almost all ω (see [2], Theorem 2.6.3, or [4]). But then

$$1_{(-\infty,u)}(S_n/\sqrt{n}) \leqslant 1_{(-\infty,u+\eta_n)}(W_n/\sqrt{n}),$$

where

$$\eta_n = \sup_{j\geqslant n}|\varepsilon_j(\omega)|\, j^{-\delta/2(2+\delta)}\to 0 \qquad \text{as } n\to\infty.$$

It follows that

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}1_{(-\infty,u)}(S_n/\sqrt{n}) \leqslant \frac{1}{\log N}\left\{\sum_{n=1}^{M-1}\frac{1}{n}+\sum_{n=M}^{N}\frac{1}{n}1_{(-\infty,u+\eta_M)}(W_n/\sqrt{n})\right\}$$

$$\leqslant (1+\log M)/\log N + \Phi(u+\eta_M)+\varepsilon/2 \leqslant \Phi(u)+\varepsilon$$

for sufficiently large N and suitable $M = M(N)$. Correspondingly the left-hand sum can be bounded below, and (2) is also established for X_n with a general distribution for our special $a(x)$.

2.2. Secondly, we consider $a(x) = e^{\gamma x^2}$, $\gamma < 1/4$. We remark that $\mathrm{E}a(W_n/\sqrt{n}) = 1/\sqrt{1-2\gamma}$ and consider

$$h_{jn} = \mathrm{E}\left\{\left(a(W_j/\sqrt{j})-1/\sqrt{1-2\gamma}\right)\left(a(W_n/\sqrt{n})-1/\sqrt{1-2\gamma}\right)\right\}$$

for $j < n$. Then

$$h_{jn} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{\gamma x^2 + \gamma(x\sqrt{j/n} + y\sqrt{(n-j)/n})^2 - x^2/2 - y^2/2\}\, dx\, dy$$
$$-1/(1-2\gamma)$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{\frac{\gamma}{n}(x^2 j + 2xy\sqrt{j(n-j)} - y^2 j) - 1\right\}$$
$$\times \exp\left(\left(\gamma - \frac{1}{2}\right)(x^2+y^2)\right) dx\, dy,$$

$$|h_{jn}| \leqslant C_4 \sqrt{\frac{j}{n-j}}, \tag{11}$$

since $|e^a - 1| \leqslant |a|(e^a + 1)$ and

$$x^2 j + 2xy\sqrt{j(n-j)} - y^2 j \leqslant x^2 n. \tag{12}$$

Thus as in the first step we arrive at

$$\frac{1}{\log N_k} \sum_{n=1}^{N_k} \frac{1}{n}(\exp(\gamma W_n^2/n) - 1/\sqrt{1-2\gamma}) \to 0 \qquad (k \to \infty)$$

with probability one. On the other hand, by the law of the iterated logarithm, we have

$$\exp(\gamma W_n^2/n) \leqslant (\log n)^{\gamma + 1/4} \text{ a.s.} \tag{13}$$

for sufficiently large n and, consequently,

$$\frac{1}{\log N_k} \sum_{n=N_k+1}^{N_{k+1}} \frac{1}{n}(\exp(\gamma W_n^2/n) - 1/\sqrt{1-2\gamma}) = O(k^{2\gamma - 1/2}) \to 0.$$

Thus the second step is also completed if the X_n are normally distributed. In the general case we conclude from (10) and from the law of the iterated logarithm that

$$|\exp(\gamma S_n^2/n) - \exp(\gamma W_n^2/n)| \leqslant \frac{2\gamma}{n} |S_n - W_n| \max\{|S_n| \exp(\gamma S_n^2/n), |W_n| \exp(\gamma W_n^2/n)\}$$
$$\leqslant 4\gamma |\varepsilon_n(\omega)| n^{-\delta/2(2+\delta)} \sqrt{\log\log n \cdot \log n} = O(n^{-\varepsilon}),$$

where $\varepsilon > 0$. Thus (2) is established again for the special $a(x)$.

2.3. Finally, we let $a(x)$ fulfil the assumptions of Theorem 1. We introduce an auxiliary function $a_1(x)$ which vanishes for $|x| > K$ and is in each of the intervals $-K + 2iK/L \leqslant x < -K + 2(i+1)K/L$, $i = 0, 1, \ldots, L-1$, equal to the supremum of $a(x) - e^{\gamma x^2}$ in these intervals. We put $a_2(x) = a_1(x) + e^{\gamma x^2}$ and choose first K and then L large enough such that

$$\int_{-\infty}^{\infty} a_2(x)\,d\Phi(x) \leqslant \int_{-\infty}^{\infty} a(x)\,d\Phi(x)+\varepsilon/2.$$

This is possible since $a(x)$ is continuous a.e. and, consequently, Riemann–Stieltjes integrable with respect to $\Phi(x)$. Obviously, $a(x) \leqslant a_2(x)$ for all real x. On the other hand, $a_2(x)$ is a finite linear combination of the special functions considered in the first and second steps, respectively. Therefore

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}a(S_n/\sqrt{n}) \leqslant \frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}a_2(S_n/\sqrt{n})$$
$$\leqslant \int_{-\infty}^{\infty} a_2(x)\,d\Phi(x)+\varepsilon/2 \leqslant \int_{-\infty}^{\infty} a(x)\,d\Phi(x)+\varepsilon$$

for almost all ω and sufficiently large N. Replacing $a(x)$ by $-a(x)$ we obtain the assertion of Theorem 1. ∎

3. Proof of Theorem 2. The proof of Theorem 2 is quite analogous to that of Theorem 1.

3.1. We assume that $a(x) = 1_{(-\infty,u)}(x)$. From (9) it follows that

$$\mathrm{E}\{\sum_{k=1}^{K}\left(1_{(-\infty,u)}(W_{n_k}/\sqrt{n_k})-\Phi(u)\right)\}^2 \leqslant 2\sum_{k=1}^{K}\sum_{j=1}^{k}|g_{n_j n_k}|$$
$$\leqslant 2C_1\sum_{k=1}^{K}\sum_{j=1}^{k-1}\sqrt{\frac{c^{j^\alpha}}{c^{k^\alpha}-c^{j^\alpha}}}+2C_1^*K.$$

In the case of $0<\alpha\leqslant 1$ we apply the simple estimate

$$\frac{c^{j^\alpha}}{c^{k^\alpha}-c^{j^\alpha}} \leqslant \frac{1}{\alpha\log c}\frac{k^{1-\alpha}}{k-j}$$

and obtain

$$\mathrm{E}\left\{\frac{1}{K}\sum_{k=1}^{K}\left(1_{(-\infty,u)}(W_{n_k}/\sqrt{n})-\Phi(u)\right)\right\}^2 = O(K^{-\alpha}\log K).$$

We put $K_k = [k^{\beta/\alpha}]$, $\beta = 8/(4\gamma+3) > 2$, and arrive at

$$\frac{1}{K_k}\sum_{k=1}^{K_k}\left(1_{(-\infty,u)}(W_{n_k}/\sqrt{n_k})-\Phi(u)\right)\to 0 \qquad (k\to\infty)$$

with probability one. In the case of $\alpha>1$ we have

$$\frac{c^{j^\alpha}}{c^{k^\alpha}-c^{j^\alpha}} \leqslant \frac{1}{\alpha\log c(k-j)}$$

and we can reason as in the case $\alpha = 1$. Further

$$\frac{1}{K_k}\sum_{j=K_k+1}^{K_{k+1}}\ldots \leqslant \frac{C_7}{k}\to 0 \qquad (k\to\infty)$$

and, consequently,

$$\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}\left(1_{(-\infty,u)}(W_{n_k}/\sqrt{n_k})-\Phi(u)\right)=0 \text{ a.s.}$$

The transition from W_{n_k} to S_{n_k} can be performed as in the first step of the proof of Theorem 1.

3.2. We assume that $a(x)=e^{\gamma x^2}$. From (11) it follows that

$$\frac{1}{K_k}\sum_{j=1}^{K_k}\left(a(W_{n_j}/\sqrt{n_j})-1/\sqrt{1-2\gamma}\right)\to 0 \qquad (k\to\infty)$$

with probability one. Further by (13) we have

$$\frac{1}{K_k}\sum_{j=K_k+1}^{K_{k+1}}\left(a(W_{n_j}/\sqrt{n_j})-1/\sqrt{1-2\gamma}\right)=O(k^{\beta(\gamma+1/4)-1})\to 0$$

and, consequently,

$$\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}\left(a(W_{n_k}/\sqrt{n_k})-\Phi(u)\right)=0 \text{ a.s.}$$

The transition from W_{n_k} to S_{n_k} can be performed as in the second step of the proof of Theorem 1.

3.3. In the general case there are only minor differences to the third step of the proof of Theorem 1 which can be suppressed here.

4. Proof of Theorem 3. We put

$$V_n=\sup_{0\leqslant t\leqslant n} W(t),$$

where $W(t)$ is a Wiener process.

4.1. We assume that $a(x)=1_{(-\infty,u)}(x)$, $u>0$, and estimate the quantities

$$g_{jn}^{*}=\mathrm{E}\left\{\left(1_{(-\infty,u)}(V_j/\sqrt{j})-2\Phi(u)+1\right)\left(1_{(-\infty,u)}(V_n/\sqrt{n})-2\Phi(u)+1\right)\right\}$$

for $j<n$. Then

$$V_n=\max\{V_j,\ W(j)+V_{j,n}\},\qquad \text{where } V_{j,n}=\sup_{j\leqslant t\leqslant n}\{W(t)-W(j)\}.$$

Now the random vector $(V_j/\sqrt{j},\ W(j)/\sqrt{j})$ is distributed with the density

$$p(x,y)=\begin{cases}\dfrac{2}{\sqrt{2\pi}}(2x-y)\exp\{-(2x-y)^2/2\} & \text{if } x\geqslant 0,\ y\leqslant x,\\ 0 & \text{otherwise}\end{cases}$$

(cf. [1], equality (11.11), p. 79). Hence $(V_j/\sqrt{j},\ W(j)/\sqrt{j},\ V_{j,n}/\sqrt{n-j})$ is distributed with the density

$$p(x, y, z) = \begin{cases} \frac{2}{\pi}(2x-y)\exp\{-(2x-y)^2/2-z^2/2\} & \text{if } x \geqslant 0,\ y \leqslant x,\ z \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

But if $V_j/\sqrt{j} = x,\ W(j)/\sqrt{j} = y,\ V_{j,n}/\sqrt{n-j} = z$, then

$$V_n = \max\{x\sqrt{j},\ y\sqrt{j}+z\sqrt{n-j}\},$$

and therefore

$$g_{jn}^* = \int_{x=0}^{u} \int_{y=-\infty}^{x} \int_{z=0}^{(u\sqrt{n}-y\sqrt{j})/\sqrt{n-j}} p(x, y, z)\,dx\,dy\,dz - (2\Phi(u)-1)^2$$

$$= 2\int_{x=0}^{u} \int_{y=-\infty}^{x} \left\{\Phi\left(\frac{u\sqrt{n}-y\sqrt{j}}{\sqrt{n-j}}\right)-\Phi(u)\right\} p(x, y)\,dx\,dy,$$

$$|g_{jn}^*| \leqslant C_5\sqrt{\frac{j}{n-j}}. \tag{14}$$

We obtain

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n}\left(1_{(-\infty,u)}(V_n/\sqrt{n})-2\Phi(u)+1\right) = 0 \text{ a.s.}$$

Now we have

$$\max_{0\leqslant k\leqslant n}\ \sup_{0\leqslant t\leqslant 1} |W(k+t)-W(k)| \leqslant 2\sqrt{\log n}$$

for sufficiently large n (cf., e.g., [2], Theorem 1.2.1); choose $a_T = 1$. From this and from (10) we find that

$$M_n - V_n = \varepsilon_n^*(\omega)\,n^{1/(2+\delta)} \tag{15}$$

holds, where $\varepsilon_n^*(\omega)\to 0$ as $n\to\infty$ for almost all ω. Thus we can finish as in 2.1.

4.2. We assume that $a(x) = e^{\gamma x^2}$ and remark that $\mathrm{E}a(V_n/\sqrt{n}) = 1/\sqrt{1-2\gamma}$. We consider

$$h_{jn}^* = \mathrm{E}\left\{\left(a(V_j/\sqrt{j})-1/\sqrt{1-2\gamma}\right)\left(a(V_n/\sqrt{n})-1/\sqrt{1-2\gamma}\right)\right\}$$

for $j < n$. Clearly,

$$V_n = \begin{cases} x\sqrt{j} & \text{if } z \leqslant z^* = (x-y)\sqrt{j/(n-j)}, \\ y\sqrt{j}+z\sqrt{n-j} & \text{if } z > z^*. \end{cases}$$

Thus we get

$$h_{jn}^* = \iiint_{z>z^*} \exp\left\{\frac{\gamma}{n}\left(x^2 n + y^2 j + 2yz\sqrt{j(n-j)} + z^2(n-j)\right)\right\} p(x, y, z)\, d\tau$$

$$+ \iiint_{z\leqslant z^*} \exp\left\{\frac{\gamma}{n} x^2(n+j)\right\} p(x, y, z)\, d\tau - 1/(1-2\gamma)$$

$$= \iiint_{z>z^*} e^{\gamma(x^2+z^2)} \left\{\exp\left(\frac{\gamma}{n}\left(y^2 j + 2yz\sqrt{j(n-j)} - z^2 j\right)\right) - 1\right\} p(x, y, z)\, d\tau$$

$$+ \iiint_{z\leqslant z^*} e^{\gamma x^2}\left(e^{\gamma x^2 j/n} - e^{\gamma z^2}\right) p(x, y, z)\, d\tau,$$

where $d\tau = dx\,dy\,dz$. It follows that

$$|h_{nj}^*| \leqslant C_6 \sqrt{\frac{j}{n-j}} \tag{16}$$

on account of $|e^a - 1| \leqslant |a|(e^a + 1)$, (12), and

$$\int_0^\infty \int_{-\infty}^x \int_0^\infty (y^2 + z^2) \exp\{\gamma(x^2 + y^2 + z^2)\}\, p(x, y, z)\, d\tau < \infty,$$

where the last estimate holds since $y^2 \leqslant (2x-y)^2$ in the domain of integration. If the law of the iterated logarithm for V_n is applied, then we arrive at

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n}\left(\exp(\gamma V_n^2/n) - 1/\sqrt{1-2\gamma}\right) = 0 \ \text{ a.s.}$$

In the general case we must apply (15) and the law of the iterated logarithm for the M_n.

4.3. For general $a(x)$ we can conclude as in 2.3. Thus (7) is proved. In order to prove (8) we can proceed as in the proof of Theorem 2 (cf. Section 3).

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