Interpretations in Philosophical Logic

Albert Visser

Theoretical Philosophy, Department of Philosophy,
Faculty of Humanities Utrecht University

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Overview

Comparing Theories

The Predicative Frege Hierarchy

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Why Compare Theories?

Why compare theories in a systematic way?

- To explicate intuitions of sameness.
- To transfer information from one theory to another: consistency, essential undecidability.
- Comparison of strength.
- To provide a philosophical reduction of ontologies.

What is a Relative Translation?

A *relative translation* $\tau : \Sigma \to \Theta$ is a pair $\langle \delta, F \rangle$.

- δ is Θ-formula
- F associates to R of Σ of arity n a Θ-formula $F(R)$ with variables among $v_0, \ldots, v_{n-1}$.

Induced extension mapping:

- $(R(y_0, \cdots, y_{n-1}))^\tau := F(R)(y_0, \cdots, y_{n-1})$;
- $(\cdot)^\tau$ commutes with propositional connectives;
- $(\forall y\, A)^\tau := \forall y\, (\delta(y) \to A^\tau)$;
- $(\exists y\, A)^\tau := \exists y\, (\delta(y) \wedge A^\tau)$.

Variants: sorted, parameters, multidimensional, piecewise.

What is a Relative Interpretation?

An interpretation K is of the form $\langle U, \tau, V \rangle$, where, for all U-sentences A, we have: $U \vdash A \Rightarrow V \vdash A^\tau$.

We write:
$K : U \to V$, or $U \xrightarrow{K} V$, or $K : V \rhd U$, or $K : U \lhd V$.

Here are various notions of sameness for $K, K' : U \to V$:

same(1) V proves that they are the same.

same(2) V proves that they are isomorphic via a definable isomorphism.

same(3) In every model of V, the internal model defined by K is isomorphic to the internal model defined by K'.

same(4) For all sentences A of U, we have: $V \vdash A^K \leftrightarrow A^{K'}$.

same(5) Always.

Examples

Interpretations are everywhere dense in Mathematics.

- Arithmetic in Set theory
- Hyperbolic Geometry in Eucidean Geometry
- Elementary Syntax in Arithmetic
- True Arithmetic in a non-abelian Group

Categories

Each of the possibilities of identification of interpretations n, gives rise to a category INT_{n-1}.

- U and V are *synonymous* or *definitionally equivalent* iff they are isomorphic in INT_0.
- U and V are *bi-interpretable* iff they are isomorphic in INT_1.

We have the contravariant mod functor from INT_0 to CLASS that sends $K : U \to V$ to the map that associates to each model $\mathcal{M}$ of V the the internal model $\mathrm{MOD}(K)(\mathcal{M})$ defined by K.

Synonymy

Synonymy is the strictest extensional relation of sameness known apart from identity.

- Point-and-Line versions of Elementary Geometry are synonymous with Point-Only versions.
- S^1_2 is synonymous with an appropriate theory of strings: Ferreira Arithmetic.
- PA is synonymous with $\mathrm{ZF}^- + \neg\mathrm{INF} + \mathrm{TC}$. (Kay and Wong, 2006)
- ZF is a synonymous with an appropriate version of ZF enriched with a countable set of urelements. (Löwe, 2006)
- $I\Delta_0$ is *not* synonymous with Q. (Visser 2007, Friedman 2007) Friedman: these theories are not weakly bi-interpretable.

Are Euclidean plane geometry and hyperbolic plane geometry synonymous? If not, are they bi-interpretable?

Ontological Reduction

In general, interpretations do not provide an ontological reduction.

Interpretations need not map standard models to standard models modulo isomorphism. Positive examples:

- PA in ZF via the von Neumann interpretation.
- $\mathsf{ZF}^- + \neg\mathsf{INF} + \mathsf{TC}$ in PA via the Ackermann interpretation.
- Hyperbolic into Euclidean Geometry via the Beltrami-Poincaré interpretation.

Negative examples:

- $\mathsf{PA} + \mathsf{incon}(\mathsf{PA})$ in PA, via any interpretation.
- PA in PA via any restricted interpretation.

Is there a real life example of theories U and V with conventional standard models, where U is interpretable in V, but where no interpretation maps the standard model of V to the standard model of U?

Comparing Strength

- $(\mathrm{Q} + \mathrm{con}(U)) \rhd U$
- $U \not\rhd (\mathrm{Q} + \mathrm{con}(Q))$.
- $\mathrm{Q} + \mathrm{con}(\mathrm{Q})$ is mut. interpretable with $I\Delta_0 + \mathrm{EXP}$.
- $\mathrm{ZF} \rhd \mathrm{PA}$.

Conservativity

U is Γ-conservative over V, or $V \rhd_{\mathrm{cons},\Gamma} U$.

Conservativity is not coordinate-free!

$$\begin{tikzcd}
 & \text{basic} \arrow[dl, "K"'] \arrow[dr, "M"] & \\
U & & V
\end{tikzcd}$$

$$K^{-1}[U] \cap \Gamma \quad \subseteq \quad M^{-1}[V]$$

Examples

- GB is conservative over ZF, for the language of ZF, with respect to EMB and ID.
- ZF is conservative over Q , for the language of arithmetic, w.r.t. a faithful interpretation of Q in ZF and ID.

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The Hierarchy Defined

$\mathrm{PV} := \mathrm{P}^1\mathrm{V}$

$\mathrm{P}^1 1)$ $\vdash \exists X^0 \forall x\, (X^0 x \leftrightarrow A(x, \vec{y}, \vec{Y}^0))$,
where A does not contain X and does not contain bound concept variables of degree 0.

$\mathrm{P}^1 2)$ $\vdash \ddagger^0 X^0 = \ddagger^0 Y^0 \leftrightarrow \forall z\, (X^0 z \leftrightarrow Y^0 z)$.

$\mathrm{P}^{n+2}\mathrm{V}$

$\mathrm{P}^{n+2} 1)$ $\vdash \exists X^{n+1} \forall x\, (X^{n+1} x \leftrightarrow A(x, \vec{y}, \vec{Y}^0, \ldots, \vec{Y}^{n+1}))$,
where A does not contain X and does not contain bound concept variables of degree $n+1$.

$\mathrm{P}^{n+2} 2)$ $\vdash \ddagger^{n+1} X^{n+1} = \ddagger^{n} Y^{n} \leftrightarrow \forall z\, (X^{n+1} z \leftrightarrow Y^{n} z)$.

$\mathrm{P}^{n+2} 3)$ $\vdash \ddagger^{n+1} X^{n+1} = \ddagger^{n+1} Y^{n+1} \leftrightarrow \forall z\, (X^{n+1} z \leftrightarrow Y^{n+1} z)$.

From Consistency to Comprehension

We can construct an interpretation $\mathcal{H} : (\mathsf{Q} + \mathsf{con}(U)) \rhd U$, using the Henkin-Feferman construction.

We can extend this interpretation to an interpretation of U plus predicative comprehension over U by letting one-place formulas play the role of concepts.

We can enrich this last interpretation to an interpretation that also provides a Frege function in case U proves the infinity of its domain in a sufficiently convenient way.

From Comprehension to Consistency

Suppose U provides sufficient coding machinery. Then, U plus predicative comprehension over U proves the consistency of U.

We do this by building a truth predicate for the language of U.

So, under reasonable conditions we have:
Consistency $\approx$ Predicative comprehension plus Frege function.

The Result

Let:

- $\mathrm{EA} := I\Delta_0 + \mathrm{EXP}$,
- $\mathrm{EA}^+ := I\Delta_0 + \mathrm{SUPEXP}$,
- $U_n := U + \mathrm{con}^n(U)$,
- $U_\omega := \bigcup_n U_n$.

We find:

- $\mathrm{P}^{\mathrm{n+1}}\mathrm{V} \equiv \mathrm{Q}_n$.
- $\mathrm{P}^\omega\mathrm{V} \equiv_{\mathrm{loc}} \mathrm{Q}_\omega$.
- $\mathrm{P}^{\mathrm{2n+1}}\mathrm{V} \equiv \mathrm{EA}_n$.
- $\mathrm{P}^\omega\mathrm{V} \equiv_{\mathrm{loc}} \mathrm{Q}_\omega \equiv \mathrm{EA}_\omega$.
- EA^+ proves the equiconsistency of $\mathrm{P}^\omega\mathrm{V}$ and EA^+.