The infinitesimal subgroup of a definably compact group

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The goal

To describe recent work on definable groups in o-minimal structures via the solution to Pillay's Conjecture.

The conjecture suggests a beautiful connection between the "lattice of formulas" in definable groups on one hand and compact Real Lie groups on the other.

An example

Consider the circle group:

$$G = SO(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL(2,\mathbb{R}) : \ a^2 + b^2 = 1 \right\}$$

G is a 1-torus. Namely,

$$G \simeq \mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\},$$

with its complex field-multiplication.

G and $\mathbb{T}^1$, together with their group operations, are definable in the real field $\overline{\mathbb{R}} = \langle \mathbb{R}, <, +, \cdot, 0, 1 \rangle$.

Consider a κ-saturated real closed field $\mathcal{R} \succ \overline{\mathbb{R}}$ (κ large). $G(\mathcal{R}) = SO(2, \mathcal{R})$.

$SO(2, \mathbb{R})$ is compact $\Rightarrow$ the *standard-part mapping* induces a group-homomorphism $st : SO(2, \mathcal{R}) \to SO(2, \mathbb{R})$. Namely,

$$st \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} st(a) & st(b) \\ st(-b) & st(a) \end{pmatrix}.$$

$$ker(st) = \mu(I) = \bigcap_{n \in \mathbb{N}} \{A : |A - I| < 1/n\},$$

the intersection of countably many definable sets in $\mathcal{R}$.

$\Rightarrow$ $ker(st)$ is <u>type-definable</u>.

Two topologies on $G(\mathcal{R})/\mu(I) \simeq SO(2,\mathbb{R})$

E-topology: The *Euclidean topology* of $SO(2,\mathbb{R})$.

L-topology: The *Logic Topology*:

$F \subseteq SO(2,\mathbb{R})$ is L-closed

iff

$st^{-1}(F) \subseteq G(\mathcal{R})$ is *type-definable* (in the ordered field structure).

Type-definable : The intersection of $< \kappa$-many definable sets.

Fact *The Euclidean topology=Logic topology on* $G(\mathbb{R}) = SO(2, \mathbb{R})$.

Proof Assume $F \subseteq G(\mathbb{R})$ is L-closed.

$\Rightarrow st^{-1}(F) = p(\mathcal{R})$ for a type
$p(x) = \{\phi_i(x) : i \in I\}$.

Fix $g \in Cl_E(F)$.
For all $n \in \mathbb{N}\ \exists\, g' \in F$, $|g' - g| < 1/n$.

$\Rightarrow$ For all $n \in \mathbb{N}\ \exists h \in st^{-1}(g')$ (hence $h \models p(x)$), $|h - g| < 1/n$.

$\Rightarrow \exists h \models p(x)$, for all $n \in \mathbb{N}$ $|h - g| < 1/n$. (saturation)

$\Rightarrow h \in p(\mathcal{R})$ and $st(h) = g$
$\Rightarrow p(\mathcal{R}) \cap st^{-1}(g) \neq \emptyset$
$\Rightarrow st^{-1}(F) \cap st^{-1}(g) \neq \emptyset$
$\Rightarrow g \in F$.

$\Rightarrow F$ is E-closed.

Assume $F \subseteq G(\mathbb{R})$ is E-closed.

In $\overline{\mathbb{R}}$: For every $g \in G(\mathbb{R}) \setminus F$ there is $n_g \in \mathbb{N}$ such that $|g - F| > 1/n_g$ (because F is compact).

Claim: $st^{-1}(F) = p(\mathcal{R})$ for the type:

$$p(x) = \{x \in G \,\&\, |x - g| \geqslant 1/n_g \ : \ g \in G(\mathbb{R}) \setminus F\}.$$

$\subseteq$: Assume that $st(h) = g' \in F$.

$\Rightarrow$ For every $g \in G(\mathbb{R}) \setminus F$, $|g' - g| > 1/n_g$.
$\Rightarrow |h - g| > 1/n_g$.
$\Rightarrow h \models p(x)$.

$\supseteq$: Assume $h \notin st^{-1}(F)$.
$\Rightarrow st(h) = g \in G(\mathbb{R}) \setminus F$, $\Rightarrow |h - g| < 1/n_g$.

$\Rightarrow h \notin p(\mathcal{R})$.

$\Rightarrow st^{-1}(F) = P(\mathcal{R}) \Rightarrow F$ is L-closed. $\square$

Note: The type defining $st^{-1}(F)$ uses $2^{\aleph_0}$-many formulas.

Remarks and definitions:

1. The infinitesimal group $\mu(I)$ is called *"a type-definable subgroup"* of G. Namely, it is the intersection of a collection of small cardinality of definable sets in $\mathcal{R}$. "Small" relative to the saturation of $\mathcal{R}$.

2. The quotient group $G(\mathcal{R})/\mu(I)$ ($\simeq SO(2, \mathbb{R})$) is independent of $\mathcal{R}$. I.e. every coset, even in elementary extensions, is already represented in $\mathcal{R}$. Hence, $\mu(I)$ is called a type-definable subgroup *"of bounded index"*.
Equivalently, the cardinality of $G(\mathcal{R})/\mu(I)$ is small.

A negative example The infinitesimals $\mu(0)$ in $\langle \mathcal{R}, <, + \rangle$ are **not** of bounded index. The quotient is **not** $\langle \mathbb{R}, + \rangle$!

Remarks continued

3. The Logic topology on $G(\mathcal{R})/\mu(I)$ is **not** the quotient topology with respect to the topology of the real closed field, because $\mu(I)$ is open in this topology (so the quotient topology is discrete).

4. The same process will work if we start with any compact Hausdorff topological space, with a definable basis for the topology, instead of $SO(2, \mathbb{R})$.

Generalizing the example

$\mathcal{R}$	$\longrightarrow$	a sufficiently saturated o-minimal structure $\mathcal{M}$.
$SO(2,\mathcal{R})$	$\longrightarrow$	$\mathcal{M}$-definable group G.
$\mu(I)$	$\longrightarrow$	$G^{00} :=$ the smallest type definable subgroup of G of bounded index.
$\frac{SO(2,\mathcal{R})}{\mu(I)}$	$\longrightarrow$	a compact Lie group.

Pillay's idea

Every (definably compact) group in any o-minimal structure should have a canonical type-definable infinitesimal subgroup, whose quotient, equipped with the Logic topology, is a compact Lie group.

Pillay's Conjecture (PC)

Let G be a definable group in a κ-saturated o-minimal structure $\mathcal{M}$. ($\kappa > 2^{\aleph_0}$)
Then:

1. G has a smallest type-definable normal subgroup of bounded index, call it G^{00}.

2. G/G^{00}, equipped with a Logic topology, is isomorphic to a compact Real Lie group.

3. If G is definably compact then

$$\dim_{Lie}(G/G^{00}) = \dim_{\mathcal{M}}(G).$$

The status of PC

Items 1-2 : A theorem.

Item 3 : A theorem when:

(1) $\mathcal{M}$ expands a real closed field, or

(2) $\mathcal{M}$ is an ordered vector space over an ordered division ring.

Unknown in, say, expansions of ordered groups.

Two more examples

Example 1

$\mathcal{R}$ still a large real closed field.

$\alpha \in \mathcal{R}$ infinitesimal > 0.

$$G = \langle [0, \alpha), + \; mod \, \alpha \rangle.$$

Now, the usual $\mu(0)$ in $\mathcal{R}$ does not work (it contains the whole group).

Take

$$G^{00} = \mu_\alpha(0) = \bigcap_{n \in \mathbb{N}} [0, \frac{\alpha}{n}) \cup (1 - \frac{\alpha}{n}, \alpha).$$

Again, $G/G^{00} \simeq SO(2, \mathbb{R})$.

G^{00} is a canonical to G!.

Example 2 : A nontrivial example

Let $\mathcal{R}$ be a large real closed field, A an abelian variety defined over $\mathcal{R}$, $G = A(\mathcal{R})$, $\dim A = n$.

By **PC**, there is a surjective homomorphism $\sigma : A(\mathcal{R}) \to \mathbb{T}^n$ such that

(1) $ker(\sigma)$ is type-definable (in the language of real closed fields),

(2) Any other homomorphism from $A(\mathcal{R})$ onto a compact Lie group, with a type-definable kernel, factors through σ.

$ker(\sigma)$ is a "canonical infinitesimal subgroup" of $A(\mathcal{R})$.

Plan

I : O-minimal structures.

II : A survey of definable groups.

III : Some aspects of the proof. Connections to NIP and measure.

O-minimal structures

Defn. A linearly ordered structure $\mathcal{M} = \langle M, <, \cdots \rangle$, is o-minimal if every definable subset of M is of the form

$$\bigcup_{i=1}^{k} (a_i, b_i) \cup \underbrace{F}$$

with $a_i, b_i \in M \cup \{\pm\infty\}$.

Negative Examples

- $\langle \mathbb{Z}; <, + \rangle$ (why? $2\mathbb{Z}$)
- $\langle \mathbb{Q}; <, +, \cdot \rangle$ (why? The set of squares)
- $\langle \mathbb{R}; <, +, \cdot, \sin x \rangle$ (why? Zeroes of $\sin x$).
- $\langle \gamma; < \rangle$, γ an ordinal $\geqslant \omega^2$ (why?).

Positive examples

- $\langle \mathbb{Q}, < \rangle$ and $\langle \mathbb{Z}, < \rangle$ (QE)
- Ordered vector spaces over ordered division rings (by QE)
- Real closed fields (by Tarski's QE).
- $\mathbb{R}_{exp}\langle \mathbb{R}; <, +, \cdot, e^x \rangle$ (by Wilkie),
- Expansions of $\mathbb{R}_{exp}$ by local functions for all converging power series near 0.
- The field of Puiseux series, with local functions for **all** formal real power series (Lipshitz-Robinson).

Some properties of an o-minimal $\mathcal{M}$

Pillay-Steinhorn and Knight-Pillay-Steinhorn

Note: There is a definable basis for $<$-topology on M and product topology on M^n.

Warning: If $\mathcal{M}$ sufficiently saturated then it is not locally compact, locally connected. **No** converging infinite sequences.

(1) Every definable subset of M^n has finitely many *definably connected* components.

(2) Definable subsets of M^n have dimension in $\mathbb{N}$ (a-la manifold dimension). Tuples in M^n have dimension (a-la transcendence degree).

Properties of an o-minimal $\mathcal{M}$ (cont.)

(4) If $X \subseteq M^n$ is definable then

$$\dim(Cl(X) \setminus X) < \dim X.$$

(5) Definable functions are continuous (C^n) outside a set of small dimension.

(6) If $\mathcal{M} \equiv \mathcal{N}$ then $\mathcal{N}$ also o-minimal.

Definable groups

$\mathcal{M}$ an o-minimal structure.

Defn. A group $\langle G, * \rangle$ is *definable in* $\mathcal{M}$ if $G \subseteq M^n$ is a definable set and $*$ a definable function.

Examples

• (1-dimensional)

1. $\langle \mathbb{Q}, + \rangle$, definable in $\langle \mathbb{Q}, <, + \rangle$.
2. $\langle R^{>0}, \cdot \rangle$ in a real closed field R.
3. $\langle [0, a), +(mod\, a) \rangle$
(in a real closed field R, or an ordered v.s.)

• $G(R)$, G a real algebraic group over a real closed field R.

• $G(K)$, G an algebraic group over an alg. closed field K of char. zero (definable in any maximal real closed sub-field $R \subseteq K$).

More examples

- Compact real analytic Lie groups (definable in $\mathbb{R}_{an}$).

- **A group not isomorphic to any semi-algebraic group**

$$G = \left\{ \begin{pmatrix} e^t & te^t & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix} t > 0, u, v \in \mathbb{R} \right\}$$

A solvable group, definable in $\langle \mathbb{R}, <, +, \cdot, e^x \rangle$.

Negative examples

- $\langle \mathbb{Z}, + \rangle$ is **not** definable in any o-minimal structure.

- $\langle \mathbb{Z}_2^\omega, + \rangle$ is **not** definable in any o-minimal structure.

- Any finitely generated free group.

Some properties of definable groups

Pillay

G definable in an o-minimal $\mathcal{M}$.

1. G admits a definable manifold-like structure w.r. to M^n, with finitely many definable charts, making it into a topological group.

2. G is definably-connected-by-finite.

$G^0 :=$ the component containing the identity is the smallest definable subgroup of finite index.

3. G has no infinite descending chain of definable subgroups. (DCC on definable subgroups).

A sample of proofs

Fact 1 $H \subseteq G$ a definable subgroup $\Rightarrow H$ is closed.

Proof Let $Fr(H) = Cl(H) \setminus H$ (definable!).

Every $h \in H$ acts as a homeomorphism of G and $hH = H$.

$\Rightarrow hFr(H) = Fr(H) \Rightarrow H \cdot Fr(H) = Fr(H)$.

$\Rightarrow$ If $Fr(H) \neq \emptyset$ then $\dim H \leqslant Fr(H)$.

But, by o-minimality, $\dim Fr(H) < \dim H$

$\Rightarrow Fr(H) = \emptyset \Rightarrow H$ is closed. $\square$

Fact 2 $\dim H = \dim G \Rightarrow G^0 \subseteq H$.

Proof If $\dim H = \dim G$ then G/H is finite (dimension properties).

H is closed $\Rightarrow H$ is open. $\Rightarrow H$ is clopen.

$\Rightarrow H$ contains G^0 (G^0 is definably connected.) $\square$

A sample of proofs (cont.)

Fact 3 G has no infinite descending chain of definable subgroups.

Proof Assume

$$G \supseteq H_1 \supseteq H_2 \supseteq \cdots$$

are definable subgroups.

Take H_k with minimal

$$\langle \dim H_k, [H_k : H_k^0] \rangle.$$

$\Rightarrow H_{k+1} = H_k$. □

Definably compact sets

What is a good notion of compactness?

If $\mathcal{M}$ is sufficiently saturated then **no** infinite subset of M^n is compact, and **no** infinite sequence in M^n can converge.

E.g. consider the interval $[0, 1]$, α an infinitesimal. The family

$$\mathcal{U} = \{(x - \alpha, x + \alpha) : x \in [0, 1]\}$$

is a uniformly definable open cover but it has no finite subcover.

Defn. A definable group G is *definably compact* if every definable continuous $f : (a, b) \to G$ has a limit point in G (with respect to the group topology), as t tends to a and to b.

Examples

1. If $G \subseteq Gl(n, \mathcal{R})$ then G is definably compact iff it is closed and bounded.

(So, over $\mathbb{R}$: definably compact $\Leftrightarrow$ compact)

2. Compact real Lie groups are definably compact (in any o-minimal structure).

3. $A(R) =$ the R-points of an Abelian variety over a real closed field R is definably compact.

4. $\langle [0, a), + \, mod \, a \rangle$.

5. $GL(n, \mathcal{R})$ is not definably compact.

Definable compactness and Torsion

Assume G is abelian.

Theorem 1 (P-Steinhorn, Beraarducci-Otero, Edmundo) G has torsion iff there are definable $H_1 \subsetneq H_2 \subseteq G$ such that H_2/H_1 is definably compact.

Theorem 2 (Edmundo-Otero) ($\mathcal{M}$ expands a real closed field). If G is definably compact, definably connected, abelian, $\dim G = n$, then

$$\forall k \; G[k] \simeq (\mathbb{Z}/k\mathbb{Z})^n \simeq \mathbb{T}^n[k]$$

In particular, $G[k]$ is finite, but this holds even without compactness.

Remark Unlike the category of compact abelian Lie groups, it is not true that G is definably isomorphic to $\mathbb{T}^n$!!!

Definable compactness and simple groups

Defn. G is definably simple if it is non-abelian, and has no definable normal subgroups.

Theorem (Pe-Pillay-Starchenko) If G is definably simple then it is definably isomorphic to a semialgebraic group over the real algebraic numbers.

Theorem (Pe-Pi-S, Pe-S) If G is definably compact then

$$G/Z(G) \simeq S_1 \times \cdots \times S_k$$

(an almost direct product), each S_i definably simple.

Type-definable subgroups

Examples

1. If $G = \langle \mathcal{R}, + \rangle$ then

$$\mu(0) = \bigcap_{n \in \mathbb{N}} \{x \in \mathcal{R} : -1/n < x < 1/n\}$$

is a type-definable subgroup, **of unbounded index**. G has **no** nontrivial type-definable subgroup of bounded index.

2. If $G = H(\mathcal{R})$, H a compact simple real algebraic group (e.g. $H = SO(3, \mathbb{R})$) then G has no **definable** normal subgroup.

But $\mu(I)$ is a **type-definable** normal subgroup of G.

Returning to Pillay's Conjecture:

DCC on subgroups of bounded index

(Berarducci-Otero-P-Pillay)

G definable in an o-minimal structure.

Theorem

1. If

$$G \supset H_1 \supset H_2 \supset H_3 \cdots$$

a chain of type-definable subgroups of bounded index then $H_k = H_{k+1}$ for k large enough.

In particular, G has a smallest type-definable subgroup of bounded index, denoted by G^{00}.

2. G/G^{00}, with the Logic topology, is a Lie group. If G is definably connected then G/G^{00} is connected.

(**Shelah** : If G has the Non Independence Property then G^{00} exists.)

Example: A group G without G^{00}

$\mathcal{M} = \langle G; P\rangle$ a two-sorted structure.

$G = \langle \mathbb{Z}^\omega, +\rangle$;

$P = \mathbb{N}$, with a predicate

$$R = \{(\vec{a}, n) \in G \times P : a_i = 0\}.$$

For all $n \in P$, R_n is subgroup of index 2.

For all $g \in G$ there is $n \in P$, $g \notin R_n$ and $[G : R_n] = 2$.

$\Rightarrow$ In models of $\mathcal{M}$: for every subgroup $H \subseteq G$ there is $b \in P$ such that $[H : (H \cap R_b)] = 2$.

$\Rightarrow$ There is no minimal type-definable subgroup of bounded index.

Two corollaries of B-O-P-P

Assume $\langle G, +\rangle$ is abelian

Corollary 1 G^{00} is divisible.

Proof Take $k \in \mathbb{N}$. $G[k] = ker(g \mapsto kg)$ is finite $\Rightarrow \dim(kG) = \dim G$.
$\Rightarrow [G : kG]$ is finite (dimension).

We have, $[G : G^{00}] \geqslant [kG : kG^{00}]$.

$\Rightarrow [G : kG^{00}]$ is bounded index

$\Rightarrow kG^{00} = G^{00}$ (minimality). $\square$

Corollary 2 If $H \subseteq G$ is type-definable, bounded index, torsion-free then $H = G^{00}$.

If G is torsion-free then $G = G^{00}$.

Proof Assume $H \neq G^{00}$. By BOPP, $H/G^{00} \subseteq G/G^{00}$ is a compact Lie group $\Rightarrow$ H/G^{00} has torsion. Since G^{00} divisible, $\Rightarrow H$ has torsion. Contradiction. $\square$

What is left to prove? PC (3)

G def. compact $\Rightarrow \dim_{Lie}(G/G^{00}) = \dim_{\mathcal{M}} G$

The abelian case

Assumption: $\mathcal{M}$ expands a r.cl.field.

Enough to see: G^{00} is torsion-free.

Why is it enough?

If G^{00} is torsion-free (and it is divisible) then for every k, $(G/G^{00})[k] \simeq G[k]$.

Let $n = \dim_{\mathcal{M}} G$.

By Edmundo-Otero, $G[k] \simeq \mathbb{T}^n[k]$, hence

$$(G/G^{00})[k] \simeq \mathbb{T}^n[k].$$

G/G^{00} and $\mathbb{T}^n$ are abelian connected Lie groups. $\Rightarrow G/G^{00} \simeq \mathbb{T}^n$

$\Rightarrow \dim_{Lie}(G/G^{00}) = \dim_{\mathcal{M}} G$.

Goal (for G abelian): G^{00} is torsion free.

Generic Sets (P-Pillay)

Defn. $X \subseteq G$ is *left generic* if $G = \bigcup_{i=1}^{k} g_i X$ for some $k \in \mathbb{N}$.

Examples 1. In $\langle \mathbb{R}, + \rangle$: $(-\infty, b) \cup (a, +\infty)$ is generic. $(a, +\infty)$ is not generic.

2. In the circle group $\mathbb{T}^1$: Every non-infinitesimal "arc" is generic.

3. If $\dim(G \setminus X) < \dim G$ then X is left generic.

Generic sets and s.g's of bounded index

Assume $H = \bigcap_{i \in I} X_i$ *is type-definable, of bounded index.*

$\Rightarrow$ for each $i \in I$, boundedly-many left (right) translates of X_i cover G.

$\Rightarrow$ (compactness) **finitely** many translates of X_i cover G

$\Rightarrow$ each X_i is left (and right) generic.

Theory of generic sets
G abelian.

Goal: The definable non-generic subsets of G form an ideal.

Assume $X \subseteq G$ is non-generic, definable over a small model $\mathcal{M}_0$.

$\Rightarrow$ for all k, $\forall h_1, \ldots, h_k \exists g\ g \notin \bigcup_{i=1}^{k} h_i + X$,

$\Rightarrow$ for all k, $\forall h_1, \ldots, h_k \exists g \bigwedge_{i=1}^{k} (-h_i \notin X - g)$.

$\Rightarrow$ for all k, $\forall h_1, \ldots, h_k \exists g\ \bigwedge_{i=1}^{k} (h_i \notin X + g)$.

$\Rightarrow$ (M_0 is small and $\mathcal{M}$ is saturated)

$$\exists g\ \forall h \in M_0\ h \notin X + g.$$

$\Rightarrow$ $(X + g) \cap M_0 = \emptyset$.

Why does it help?

Dolich's work

Theorem (Dolich) Assume $X(a) \subseteq M^n$ is a definably compact, a-definable set, $\mathcal{M}_0 \preceq \mathcal{M}$ small. If $\{X(a') : a' \equiv_{M_0} a\}$ is finitely satisfiable then

$$X(a) \cap M_0 \neq \emptyset.$$

Corollary G def. compact and abelian $\Rightarrow$ the non-generic sets form an ideal.

Proof Assume $X \subseteq G$ is not generic.

$\Rightarrow Cl(X)$ is non-generic. $\Rightarrow$ W.m.a X is closed in G, $\Rightarrow X$ is definably compact. As we saw, there is g, $(X + g) \cap M_0 = \emptyset$.

By Dolich, $\exists g_1, \ldots, g_k \bigcap_{i=1}^k X + g_i = \emptyset$.

$\Rightarrow \bigcup_{i=1}^k (G \setminus X) + g_i = G$

$\Rightarrow G \setminus X$ is generic.

$\cdots \Rightarrow$ the non-generic sets form an ideal.

Non-generic "means" small

Group, measure and the NIP

(Hrushovski-P-Pillay)

Measure-NIP is due to Keisler

Defn.(Shelah) A theory T has NIP if for every indiscernible sequence $\langle a_i : i \in \omega \rangle$ over $\emptyset$, for every c, and a formula $\phi(x, y)$, $\exists k \in \mathbb{N}$:

Either $\forall i \geqslant k\ \phi(c, a_i)$ **or** $\forall i \geqslant k\ \neg\phi(c, a_i)$.

A negative example The Theory of Random graphs: no NIP.

Positive examples O-minimal theories, stable theories have NIP.

Fact If T has NIP and $\langle a_i : i < \omega \rangle$ is indiscernible, $\phi(x, y)$ a formula, then the type
$\{\phi(x, a_{2j}) \triangle \phi(x, a_{2j+1}) : j < \omega\}$
is inconsistent.
(where $\phi \triangle \psi := (\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)$).

G definably compact and abelian

Goal: G^{00} is torsion-free.

Defn. $X, Y \subseteq G$ definable. Say $X \approx_{ng} Y$ if $X \triangle Y$ is non-generic in G.

Fact $\approx_{ng}$ is an equivalence relation.

Theorem $Def(G)/\approx_{ng}$ is bounded. I.e., $\exists$ a small $\mathcal{M}_0 \preceq \mathcal{M}$ s.t. every $\approx_{ng}$-class has a representative definable over $\mathcal{M}_0$.

Measure

Recall: An abelian group is *amenable*. I.e., it admits a left-invariant, finitely additive probability measure μ.

Proof of Theorem

Assume $\approx_{ng}$ is unbounded.

$\Rightarrow$ there is $\phi(x, y)$, and unboundedly many b_i's, such that $\phi(G, b_i)\triangle\phi(G, b_j)$ is generic.

$\Rightarrow$ $\exists$ a fixed n and an indiscernible $\langle a_i : i < \omega\rangle$ such that $\forall i \neq j$, the set $\phi(G, a_i)\triangle\phi(G, a_j)$ is n-generic.

Let
$\mathcal{F} = \{\overbrace{\phi(G, a_{2j})\triangle\phi(G, a_{2j+1})} : j < \omega\}$.
By NIP, $\exists k\ \forall j_1, \ldots, j_k$,

$$Y_{j_1} \cap \cdots \cap Y_{j_k} = \emptyset.$$

However, $\forall j\ \mu(Y_j) \geqslant 1/n$, and $\mu(G) = 1$.

$\Rightarrow$ it is impossible that every k sets in $\mathcal{F}$ intersect trivially. Contradiction.

$\Rightarrow Def(G)/\approx_{ng}$ is bounded. $\square$

The non-generic stabilizer

G definably compact and abelian.

Defn. For X definable, let
$Stab_{ng}X = \{g \in G : X \approx_{ng} g + X\}$.

Theorem $Stab_{ng}X$ is a type-definable subgroup of bounded index.

Proof (1) A subgroup: $g, h \in Stab_{ng}X \Rightarrow$
$X \triangle (g+h) + X \subseteq$
$(\underbrace{X \triangle h + X}) \cup ((\underbrace{h+X) \triangle h + (g+X)})$

(2) Type-definable:
$Stab_{ng}X = \bigcap_{n \in \mathbb{N}} X$ is "not n-generic".

(3) Bounded index:
$|G/Stab_{ng}X| \leqslant |Def(G)/\approx_{ng}|$
via $\qquad g \mapsto g + X$.

Theorem G abelian and definably compact $\Rightarrow \dim_{Lie} G/G^{00} = \dim G$.

Proof Enough to see: G^{00} is torsion-free.

Given $k \in \mathbb{N}$, by o-minimality there is a definable $X \subseteq G$ such that

$$G = \bigcup_{g \in G[k]} X + g,$$

and for $g \neq h \in G[k]$, $X + g \cap X + h = \emptyset$.

$\Rightarrow X$ is generic and $G[k] \cap Stab_{ng} X = \{0\}$.

$Stab_{ng} X$ type-definable of bounded index $\Rightarrow G^{00} \subseteq Stab_{ng} X$ (G^{00} is the smallest one).

$\Rightarrow G^{00} \cap G[k] = \{0\}$

$\Rightarrow G^{00}$ is torsion-free.

$\Rightarrow$ **PC** for abelian groups. $\square$

PC for arbitrary definably compact G

Case I *G is definably simple*: G is definably isomorphic to a semi-algebraic group H over $\mathbb{R}^{alg}$. In this case $H^{00} = \mu(I)$ and $H/H^{00} \simeq H(\mathbb{R})$. $\Rightarrow$ **PC** for G.

Case II *G has finite center*: G is an almost direct product of definably simple groups $\Rightarrow$ (by I) **PC** for G.

Case III *The general case* : $G/Z(G)$ is an almost direct product of definably simple groups. Using **PC** for $Z(G)$ and for $G/Z(G)$, the problem reduces to showing:

$$G^{00} \cap Z(G) = Z(G)^{00}.$$

Towards that goal one develops a more general theory of measure and generic sets for arbitrary groups. $\Rightarrow$ **PC** for G.

End of PC for groups in o-minimal expansions of real closed fields.

One corollary of our work is:

Theorem If G is definably compact then G admits a left-invariant, finitely additive real probability measure μ such that for every definable $X \subseteq G$,

$$\mu(X) = 0 \text{ iff } X \text{ is non-generic.}$$

(See also more recent work on measure and groups by Hrushovski and Pillay).

Some open questions

G definably compact.

Conjecture 1 The torsion points are "generically dense": Every definable generic set contains a torsion point.

Conjecture 2 The measure μ above commutes with the usual Haar measure on G/G^{00}. Namely, with $\pi : G \to G/G^{00}$ and $X \subseteq G$ definable, we have:

$$\mu(X) = \text{Haar measure of } \pi(X).$$

Conjecture 3 (*Compact domination*) For every definable $X \subseteq G$ and every $h \in G/G^{00}$ outside a set of measure zero, either $\pi^{-1}(h) \subseteq X$ or $\pi^{-1}(h) \subseteq G \setminus X$.

Fact

$$2 \Leftrightarrow 3 \Rightarrow 1.$$

Related developments

(1) (Eleftheriou, Onshuus) **PC** and compact domination holds in ordered vector spaces over ordered division rings. (The results could be useful for algebraically closed valued fields).

(2) (Otero-P) If G definably compact and $\dim(G \setminus X) < \dim G$ then X contains a torsion point.

(3) (Petrykovski) Weaker notions of genericity for groups in o-minimal structures.

Bibliography

A. Pillay, *Type-definability, compact Lie groups, and o-minimality*, J. Math. Logic, 4 (2004), 147-162.

A. Berarducci, M. Otero, Y. Peterzil, and A. Pillay, *A descending chain condition for groups in o-minimal structures*, APAL 134 (2005), 303-313..

Y. Peterzil and A. Pillay, *Generic sets in definably compact groups*, Fundamenta Mathematicae 193 (207), 153-170

E. Hrushovski, Y. Peterzil abd A. Pillay, *Groups measure and the NIP*, to appear in JAMS.