A remark on a characterization of non-forking in generic structures

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Main Results

Let L be a countable relational language and $\mathbf{K}$ a class of finite L-structures with $\delta \geq 0$ that has finite closures.

Theorem

Let M be a saturated $\mathbf{K}$-generic structure. Let $B, C \leq \mathcal{M}$ with $A = B \cap C$. Then the following are equivalent:
(i) $B \downarrow_A C$;
(ii) $B \perp_{\mathbf{acl}(A)} C$ and $BC \cup \mathbf{acl}(A) \leq \mathcal{M}$.

The following corollary is a partial answer to Baldwin's question.

Corollary

There is no saturated $\mathbf{K}$-generic structure that is superstable but not ω-stable.

Notation and Definition

- $L = \{R_1, R_2, R_3, \cdots\}$
- $A, B, C, \ldots$ are L-structures
- $R_i^A = \{\bar{e} : A \models R_i(\bar{e})\}$
- A **predimension**: $\delta(A) = |A| - \sum_i \alpha_i |R_i^A| \quad (0 < \alpha_i \leq 1)$
- $A \leq B \quad \Leftrightarrow \quad \delta(X/A) \geq 0$ for any $X \subset B - A$
- $\mathbf{K}^*$ is a class of all finite L-structures with $\delta \geq 0$
- Fix a subclass $\mathbf{K} \subset \mathbf{K}^*$ closed under "substructures"

Notation and Definition

Warning

In my talk, the notion "substructure" will be used in the following sense: A is a **substructure** of B, if the universe of A is contained in that of B, and $R^A \subset R^B|A$ for every $R \in L$. (A generalization of "subgraph")

Definition (Generic)

A countable L-structure M is **$\mathbf{K}$-generic**, if
(1) Any finite $A \subset M$ belongs to $\mathbf{K}$;
(2) For any $A \leq B \in \mathbf{K}$ with $A \leq M$ there is $B' \cong_A B$ with $B' \leq M$;
(3) $M = \bigcup_i A_i$ for some $A_0 \leq A_1 \leq \cdots \in \mathbf{K}$.

Definition (Free)

Let $A = B \cap C$. Then B and C are **free** over A (in symbol, $B \perp_A C$), if $R^{B \cup C} = R^B \cup R^C$ for any $R \in L$.

Fact

Fact (Wagner,...)

Let M be a saturated $\mathbf{K}$-generic structure. Let $B, C \leq \mathcal{M}$ with $A = B \cap C$ algebraically closed. Then the following are equivalent:

(i) $B \downarrow_A C$;

(ii) $B \perp_A C$ and $BC \leq \mathcal{M}$.

Note

In the above fact, one cannot omit the condition that A is algebraically closed. (There is an example.)

Theorem

Fact (Wagner, ...)

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Theorem

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(i) $B \downarrow_A C$;
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Lemma

Lemma

Let $A \leq C \leq \mathcal{M}$ with $A = \mathbf{acl}(A) \cap C$. Then $\mathbf{acl}(A) \perp_A C$ and $\mathbf{acl}(A) \cup C \leq \mathcal{M}$.

Outline of Proof

Let $B = \mathbf{acl}(A)$. For simplicity, we assume that B is finite and $\mathbf{mult}(B/A) = 1$.

1. Since $\mathbf{K}$ is closed under ”substructures”, there is $B' \cong_A B$ such that $B' \perp_A C$ and $B'C \in \mathbf{K}$.
2. By genericity, we can assume $B' \leq B'C \leq \mathcal{M}$.
3. Since $B, B' \leq \mathcal{M}$, we have $\mathbf{tp}(B'/A) = \mathbf{tp}(B/A)$.
4. Then $B' = B$ by $\mathbf{mult}(B/A) = 1$.
5. Hence $B \perp_A C$ and $BC \leq \mathcal{M}$.

Using this lemma, we will sketch out the proof of our theorem.

Theorem

Let M be a saturated $\mathbf{K}$-generic structure. Let $B, C \leq \mathcal{M}$ with $A = B \cap C$. Then the following are equivalent:

(i) $B \downarrow_A C$;

(ii) $B \perp_{\mathrm{acl}(A)} C$ and $BC \cup \mathrm{acl}(A) \leq \mathcal{M}$.

Outline of Proof of Theorem

(i) ⇒ (ii)

1. Suppose $B \downarrow_A C$.
2. Let $A^* = \mathrm{acl}(A)$ and $B^* = \mathrm{acl}(B)$, and take sufficiently saturated $C^* \supset C$ with $B^* \downarrow_{A^*} C^*$.
3. Take small $D \subset C^*$ such that
$$B^* \perp_{A^* D} C^*, \qquad \mathrm{cl}(B^* D) \cup C^* \leq \mathcal{M}.$$
4. By saturation of C^*, take small $E \subset C^*$ with $E \downarrow_{A^*} D$ such that
$$B^* \perp_{A^* E} C^*, \qquad \mathrm{cl}(B^* E) \cup C^* \leq \mathcal{M}.$$
5. By 3 and 4, we have $B^* \perp_{A^*} C^*$ and $B^* C^* \leq \mathcal{M}$.
6. Clearly $B \perp_{A^*} C$.
7. By lemma, we have $BA^*, CA^* \leq \mathcal{M}$.
8. By 5 and 7, we have $BCA^* \leq \mathcal{M}$.

Outline of Proof of Theorem

(ii) ⇒ (i)

1. Suppose $B \perp_{\mathrm{acl}(A)} C$ and $BC \cup \mathrm{acl}(A) \leq \mathcal{M}$.
2. Take $B' \models \mathrm{tp}(B/\mathrm{acl}(A))$ with $B' \downarrow_A C$.
3. Since (i) ⇒ (ii) has been proved, we get $B' \perp_{\mathrm{acl}(A)} C$ and $B'C \cup \mathrm{acl}(A) \leq \mathcal{M}$.
4. So $B'C \cong_{\mathrm{acl}(A)} BC$.
5. Thus we have $\mathrm{tp}(B'/C \cup \mathrm{acl}(A)) = \mathrm{tp}(B/C \cup \mathrm{acl}(A))$.
6. Hence $B \downarrow_A C$.

Corollary

Question (Baldwin)

Is there any ”generic” structure that is superstable but not ω-stable ?

This corollary is a partial answer to Baldwin’s question.

Corollary

There is no saturated $\mathbf{K}$-generic structure that is superstable but not ω-stable.

Corollary

Outline of Proof

1. Take a superstable generic structure M.
2. Note that the theory T is small.
3. So, to show that T is ω-stable, it is enough to prove that, for any $p \in S(M)$ there is finite $A \subset M$ with $p|A$ stationary.
4. Let $b_0 \models p$.
5. By superstablity, there is finite $A \leq M$ with $b_0 \downarrow_A M$.
6. Take any $b_1 \models p$ with $b_1 \downarrow_A M$.
7. By theorem, $\mathrm{cl}(b_i A) \perp_{\mathrm{acl}(A)} M$ and $\mathrm{cl}(b_i A) M \leq \mathcal{M}$.
8. By lemma, $\mathrm{cl}(b_i A) \perp_A \mathrm{acl}(A)$, and therefore $\mathrm{cl}(b_i A) \perp_A M$.
9. By 7 and 8, $\mathrm{tp}(b_0/M) = \mathrm{tp}(b_1/M)$, and hence $p|A$ is stationary.