Model completeness and semisimple groups
(joint with Daniel Max Hoffmann, Chieu-Minh Tran, Jinhe Ye)

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Linear algebraic groups and group schemes over $\mathbb{Z}$

- We are interested in algebraic linear groups.
- They are subgroups of $\mathrm{GL}_n(K)$ (K is a field) given by systems of polynomial equations in n^2 variables.
- For example: $\mathrm{SL}_n(K)$, $T_n(K), \mathrm{UT}_n(K), \ldots$.
- In all these examples, the defining polynomials are over $\mathbb{Z}$.
- Hence, $\mathrm{GL}_n, \mathrm{SL}_n$, $T_n, \mathrm{UT}_n, \ldots$ are actually *functors* from the category of fields (or even commutative rings with 1) to the category of groups.
- Such functors (given by polynomials over $\mathbb{Z}$) are called (affine) group schemes over $\mathbb{Z}$.

Our problem (Daniel Max Hoffmann, P.K., Chieu-Minh Tran, Jinhe Ye)

- We say that a structure M is model complete, if $\mathrm{Th}(M)$ is model complete (no parameters).
- Let G be a group scheme over $\mathbb{Z}$, so G is a functor

$$G : \text{Fields} \longrightarrow \text{Groups}$$

  given by polynomials over $\mathbb{Z}$ in n^2 variables as above.
- Let K be a model complete field, e.g.:
  - $\mathbb{Q}_p, \mathbb{R}, \mathbb{C}$;
  - curve excluding fields (Johnson-Ye);
  - perfect PAC e-free fields with e-free field of absolute numbers (Jarden-Wheeler);
  - the fraction field of Witt vectors over a model complete field of positive characteristic (pointed out by F. Jahnke).

Problem

When the group $G(K)$ is model complete?

Multiplicative group is bad

Fact

The group $\mathbb{Q}_p^*$ is not model complete ($p > 2$).

Proof.

There is no primitive p-th root of unity in $\mathbb{Q}_p$ ($p > 2$), so:

$$\mathbb{Q}_p^* \cong \left(\mathbb{Q}_p^*\right)^p < \mathbb{Q}_p^*.$$

However, we also have:

$$\left(\mathbb{Q}_p^*\right)^p \not\prec \mathbb{Q}_p^*,$$

since $p^p \in \left(\mathbb{Q}_p^*\right)^p$ but $p \notin \left(\mathbb{Q}_p^*\right)^p$ ($v_p(p) = 1 \notin p\mathbb{Z}$). □

- Minh asked Lou one day:
  Is the group $\mathrm{SL}_2(\mathbb{R})$ *model complete?*
  It turned out to be unknown.
- Afterwards Daniel, Minh, and Vincent worked on the model completeness of $\mathrm{SL}_2(\mathbb{R})$.
- They used root systems and the KAN-decomposition.

Remark

The field $\mathbb{R}$ is model complete and $\mathbb{R}$ is bi-interpretable with the group $\mathrm{SL}_2(\mathbb{R})$ (Maltsev, Simon Thomas),
but it does *not* formally imply that $\mathrm{SL}_2(\mathbb{R})$ is model complete because of:

1. parameters,
2. possible quantifiers in the interpretations,
3. *any* theory is bi-interpretable with its Morleyization which has QE, so...

Borel-Tits Theorem

- Daniel gave a talk about it at Sirince, Turkey (May 2023, 22nd and last *Antalya Algebra Days*).
- During this talk, I realized that the following classical theorem may be useful.

Theorem (Borel-Tits)

Let H, G be simple algebraic groups defined over fields L, M respectively. Assume that H is simply connected or G is adjoint. Let $\alpha : H(L) \to G(M)$ be a group homomorphism with Zariski dense image (in $G(M^{\mathrm{alg}})$). Then there exist:

- *a field homomorphism $\varphi : L \to M$ giving $\varphi_H : H(L) \to H(M)$,*
- *an isogeny $\beta : {}^{\varphi}H \to G$ giving $\beta_M : H(M) \to G(M)$*

such that $\alpha = \beta_M \circ \varphi_H$.

It is quite an amazing rigidity statement. The terms in red will be explained on the next slide.

Definitions

- By a simple algebraic group over a field K, we mean an algebraic group G over K such that any proper normal subgroup of $G(K^{\mathrm{alg}})$ is finite.
- By a semisimple algebraic group over a field K, we mean an algebraic group G over K such that any normal commutative subgroup of $G(K^{\mathrm{alg}})$ is finite.
- An isogeny is an algebraic group epimorphism with a finite kernel.
- A semisimple algebraic group G is simply connected if for any isogeny $\alpha : \widetilde{G} \to G$, where $\widetilde{G}$ is a connected algebraic group, the map $\alpha_{K^{\mathrm{alg}}} : \widetilde{G}(K^{\mathrm{alg}}) \to G(K^{\mathrm{alg}})$ is one-to-one.
- A semisimple algebraic group G is adjoint if $Z(G)$ is trivial.

Fact

If G is semisimple, then there is a semisimple simply connected algebraic group G_{sc} and an isogeny $G_{\mathrm{sc}} \to G$ ("universal cover").

Examples/Remarks I

- Fix $n \geqslant 2$.
- SL_n is simple and simply connected.
- The quotient morphism of group schemes

$$\pi : \mathrm{SL}_n \to \mathrm{PGL}_n = \mathrm{SL}_n / Z(\mathrm{SL}_n)$$

  is an isogeny (an epimorphism with finite kernel).
- PGL_n is simple and adjoint.
- If G is over $\mathbb{C}$ then:
  G is simply connected iff $\pi_1(G(\mathbb{C})) = 0$.
- Note that

$$\pi_1(\mathrm{SL}_2(\mathbb{C})) = 0, \quad \pi_1(\mathrm{SL}_2(\mathbb{R})) = \mathbb{Z}.$$

Examples/Remarks II

- The situation gets complicated on the level of rational points. Let K be a field.
- The corresponding group homomorphism

$$\pi_K : \mathrm{SL}_n(K) \to \mathrm{PGL}_n(K)$$

is usually not onto. Its image is $\mathrm{PSL}_n(K)$ and we have:

$$\mathrm{PGL}_n(K)/\mathrm{PSL}_n(K) \cong K^*/(K^*)^n.$$

- $\mathrm{PSL}_n(K)$ is the commutator group of $\mathrm{PGL}_n(K)$.
- The PSL_n functor is *not* an algebraic group, but it is a Chevalley group (functor).
- Usually, $\mathrm{SL}_n(K), \mathrm{PGL}_n(K)$ are not simple groups, but $\mathrm{PSL}_n(K)$ is simple.

Chevalley-Demazure group schemes

- Let $G_{\mathbb{C}}$ be a connected complex semisimple group.
- Chevalley showed that there is a uniquely determined affine group scheme G over $\mathbb{Z}$ such that:
  1. $G(\mathbb{C}) \cong G_{\mathbb{C}}$,
  2. For any algebraically closed field K, $G(K)$ is a connected semisimple algebraic group "defined and split over the prime field of K". (In short: "best possible".)
- We call G the Chevalley-Demazure group scheme.
- We say that G is simple (simply connected, etc.) if the algebraic group $G(\mathbb{C})$ is so.
- They should *not* be confused with the Chevalley groups (as in the previous slide)!

Chevalley-Demazure simple group schemes

We will be interested in Chevalley-Demazure simple group schemes which are classified below.

Φ	$\Lambda(\Phi)$	G_{sc}	G_{ad}	in between
$A_{n-1},\ n \geq 2$	Z_n	SL_n	PGL_n	$\mathrm{SL}_n/Z_d\ (d\|n)$
$B_n,\ n \geq 2$	Z_2	Spin_{2n+1}	SO_{2n+1}	–
$C_n,\ n \geq 2$	Z_2	Sp_{2n}	PCSp_{2n}	–
$D_n,\ n \geq 3$ odd	Z_4	Spin_{2n}	PCO°_{2n}	SO_{2n}
$D_n,\ n \geq 4$ even	$Z_2 \times Z_2$	Spin_{2n}	PCO°_{2n}	$\mathrm{SO}_{2n}, \mathrm{HSpin}_{2n}$
G_2	1	G_2		–
F_4	1	F_4		–
E_6	Z_3	$(E_6)_{sc}$	$(E_6)_{ad}$	–
E_7	Z_2	$(E_7)_{sc}$	$(E_7)_{ad}$	–
E_8	1	E_8		–

Segal-Tent and Simon Thomas

Theorem (Segal-Tent, Simon Thomas)

Let G be a Chevalley-Demazure simple and simply connected group scheme and K be a field. Then we have the following.

1. *If N is a group and $N \equiv G(K)$, then there is a field M such that $N \cong G(M)$.*
2. *If M is a field such that $G(K) \equiv G(M)$, then $K \equiv M$.*

- This result (more generally, its bi-interpretability version) was shown by Segal-Tent in the case of G of rank at least 2 (so, excluding SL_2).
- In his PhD thesis, Simon Thomas showed it for SL_2 as well (but he worked in the context of Chevalley groups...).

Consequence of Borel-Tits

There is a density assumption in Borel-Tits, which is of course necessary, since the existence of an isogeny $H \to G$ implies $\dim(H) = \dim(G)$.

However, we still have the following.

Corollary of Borel-Tits

Suppose that K, L are infinite, G is simple algebraic and $f : G(K) \to G(L)$ is a monomorphism. Then the image of f is Zariski dense.

Proof.

- We use Borel-Tits for f composed with the "universal cover" map $G_{\mathrm{sc}} \to G$.
- If the image of f is not Zariski dense, then there is an isogeny from G_{sc} to a quotient of a proper algebraic subgroup of G, which contradicts the dimension equality above.

□

Theorem

Let G be a simply connected simple Chevalley-Demazure group scheme (e.g. $G = \mathrm{SL}_2$) and K be a model complete field. Then $G(K)$ is model complete.

- For the proof, let $H \equiv G(K) \equiv N$ and $f : H \to N$ be a monomorphism. We need to show that f is elementary.
- By Segal-Tent, there are fields L, M such that:

$$H \cong G(L), \quad N \cong G(M), \quad L \equiv K \equiv M.$$

  Therefore, we can assume that $f : G(L) \to G(M)$.
- By Borel-Tits (and Corollary), there is a field homomorphism $\varphi : L \to M$ (necessarily elementary!) and an isogeny $\beta : G \to G$ such that $f = \beta_M \circ \varphi_G$.

Proof for simple simply connected II

- Since G is simply connected, the map
$$\beta_{M^{\mathrm{alg}}} : G(M^{\mathrm{alg}}) \to G(M^{\mathrm{alg}})$$
is one-to-one.
- Hence, there is an isogeny $\beta' : G \to {}^{\mathrm{Fr}^i}G$ such that $\beta' \circ \beta = \mathrm{Fr}^i_G$.
- Since M is model complete, M is perfect, so
$$\left(\mathrm{Fr}^i_G\right)_M : G(M) \cong {}^{\mathrm{Fr}^i}G(M).$$
- Hence, β_M is an isomorphism as well.
- Since $f = \beta_M \circ \varphi_G$, we can assume that $f = \varphi_G$ which is elementary, since $\varphi : L \to M$ is elementary.

Proof for products of simple simply connected I

Lemma 1 (Feferman-Vaught)

If $M_1 \preccurlyeq N_1, M_2 \preccurlyeq N_2$ then $M_1 \times M_2 \preccurlyeq N_1 \times N_2$.

Lemma 2 (Ziegler)

Suppose that $M \equiv M_1 \times M_2$ and M is special (kind of saturated, they exist). Then there are L-structures N_1, N_2 such that

$$M_1 \equiv N_1, \quad M_2 \equiv N_2, \quad M \cong N_1 \times N_2.$$

- For proof of Lemma 2, take $T := \mathrm{Th}(M_1 \times M_2)$ and a new language L', which is L expanded by two extra binary relations.
- Let T' be an L'-theory whose models are models of T which are products of models of the theories of M_1 and M_2.
- By Casanovas' theorem (special models are expandable), M can be expanded to an L'-structure which is a model of T', so we obtain N_1, N_2 as in the statement.

Proof for products of simple simply connected II

Theorem

Suppose that:

1. *The structures* $M_1, \ldots, M_n$ *are model complete.*
2. *For any* $M_1' \equiv M_1 \equiv M_1'', \ldots, M_n' \equiv M_n \equiv M_n''$ *and any*

$$\Psi : M_1' \times \ldots \times M_n' \hookrightarrow M_1'' \times \ldots \times M_n'',$$

there is $\sigma \in \mathrm{Sym}(n)$ *and* $\Psi_i : M_i' \hookrightarrow M_{\sigma(i)}''$ *such that*

$$\Psi = \mathrm{coordinate}\left(\sigma^{-1}\right) \circ (\Psi_1 \times \ldots \times \Psi_n)$$

and for each i*, we have* $M_i' \equiv M_{\sigma(i)}''$.

Then, $M_1 \times \ldots \times M_n$ *is model complete.*

Lemmas 1. and 2. give a proof, since model completeness can be checked on special models.

Proof for products of simple simply connected III

Theorem

The assumptions of the previous theorem hold in the following cases.

1. $M_i := G_i(K_i)$, *where all* G_i*'s are simple and simply connected.*
2. $M_i := G_i(K_i)$, *where all* G_i*'s are simple and adjoint.*
3. $M_i := G_i(K_i)'$, *where all* G_i*'s are simple and adjoint.*

Corollary

If $G_1, \ldots, G_n$ are simple and simply connected Chevalley-Demazure group schemes and K is model complete, then $G_1(K) \times \ldots \times G_n(K)$ is model complete.

Example (thanks to Martin Hils!)

$\mathbb{Z}/2\mathbb{Z}$ and C_{2^∞} are model complete, but $\mathbb{Z}/2\mathbb{Z} \times C_{2^\infty}$ is not.

Quotients

- Let G be a semisimple algebraic group.
- As stated before, there exists a semisimple simply connected algebraic group G_{sc} and an isogeny $G_{\mathrm{sc}} \to G$.
- Similarly, $G/Z(G) =: G_{\mathrm{ad}}$ is adjoint and the quotient morphism is an isogeny.
- We would like to show that $G(K)$ is model complete for a model complete K.
- There are two problems:
  1. How model completeness behaves with respect to finite quotients/extensions (finite extensions of model complete groups need not be model complete!).
  2. On the level of rational points, they are not really quotients!

Model completeness of Chevalley groups

- Let G be a semisimple Chevalley-Demazure group scheme.
- Then G is a finite central quotient of a finite product of simple and simply connected Chevalley-Demazure group schemes.

Theorem

If K is a model complete field, then $G(K)'$ (the commutator group) is model complete.

- As mentioned above, $G(K)'$ is the corresponding Chevalley group.
- A sketch of the proof is on the next three slides.

Proof for Chevalley I: set-up and coding

- We have the "universal cover" isogeny $\pi : G_{\mathrm{sc}} \to G$ and

$$G(K)' = \mathrm{im}(\pi_K : G_{\mathrm{sc}}(K) \to G(K)).$$

  Let $N(K) := \ker(\pi_K)$ (finite and central).
- We also have $G_{\mathrm{sc}} \cong S_1 \times \ldots \times S_n$, where $S_1, \ldots, S_n$ are simple and simply connected. Assume for simplicity that $n = 2$.
- We code the above in a first-order way using a bigger language L'_c: the language of groups together with a function symbol $c : G(K)' \times G(K)' \to N(K)$ and two binary relation symbols.
- The theory T'_c says that c is a 2-cocycle, the group given by c is isomorphic to $H_1 \times H_2$, and

$$H_1 \equiv S_1(K), \quad H_2 \equiv S_2(K).$$

Proof for Chevalley II: expanding

- Take $G_1 \equiv G(K)' \equiv G_2$ and a monomorphism $f : G_1 \to G_2$.
- We can assume that G_1, G_2 are special, so, by Casanovas again, they expand to models of T_c'.
- By Segal-Tent, we have that:

$$G_1 \cong (S_1(L_1) \times S_2(L_2)) / N(K), \; G_2 \cong (S_1(M_1) \times S_2(M_2)) / N(K)$$

and $L_1 \equiv L_2 \equiv K \equiv M_1 \equiv M_2$. Assume for simplicity that $L_1 = L_2 = M = M_1 = M_2$, so $G_1 = G(M)' = G_2$.

- It can be shown that f maps center onto center, so it induces a monomorphism:

$$f_{\mathrm{ad}} : G_{\mathrm{ad}}(M)' = (S_1)_{\mathrm{ad}}(M)' \times (S_2)_{\mathrm{ad}}(M)' \to G_{\mathrm{ad}}(M)'.$$

- Such monomorphisms are understood (a theorem few slides ago), so we can assume that $f_{\mathrm{ad}} = f_1 \times f_2$.

Proof for Chevalley III: lifting and conclusion

Each $f_i : (S_i)_{\mathrm{ad}}(M)' \to (S_i)_{\mathrm{ad}}(M)'$ has a description as in Borel-Tits, hence f_i and f_{ad} lift to

$$\widetilde{f_i} : S_i(M) \to S_i(M), \quad f_{\mathrm{sc}} := \widetilde{f_1} \times \widetilde{f_2} : G_{\mathrm{sc}}(M) \to G_{\mathrm{sc}}(M).$$

By the result about products, f_{sc} is elementary.
We have:

$$\begin{array}{ccc}
G_{\mathrm{sc}}(M) & \xrightarrow{\ f_{\mathrm{sc}}\ } & G_{\mathrm{sc}}(M) \\
\downarrow & & \downarrow \\
G(M)' & \xrightarrow{\ f\ } & G(M)' \\
\downarrow & & \downarrow \\
G_{\mathrm{ad}}(M)' & \xrightarrow{\ f_{\mathrm{ad}}\ } & G_{\mathrm{ad}}(M)'
\end{array}$$

The upper square *need not* commute, but after little corrections, f_{sc} factors to $f' : G(M)' \to G(M)'$ which is elementary.
We can show that $\mathrm{im}(f) = \mathrm{im}(f')$, so f is elementary as well.

General semisimple case (in progress)

- We take G and K as before and consider $G(K)$ (instead of $G(K)'$).
- We take one more language L''_c, which is L'_c expanded by the constant symbols living in $S_i(K)$ needed for the Segal-Tent bi-interpretations between each $S_i(K)$ and K ($i = 1, 2$).
- There is an L''_c-theory T''_c saying that

$$G(K) = \bigcup_{L/K} G(K) \cap G(L)',$$

where L/K is a Kummer extension such that

$$[L : K] \leqslant 6 \cdot \mathrm{rank}(G),$$

so these extensions form a definable family!
- The previous proof needs to be adjusted to take into account these definable families. Work in progress...

Further directions

- The case of $G = \mathrm{UT}_3$ (Heisenberg) is also done. Maybe the arguments work for a large class/any unipotent group.
- There are some very conjectural algebraic criteria on a group scheme G over $\mathbb{Z}$ which should be equivalent to:
  "for each model complete field K, the group $G(K)$ is model complete".
- We conjecture that if G is *any* group definable in an algebraically closed field K, then G is model complete.